The following properties can all be derived from the definition of absolute value.

Properties of Absolute Value

In these properties, a and b represent arbitrary real numbers.

1. $|a| \geq 0$ (The absolute value of a number is never negative.)
2. $|-a| = |a|$
3. $a \leq |a|$
4. $|ab| = |a||b|$
5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$
6. $|a + b| \leq |a| + |b|$ (This is called the **triangle inequality**, as it is a reflection of the fact that one side of a triangle is never longer than the sum of the other two sides.)

Example 7: Using Absolute Value Properties

a. $|(-3)(5)| = |-15| = 15 = |-3|\,|5|$

b. $1 = |-3 + 4| \leq |-3| + |4| = 7$

c. $7 = |-3 - 4| \leq |-3| + |-4| = 7$

d. $\left|\frac{-3}{7}\right| = \frac{|-3|}{|7|} = \frac{3}{7}$

0.1 EXERCISES

PRACTICE

Which elements of the following sets are **a.** natural numbers, **b.** whole numbers, **c.** integers, **d.** rational numbers, **e.** irrational numbers, **f.** real numbers, **g.** undefined? See Example 1.

1. $\left\{19,\ -4.3,\ -\sqrt{3},\ \frac{15}{0},\ \frac{0}{15},\ 2^5,\ -33\right\}$

2. $\left\{5\sqrt{7},\ 4\pi,\ \sqrt{16},\ 3.\bar{3},\ -1,\ \frac{22}{7},\ |-8|\right\}$

3. $\left\{5.41, |-16|, \frac{12}{3}, 0, \sqrt{4}, 2.\overline{145}, \frac{1}{4}\right\}$

4. $\left\{2\sqrt{25}, -4, 0.125, |32|, 2.1563, 6, \sqrt[3]{8}\right\}$

Plot the real numbers in the following sets on a number line. Choose the unit length appropriately for each set. See Example 2.

5. $\{-4.5, -1, 2.5\}$

6. $\{5.1, 5.2, 5.8\}$

7. $\{-24, 2, 15\}$

8. $\left\{0, \frac{1}{2}, \frac{5}{6}\right\}$

Select all of the symbols from the set $\{<, \leq, >, \geq\}$ that can be placed in the blank to make each statement true. See Example 3.

9. $12 \_\_\_\_ 14$

10. $-3.4 \_\_\_\_ -3.5$

11. $-102 \_\_\_\_ 9$

12. $3 \_\_\_\_ 3$

13. $-50 \_\_\_\_ -45$

14. $-\frac{1}{4} \_\_\_\_ -\frac{1}{3}$

15. $0.0087 \_\_\_\_ -42.9$

16. $\frac{2}{16} \_\_\_\_ 0.125$

17. $-7 \_\_\_\_ -9$

18. $-8 \_\_\_\_ 2$

Write each statement as an inequality, using the appropriate inequality symbol. See Example 3.

19. "$2a+b$ is strictly greater than c"

20. "2 is less than or equal to x"

21. "9 is greater than or equal to 7"

22. "7 is less than or equal to 9"

23. "$x+5$ is strictly less than 3"

24. "$2c$ is no more than $3d$"

25. "9 is no less than 8"

26. "$6+x$ is greater than or equal to $4x$"

Describe each of the following sets using set-builder notation. There may be more than one correct way to do this. See Example 4.

27. $\{-6, -3, 0, 3, 6, 9\}$

28. $\{5, 6, 7, \ldots, 105\}$

29. $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$

30. $\{1, 2, 4, 8, 16, 32, \ldots\}$

31. $\left\{\ldots, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$

32. $\{0, 1, 2, 3, 4, 5, \ldots\}$

Write each set as an interval using interval notation. See Example 5.

33. $x < 15$

34. $-9 \leq x \leq 6$

35. $2.5 < x \leq 3.7$

36. $\{x \mid -3 \leq x < 19\}$

37. $\{x \mid x < 4\}$

38. The positive real numbers

39. $\left\{x \,\middle|\, -\frac{1}{2} < x < \frac{2}{5}\right\}$

40. $\{x \mid 1 \leq x \leq 2\}$

41. The nonnegative real numbers

Graph the following intervals.

42. $[5, 14)$

43. $[-9, -1]$

44. $(0, 2)$

45. $(-3, 18]$

46. $(-\infty, 7]$

47. $(25, \infty)$

Evaluate the absolute value expressions. See Examples 6 and 7.

48. $-|-11|$

49. $|3-7|$

50. $-|4-9|$

51. $\left|\sqrt{3}-\sqrt{5}\right|$

52. $\sqrt{|-4|}$

53. $-\left|-4-|-11|\right|$

54. $\left|-\sqrt{2}\right|$

55. $\dfrac{|-x|}{|x|}\ (x \neq 0)$

56. $\left|(-7)(-5)\right|$

57. $-\left|\sqrt{16}-5\right|$

58. $\left|2-\sqrt{7}\right|$

59. $-\left|-\sqrt{|-9|}-|-9|\right|$

Find the distance on the real number line between each pair of numbers given. See Example 6.

60. $a = 8, b = 3$

61. $a = 6, b = 14$

62. $a = 5, b = 5$

63. $a = 4, b = -2$

64. $a = -7, b = 7$

65. $a = -12, b = -1$

APPLICATIONS

66. Jess, Stan, Nina, and Michele are in a marathon. Twenty-five minutes after beginning, Jess has run 3.4 miles, Stan has run 4 miles, Nina has run 2.25 miles, and Michele has walked 1.6 miles. Using 0 as the beginning point, plot each competitor's location on a real number line using an appropriate interval.

