9.8 EXERCISES

PRACTICE

Evaluate each of the following expressions. Round your answers to two decimal places if necessary. See Example 1.

1. $\sinh 0$

2. $\tanh 0$

3. $\operatorname{sech} 0$

4. $\sinh(\ln 2)$

5. $\tanh(\ln 2)$

6. $\cosh(-2)$

7. $\coth(-2)$

8. $\operatorname{csch}(-2)$

9. $\cosh 5$

Use each of the given equations to classify the hyperbolic function as even or odd. Then use the definition of the function to prove your assertion.

10. $\sinh(-x) = -\sinh x$

11. $\cosh(-x) = \cosh x$

12. $\tanh(-x) = -\tanh x$

13. $\coth(-x) = -\coth x$

14. $\operatorname{sech}(-x) = \operatorname{sech} x$

15. $\operatorname{csch}(-x) = -\operatorname{csch} x$

Verify each of the following identities. See Example 2.

16. $e^{kx} = \cosh(kx) + \sinh(kx)$

17. $e^{-kx} = \cosh(kx) - \sinh(kx)$

18. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

19. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

20. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

21. $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

22. $\sinh(2x) = 2\sinh x \cosh x$

23. $\cosh(2x) = \cosh^2 x + \sinh^2 x$

24. $\cosh^2 x = \dfrac{\cosh(2x)+1}{2}$

25. $\coth^2 x = 1 + \operatorname{csch}^2 x$

26. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

27. $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

28. $\tanh(2x) = \dfrac{2\tanh x}{1 + \tanh^2 x}$

29. $\coth(2x) = \dfrac{1+\coth^2 x}{2\coth x}$

30. $\sinh\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{\cosh x - 1}{2}}$

31. $\cosh\left(\dfrac{x}{2}\right) = \sqrt{\dfrac{\cosh x + 1}{2}}$

32. $\tanh\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}}$

33. $(\cosh x + \sinh x)^2 = \cosh(2x) + \sinh(2x)$

34. $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx), \quad n \in \mathbb{N}$

35. $(\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx), \quad n \in \mathbb{N}$

Evaluate each of the following expressions. See Example 3.

36. $\cosh^{-1} 1$

37. $\tanh^{-1} 0$

38. $\operatorname{sech}^{-1} 1$

WRITING & THINKING

Given that the hyperbolic functions can be expressed in terms of exponential functions, it's not surprising that their inverses can be expressed in terms of logarithms. For example, if we let $y = \sinh^{-1} x$, then $x = \sinh y$ and we have the following.

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$e^y - 2x - e^{-y} = 0$$

$$e^{2y} - 2xe^y - 1 = 0 \qquad \text{Multiply through by } e^y.$$

$$\left(e^y\right)^2 - 2xe^y - 1 = 0 \qquad \text{Express as a quadratic in } e^y.$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \qquad \text{Solve for } e^y.$$

$$e^y = x + \sqrt{x^2 + 1} \qquad x - \sqrt{x^2+1} < 0 \text{ but } e^y > 0, \text{ so discard } x - \sqrt{x^2+1}.$$

$$y = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \text{Take the natural logarithm of both sides.}$$

Use this procedure to verify each of the following identities.

39. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1$

40. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad -1 < x < 1$

(**Hint:** Begin by setting $y = \tanh^{-1} x$; then write the equation as $\tanh y = x$, square both sides, and apply the identity $\tanh^2 y = 1 - \operatorname{sech}^2 y$. Solve the result for $\cosh y$, apply $\cosh^{-1}$ to both sides, and apply the result of the previous exercise. Then apply some logarithmic properties.)

Given a function f, let $\frac{1}{f}$ denote its reciprocal—that is, $\left(\frac{1}{f}\right)(x)=\frac{1}{f(x)}$. The following is a useful relationship between the functions f^{-1} and $\left(\frac{1}{f}\right)^{-1}$, assuming both of these inverse functions exist.

$$\left(\frac{1}{f}\right)\left(f^{-1}\left(\frac{1}{x}\right)\right)=\frac{1}{f\left(f^{-1}\left(\frac{1}{x}\right)\right)}=\frac{1}{\frac{1}{x}}=x$$

$$\text{so}\quad \left(\frac{1}{f}\right)^{-1}(x)=f^{-1}\left(\frac{1}{x}\right).$$

Applied to hyperbolic functions, this fact indicates the following.

$$\operatorname{csch}^{-1}x=\sinh^{-1}\frac{1}{x},\qquad \operatorname{sech}^{-1}x=\cosh^{-1}\frac{1}{x},\qquad \coth^{-1}x=\tanh^{-1}\frac{1}{x}$$

Use these relationships to verify each of the following identities.

41. $\operatorname{csch}^{-1}x=\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right),\quad x\neq 0$

42. $\operatorname{sech}^{-1}x=\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\quad 0<x\leq 1$

43. $\coth^{-1}x=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\quad |x|>1$

Evaluate each of the following expressions using the formulas from Exercises 39–43. Round your answers to two decimal places.

44. $\sinh^{-1}2$

45. $\operatorname{csch}^{-1}(-3)$

46. $\cosh^{-1}5$

47. $\operatorname{sech}^{-1}(0.8)$

48. $\tanh^{-1}(-0.3)$

49. $\coth^{-1}3$