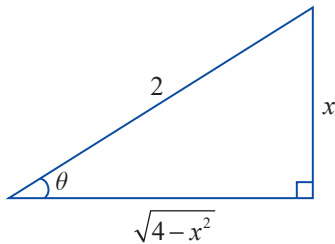


Example 6: Using Trigonometric Substitutions

Use the substitution $\sin \theta = \frac{x}{2}$ to write $\sqrt{4-x^2}$ as a trigonometric expression. Assume $0 \leq \theta \leq \frac{\pi}{2}$.

**FIGURE 3****Solution**

Although it is not necessary for the task at hand, a diagram motivating the substitution may be helpful. The triangle in Figure 3 illustrates the geometric relation between θ and the various algebraic expressions.

The suggested substitution can be rewritten as $x = 2 \sin \theta$, and so we obtain the following.

$$\begin{aligned}\sqrt{4-x^2} &= \sqrt{4-(2\sin\theta)^2} \\ &= \sqrt{4-4\sin^2\theta} \\ &= 2\sqrt{1-\sin^2\theta} \\ &= 2\sqrt{\cos^2\theta} \\ &= 2\cos\theta\end{aligned}$$

We can write $2\cos\theta$ instead of $2|\cos\theta|$ since the restriction $0 \leq \theta \leq \frac{\pi}{2}$ means $\cos\theta \geq 0$.

8.1 EXERCISES**PRACTICE**

Use trigonometric identities to simplify the expressions. There may be more than one correct answer. See Examples 1 and 2.

- $\tan x \csc x$
- $\frac{1}{\tan^2 \theta + 1}$
- $\frac{\tan t}{\sec t}$
- $\cot^2 x - \cot^2 x \cos^2 x$
- $\sin(-x) \tan x$
- $\frac{1}{\sec^2 x} + \sin x \cos\left(\frac{\pi}{2} - x\right)$
- $\sin(\alpha + 2\pi) \sec \alpha$
- $\sin t (\csc t - \sin t)$
- $\cos y (1 + \tan^2 y)$
- $\frac{1}{\cos x \csc(-x)}$
- $\frac{1 - \tan^2 x}{\cot^2 x - 1}$
- $\frac{\sin \beta \tan\left(\frac{\pi}{2} - \beta\right)}{\cos \beta}$

Use the suggested substitution to rewrite the given expression as a trigonometric expression.

See Example 6. Assume $0 \leq \theta \leq \frac{\pi}{2}$.

13. $\sqrt{x^2 + 1}$, $x = \tan \theta$

14. $\sqrt{x^2 - 16}$, $x = 4 \sec \theta$

15. $\sqrt{9 - x^2}$, $\cos \theta = \frac{x}{3}$

16. $\sqrt{4x^2 + 100}$, $\cot \theta = \frac{x}{5}$

17. $\sqrt{64 - x^2}$, $x = 8 \sin \theta$

18. $\sqrt{x^2 - 4}$, $x = 2 \csc \theta$

19. $\sqrt{x^2 + 25}$, $\tan \theta = \frac{x}{5}$

20. $\sqrt{144 - 9x^2}$, $x = 4 \cos \theta$

WRITING & THINKING

Verify the following trigonometric identities. See Examples 3, 4, and 5.

21. $(1 - \cos \theta)(1 + \cos \theta) = \sin^2 \theta$

22. $\csc x - \sin x = \cos x \cot x$

23. $\sec^2 y - \tan^2 y = \sec y \cos(-y)$

24. $(1 - \sin \beta)(\sec \beta \tan \beta) = \frac{\sin \beta}{1 + \sin \beta}$

25. $\frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \cot x$

26. $\frac{\sec^2 \theta}{\tan \theta} = \sec \theta \csc \theta$

27. $\frac{1}{\tan x} + \tan x = \frac{\sec^2 x}{\tan x}$

28. $\sin^2 t + \sin^2\left(\frac{\pi}{2} - t\right) = 1$

29. $\frac{1}{\sin(\theta + 2\pi) + 1} + \frac{1}{\csc(\theta + 2\pi) + 1} = 1$

30. $3 + \cot^2 \alpha = 2 + \csc^2 \alpha$

31. $\sin^2 x - \sin^4 x = \cos^2(-x) - \cos^4(-x)$

32. $\cot\left(\frac{\pi}{2} - \beta\right) \cot \beta = 1$

33. $\frac{\cos\left(\frac{\pi}{2} - \alpha\right)}{\csc \alpha} - 1 = \sin \alpha \cot(-\alpha) \cos(-\alpha)$

Show how the identities below follow from the first Pythagorean identity.

34. $\tan^2 x + 1 = \sec^2 x$

35. $1 + \cot^2 x = \csc^2 x$