In other words, a change of sign occurs on $[0,1]$; that is, f goes from negative to positive on $[0,1]$. The statement of the Intermediate Value Theorem is precisely the fact that a continuous function cannot do this and avoid intersecting the horizontal line $y=0$ (the x-axis). More precisely, since $L=0$ we have $-4<L<6$; that is, since 0 is a real number between $f(0)=-4$ and $f(1)=6$, by the Intermediate Value Theorem there is a c in $[0,1]$ such that $f(c)=L=0$. In other words, we have

$$c^5+9c-4=0.$$

Thus we proved the existence of a root between 0 and 1 for the given equation.

NOTE

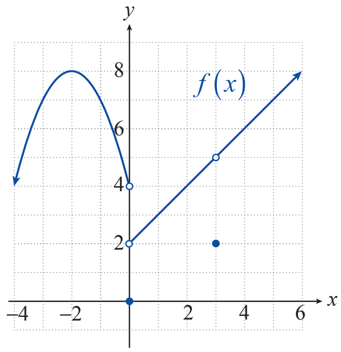
The Intermediate Value Theorem is an *existence theorem*; in other words, we proved the *existence* of a solution between 0 and 1 without actually specifying what it is.

13.5 EXERCISES

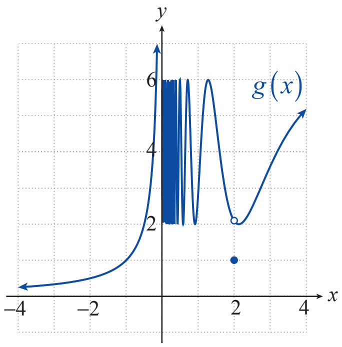
PRACTICE

Find all points of continuity as well as all points of discontinuity for the given function. For any discontinuities, identify those from the three continuity criteria that fail to hold. See Examples 1 and 2.

1.



2.



3. Sketch a graph of a function (a formula is not necessary) that has a removable discontinuity at $x=-1$, a jump discontinuity at $x=2$, but is right-continuous at $x=2$.

4. Sketch a graph of a function that has an infinite discontinuity at $x=0$ and an oscillating discontinuity at $x=5$ so that it is still left-continuous at $x=5$.

Find and classify the discontinuities (if any) of the function as removable or nonremovable. See Example 3.

5. $f(x)=\dfrac{1}{x}$

6. $g(x)=\dfrac{-2}{x-3}$

7. $h(x)=\dfrac{x^2-9}{x-3}$

8. $k(x)=\dfrac{x^2-2x}{x^2+5x-14}$

9. $u(x)=\dfrac{x^2-9}{x-2}$

10. $v(x)=\dfrac{x-1}{x^2+2x-3}$

11. $w(x)=\begin{cases} x+1 & \text{if } x\le 0 \\ \frac{1}{2}x^2+1 & \text{if } x>0 \end{cases}$

12. $f(x)=\begin{cases} \frac{1}{2}x-2 & \text{if } x\le 4 \\ x^3+1 & \text{if } x>4 \end{cases}$

13. $g(x)=\begin{cases} \tan x & \text{if } x<\frac{\pi}{2} \\ \cos x & \text{if } x\ge\frac{\pi}{2} \end{cases}$

14. $h(x)=\begin{cases} \cos x & \text{if } x\le 0 \\ \tan x+1 & \text{if } x>0 \end{cases}$

15. $F(u)=\dfrac{u-4}{\sqrt{u}-2},\quad u\ge 0$

16. $G(s)=\dfrac{s}{\sqrt{s+4}-2},\quad s\ge -4$

17. $H(t)=\dfrac{t}{\sqrt{t^2+2}}$

18. $u(x)=\cos\left(\dfrac{1-x^2}{1-x}\right)$

19. $v(t)=|\sin t|$

20. $K(x)=|x+2|+|x-1|$

21. $F(t)=\dfrac{t}{t^2-1}$

22. $G(t)=\dfrac{t}{t^2+1}$

23. $H(x)=|x+2|$

24. $k(x)=\dfrac{|x-4|}{x-4}$

25. $s(x)=[\![x+2]\!]$

26. $t(x)=4-[\![x]\!]$

27. $u(z)=[\![z^2]\!]$

28. $v(x)=x[\![x]\!]$

29. $w(x)=x\left[\!\!\left[\dfrac{1}{x}\right]\!\!\right]$

Use the ε-δ definition to prove that the function is continuous. See Example 5.