67. Freddie, Sarah, Elizabeth, JR, and Aubrey are trying to line up by height for a photo shoot. JR is the tallest and Elizabeth is the shortest. Freddie is taller than Sarah, and Sarah is taller than Aubrey. Express their line-up using appropriate inequality symbols.

68. Sue boards an eastbound train in Center Station at the same time Joy boards a westbound train in Center Station. After riding the Straight Line for 20 minutes, Sue's train has traveled 13 miles east, while Joy's train (also on the Straight Line) has traveled 7 miles west. Find the distance between the two trains at this time. (Assume the Straight Line is true to its name and that the tracks lie literally along a straight line.)

69. The admission prices at the local zoo are as follows.

Admission Prices

Children under 2	free
Children under 12	\$3
Adults	\$7
Seniors (65 and up)	\$5

Express the age range for each of these prices in set-builder notation and interval notation.

70. A particular fudge recipe calls for at least 3 but no more than 4 cups of sugar and at least $\frac{1}{2}$ but no more than $\frac{2}{3}$ of a cup of walnuts. Express the amount of sugar and nuts needed in both set-builder and interval notation.

WRITING & THINKING

71. Can a natural number be irrational? Explain.

72. Are all whole numbers also integers? Are all integers also whole numbers? Explain your answers.

73. In your own words, define absolute value.

74. Write a short paragraph explaining the similarities and differences between $>$ and $\geq$.

TECHNOLOGY

Select all of the symbols from the set $\{<, \leq, >, \geq\}$ that can be placed in the blank to make each statement true. Use a graphing utility to check your answers.

75. $-2.9 \underline{\quad\quad} -3.1$

76. $2.1 \underline{\quad\quad} -5.5$

77. $100 \underline{\quad\quad} -4$

78. $0.001 \underline{\quad\quad} -99.8$

79. $\frac{1}{3} \underline{\quad\quad} \frac{1}{4}$

80. $-\frac{1}{5} \underline{\quad\quad} -\frac{3}{4}$

c. $[3, 4) \cap (4, 9) = \varnothing$

These two intervals have no elements in common, so their intersection is the empty set.

d. $(-\infty, 4] \cup (-1, \infty) = (-\infty, \infty)$

The union of these two intervals is the entire set of real numbers.

Example 8: Union and Intersection

Simplify each of the following set expressions.

a. $\{1, 2\} \cup \{0, 3\}$

b. $\{x, y, z\} \cap \{w, x\}$

c. $\mathbb{Z} \cup \mathbb{R}$

d. $\mathbb{Z} \cap \mathbb{R}$

Solutions

a. The union of the two sets consists of all elements in either set: $\{0, 1, 2, 3\}$.

b. The intersection consists only of elements in both sets: $\{x\}$.

c. Since the integers are all also real numbers, the union of these two sets is simply the set of real numbers $\mathbb{R}$. We say that $\mathbb{Z}$ is *contained* in $\mathbb{R}$.

d. Similarly, since all integers are also real numbers, the integers are the elements contained in both sets. Thus, the intersection is $\mathbb{Z}$.

0.2 EXERCISES

PRACTICE

Identify the components of the algebraic expressions, as indicated. See Example 1.

1. Identify the terms in the expression $3x^2y^3 - 2\sqrt{x+y} + 7z$.
2. Identify the coefficients in the expression $3x^2y^3 - 2\sqrt{x+y} + 7z$.
3. Identify the factors in the term $-2\sqrt{x+y}$.
4. Identify the terms in the expression $x^2 + 8.5x - 14y^3$.
5. Identify the coefficients in the expression $x^2 + 8.5x - 14y^3$.
6. Identify the factors in the term $8.5x$.
7. Identify the terms in the expression $\frac{-5x}{2yz} - 8x^5y^3 + 6.9z$.
8. Identify the coefficients in the expression $\frac{-5x}{2yz} - 8x^5y^3 + 6.9z$.
9. Identify the factors in the term $\frac{-5x}{2yz}$.

Evaluate the following algebraic expressions for the given values of the variables. See Example 2.

10. $3x^3+5x-2$ for $x=-3$

11. $-8(2x-y)+4x^2$ for $x=3$ and $y=4$

12. $\sqrt{2x}+\frac{3x}{4}$ for $x=8$

13. $3x^2y^3-2\sqrt{x+y}+7z$ for $x=-1, y=2,$ and $z=-2$

14. $-3\pi y+8x+y^3$ for $x=2$ and $y=-2$

15. $\frac{|x|\sqrt{2}}{x^3y^2}-\frac{3y}{x}$ for $x=-3$ and $y=2$

16. $y\sqrt{x^3-2}+\sqrt{x-2y}-3y$ for $x=3$ and $y=-\frac{1}{2}$

17. $\left|-x^2+2xy-y^2\right|$ for $x=-3$ and $y=-5$

18. $\frac{1}{32}x^2y^3+y\sqrt{x}-7y$ for $x=4$ and $y=2$

19. $6x^2+3\pi y+y^2$ for $x=3$ and $y=2$

20. $|x-9y|-(8z-8)$ for $x=-3, y=1,$ and $z=5$

21. $\frac{x^2y^3}{8z}-\frac{|2xy|}{8z}$ for $x=2, y=-1,$ and $z=3$

22. $5\sqrt{x+6}-8y^2$ for $x=10$ and $y=-2$

Identify the property that justifies each of the following statements. If one of the cancellation properties is being used to transform an equation, identify the quantity that is being added to both sides or the quantity by which both sides are being multiplied. See Examples 3 and 5.

