

8.3 Slope-Intercept Form

A The Meaning of Slope

If you ride a bicycle up a mountain road, you certainly know when the **slope** (a measure of steepness called the **grade** for roads) increases because you have to pedal harder. The contractor who built the road was aware of the slope because trucks traveling the road must be able to control their downhill speed and be able to stop in a safe manner. A carpenter given a set of house plans calling for a roof with a **pitch** of 7 : 12 knows that for every 7 feet of rise (vertical distance) there are 12 feet of run (horizontal distance). That is, the ratio of rise to run is $\frac{\text{rise}}{\text{run}} = \frac{7}{12}$.



Figure 1

Note that this ratio can be in units other than feet, such as inches or meters. (See Figure 2.)

$$\frac{\text{rise}}{\text{run}} = \frac{7 \text{ inches}}{12 \text{ inches}} = \frac{3.5 \text{ feet}}{6 \text{ feet}} = \frac{14 \text{ feet}}{24 \text{ feet}}$$

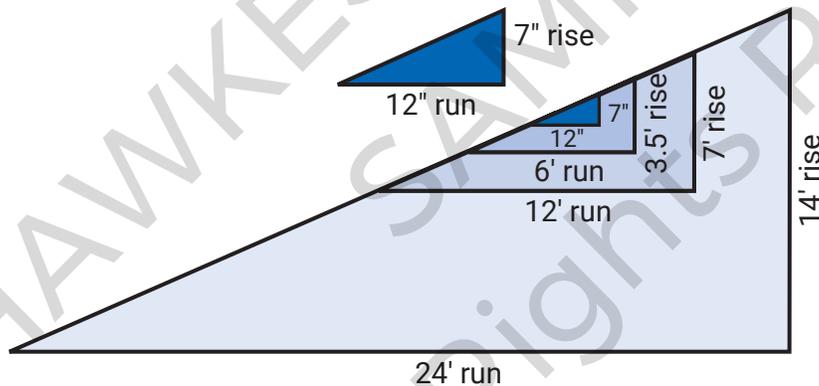


Figure 2

For a line, the **ratio of rise to run** is called the **slope of the line**. The graph of the linear equation $y = \frac{1}{3}x + 2$ is shown in Figure 3. What do you think is the slope of the line? Do you think that the slope is positive or negative? Do you think the slope might be $\frac{1}{3}$? $\frac{3}{1}$? 2?

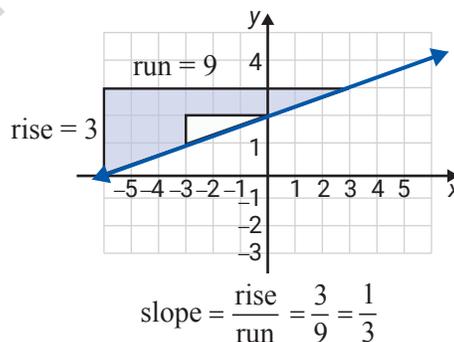
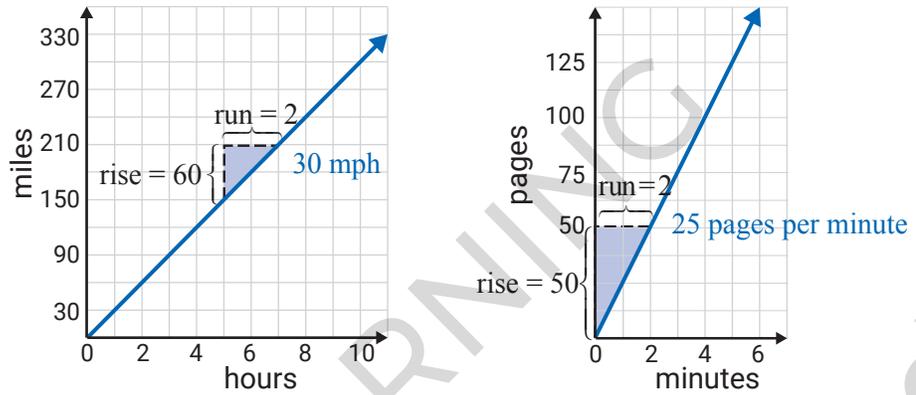


Figure 3

Objectives

- Interpret the slope of a line as a rate of change.
- Find the slope of a line given two points on the line.
- Find the slopes of horizontal and vertical lines.
- Graph a linear equation by finding the slope and y-intercept.
- Use slope-intercept form to write the equation of a line given its slope and y-intercept.

The concept of slope also relates to situations that involve **rate of change**. For example, the graphs in Figure 4 illustrate slope as miles per hour that a car travels and as pages per minute that a printer prints. (**Note:** Be sure to look at the scales on the axes when reading a graph.)



slope = rate of change

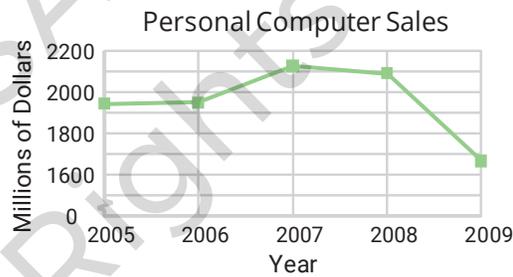
$$= \frac{60 \text{ miles}}{2 \text{ hours}} = 30 \text{ mph}$$

slope = rate of change

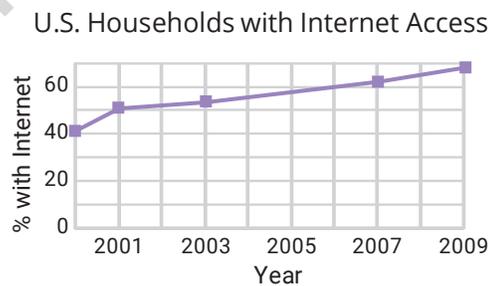
$$= \frac{50 \text{ pages}}{2 \text{ minutes}} = 25 \text{ pages per minute}$$

