If $h > 0$, since in this case $|h| = h$, the difference quotient becomes

$$\frac{|0+h|}{h} = \frac{|h|}{h} = \frac{h}{h} = 1,$$

regardless of the value of h. On the other hand, if $h < 0$, then $|h| = -h$, so

$$\frac{|0+h|}{h} = \frac{|h|}{h} = \frac{-h}{h} = -1,$$

and again, h can be any negative real number.

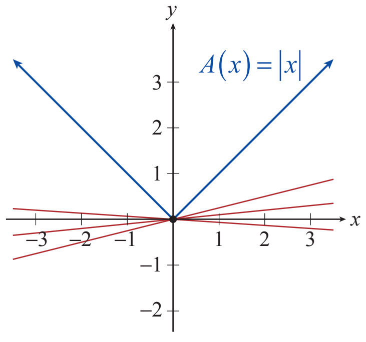


FIGURE 6

Notice that our results indicate that any secant line corresponding to a positive h-value will coincide with the right branch of the graph, while all secant lines obtained from $h < 0$ coincide with the left branch.

What does all this mean? There is no single real number being approached by the difference quotients or slopes as h is getting smaller. Graphically, this means that there is no tangent, no single line that would "best capture" the trend of the graph, none that would best align itself to the graph of $A(x)$ at $(0,0)$. The graph has a "sharp turn" there, making it impossible for a tangent line to exist, as shown in Figure 6. We also express this fact by saying that the graph is not "smooth" at $x = 0$.

13.1 EXERCISES

PRACTICE

Construct and simplify the difference quotient at c with increment h. See Example 1.

1. $f(x) = -7x + 5$

2. $g(x) = 2x + 13$

3. $p(x) = x^2 - 5x + 3$

4. $g(x) = 9x^2 - 17$

5. $f(x) = \dfrac{1}{6-x}$

6. $f(x) = \dfrac{-x}{3x+1}$

7. $g(x) = \ln x$

8. $p(x) = \dfrac{x^2-1}{x+3}$

9. $f(x) = 3\sqrt{x-2}$

Use the difference quotient of each function for one appropriate value of c to determine the average rate of change of the function over each of the given intervals. See Example 2.

10. $f(x) = -7x + 5$ **a.** $[5,7]$ **b.** $[5,5.5]$ **c.** $[4.9,5]$

11. $g(x) = 2x + 13$ **a.** $[-3,-1]$ **b.** $[-1,0]$ **c.** $[-1.0001,-1]$

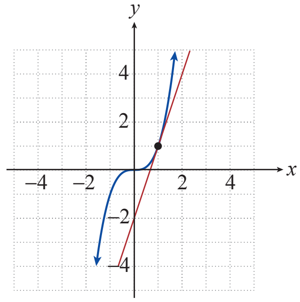
12. $p(x) = x^2 - 5x + 3$ **a.** $[1,2]$ **b.** $[0.9,1]$ **c.** $[1,1.0001]$

13. $f(x)=\dfrac{1}{6-x}$ **a.** $[2,3]$ **b.** $[3,3.1]$ **c.** $[2.999,3]$

14. $f(x)=3\sqrt{x-2}$ **a.** $[3,4]$ **b.** $[2.9,3]$ **c.** $[3,3.0001]$

Estimate the slope of the tangent line shown in the given graph.

15.



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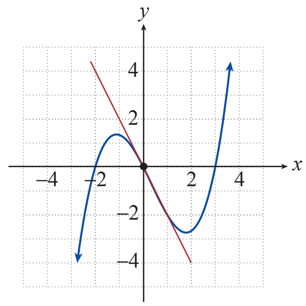

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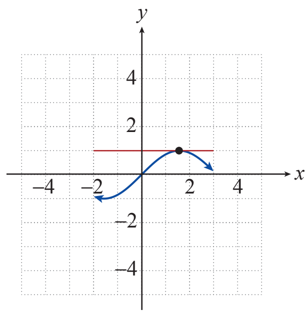

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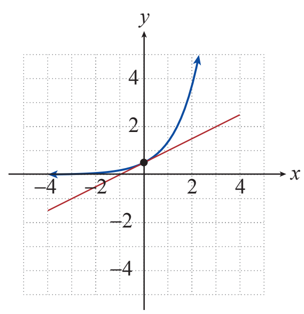

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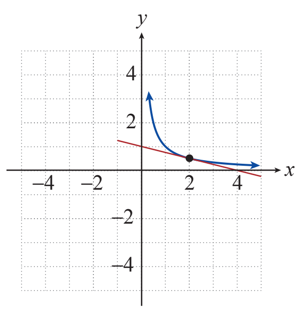

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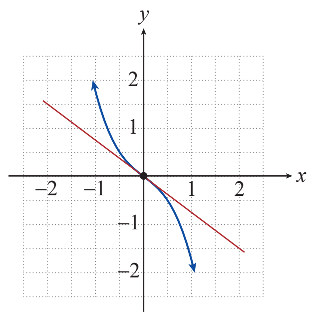


Use difference quotients to approximate the slope of the tangent to the graph of the function at the given point. Use at least five different *h*-values that are decreasing in magnitude. (Answers will vary.)