23. $(x-y)(z^2)=(z^2)(x-y)$

24. $3-7=-7+3$

25. $(3x+2)+z=3x+(2+z)$

26. $4(y-3)=4y-12$

27. $-3(4x^6z)=(-3)(4)(x^6z)=-12x^6z$

28. $4+(-3+x)=(4-3)+x=1+x$

29. $-2(4-x)=-8+2x$

30. $(x+y)\left(\frac{1}{x+y}\right)=1$

31. $(-5+1)(7^7)=(7^7)(-5+1)$

32. $-5(-7x^8y^4z)=[(-5)(-7)](x^8y^4z)$

33. $25x^3=10y \Leftrightarrow 5x^3=2y$

34. $-14y=7 \Leftrightarrow y=-\frac{1}{2}$

35. $14-x=2x \Leftrightarrow 14=3x$

36. $5+3x-y=2x-y \Leftrightarrow 5+x=0$

37. $x^2z=0 \Rightarrow x^2=0$ or $z=0$

38. $(a+b)(x)=0 \Rightarrow a+b=0$ or $x=0$

39. $\frac{x}{6}+\frac{y}{3}-2=0 \Leftrightarrow x+2y-12=0$

40. $(x-3)(x+2)=0 \Rightarrow x-3=0$ or $x+2=0$

41. $21x^4 = 15y^4z \Leftrightarrow 7x^4 = 5y^4z$

42. $6x + \frac{25}{4}y^9 - z = \frac{1}{4}y^9 - z \Leftrightarrow 6x + 6y^9 = 0$

Evaluate each of the following expressions. Be sure to use the correct order of operations. See Example 6.

43. $2 + 3 - 4 \div 8 + (-1)^2$

44. $\dfrac{-2\left(13 - \sqrt{9} + 2\right)}{14 - 4 \div 2}$

45. $-3^2 - 2 \div 2$

46. $\left(-3^2 - 2\right) \div 2$

47. $\dfrac{\sqrt{\sqrt{81} + 4^2}}{10(4 - 7 \div 2)}$

48. $4\pi + 6^{\sqrt{5 - \frac{2}{2}}} - 3\pi\left[8 - 15 \div (2 + 3)\right]$

49. $4 - 10 \cdot (-1) \div 5 + (-8)^2$

50. $-3^2 + 2 \cdot \sqrt{2 + 1 \cdot 2} - 7\pi$

51. $1 \div 6 + 3^{\sqrt{2^2}} - (-4 \cdot 2)$

52. $\dfrac{8 - 9 \cdot 5 - 7}{-4\left(-9 - 5 \div (2 + 4)\right)}$

53. $-3 + 6 \cdot 1 \div 5 + (-3)^3$

54. $-5^2 + 4 \cdot \sqrt{2 + 7 \cdot 2} - 2\pi$

55. $9 \div 2 + 2^{\sqrt{2^4}} - (1 \cdot 2)$

56. $\dfrac{4 + 3 \cdot 8 - 6}{-5\left(3 - 8 \div (2 + 5)\right)}$

Use a calculator to evaluate each of the following expressions. Be sure to use the correct order of operations. Round your answers to two decimal places. See Example 6.

57. $(-3.28)^2 + 4 \cdot \sqrt{2 + 7 \cdot 3} - 2\pi$

58. $2.66 - 7 \cdot 4 \div 5 + (2 \div 3)^2$

59. $\dfrac{7.6 - 5.2 \cdot 9.8 - 8.1}{-3.22\left(11 - 6 \div (-1.45 + 6.32)\right)}$

60. $7 \div 4.6 + 2.4^{\sqrt{3}} - (1.23 \cdot 2)^4$

Translate each of the following directions into an algebraic expression.

61. Begin with 3. Add 7, and multiply the result by 3. Subtract 5. Take the square root, raise the result to the 3rd power, and then multiply by $-\frac{1}{5}$.

62. Begin with –6. Add 4, raise the result to the 3rd power, multiply by –2, and take the fourth root of the result.

63. Begin with x. Subtract 4, and take the third root of the result. Divide by 2, and square the result.

Simplify the following set expressions. See Examples 7 and 8.

64. $[-7, 7) \cup (2, 5)$

65. $(-5, 2] \cup (2, 4]$

66. $(-5, 2] \cap (2, 4]$

67. $[3, 5] \cap [2, 4]$

68. $(-\infty, 4] \cup (0, \infty)$

69. $(-\infty, \infty) \cap [-\pi, 21)$

70. $[2, \infty) \cap (-4, 7) \cap (-3, 2]$

71. $(3, 5] \cup [5, 9]$

72. $[-\pi, 2\pi) \cap [0, 4\pi]$

73. $\mathbb{Q} \cap \mathbb{Z}$

74. $\mathbb{N} \cup \mathbb{R}$

75. $\mathbb{N} \cup \mathbb{Z} \cap \mathbb{Q}$

76. $(-4.8, -3.5) \cap \mathbb{Z}$

APPLICATIONS

77. At the beginning of the month, your checking account contains \$128. For your birthday, your mother deposits \$50 and your grandmother deposits \$25. After you write three checks for \$17, \$23, and \$62, you make a deposit of \$41. At the end of the month, your bank removes half of the balance to put in your savings account and then charges you a \$5 fee for doing so. How much do you have remaining in your checking account?

78. A particular liquid boils at 268 °F. Given the formula $C = \frac{5}{9}(F - 32)$ for converting temperatures from Celsius (C) to Fahrenheit (F), find the boiling point of this liquid in the Celsius scale. Round your answer to two decimal places.