Figure 4

In general, the ratio of a change in one variable (say y) to a change in another variable (say x) is called the **rate of change of y with respect to x** . Figure 5 shows how the rate of change (the slope) can change over periods of time.



Source: Consumer Electronics Association



Source: U.S. Dept. of Commerce

Figure 5

B Calculating the Slope

Consider the line $y = 2x + 3$, and two points on the line $P_1(-2, -1)$ and $P_2(2, 7)$ as shown in Figure 6.

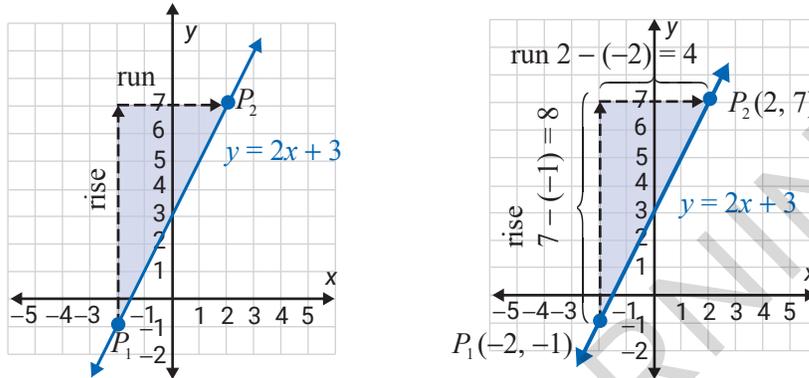


Figure 6

For the line $y = 2x + 3$ and using the points $(-2, -1)$ and $(2, 7)$ that are on the line,

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}} = \frac{7 - (-1)}{2 - (-2)} = \frac{8}{4} = 2.$$

From similar illustrations and the use of subscript notation, we can develop the following formula for the slope of any line.

Slope

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on a line. The **slope** can be calculated as follows.

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note: The letter m is standard notation for representing the slope of a line.

FORMULA

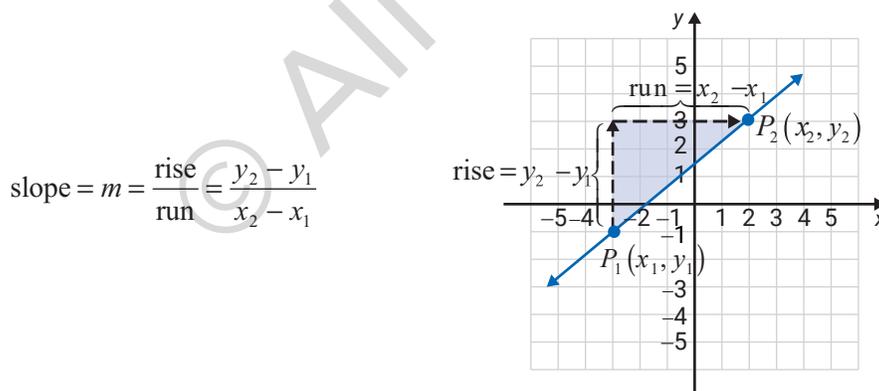


Figure 7

Note

In the notation P_1 , 1 is called a **subscript** and P_1 is read "P sub 1." Similarly, P_2 is read "P sub 2." Subscripts are used in labeling and are not used in calculations.

1. Find the slope of the line that contains the points $(-1, 2)$ and $(3, 5)$, and then graph the line.

Example 1 Finding the Slope of a Line

Find the slope of the line that contains the points $(-1, 2)$ and $(3, 5)$, and then graph the line.

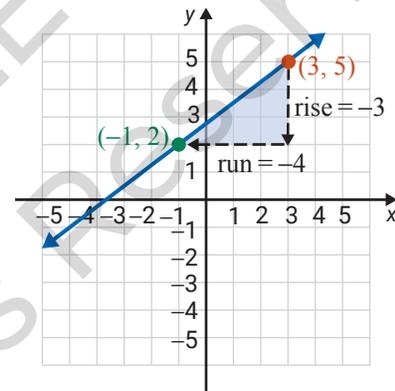
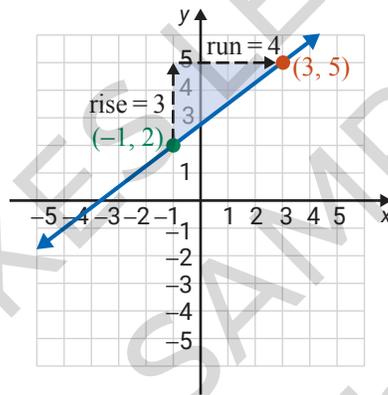
Solution

For (x_1, y_1) , use $(-1, 2)$ and
for (x_2, y_2) , use $(3, 5)$.

$$\begin{aligned}\text{slope} = m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 2}{3 - (-1)} \\ &= \frac{3}{4}\end{aligned}$$

Or, for (x_1, y_1) use $(3, 5)$ and
for (x_2, y_2) , use $(-1, 2)$.

$$\begin{aligned}\text{slope} = m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 5}{-1 - 3} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4}\end{aligned}$$



Now work margin exercise 1.