21. $f(x)=1-2x;\quad (1,-1)$

22. $g(x)=\dfrac{5}{4}x-8;\quad (8,2)$

23. $h(x)=\dfrac{1}{3}x^2-1;\quad (3,2)$

24. $F(x)=3+x-\dfrac{x^2}{2};\quad (4,-1)$

25. $G(x)=\dfrac{1}{4}x^3-x+1;\quad (-2,1)$

26. $k(x)=10-x^{\frac{3}{2}};\quad (4,2)$

27. $H(x)=\ln x+1;\quad (e,2)$

28. $u(x)=\cos x;\quad \left(\dfrac{\pi}{2},0\right)$

29. $v(x)=\log(2x)-1;\quad (5,0)$

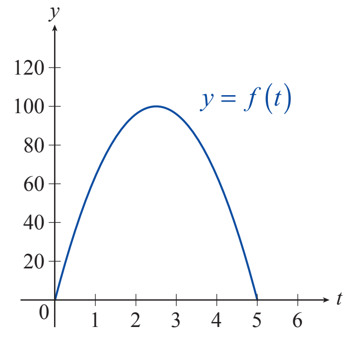
30. $w(x)=\tan x;\quad (0,0)$

31. $p(x)=-x^4+1;\quad (1,0)$

32. $q(x)=x^5-x+3;\quad (0,3)$

APPLICATIONS

33. An arrow is shot into the air and its height in feet after t seconds is given by the function $f(t) = -16t^2 + 80t$. The graph of the curve $y = f(t)$ is shown.



a. Find the height of the arrow when $t = 2$ seconds.
b. Find the instantaneous velocity of the arrow when $t = 2$ seconds.
c. Find the slope of the line tangent to the curve at $t = 2$ seconds.
d. Find the time it takes the arrow to reach its peak.

34. Suppose that a sailboat is observed, over a period of 5 minutes, to travel a distance from a starting point according to the function $s(t) = t^3 + 60t$, where t is time in minutes and s is the distance traveled in meters.

a. How far will it travel during the first 6 seconds?
b. What is the average velocity during the first 6 seconds?
c. Estimate how fast the boat is moving at the starting point.
d. Estimate how fast the boat is moving at the end of 3 minutes.

35. A model rocket is fired vertically upward. The height after t seconds is $h(t) = 192t - 16t^2$ feet.

a. What will be its height at the end of the first second?
b. What is the average velocity of the rocket during the first second?
c. Estimate the instantaneous velocity at $t = 0$ seconds.
d. Estimate the instantaneous velocity at $t = 4$ seconds.
e. When will the velocity be 0? (**Hint:** You may want to start with the initial velocity you found in part a. and use the fact that under the influence of gravity, when air resistance is ignored, vertical upward velocity decreases by 32 ft/s every second. Once you have a guess, test it by a table of difference quotients.)

36. A particle moving in a straight line is at a distance of $s(t) = 2.5t^2 + 18t$ feet from its starting point after t seconds, where $0 \le t \le 12$. Estimate the instantaneous velocity at **a.** $t = 6$ seconds and **b.** $t = 9$ seconds.

37. The distance, in meters, traveled by a moving particle in t seconds is given by $d(t) = 3t(t+1)$. Estimate the instantaneous velocity at **a.** $t = 0$ seconds, **b.** $t = 2$ seconds, and **c.** at time t_0. (**Hint:** Write the difference quotient corresponding to $t = t_0$, simplify, and try to find the value being approached by the expression as h decreases.)

38. The distance, in meters, traveled by a moving particle in t seconds is given by $d(t) = t^2 - 3t$. Estimate the instantaneous velocity at **a.** $t = 0$ seconds, **b.** $t = 4$ seconds, and **c.** at time t_0. (See the hint given in part c. of the previous problem.)

39. After start, on a straight stretch of the track, a race car's velocity changes according to the function $v(t) = -1.8t^2 + 18t$, when $0 \le t \le 10$, t is measured in seconds, and $v(t)$ is measured in meters per second.

a. When does peak velocity occur and what is it? (**Hint:** The graph of $v(t)$ may be helpful.)
b. When does peak deceleration occur?
c. Use difference quotients to estimate peak deceleration. Approximately what multiple of $g \approx 9.81\ \text{m/s}^2$ have you obtained?

40. If we ignore air resistance, a falling body will fall $16t^2$ feet in t seconds.

a. How far will it fall between $t = 2$ and $t = 2.1$?
b. What is its average velocity between $t = 2$ and $t = 2.1$?
c. Estimate its instantaneous velocity at $t = 2$.

TECHNOLOGY

Use a graphing utility to graph $f(x)$ along with three secant lines at the indicated *x*-value, corresponding to the difference quotients with *h*-values of 0.2, 0.1, and 0.01, respectively. Can you come up with a possible equation for the tangent? Use technology to test your conjecture.

41. $f(x) = x^2; \quad x = 2$

42. $f(x) = -x^3 + x + 1; \quad x = \frac{\sqrt{3}}{3}$

43. $f(x) = \sin x + \cos x; \quad x = 0$

44. $f(x) = 3\sqrt{x}; \quad x = 4$

Use a graphing utility to graph the given function $f(x)$ along with $D(x) = \frac{f(x + 0.001) - f(x)}{0.001}$ in the same coordinate system. Explain how the function values of $D(x)$ are reflected on the graph of $f(x)$.

45. $f(x) = x^4$

46. $f(x) = x(3 - x)$

47. $f(x) = \sin x$

48. $f(x) = \ln x$

Use a graphing utility to find the *x*-values at which the graph of $f(x)$ does not have a tangent line. Explain.

49. $f(x) = -|x - 1| + 1$

50. $f(x) = |x^2 - 4|$

51. $f(x) = |\ln x|$

52. $f(x) = (x - 1)^{\frac{2}{3}}$