30. $f(x)=\dfrac{1}{x}$

31. $g(x)=3x-2$

32. $F(x)=x^3$

33. $G(x)=\sqrt{x}$

Use the theorems of this section to describe the continuity of the function. See Example 7.

34. $F(x)=\sqrt{\dfrac{x}{x^2+7x+12}}$

35. $G(x)=\sqrt{\dfrac{x^4-x^3-11x^2+9x+18}{2x^3+x}}$

36. $H(x)=\cos\left(\dfrac{2\ln(x-3)+1}{\sqrt[3]{x^2-2x-15}}\right)$

37. $f(x)=\arctan\left(\dfrac{x}{\sqrt{3-x^2}}\right)$

38. $g(x)=\ln(\arcsin(\pi x+1))$

39. $h(x)=\dfrac{\csc(\pi x+1)}{\sin(\pi e^{x+2})}$

Use the alternate formulation of continuity to prove that the function is continuous. See Example 8.

40. $f(x)=3x-5x^2$

41. $g(x)=\cos x$

42. $h(x)=\tan x$

43. $k(x)=e^x$

Identify the removable discontinuities and define the continuous extension of the function. See Example 10.

44. $f(x)=\dfrac{x^2+x-12}{x-3}$

45. $g(x)=\dfrac{x^3-2x^2-x+2}{x^2-3x+2}$

46. $h(x)=\dfrac{x-1}{\sqrt{x}-1}$

47. $F(x)=\dfrac{\sqrt{x+1}-2}{x-3}$

48. $G(x)=2^{-\frac{1}{x^2}}$

49. $H(x)=x\cos\left(\dfrac{\pi}{x}\right)$

Describe the continuity of the function on the given closed interval.

50. $S(x)=\sqrt{16-x^2}$ on $[-4,4]$

51. $T(x)=\left[\!\left[\dfrac{x}{3}\right]\!\right]$ on $[0,3]$

52. $U(x)=\begin{cases}\dfrac{1}{x^2-9} & \text{if } |x|<3\\ 0 & \text{if } |x|=3\end{cases}$ on $[-3,3]$

53. $V(x)=\begin{cases}x^2\sin\left(\dfrac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ on $\left[0,\dfrac{1}{\pi}\right]$

Find the value of *a* (or the values of *a* and *b*, where applicable) such that *f* is continuous on the entire real line.

54. $f(x)=\begin{cases}0 & \text{if } x\le 0\\ ax & \text{if } 0<x<1\\ 2x+3 & \text{if } x\ge 1\end{cases}$

55. $f(x)=\begin{cases}x^3 & \text{if } x\le 3\\ ax^2 & \text{if } x>3\end{cases}$

56. $f(x)=\begin{cases}-x^2 & \text{if } x<1\\ ax+b & \text{if } 1\le x\le 3\\ (x-3)^2+2 & \text{if } x>3\end{cases}$

57. $f(x)=\begin{cases}\cos x & \text{if } x\le 0\\ -(x-a)^2+b & \text{if } 0<x<2\\ \dfrac{1}{2}x-2 & \text{if } x\ge 2\end{cases}$

Decide whether the Intermediate Value Theorem applies to the given function on the indicated interval. If so, find *c* as guaranteed by the theorem. If not, find the reason. See Example 11.

58. $f(x)=-x^2+x+3$ on $[0,3]$; $f(c)=1$

59. $g(x)=2x^3-x^2-1$ on $[-1,2]$; $g(c)=0$

60. $h(x)=\dfrac{x}{x+2}$ on $[0,4]$; $h(c)=\dfrac{1}{2}$

61. $F(x)=\dfrac{2x}{x-1}$ on $[0,2]$; $F(c)=2$

62. $G(x)=[\![x-2]\!]$ on $[-2,2]$; $G(c)=-\dfrac{1}{2}$

63. $H(x)=\sin\left(\dfrac{3x+2}{2}\right)$ on $\left[-\dfrac{2}{3},\dfrac{\pi-2}{3}\right]$; $H(c)=\dfrac{1}{2}$

Use the Intermediate Value Theorem to prove that the given equation has a solution on the indicated interval. See Example 11.