As we see in Example 1, the slope is the same even if the order of the points is reversed. The important part of the procedure is that the coordinates must be subtracted in the same order in both the numerator and the denominator.

In general,

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

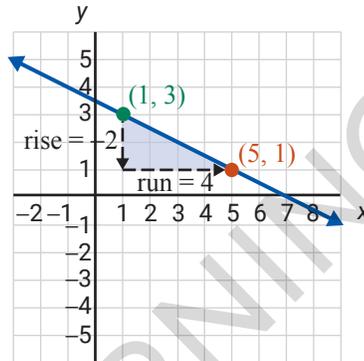
Example 2 Finding the Slope of a Line

Find the slope of the line that contains the points $(1, 3)$ and $(5, 1)$, and then graph the line.

Solution

For (x_1, y_1) , use $(1, 3)$ and for (x_2, y_2) , use $(5, 1)$.

$$\begin{aligned}\text{slope} = m &= \frac{1-3}{5-1} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2}\end{aligned}$$



2. Find the slope of the line that contains the points $(0, 5)$ and $(4, 2)$, and then graph the line.

Now work margin exercise 2.**Positive and Negative Slope**

Lines with **positive slope go up** (increase) as we move along the line from left to right.

Lines with **negative slope go down** (decrease) as we move along the line from left to right.

DEFINITION**C Slopes of Horizontal and Vertical Lines**

In Section 8.1, we discussed the graphs of horizontal lines ($y = b$) and vertical lines ($x = a$), but the slopes of lines of these types were not discussed.

To find the slope of a horizontal line, such as $y = 3$, find two points on the line and substitute into the slope formula. Note that any two points on the line will have the same y -coordinate—namely, 3. Two such points are $(-2, 3)$ and $(5, 3)$. Using the two points in the formula for slope gives the following.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 3}{5 - (-2)} = \frac{0}{7} = 0$$

For any horizontal line, all of the y -values will be the same. Consequently, the formula for slope will always have 0 in the numerator. Therefore, **the slope of every horizontal line is 0**. (See Figure 8.)

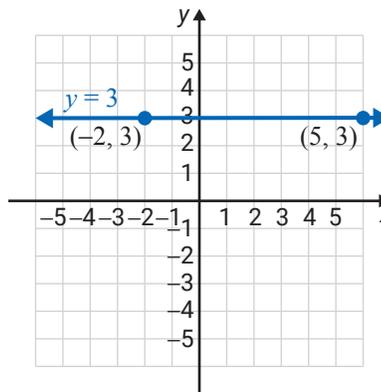


Figure 8

To find the slope of a vertical line, such as $x = 4$, find two points on the line and substitute into the slope formula. Now any two points on the line will have the same x -coordinate—namely, 4. Two such points are $(4, 1)$ and $(4, 6)$. The slope formula gives the following.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{4 - 4} = \frac{5}{0}, \text{ which is undefined.}$$

(Remember, division by 0 is undefined; thus, the slope is undefined.)

For any vertical line, all of the x -values will be the same. Consequently, the formula for slope will always have 0 in the denominator. Therefore, **the slope of every vertical line is undefined.** (See Figure 9.)

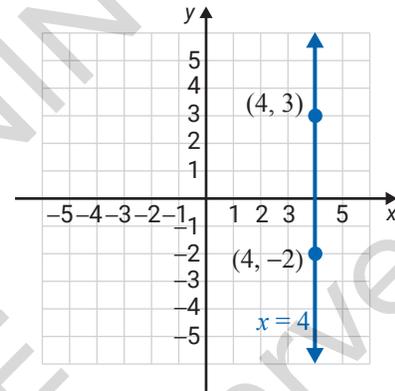


Figure 9

Horizontal and Vertical Lines

The following two general statements are true for horizontal and vertical lines.

1. For **horizontal lines** (of the form $y = b$), the **slope is 0**.
2. For **vertical lines** (of the form $x = a$), the **slope is undefined**.

DEFINITION

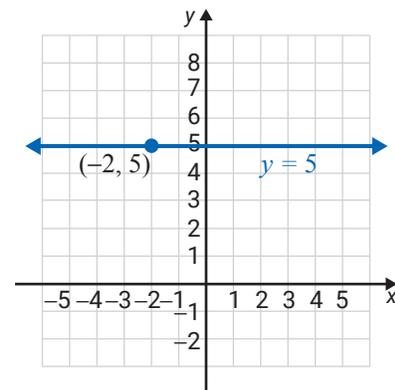
3. Find the equation and slope of the horizontal line through the point $(3, -2)$.

Example 3 Finding the Slope of a Horizontal Line

Find the equation and slope of the horizontal line through the point $(-2, 5)$.

Solution

The equation is $y = 5$ and the slope is 0.