79. Stephen received \$75 as a gift from his aunt. With this money, he decided to start saving to buy the newest gaming console, which costs \$398 after tax. After working two weeks at his part-time job, he got one check for \$123 and a second check for \$98. How much more does Stephen need to save to buy his gaming console?

80. Body mass index, abbreviated BMI, is one way doctors determine an adult's weight status. A BMI below 18.5 is considered underweight, the range 18.5–24.9 is normal, the range 25.0–29.9 is overweight, and a BMI above 30.0 indicates obesity. The formula used to determine BMI is $\text{BMI} = 703\left(\frac{\text{weight in pounds}}{(\text{height in inches})^2}\right)$. Derek weighs 180 lbs and is 73 inches tall. Use this formula to determine Derek's BMI and weight status. Round your answer to one decimal place.

81. The Du Bois Method provides a formula used to estimate your body's surface area in meters squared: $\text{BSA} = 0.007184h^{0.725}w^{0.425}$, where h is height in centimeters and w is weight in kilograms. Assume Juan is 193 cm tall and weighs 88 kg. Use the Du Bois Method to estimate his body's surface area in square meters. Round your answer to two decimal places.

82. Samantha drops a tennis ball from the top of the mathematics building. If it takes the ball 3.42 seconds to hit the ground, use the formula $\text{distance} = \frac{1}{2}(\text{acceleration})(\text{time})^2$ to find the height of the building, which is equivalent to the distance the ball falls. Use the value of $32\ \text{ft/s}^2$ for the acceleration of a falling object. Round your answer to the nearest foot.

WRITING & THINKING

83. Choose a number. Multiply it by 3 and then add 4. Now multiply by 2 and subtract 8. Finally divide by 6. What do you notice about your final answer? Explain why you got this as a result.

84. Use your knowledge of the order of operations to check the following problem for accuracy. Explain any errors you find.

$$\begin{aligned} -8 \div 4 + 2^3 - (3 \cdot 2) &= -8 \div 4 + 2^3 - (6) \\ &= -8 \div 4 + 8 - 6 \\ &= -8 \div 4 + 2 \\ &= -8 \div 6 \\ &= \frac{-4}{3} \end{aligned}$$

85. A mnemonic is a device used to recall particular information. For example, "*My Very Educated Mother Just Served Us Nachos*" is often used to recall the order of the planets in our solar system: *My* = Mercury, *Very* = Venus, *Educated* = Earth, and so on. Come up with your own mnemonic for remembering the order of operations.

86. After taking a poll in her town, Sally began grouping the citizens into various sets. One set contained all the citizens with brown hair and another set contained all the citizens with blue eyes. What do you know about the citizens who would be listed in the union of these two sets? What do you know about the citizens who would be listed in the intersection of these two sets?

87. In your own words, explain the difference between a union and an intersection of two sets.

TECHNOLOGY

Use a graphing utility to evaluate the following algebraic expressions.

88. $\sqrt{x^4 y - z} + \dfrac{x - y^3}{z^2}$ for $x = -3$, $y = 2$, and $z = -2$

89. $\dfrac{(x - pq^2)^3}{2q^3}$ for $x = -5$, $p = 2$, and $q = -3$

90. $\dfrac{|x^2 - y^3| - 4x}{3y^5}$ for $x = 2$ and $y = 3$

91. $\sqrt{p^3 q - q^3} - |p + q^2|$ for $p = -5$ and $q = 2$

⚠ CAUTION

Many errors can be made in applying the properties of exponents as a result of forgetting the exact form of the properties. The first column below contains examples of some common errors. The second column contains the corrected statements.

Incorrect	Correct
$x^2x^5 = x^{10}$	$x^2x^5 = x^{2+5} = x^7$
$2^4 2^3 = 4^7$	$2^4 2^3 = 2^{4+3} = 2^7$
$(3+4)^2 = 3^2 + 4^2$	$(3+4)^2 = 7^2$
$(x^2+3y)^{-1} = \frac{1}{x^2} + \frac{1}{3y}$	$(x^2+3y)^{-1} = \frac{1}{x^2+3y}$
$(3x)^2 = 3x^2$	$(3x)^2 = 3^2x^2 = 9x^2$
$\frac{x^5}{x^{-2}} = x^3$	$\frac{x^5}{x^{-2}} = x^{5-(-2)} = x^7$

c. $$\frac{\left(-2x^3y^{-1}\right)^{-3}}{\left(18x^{-3}\right)^0\left(xy\right)^{-2}} = \frac{\left(-2\right)^{-3}x^{-9}y^3}{x^{-2}y^{-2}}$$

Begin by applying Property 4 in the numerator, Property 5 in the denominator.

$$= \left(-2\right)^{-3}x^{-9-(-2)}y^{3-(-2)}$$

Then simplify using Property 2.

$$= \left(-2\right)^{-3}x^{-7}y^5$$

$$= \frac{y^5}{-8x^7}$$

Unlike the previous example, Property 3 gets applied at the very end.

$$= -\frac{y^5}{8x^7}$$

d. $$\left(7xz^{-2}\right)^2\left(5x^2y\right)^{-1} = \frac{49x^2z^{-4}}{5x^2y}$$

$$= \frac{49}{5yz^4}$$

Note that the variable x no longer appears in the expression.

0.3 EXERCISES

PRACTICE

Simplify each of the following expressions, writing your answer with only positive exponents. See Examples 1 and 2.

