Polynomial long division (Section 5.2) then gives us the following result.

$$\frac{3x^5+6x^4+9x^3+7x^2+4x-1}{\left(x^2+x+1\right)^2}=3x+\frac{x^2+x-1}{\left(x^2+x+1\right)^2}$$

Now we need to decompose the fractional part.

Note that x^2+x+1 is irreducible. This follows from the use of the quadratic formula to solve the equation $x^2+x+1=0$; since the solutions of this equation are complex numbers, the linear factors of x^2+x+1 contain complex coefficients. The partial fraction decomposition thus has the following form.

$$\frac{x^2+x-1}{\left(x^2+x+1\right)^2}=\frac{A_1x+B_1}{x^2+x+1}+\frac{A_2x+B_2}{\left(x^2+x+1\right)^2}$$

Clearing the equation of fractions gives us the following.

$$x^2+x-1=\left(A_1x+B_1\right)\left(x^2+x+1\right)+\left(A_2x+B_2\right)$$
$$x^2+x-1=A_1x^3+\left(A_1+B_1\right)x^2+\left(A_1+B_1+A_2\right)x+\left(B_1+B_2\right)$$

From this polynomial equation we derive the following system.

$$\begin{cases} 0=A_1 \\ 1=A_1+B_1 \\ 1=A_1+B_1+A_2 \\ -1=B_1+B_2 \end{cases}$$

This system is easily solved, considering the equations in the order presented, and gives us $A_1=0, B_1=1, A_2=0,$ and $B_2=-2.$ The following answer results.

$$\frac{3x^5+6x^4+9x^3+7x^2+4x-1}{\left(x^2+x+1\right)^2}=3x+\frac{1}{x^2+x+1}+\frac{-2}{\left(x^2+x+1\right)^2}$$

11.6 EXERCISES

PRACTICE

Write the form of the partial fraction decomposition of each of the following rational functions. In each case, assume the degree of the numerator is less than the degree of the denominator. See Examples 1 and 2.

1. $f(x)=\frac{p(x)}{x^2-x-6}$

2. $f(x)=\frac{p(x)}{x^2-2x-24}$

3. $f(x)=\frac{p(x)}{x^3+11x^2+40x+48}$

4. $f(x)=\frac{p(x)}{(x+5)^2\left(x^2+3\right)^2}$

5. $f(x)=\dfrac{p(x)}{(x+3)(x^2-4)}$

6. $f(x)=\dfrac{p(x)}{(x^2+5)(x^2+3x-4)}$

Match each rational expression with the form of its decomposition. The decompositions are labeled a.–h.

7. $\dfrac{2x-1}{(x+2)^3(x-2)}$

8. $\dfrac{2x-1}{x^2(x^2-4)}$

9. $\dfrac{2x-1}{x^3-4x^2+4x}$

10. $\dfrac{2x-1}{x^4-16}$

11. $\dfrac{2x-1}{x^3(x-2)^2}$

12. $\dfrac{2x-1}{x^5+2x^4+4x^3+8x^2}$

13. $\dfrac{2x-1}{x^3(x-2)}$

14. $\dfrac{2x-1}{x^5+6x^4+12x^3+8x^2}$

a. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-2}+\dfrac{E}{(x-2)^2}$

b. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+2}+\dfrac{D}{x-2}$

c. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-2}$

d. $\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}+\dfrac{D}{x-2}$

e. $\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x+2}+\dfrac{D}{x-2}$

f. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+2}+\dfrac{Dx+E}{x^2+4}$

g. $\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}+\dfrac{D}{x}+\dfrac{E}{x^2}$

h. $\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

Find the partial fraction decomposition of each of the following rational functions. See Examples 3, 4, and 5.

15. $f(x)=\dfrac{3x^2+4}{x^3-4x}$

16. $f(x)=\dfrac{2x}{x^3+7x^2-6x-72}$

17. $f(x)=\dfrac{4x+2}{(x^3+8x)(x^2+2x-8)}$

18. $f(x)=\dfrac{5}{x^2+3x-4}$

19. $f(x)=\dfrac{5x}{x^2-6x+8}$

20. $f(x)=\dfrac{6x^2-4}{(x^2+3)(x+6)(x+5)}$

21. $f(x)=\dfrac{6x}{x^3+8x^2+9x-18}$

22. $f(x)=\dfrac{12x^2+x-1}{x^4+7x^3+5x^2-31x-30}$

23. $f(x)=\dfrac{1}{x^2-1}$

24. $f(x)=\dfrac{x+3}{(x^2+3)(x^2+x-6)}$

25. $f(x)=\dfrac{x^2-4}{(x^4-16)(x^2+2x-8)}$

26. $f(x)=\dfrac{x+1}{x^3-x}$

27. $f(x)=\dfrac{x+3}{x^2-4}$

28. $f(x)=\dfrac{x}{x^3+6x^2+11x+6}$

29. $f(x) = \dfrac{x}{x^4 - 16}$

30. $f(x) = \dfrac{5}{x^2 - 6x + 8}$

31. $f(x) = \dfrac{2x+3}{(x^2-9)(x^2+4x-12)}$

32. $f(x) = \dfrac{x}{x^2 - 7x + 12}$

33. $f(x) = \dfrac{x^2}{x^3 + 5x^2 + 3x - 9}$

34. $f(x) = \dfrac{2x}{(x+4)(x^2-2x-3)}$

35. $f(x) = \dfrac{2x}{x^2 - 9}$

36. $f(x) = \dfrac{4x+3}{(x^2-9)(x^2-2x-24)}$

37. $f(x) = \dfrac{x^2}{x^3 + 4x^2 - 12x}$

38. $f(x) = \dfrac{2}{x^3 + 7x^2 - 8x}$

Write the partial fraction decomposition for the rational expression. You may check your answer by assigning a value to the constant *a* and graphing the result.

39. $\dfrac{1}{x(x+a)}$

40. $\dfrac{1}{x(a-x)}$

41. $\dfrac{1}{a^2 - x^2}$

42. $\dfrac{1}{(x+1)(a-x)}$

43. $\dfrac{1}{(x+a)(x+1)}$

TECHNOLOGY

Using a graphing utility, determine whether the partial fraction decomposition is true or false by graphing the left and right side of the equation on the same coordinate plane.

44. $\dfrac{x+7}{x^2 - x - 6} = \dfrac{2}{x-3} - \dfrac{1}{x+2}$

45. $\dfrac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \dfrac{6}{x} - \dfrac{1}{x+1} + \dfrac{9}{(x+1)^2}$

46. $\dfrac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \dfrac{2}{x} - \dfrac{1}{x-1} + \dfrac{3}{x+2}$

47. $\dfrac{2x+4}{x^3 - 2x^2} = -\dfrac{2}{x} - \dfrac{2}{x^2} + \dfrac{2}{x-2}$

48. $\dfrac{1}{x^2 + x - 2} = \dfrac{1}{x-1} + \dfrac{1}{x+2}$

49. $\dfrac{6x^2 + 2}{x^2(x-3)^3} = \dfrac{4}{x} + \dfrac{2}{x^2} + \dfrac{3x}{x-3} + \dfrac{4}{(x-3)^2} + \dfrac{6x}{(x-3)^3}$