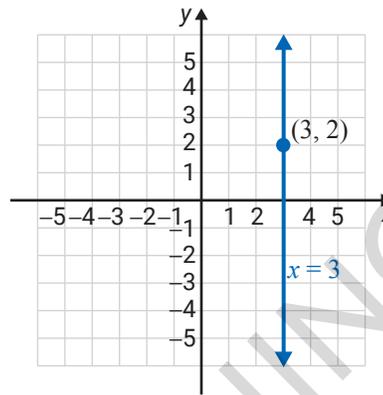
Now work margin exercise 3.

Example 4 Finding the Slope of a Vertical Line

Find the equation and slope of the vertical line through the point $(3, 2)$.

Solution

The equation is $x = 3$ and the slope is undefined.



4. Find the equation and slope of the vertical line through the point $(2, 4)$.

Now work margin exercise 4.

D Slope-Intercept Form

There are certain relationships between the coefficients in the equation of a line and the graph of that line. For example, consider the equation

$$y = 5x - 7.$$

First, find two points on the line and calculate the slope. The points $(0, -7)$ and $(2, 3)$ both satisfy the equation.

$$\text{slope} = m = \frac{3 - (-7)}{2 - 0} = \frac{10}{2} = 5$$

Observe that the slope, $m = 5$, is the same as the coefficient of x in the equation $y = 5x - 7$. This is not just a coincidence. In fact, if a linear equation is solved for y , then the coefficient of x will always be the slope of the line.

The Slope m

For an equation in the form $y = mx + b$, the slope of the line is m .

DEFINITION

For the line $y = mx + b$, the point where $x = 0$ is the point where the line crosses the y -axis. Recall that this point is called the **y -intercept**. By letting $x = 0$, we get

$$\begin{aligned} y &= mx + b \\ y &= m \cdot 0 + b \\ y &= b. \end{aligned}$$

Thus, the point $(0, b)$ is the y -intercept. The concepts of slope and y -intercept lead to the following definition.

Slope-Intercept Form

$y = mx + b$ is called the **slope-intercept** form for the equation of a line, where m is the **slope** and $(0, b)$ is the **y-intercept**.

DEFINITION

As illustrated in Example 5, an equation in **standard form**

$$Ax + By = C \text{ with } B \neq 0$$

can be written in slope-intercept form by solving for y .

5. Find the slope and y-intercept of $-4x + 2y = 12$, and graph the line.

Example 5 Using Slope and the y-Intercept to Graph a Line

Find the slope and y-intercept of $-2x + 3y = 6$, and graph the line.

Solution

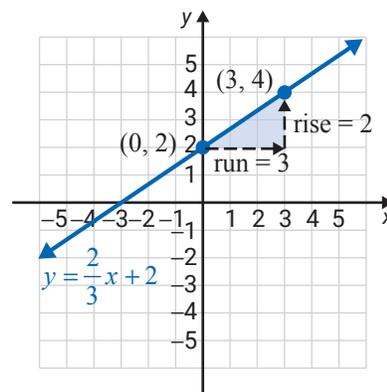
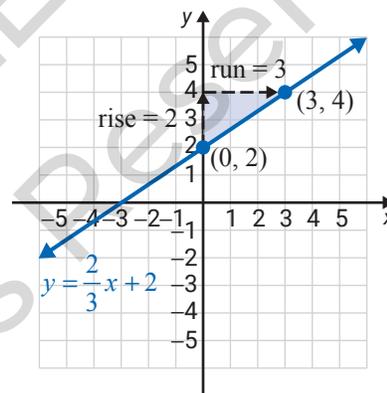
Solve for y .

$$\begin{aligned} -2x + 3y &= 6 \\ 3y &= 2x + 6 \\ \frac{3y}{3} &= \frac{2x}{3} + \frac{6}{3} \\ y &= \frac{2}{3}x + 2 \end{aligned}$$

Thus, $m = \frac{2}{3}$, which is the slope, and b is 2, making the y-intercept $(0, 2)$.

As shown in the graph, if we “rise” 2 units up and “run” 3 units to the right **from the y-intercept** $(0, 2)$, we locate another point, $(3, 4)$. The line can be drawn through these two points.

Note: As shown in the second graph, we could also first “run” 3 units right and “rise” 2 units up from the y-intercept to locate the point $(3, 4)$ on the graph.



Now work margin exercise 5.

Example 6 Using Slope and the y -Intercept to Graph a Line

Find the slope and y -intercept of $x + 2y = -6$, and graph the line.

Solution

Solve for y .

$$x + 2y = -6$$

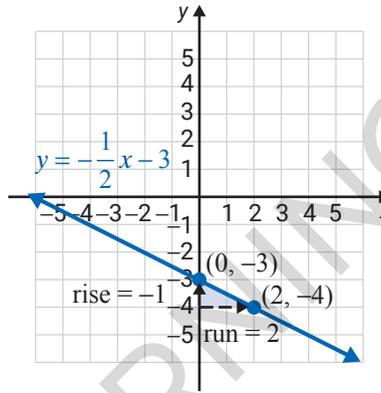
$$2y = -x - 6$$

$$\frac{2y}{2} = \frac{-x}{2} - \frac{6}{2}$$

$$y = -\frac{1}{2}x - 3$$

Thus, $m = -\frac{1}{2}$, which is the slope, and $b = -3$, making the y -intercept $(0, -3)$.

We can treat $m = -\frac{1}{2}$ as $m = \frac{-1}{2}$ and the “rise” as -1 and the “run” as 2 . Moving from $(0, -3)$ as shown in the graph, we locate another point $(2, -4)$ on the graph and draw the line.