1. $(-2)^4$ **2.** -2^4 **3.** -3^2 **4.** $(-3)^2$

5. $3^2 \cdot 3^2$ **6.** $2^3 \cdot 3^2$ **7.** $4 \cdot 4^2$ **8.** $(-3)^3$

9. $\frac{8^2}{4^3}$ **10.** $2^2 \cdot 2^3$ **11.** $\frac{7^4}{7^5}$ **12.** $n^2 \cdot n^5$

13. $\frac{x^5}{x^2}$ **14.** $\frac{y^3 \cdot y^8}{y^2}$ **15.** $\frac{3^7}{3^4 s^{-10}}$

Use the properties of exponents to simplify each of the following expressions, writing your answer with only positive exponents. See Examples 1, 2, and 3.

16. $\frac{3t^{-2}}{t^3}$ **17.** $-2y^0$ **18.** $\frac{1}{7x^{-5}}$

19. $9^0x^3y^0$ **20.** $\frac{2n^3}{n^{-5}}$ **21.** $\frac{11^{21}}{11^{19}x^{-7}}$

22. $\frac{x^7y^{-3}z^{12}}{x^{-1}z^9}$ **23.** $\frac{x^4\left(-x^{-3}\right)}{-y^0}$ **24.** $\frac{s^3}{s^{-2}}$

25. $\frac{x^{-1}}{x}$ **26.** $x^{\left(y^0\right)}x^9$ **27.** $\frac{x^2y^{-2}}{x^{-1}y^{-5}}$

28. $\dfrac{s^5 y^{-5} z^{-11}}{s^8 y^{-7}}$

29. $\dfrac{2^7 s^{-3}}{2^3}$

30. $\dfrac{3^{-5}}{\left(3^{-4} x^5 y^4\right)^2}$

31. $\dfrac{-9^0\left(x^2 y^{-2}\right)^{-3}}{3x^{-4} y}$

32. $\left[\left(2x^{-1} z^3\right)^{-2}\right]^{-1}$

33. $\dfrac{\left(3yz^{-2}\right)^0}{3y^2 z}$

34. $\left(12a^2 - 3b^4\right)^0$

35. $\dfrac{3^{-1}}{\left(3^2 xy^2\right)^{-2}}$

36. $\left[9m^2 - \left(2n^2\right)^3\right]^{-1}$

37. $\left[\left(12x^{-6} y^4 z^3\right)^5\right]^0$

38. $\dfrac{x\left(x^{-2} y^3\right)^3}{\left(2x^4\right)^{-2} y}$

39. $\dfrac{(-3a)^{-2}\left(bc^{-2}\right)^{-3}}{a^5 c^4}$

40. $\left[\left(5m^4 n^{-2}\right)^{-1}\right]^{-2}$

41. $\left(9x^{-1} z\right)^2\left(2xy^{-3}\right)^{-1}$

42. $\left(4^{-2} x^5 y^{-3} z^4\right)^{-2}$

43. $\left[\left(4a^2 b^{-5}\right)^{-1}\right]^{-3}$

44. $\left[\left(2^{-3} m^{-6} n^3\right)^3\right]^{-1}$

45. $\left[\left(3^{-1} x^{-1} y\right)\left(x^2 y\right)^{-1}\right]^{-3}$

46. $\left[\dfrac{100^0\left(x^{-1} y^3\right)^{-1}}{x^2 y}\right]^{-3}$

47. $\left(5z^6 - \left(3x^3\right)^4\right)^{-1}$

48. $\left[\dfrac{y^6\left(xy^2\right)^{-3}}{3x^{-3} z}\right]^{-2}$

WRITING & THINKING

49. In your own words, explain why $a^0 = 1$.

Apply the definition of integer exponents to demonstrate the following properties.

50. $a^n \cdot a^m = a^{n+m}$

51. $\left(a^n\right)^m = a^{nm}$

52. $(ab)^n = a^n b^n$

Solution

$$\begin{aligned}\frac{\sqrt{4x}-\sqrt{6y}}{2x-3y} &= \left(\frac{\sqrt{4x}-\sqrt{6y}}{2x-3y}\right)\left(\frac{\sqrt{4x}+\sqrt{6y}}{\sqrt{4x}+\sqrt{6y}}\right)\\ &= \frac{4x-6y}{(2x-3y)\left(\sqrt{4x}+\sqrt{6y}\right)}\\ &= \frac{2(2x-3y)}{(2x-3y)\left(2\sqrt{x}+\sqrt{6y}\right)}\\ &= \frac{2}{2\sqrt{x}+\sqrt{6y}}\end{aligned}$$

Multiply both the numerator and denominator by the conjugate of the numerator.

Note that we could have begun by simplifying the term $\sqrt{4x}$. The final answer is the same.

0.4 EXERCISES

PRACTICE

Evaluate the following radical expressions. See Example 1.

1. $-\sqrt{9}$ **2.** $\sqrt[3]{-27}$ **3.** $\sqrt{-25}$ **4.** $\sqrt[6]{-64}$

5. $-\sqrt[6]{64}$ **6.** $-\sqrt{169}$ **7.** $\sqrt[3]{-125}$ **8.** $\sqrt{-49}$

9. $\sqrt[4]{-256}$ **10.** $-\sqrt[3]{-64}$ **11.** $\sqrt[3]{-\frac{27}{125}}$ **12.** $\sqrt{\frac{25}{121}}$

13. $\sqrt[3]{\frac{-8}{64}}$ **14.** $\sqrt{\frac{1}{4}}$ **15.** $-\sqrt[3]{-8}$

16. $\sqrt[4]{\sqrt{16}-\sqrt[3]{-27}+\sqrt{81}}$ **17.** $\sqrt{\frac{\sqrt[3]{-64}}{-\sqrt{144}-\sqrt{169}}}$

18. $\sqrt{\sqrt[3]{64}+\sqrt[4]{81}+\sqrt[5]{32}}$

Simplify the following radical expressions. See Example 3.

