Skill Check Answers

1. Players = 3; Quota = 6; Total number of votes = 9 **2.** Yes, it is a winning coalition.
3. Yes, player 4 is critical **4.** Player 1 is the pivotal player **5.** $P_1 = P_2 = P_3 = P_4 = 25\%$

13.4 Exercises

✔ CONCEPT CHECK

1. A group of players joining together to vote the same way is called a _____.

2. A _____ is a player who causes a winning coalition to become a losing coalition if they were to leave.

3. A player with the power to form a winning coalition by themself is called a _____.

4. The total number of sequential coalitions is found from the following formula: _____.

PRACTICE

5. Five partners start a business, with each owning the following number of shares: P_1 owns 9 shares, P_2 owns 6 shares, P_3 owns 5 shares, P_4 owns 3 shares, and P_5 owns 2 shares. If one share is equal to one vote and they use a simple majority quota, represent this weighted voting system.

6. Using the information in Exercise 5, if the quota is a two-thirds majority, represent the weighted voting system.

7. Consider the weighted voting system $[25: 10, 7, 4, 4, 2, 2, 1, 1, 1]$.

- **a.** What is the quota for this voting system?
- **b.** What is the weight of P_5?
- **c.** If only the last six voters vote for a motion, does the motion pass?
- **d.** If P_1 and P_2 vote against a motion, does the motion pass?

8. Consider the weighted voting system $[q: 10, 7, 4, 4, 2, 2, 1, 1, 1]$.

- **a.** What is the smallest quota that requires a majority for this voting system?
- **b.** What is the largest quota for this voting system?

9. In the weighted voting system $[q: 20, 19, 15, 8, 4, 2]$, if a two-thirds majority of votes is needed, what is q?

10. Which players in the voting system $[20: 7, 7, 3, 3, 2]$ have veto power?

11. In the weighted voting system $[q: 18, 15, 12, 9, 6, 3]$, what is the largest value of q so that no voter has veto power?

12. In the weighted voting system $[q: 10, 7, 4, 4, 2, 2]$, if every voter has veto power, what is q?

13. Consider the weighted voting system $[42: 22, 16, 10, 6, 4]$. Does this voting system contain a dictator? Does this system contain a dummy player?

14. Consider the weighted voting system [20: 12, 10, 3]. Does this voting system contain a dictator? Does this system contain a dummy player?

15. Consider the weighted voting system $[42: 20, 16, 10, 6, 4]$.

a. How many voters are in the system?

b. What is the quota for the system?

c. Is the coalition $\{P_2, P_3, P_4, P_5\}$ a winning coalition?

d. What are all of the winning coalitions?

e. Are there any critical players?

f. Do any of the players have veto power?

16. Consider the weighted voting system $[42: 20, 16, 10, 6, 4]$. Determine the Banzhaf Power Index for each player.

Consider the weighted voting system [30: 12, 10, 9, 5, 4, 4]. Determine the pivotal player in each of the sequential coalitions listed.

17. $\langle P_1, P_3, P_2, P_6 \rangle$

18. $\langle P_4, P_2, P_3, P_6, P_5, P_1 \rangle$

19. Consider the weighted voting system $[7: 4, 2, 2, 2, 2]$. Find the Banzhaf Power Index for each voter. Are any of the voters dictators? Do any of the voters have veto power?

20. Find the Banzhaf Power Index for each player in the voting system [10: 7, 5, 3, 2].

21. Consider the weighted voting system, $[100: 80, 60, 30, 20]$. Calculate the Shapley-Shubik Power Index for each player in this system. Are any of the voters dictators? Do any of the voters have veto power?

APPLICATIONS

22. The United Nations Security Council consists of fifteen members of the United Nations; five permanent member countries and ten nonpermanent member countries. In order for a vote to pass, nine members, including all five of the permanent members, must be in agreement. An equivalent weighted voting system is $[39: 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. According to our previous calculations, this means there are 15! or about 1.3 trillion permutations for the members.[1]

a. Without computing all possible coalitions, determine when a nonpermanent member would be a pivotal voter.

b. What is the Shapley-Shubik Power Index for each nonpermanent member?

c. What is the Shapley-Shubik Power Index for the combined nonpermanent members?

d. What is the Shapley-Shubik Power Index for each permanent member?

e. What is the Shapley-Shubik Power Index for the combined permanent members?

23. There are 435 members of the United States House of Representatives.

a. If a simple majority is needed to pass a bill, what constitutes a winning coalition?

b. If a two-thirds majority is needed to ratify an amendment, what constitutes a winning coalition?

24. In the weighted voting system $[43: 25, 20, 19, 17]$, determine each of the following.

a. List all permutations in which P_1 is a pivotal player.

b. List all permutations in which P_2 is a pivotal player.

c. Calculate the Shapley-Shubik Power Index for each player.

d. Calculate the Banzhaf Power Index for each player.

e. Is there a player with veto power?

25. If the quota in Exercise 23 is increased to 50, how would the Shapley-Shubik Power Index change?

1 United Nations, http://www.un.org

26. A company has five shareholders and a total of 500 shares. The quota for passing a measure is the number of votes where shareholders own 251 or more shares. The number of shares owned by each shareholder is as follows: $S_1 = 200$, $S_2 = 123$, $S_3 = 120$, $S_4 = 40$, and $S_5 = 17$. Suppose there is an investor S6 that wants to buy shares and currently owns no shares:

a. What are the winning and losing coalitions? Compute the number of votes needed to make a losing coalition a winning coalition.

b. How many shares can S_1 sell to S_2 without causing any of the winning coalitions listed in part **a.** to lose or any of the losing coalitions in part **a.** to win?

c. How many shares can S_1 sell to S_5 without causing any of the winning coalitions listed in part **a.** to lose or any of the losing coalitions in part **a.** to win?

d. How many shares can S_1 sell to S_6 without causing any of the winning coalitions listed in part **a.** to lose or any of the losing coalitions in part **a.** to win?

13.4 PROJECT

POWER DYNAMICS AT PLAY

In Section 13.4, you learned about weighted voting systems and the power that each player in the system has. In this project, you'll investigate power dynamics. Power dynamics are where weighted voting systems get interesting. Players may have opposing viewpoints that motivate them to desire different outcomes. Each player wants to win and wants to know the likelihood of winning.

Consider a small company with a small number of shareholders who disagree about the direction the company should take. Each member is likely to be acutely aware of how much their vote counts and with whom they need to align to be part of a winning coalition. Let's look at a few power dynamics at play.

1. Consider a scenario where player 1 is a dictator and is interested in selling some shares to another player, but wants to remain a dictator after the sale. Is this possible in the voting system [20: 25, 5, 3, 1]? Explain why this is or isn't possible. If it is possible, how many shares can player 1 sell?

2. Consider a scenario where player 1 and player 4 rarely vote in the same manner, while player 2 and player 4 often vote the same. In the voting system [14: 12, 7, 3, 2], why would player 4 be glad to see player 1 sell two shares to player 2? (**Hint:** Consider the possible winning coalitions in each voting system.)

3. Using the same scenario as part 2, use the Banzhaf Power Index to describe in words how player 4's power changes if player 1 sells two shares to player 2.