6. Find the slope and y -intercept of $3x + 2y = -10$, and graph the line.

Now work margin exercise 6.**E Finding Equations of Lines Given the Slope and the y -Intercept****Example 7** Finding Equations Given the Slope and the y -Intercept

Find the equation of the line through the point $(0, -2)$ with slope $\frac{1}{2}$.

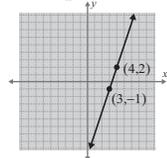
Solution

Because the x -coordinate is 0 , we know that the point $(0, -2)$ is the y -intercept. So $b = -2$. The slope is $\frac{1}{2}$. So $m = \frac{1}{2}$. Substituting in slope-intercept form, $y = mx + b$, gives the result $y = \frac{1}{2}x - 2$.

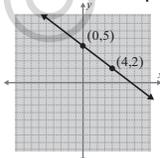
7. Find the equation of the line through the point $(0, -3)$ with a slope of $\frac{2}{3}$.

Now work margin exercise 7.**Margin Exercise Answers**

1. slope = 3



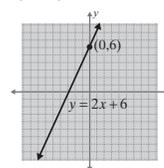
2. slope = $-\frac{3}{4}$



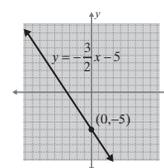
3. $y = -2$; slope is 0

4. $x = 2$; slope is undefined

5. $m = 2$; y -intercept = $(0, 6)$



6. $m = -\frac{3}{2}$; y -intercept = $(0, -5)$



7. $y = \frac{2}{3}x - 3$

8.3 Exercises

Concept Check

Fill-in-the-Blank. Complete the sentences using information found in this section.

- The slope of a line is the ratio of rise to _____.
- Another name for slope is the rate of _____.
- A line that rises (increases) from left to right has a/an _____ slope.
- The slope of every vertical line is _____.
- The slope of every horizontal line is _____.
- In the equation $y = mx + b$, m represents the _____ and $(0, b)$ represents the _____.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

- If the y -intercept and the slope of a line are given, there is enough information to write the equation of the line.
- When using the slope formula, the slope of a line changes if the order of the points is reversed.
- A line that falls (decreases) from left to right has a negative slope.
- The line that represents the equation $y = 2x + 4$ has a y -intercept of $(0, 4)$.

Practice

Find the slope of the line determined by each pair of points. See Examples 1 and 2.

- | | |
|------------------------|--|
| 1. $(2, 4); (1, -1)$ | 8. $(0, 0); (-2, -3)$ |
| 2. $(1, -2); (1, 4)$ | 9. $\left(\frac{3}{4}, \frac{3}{2}\right); (1, 2)$ |
| 3. $(-6, 3); (1, 2)$ | 10. $\left(4, \frac{1}{2}\right); (-1, 2)$ |
| 4. $(-3, 7); (4, -1)$ | 11. $\left(\frac{3}{2}, \frac{4}{5}\right); \left(-2, \frac{1}{10}\right)$ |
| 5. $(-5, 8); (3, 8)$ | 12. $\left(\frac{7}{2}, \frac{3}{4}\right); \left(\frac{1}{2}, -3\right)$ |
| 6. $(-2, 3); (-2, -1)$ | |
| 7. $(5, 1); (3, 0)$ | |

Determine whether each equation represents a horizontal line or vertical line and give its slope. Graph the line. See Examples 3 and 4.

- | | | |
|--------------|----------------|--------------------|
| 13. $y = 5$ | 16. $x = 1.7$ | 19. $-3x + 21 = 0$ |
| 14. $y = -2$ | 17. $3y = -18$ | 20. $2y + 5 = 0$ |
| 15. $x = -3$ | 18. $4x = 2.4$ | |

Write each equation in slope-intercept form. Find the slope and y -intercept, and then use them to draw the graph. See Examples 5 and 6.

- | | | |
|----------------------------|-----------------------|-----------------------|
| 21. $y = 2x - 1$ | 30. $4x + y = 0$ | 40. $3x + 6 = 6y$ |
| 22. $y = 3x - 4$ | 31. $4x + y + 3 = 0$ | 41. $5 - 3x = 4y$ |
| 23. $y = 5 - 4x$ | 32. $2x + 7y + 7 = 0$ | 42. $5x = 11 - 2y$ |
| 24. $y = 4 - x$ | 33. $2y - 8 = 0$ | 43. $6x + 4y = -8$ |
| 25. $y = \frac{2}{3}x - 3$ | 34. $3y - 9 = 0$ | 44. $7x + 2y = 4$ |
| 26. $y = \frac{2}{5}x + 2$ | 35. $2x = 3y$ | 45. $6y = -6 + 3x$ |
| 27. $x + y = 5$ | 36. $4x = y$ | 46. $4x = 3y - 7$ |
| 28. $x - 2y = 6$ | 37. $3x + 9 = 0$ | 47. $5x - 2y + 5 = 0$ |
| 29. $x + 5y = 10$ | 38. $4x + 7 = 0$ | 48. $6x + 5y = -15$ |
| | 39. $5x - 6y = 18$ | |