19. $\sqrt{9x^2}$ **20.** $\sqrt[3]{-8x^6y^9}$ **21.** $\sqrt[4]{\frac{x^8z^4}{16}}$ **22.** $\sqrt{2x^6y}$

23. $\sqrt[7]{x^{14}y^{49}z^{21}}$ **24.** $\sqrt{\frac{x^2}{4x^4y^6}}$ **25.** $\sqrt[3]{\frac{a^3b^{12}}{27c^6}}$ **26.** $\sqrt[3]{-125x^{12}y^9}$

27. $\sqrt[4]{\frac{x^{12}y^8}{16}}$ **28.** $\sqrt[3]{81m^4n^7}$ **29.** $\sqrt[5]{\frac{y^{30}z^{25}}{32x^{35}}}$ **30.** $\sqrt[5]{32x^7y^{10}}$

Simplify the following radicals by rationalizing the denominators. See Example 4.

31. $\sqrt[3]{\dfrac{4x^2}{3y^4}}$ **32.** $\dfrac{-\sqrt{3a^3}}{\sqrt{6a}}$ **33.** $\dfrac{3}{\sqrt{2}-\sqrt{5}}$ **34.** $\dfrac{10}{\sqrt{7}-\sqrt{2}}$

35. $\dfrac{3}{\sqrt{6}-\sqrt{3}}$ **36.** $\dfrac{5}{6-\sqrt{5}}$ **37.** $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{2}}$ **38.** $\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$

39. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ **40.** $\dfrac{1}{2-\sqrt{x}}$ **41.** $\dfrac{\sqrt{y}}{\sqrt{y}+2}$ **42.** $-\dfrac{\sqrt{6y^7}}{\sqrt{5y}}$

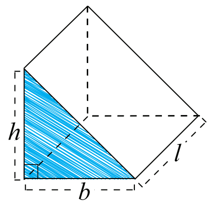
Rationalize the numerators of the following expressions. See Example 5.

43. $\dfrac{\sqrt{5}-3}{-4}$ **44.** $\dfrac{\sqrt{7}-6}{7}$ **45.** $\dfrac{3+\sqrt{y}}{6}$ **46.** $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$

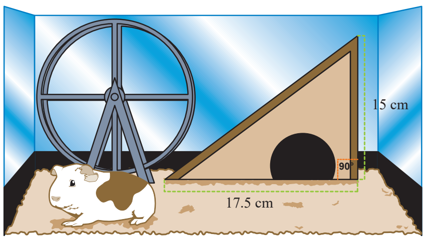
47. $\dfrac{\sqrt{13}+\sqrt{t}}{13-t}$ **48.** $\dfrac{2\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ **49.** $\dfrac{\sqrt{6}+\sqrt{y}}{\sqrt{6}-\sqrt{y}}$ **50.** $\dfrac{4\sqrt{xy}+y}{x-y}$

APPLICATIONS

51. The prism shown below is a right triangular cylinder, where the base is a right triangle. Find the surface area of the prism in terms of b, h, and l.



52. Terri has made a home for her pet guinea pig (Ralph) in the shape of a right triangular cylinder. Before she can put the new home in Ralph's cage, she must paint it with a nontoxic outer coat. If the front of the home has a base of 17.5 cm and a height of 15 cm and the length of the home is 25 cm, what is the surface area of Ralph's home, rounded to the nearest square centimeter? The small bottle of nontoxic coating will cover up to 1500 cm^2. Will the small bottle contain enough nontoxic coating to cover Ralph's home?



WRITING & THINKING

53. Explain, in your own words, why the square root of a negative number is not a real number.

0.5 EXERCISES

PRACTICE

Combine the radical expressions, if possible. See Example 1.

1. $\sqrt[3]{-16x^4} + 5x\sqrt[3]{2x}$

2. $\sqrt{27xy^2} - 4\sqrt{3xy^2}$

3. $\sqrt{7x} - \sqrt[3]{7x}$

4. $|x|\sqrt{8xy^2z^3} - |yz|\sqrt{18x^3z}$

5. $-x^2\sqrt[3]{54x} + 3\sqrt[3]{2x^7}$

6. $\sqrt[5]{32x^{13}} + 3x\sqrt[5]{x^8}$

7. $\sqrt[3]{-16z^4} + 6z\sqrt[3]{2z}$

8. $\sqrt[3]{7y} - \sqrt[4]{7y}$

9. $-x^2\sqrt[3]{16x} + 2\sqrt[3]{2x^7}$

Simplify the following expressions, writing your answer using the same notation as the original expression. See Example 2.

10. $\sqrt[3]{\sqrt[4]{x^{36}}}$

11. $\left(3x^2-4\right)^{\frac{1}{3}}\left(3x^2-4\right)^{\frac{5}{3}}$

12. $32^{\frac{-3}{5}}$

13. $81^{\frac{3}{4}}$

14. $\dfrac{(x-z)^y}{(x-z)^4}$

15. $\sqrt[7]{n^9}\cdot\sqrt[7]{n^5}$

16. $(-8)^{\frac{2}{3}}$

17. $\dfrac{x^{\frac{1}{5}}y^{\frac{-2}{3}}}{x^{\frac{-3}{5}}y}$

18. $(1024)^{-\frac{2}{5}}$

19. $(625)^{-\frac{3}{4}}$

20. $\sqrt[8]{49a^2}$

21. $\sqrt[3]{\sqrt[5]{y^{25}}}$

22. $\dfrac{(a-b)^{-\frac{2}{3}}}{(a-b)^{-2}}$

23. $\left(ax^2+by\right)^{\frac{3}{4}}\left(by+ax^2\right)^{-\frac{2}{3}}$

24. $\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^5}}$

Convert the following expressions from radical notation to exponential notation, or vice versa. Simplify each expression in the process, if possible.