64. $x^3 - 7.5x^2 + 1.2x + 1 = 0$ on $[-1, 0]$

65. $2x^3 + x + 10 = 0$ on $[-2, 1]$

66. $\cos x = x^2$ on $[0, \pi]$

67. $\ln x - \sqrt{x-2} = 0$ on $[2, 5]$

68. $\frac{5}{x^2+2} = 1$ on $[-3, -1]$

69. $\cot\left(\frac{\pi x}{4}\right) - \frac{x}{x+2} = -\frac{1}{2}$ on $[1, 3]$

APPLICATIONS

70. Suppose that the outside temperature in Columbia, SC on a summer morning at 7:00 a.m. is 74 °F, and it shoots up to 98 °F by 1:00 p.m. Assuming that temperature changes continuously, prove that sometime between 7:00 a.m. and 1:00 p.m. the temperature was exactly 88.35 °F.

71. A hermit leaves his hut at the foot of a mountain one day at 6:00 a.m. and sets out to climb all the way to the top. He arrives at 6:00 p.m. and realizes that it is too late to go back, so he sets up camp for the night. At 6:00 a.m. the following day, he starts hiking back to his hut, taking the exact same route as the day before. This time, however, it is mostly downhill, so he makes much better time and arrives home at 2:00 p.m. Prove that there is a point along the hermit's route that he passed at exactly the same time on both days. (**Hint:** Apply the Intermediate Value Theorem or the Fixed Point Theorem. See Exercise 83.)

72. A long-distance phone company charges 31 cents for the first minute and 10 cents for each additional minute or any fraction thereof. Graph the cost as a function of time, find a formula for it, and describe the significance of its discontinuities. (**Hint:** Use the greatest integer function to construct your answer.)

73. If Δt denotes the length of the time interval between two events as measured by an observer on a spaceship moving at speed v, and ΔT is the length of the same time interval as measured from Earth, then the formula relating the two quantities is given by

$$\Delta T = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where c is the speed of light. This phenomenon is called *time dilation*, and it follows from the theory of relativity. In essence, it says that a clock moving at speed v relative to an observer is perceived by the same observer to run slower.

a. Explain why we don't normally notice the time dilation effect in everyday life.

b. What is the significance of the discontinuity of ΔT (as a function of v)?

WRITING & THINKING

74. Prove the alternate formulation of continuity; that is, the statement that a function f is continuous at the point c if and only if $\lim_{h \to 0} f(c+h) = f(c)$.

75. Prove that if $f(x)$ is continuous and $f(c) > 0$, then there is a $\delta > 0$ such that $f(x) > 0$ for all $x \neq c$ in the interval $(c - \delta, c + \delta)$.

76. Prove that the Dirichlet function $\xi(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is discontinuous at every real number.

77. Prove that the function $f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x^2 & \text{if } x \text{ is irrational}\end{cases}$ is continuous only at the single point $c=0$.

78. Prove that if the functions f and g are both continuous on $\mathbb{R}$ and they agree on the rationals (i.e., $f(x)=g(x)$ for all $x \in \mathbb{Q}$), then $f=g$.

79. Prove that if f is continuous and never 0 on the interval $[a,b]$, then either $f(x)>0$ for every x in $[a,b]$, or $f(x)<0$ for every x in $[a,b]$.

80. (Existence of n^{th} roots) Prove that if b is a positive real number and n a positive integer, then there is a positive real number c such that $c^n=b$. (**Hint:** Consider the continuous function $f(x)=x^n$ on the interval $[0,b+1]$.)

81. Prove that a circle of diameter d has a chord of length c for every number c between 0 and d.

82. Use the function

$$f(x)=\begin{cases}\sin\left(\dfrac{\pi}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

to prove that the converse of the Intermediate Value Theorem is false; in other words, a function may possess the Intermediate Value Property without being continuous.

83. (Fixed Point Theorem) Prove that if the function $f:[a,b]\to[a,b]$ is continuous, then there is a number c in $[a,b]$ with $f(c)=c$ (i.e., c is "fixed," or "not being moved," by f).

Determine whether the given statement is true or false. If false, explain or provide a counterexample.

84. If f is both left- and right-continuous at c, then f is continuous at c.

85. Any function f has an interval (a,b) on which it is continuous.

86. If c is a discontinuity of f, but f does not have a vertical asymptote at c, then c is a removable or jump discontinuity.

87. If $\lim_{x\to c} f(x)=L$, and $f(c)=L$, then f is continuous at c.

88. If c is a discontinuity of f, but $\lim_{x\to c^+} f(x)$ exists, then c is a removable or jump discontinuity.

TECHNOLOGY

89–94. Use a graphing utility to solve the equations given in Exercises 64–69 to four decimal places.

95–100. Use a graphing utility to graph the functions of Exercises 34–39 and explain how the graphs support your discussions of continuity in these exercises.