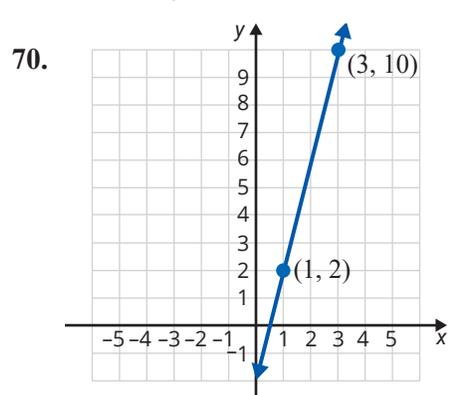
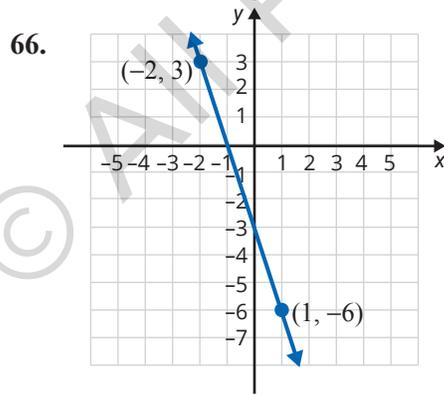
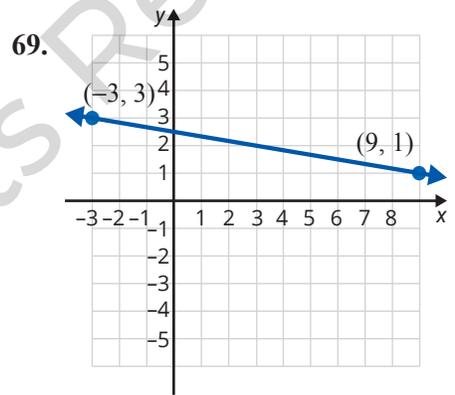
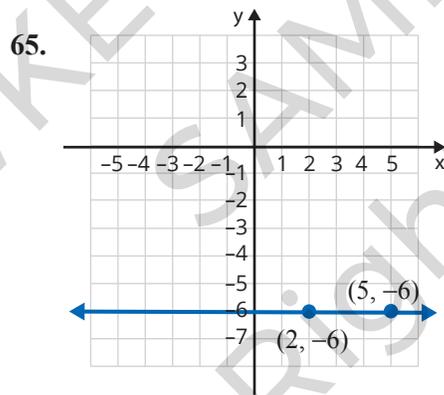
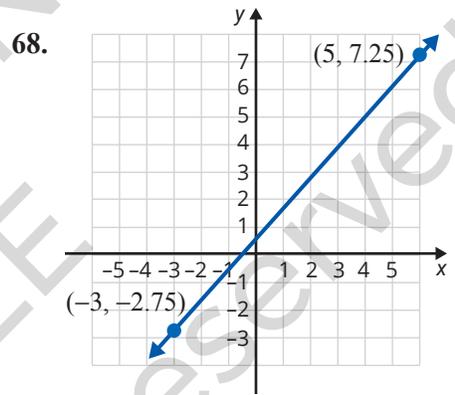
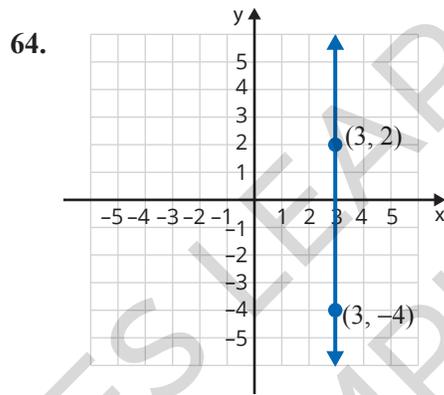
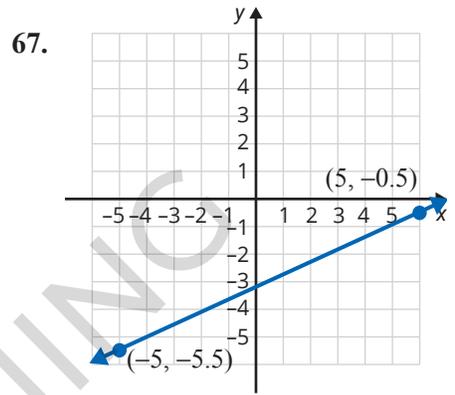
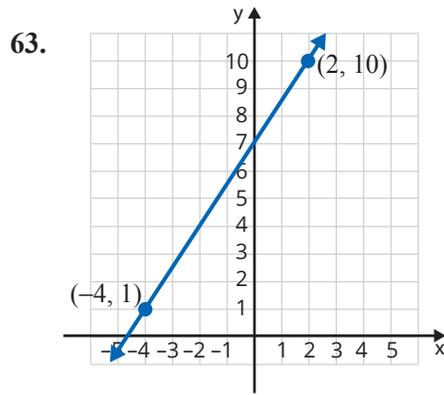
In reference to the equation $y = mx + b$, sketch the graphs of three lines for each of the two characteristics listed below.

- | | |
|-------------------------|-------------------------|
| 49. $m > 0$ and $b > 0$ | 51. $m > 0$ and $b < 0$ |
| 50. $m < 0$ and $b > 0$ | 52. $m < 0$ and $b < 0$ |

Find an equation in slope-intercept form for the line passing through the given point with the given slope. See Example 7.

