

2.3 PROJECT

EXPLORING INTERVALS: INTERSECTIONS AND UNIONS

In Section 2.3, you learned about the intersection and union of sets. In this activity, you will investigate intersections involving sets that cannot be described using roster notation.

Consider the set I_1 of all real numbers that are greater than or equal to 0 and less than or equal to 1. Using set-builder notation, we have $I_1 = \{x \mid 0 \leq x \leq 1\}$, which we can also represent using interval notation as $I_1 = [0, 1]$ or graphically as shown in Figure 1.

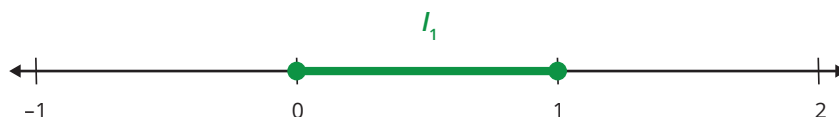


Figure 1

Now, for each natural number n , that is $n = 1, 2, 3, 4, \dots$, we can think of the interval $I_n = \left[0, \frac{1}{n}\right]$.

For example, $I_3 = \left[0, \frac{1}{3}\right]$ is the set of all real numbers greater than or equal to zero and less than or equal to $\frac{1}{3}$.

1. Determine $I_1 \cap I_2$.
2. Determine $I = I_1 \cap I_2 \cap I_3$.
3. Determine $I = I_1 \cap I_2 \cap I_3 \cap I_4$.

This sequence of intervals is called *nested intervals*, where each set in the sequence is contained within the previous one. This means that the intersection of the nested intervals is equal to the smallest interval.

Let's see what happens if we keep going with taking intersection forever. Let's consider

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cdots,$$

which is the intersection of all such intervals. If a positive number x is in I , then it has to be in every one of the intervals. Let's see if this is possible.

4. Find an interval of the form $\left[0, \frac{1}{n}\right]$ that does not contain the number 0.01.
5. Find an interval of the form $\left[0, \frac{1}{n}\right]$ that does not contain the number 0.0001.

No matter how small of a number you pick, there is always an interval in our list of nested intervals that does not contain the number you picked. We can conclude that no positive number can be in the intersection $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cdots$.

6. In fact, there is exactly one number in the intersection $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cdots$. What is the number in this intersection? Explain your reasoning.
7. What is the union $J = I_1 \cup I_2 \cup I_3 \cup I_4 \cdots$ equivalent to?