25. $\sqrt[4]{a^3}\cdot\sqrt[3]{a^9}$

26. $256^{-\frac{3}{4}}$

27. $\sqrt[12]{x^3}$

28. $\left(9y^2\right)^{\frac{3}{2}}\left(y^6\right)^{\frac{5}{3}}$

29. $\sqrt[6]{\dfrac{2}{72}}$

30. $\left(36n^4\right)^{\frac{5}{6}}$

Simplify the following expressions. See Example 3.

31. $\sqrt{5}\cdot\sqrt[4]{5}$

32. $\sqrt[4]{25}$

33. $\sqrt[16]{y^4}$

34. $\sqrt[4]{36}$

35. $\sqrt[3]{x^7}\cdot\sqrt[9]{x^6}$

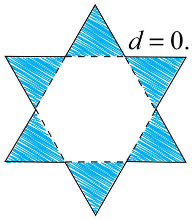
36. $\sqrt[5]{y^{16}}\cdot\sqrt[25]{y^{20}}$

37. $\sqrt[4]{7}\cdot\sqrt[16]{7}$

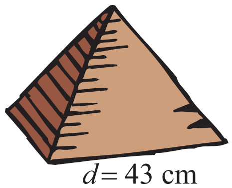
38. $\sqrt{y^4}\cdot\sqrt[6]{y^3}$

APPLICATIONS

39. A jeweler decides to construct a pendant for a necklace by simply attaching equilateral triangles to each edge of a regular hexagon. The edge length of one of the points of the resulting star is $d = 0.8$ cm. Find the formula for the area of the star in terms of d and then evaluate for $d = 0.8$ cm (rounding to three decimal places). Remember that the area of an equilateral triangle of side length d is $A = \frac{d^2\sqrt{3}}{4}$. See Example 4.



40. The pyramids in Egypt each consist of a square base and four triangular sides. For a class project, Karim constructs a model pyramid with equilateral triangles as sides. The side length is $d = 43$ cm. Find the total surface area of the pyramid (rounding to the nearest square centimeter). See Example 4.



WRITING & THINKING

41. Explain, in your own words, why exponents and roots are evaluated at the same time in the order of operations.

Apply the definition of rational exponents to demonstrate the following properties.

42. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

43. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

44. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

b. $x^2 + 12x + 36$

Solution

$x^2 + 12x + 36$ is a perfect square trinomial with $A = \frac{1}{2}(12) = 6$ and $A^2 = 6^2 = 36$:

$$x^2 + 12x + 36 = (x + 6)^2.$$

c. $x^3 - 125$

Solution

$x^3 - 125$ is the difference of two cubes with $A^3 = 5^3$:

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25).$$

0.6 EXERCISES

PRACTICE

In Exercises 1–14, perform the indicated operation and simplify the expressions.

1. $(x+3x^2)+(5-x^2)$

2. $(x^2+2x-4)+(x^2-4)$

3. $(8a^2+5a+2)+(-3a^2+9a-4)$

4. $(3x^2+5x-4)+(2x^2-2x+4)$

5. $(2x^2+3x+8)-(x^2-5x+6)$

6. $(4x^3-7x^2+3x)-(-2x^3+5x-1)$

7. $(8x^2+9)-(4x^2-3x-2)$

8. $(y^3+4y^2-7)-(3y^3+y^2+2y+1)$

9. $(a^2-3ab+b^2)+(2a^2-5ab-b^2)$

10. $(7x^2-2xy+3y^2)+(-3x^2-2xy+5y^2)$

11. $(-3x^2-2xy+5y^2)-(4x^2+3xy)$

12. $(5x^2-3xy+7y^2)-(6x^2-9xy+8y^2)$

13. $2x^2(3x^2+5x-1)$

14. $-4y^2(2y^2+5y-4)$

Fill in the missing expressions in Exercises 15–17.

15. $(2x+3y)^2 = 4x^2 + \_\_\_\_ + 9y^2$

16. $(9x-5y)^3 = 729x^3 - 3(\_\_\_\_)x^2y + 3(\_\_\_\_)xy^2 - 125y^3$

17. $(3x^2+8y)^2 = 9(\_\_\_\_) + 48(\_\_\_\_) + 64(\_\_\_\_)$

In Exercises 18–33, find the products.

18. $(3x-8)(x-5)$

19. $(7x+6)(2x-3)$

20. $(5x+11)(3x-4)$

21. $(3x-4)(4x-3)$

22. $(3x+1)^2$

23. $(4x-3)^2$

24. $(7x-4y)^2$

25. $(3x+2y)^2$

26. $(4x-5)(4x+5)$

27. $(6x+y)(6x-y)$

28. $3x^2(1+3x)$

29. $2x(x^2+3x-4)$

30. $(x+2)(x^2-2x+4)$

31. $(x+3)(x^2-3x+9)$

32. $(y-5)(y^2+5y+25)$

33. $(x+2y)(x^2-2xy+4y^2)$

In Exercises 34–39, simplify each expression.

34. $5a+2(a-3)-(3a+7)$

35. $11+[3x-2(1+5x)]$

36. $3y-[5-7(y+2)-6y]$

37. $10t-[8-5(3-2t)-7t]$

38. $x(x-5)+[6x-x(4-x)]$

39. $x(2x+1)-[5x-x(2x+3)]$

Fill in the missing expressions in Exercises 40 and 41.