- | | | |
|--------------------------------|----------------------|---------------------------------|
| 53. $(0, 3); m = -\frac{1}{2}$ | 57. $(0, -5); m = 4$ | 61. $(0, -3); m = -\frac{5}{6}$ |
| 54. $(0, 2); m = \frac{1}{3}$ | 58. $(0, 9); m = -1$ | 62. $(0, -1); m = -\frac{3}{2}$ |
| 55. $(0, -3); m = \frac{2}{5}$ | 59. $(0, -4); m = 1$ | |
| 56. $(0, -6); m = \frac{4}{3}$ | 60. $(0, 6); m = -5$ | |

The graph of a line is shown with two points labeled. Find **a.** the slope, **b.** the y -intercept (if there is one), and **c.** the equation of the line in slope-intercept form.



Points are said to be **collinear** if they lie on a straight line. If points are collinear, then the slope of the line through any two of them must be the same (because the line is the same line). Use this idea to determine whether or not the three points in each of the sets are collinear.

71. $\{(-1, 3), (0, 1), (5, -9)\}$

74. $\{(-1, -7), (1, 1), (2.5, 7)\}$

72. $\{(-2, -4), (0, 2), (3, 11)\}$

75. $\left\{\left(\frac{2}{3}, \frac{1}{2}\right), \left(0, \frac{5}{6}\right), \left(-\frac{3}{4}, \frac{29}{24}\right)\right\}$

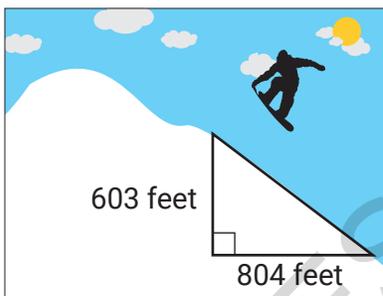
73. $\{(-2, 0), (0, 30), (1.5, 5.25)\}$

76. $\left\{\left(\frac{3}{2}, -\frac{1}{3}\right), \left(0, \frac{1}{6}\right), \left(-\frac{1}{2}, \frac{3}{4}\right)\right\}$

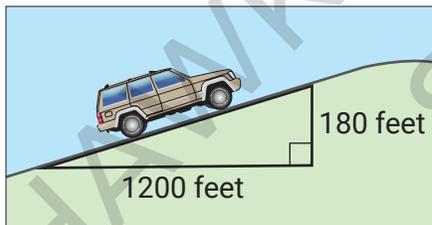
Applications

Solve.

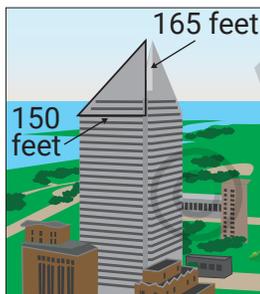
77. Find the slope of the ski slope.



78. Find the slope of the road.



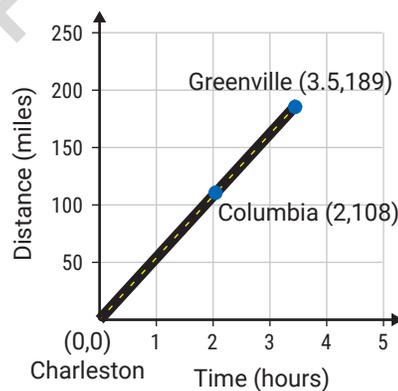
79. Find the slope of the roof of the skyscraper.



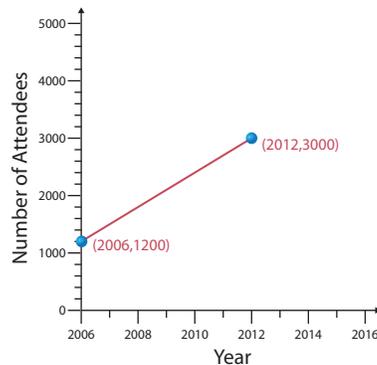
80. Find the slope of the larger sail on the sailboat.



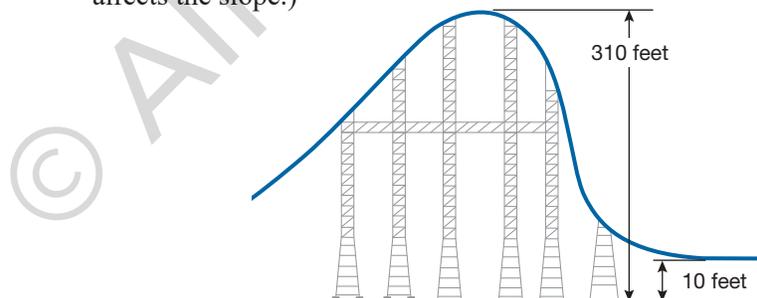
81. **Travel:** A car travels from Charleston to Greenville. Its distance related to time traveled is given on the following graph. Find the average speed of the car in miles per hour from Columbia to Greenville.



- 82. Festivals:** The attendance at Smithville's Spring Festival has been increasing steadily as shown in the graph. Find the average increase in attendees per year. How many people do you predict will attend the festival in 2016?



- 83. Purchases:** John bought his new car for \$35,000 in the year 2014. He knows that the value of his car has depreciated linearly. If the value of the car in 2017 was \$23,000, what was the annual rate of depreciation of his car? Show this information on a graph. (When graphing, use years as the x -coordinates and the corresponding values of the car as the y -coordinates.)
- 84. Cell Phones:** The number of people in the United States with mobile cellular phones was about 198 million in 2011 and about 232 million in 2016. If the growth in the usage of mobile cellular phones was linear, what was the approximate rate of growth per year from 2011 to 2016. Show this information on a graph. (When graphing, use years as the x -coordinates and the corresponding numbers of users as the y -coordinates.)¹
- 85. Amusement Parks:** The Millennium Force roller coaster at Cedar Point in Sandusky, Ohio, has been voted the Best Steel Coaster by Golden Ticket seven times since 2001. The Millennium Force is known for its steep slope on the first hill and speeds up to 93 miles per hour. The descent from the first hill starts at 310 feet above the ground and ends 10 feet above the ground. The descent runs approximately 53 feet. Find the slope of this hill to the nearest hundredth. (**Hint:** The slope is running downhill. Consider how that affects the slope.)



¹ Source: <https://www.statista.com/statistics/231612/number-of-cell-phone-users-usa/>

- 86. Construction:**  The grade, or slope, of a road is commonly given as a percentage. The grade can be determined by multiplying the slope by 100. Calculate the slope of each road or track and then determine its grade. Round each percent to the nearest hundredth if necessary.
- A road increases in height 5 feet for every 120 feet of run.
 - The railway line with the steepest grade that does not run on a track system is the Lisbon tramway network in Portugal which has a section that increases 5 feet in height for every 37 feet of run.
 - A road on the Route des Crêtes (Route of the Ridges) in France has an elevation that increases in height 450 feet over 1500 feet.
- 87. Painting:** Jared sells paintings at an open-air market. He starts his work day with \$30 and sells each painting for \$15. Jared wants to create a linear equation to model this situation where y is the amount of money Jared has at the end of the work day and x is the number of paintings sold.
- The slope, or rate of change, is the increase in the amount of money Jared makes when he sells a painting. Determine the value of the slope and list the units for both variables.
 - The y -coordinate of the y -intercept of this equation is the amount of money Jared has before he sells any paintings. What is the y -intercept?
 - Write a linear equation in slope-intercept form to model this situation using the answers from Parts **a.** and **b.**
 - Graph the equation from Part **c.**
 - Are there any solutions to the equation which do not make sense in the context of the problem? Explain why.
 - Use the graph to determine the amount of money Jared will have after selling 4 paintings.
- 88. Internet:** The given table shows the estimated number of internet users from 2010 to 2014. The number of users for each year is shown in millions.
- Plot these points on a graph.
 - Connect the points with line segments.
 - Find the slope of each line segment.
 - Interpret each slope as a rate of change.

Year	Internet users (in millions)
2010	222
2011	218
2012	250
2013	267
2014	279

Source: <https://www.statista.com/statistics/276445/number-of-internet-users-in-the-united-states/>

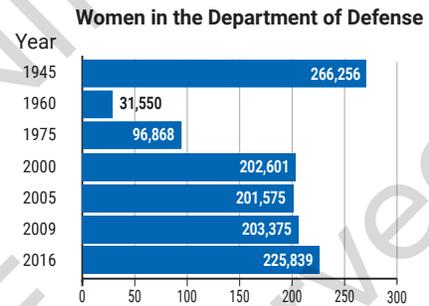
89. **Population:** The following table shows the urban growth from 1850 to 2000 in New York, NY.

Year	Population
1850	515,547
1900	3,437,202
1950	7,891,957
2000	8,008,278

Source: U.S. Census Bureau

- Plot these points on a graph.
- Connect the points with line segments.
- Find the slope of each line segment.
- Interpret each slope as a rate of change.

90. **Military:** The following graph shows the number of female active duty military personnel over a span from 1945 to 2016. The number of women listed includes both officers and enlisted personnel from the Army, the Navy, the Marine Corps, and the Air Force.



Source: U.S. Dept. of Defense

- Plot these points on a graph.
- Connect the points with line segments.
- Find the slope of each line segment.
- Interpret each slope as a rate of change.

91. **Marriage:** The following graph shows the rates of marriage per 1000 people in the U.S. over a span from 1940 to 2016.



Source: U.S. National Center for Health Statistics

- Plot these points on a graph.
- Connect the points with line segments.
- Find the slope of each line segment.
- Interpret each slope as a rate of change.

Writing & Thinking

- Explain in your own words why the slope of a horizontal line must be 0.
 - Explain in your own words why the slope of a vertical line must be undefined.

93. a. Describe the graph of the line $y = 0$.
b. Describe the graph of the line $x = 0$.
94. In the formula $y = mx + b$, explain the meaning of m and the meaning of b .
95. The slope of a road is called a **grade**. A steep grade is cause for truck drivers to have slow speed limits in mountains. What do you think that a “grade of 12%” means? Draw a picture of a right triangle that would indicate a grade of 12%.

Collaborative Learning

96. The class should be divided into teams of 2 or 3 students. Each team will need access to a digital camera, a printer, and a ruler.
- a. Take pictures of 8 things with a defined slope. (**Suggestions:** A roof, a stair railing, a beach umbrella, a crooked tree, etc. Be creative!)
- b. Print each picture.
- c. Use a ruler to draw a coordinate system on top of each picture. You will probably want to use increments of in. or cm, depending on the size of your picture.
- d. Identify the line in each picture whose slope you are calculating and then use the coordinate systems you created to identify the coordinates of two points on each line.
- e. Use the points you just found to calculate the slope of the line in each picture.
- f. Share your findings with the class.