40. $11x^2-99y^2=11(x-\_\_\_\_)(x+\_\_\_\_)$

41. $16x^3+54y^3=2(2x+\_\_\_\_)(\_\_\_\_x^2-\_\_\_\_xy+9y^2)$

In Exercises 42–60, factor each expression completely. (Each factor should have integer coefficients.)

42. $x^2+6x-27$

43. $s^2-5s-14$

44. $x^2+27x+50$

45. $x^2+11x-26$

46. $2x^2-98$

47. $4b^3-64b$

48. $9y^3-16y$

49. $27a^2-12$

50. $x^2+6xy+9y^2$

51. x^6-1

52. x^5-x^3

53. $25x^8-16$

54. $125y^6-27z^3$

55. $2t^3+16y^3$

56. $1600x^2+880xy+121y^2$

57. s^4-1

58. $3a^2+12ab+12b^2$

59. $100xy^2+200xy+100x$

60. $x^{21}-x^{19}$

WRITING & THINKING

61. If you were teaching Algebra I to ninth grade students, how would you explain the difference between a variable and a constant in an algebraic expression?

c. $5(x+1)-6x=-x+5$

Solution

$$\begin{aligned} 5(x+1)-6x &= -x+5 && \text{Use the distributive property.} \\ 5x+5-6x &= -x+5 && \text{Simplify.} \\ -x+5 &= -x+5 && \text{Add } x \text{ to both sides.} \\ 5 &= 5 \end{aligned}$$

The last equation is always true. Therefore, the original equation is an identity and has an infinite number of solutions. Every real number is a solution.

1.1 EXERCISES

PRACTICE

Solve each equation. See Examples 1 through 4.

1. $3x+11=2$

2. $3x+10=-5$

3. $5x-4=6$

4. $4y-8=-12$

5. $6x+10=22$

6. $3n+7=19$

7. $9x-5=13$

8. $2x-4=12$

9. $1-3y=4$

10. $5-2x=9$

11. $14+9t=5$

12. $5+2x=-7$

13. $-5x+2.9=3.5$

14. $3x+2.7=-2.7$

15. $10+3x-4=18$

16. $5+5x-6=9$

17. $15=7x+7+8$

18. $14=9x+5+8$

19. $5y-3y+2=2$

20. $6y+8y-7=-7$

21. $x-4x+25=31$

22. $3y+9y-13=11$

23. $-20=7y-3y+4$

24. $-20=5y+y+16$

25. $4n-10n+35=1-2$

26. $-5n-3n+2=34$

27. $3n-15-n=1$

28. $2n+12+n=0$

29. $5.4x-0.2x=0$

30. $0=5.1x+0.3x$

31. $\frac{1}{2}x+7=\frac{7}{2}$

32. $\frac{3}{5}x+4=\frac{9}{5}$

33. $\frac{1}{2}-\frac{8}{3}x=\frac{5}{6}$

34. $\frac{2}{5}-\frac{1}{2}x=\frac{7}{4}$

35. $\frac{3}{2}=\frac{1}{3}x+\frac{11}{3}$

36. $\frac{11}{8}=\frac{1}{5}x+\frac{4}{5}$

37. $\frac{7}{2}-5-\frac{5}{2}x=9$

38. $\frac{8}{3}+2-\frac{7}{3}x=6$

39. $\frac{5}{8}x-\frac{1}{4}x+\frac{1}{2}=\frac{3}{10}$

40. $\frac{1}{2}x+\frac{3}{4}x-\frac{5}{3}=\frac{5}{6}$

41. $\frac{y}{2}+\frac{1}{5}=3$

42. $\frac{y}{3}-\frac{2}{3}=7$

43. $\frac{7}{8}=\frac{3}{4}x-\frac{5}{8}$

44. $\frac{1}{10}=\frac{4}{5}x+\frac{3}{10}$

45. $\frac{y}{7}+\frac{y}{28}+\frac{1}{2}=\frac{3}{4}$

46. $\frac{5y}{6}-\frac{7y}{8}-\frac{1}{12}=\frac{1}{3}$

47. $x+1.2x+6.9=-3.0$

48. $3x-0.75x-1.72=3.23$

49. $10=x-0.5x+32$

50. $33=y+3-0.4y$

51. $2.5x+0.5x-3.5=2.5$

52. $4.7-0.5x-0.3x=-0.1$

53. $6.4+1.2x+0.3x=0.4$

54. $5.2-1.3x-1.5x=-0.4$

55. $-12.13=2.42y+0.6y-13.64$

56. $-7.01=1.75x+3.05x-8.45$

57. $-0.4x+x+17.2=18.1$

58. $y-0.75y+13.76=14.66$

59. $0=17.3x-15.02x-0.456$

60. $0=20.5x-16.35x+0.1245$

61. $3x+2=x-8$

62. $5x+1=2x-5$

63. $4n-3=n+6$

64. $6y+3=y-7$

65. $3y+18=7y-6$

66. $2y+5=8y+10$

67. $3x+11=8x-4$

68. $9x+3=5x-9$

69. $14n=3n$

70. $1.6x=0.8x$

71. $6y-2.1=y-2.1$

72. $13x+5=2x+5$

73. $2(z+1)=3z+3$

74. $6x-3=3(x+2)$

75. $16y+23y-3=16y-2y+2$

76. $5x-2x+4=3x+x-1$

77. $0.25+3x+6.5=0.75x$

78. $0.9y+3=0.4y+1.5$

79. $6.5 + 1.2x = 0.5 - 0.3x$

80. $x - 0.1x + 0.8 = 0.2x + 0.1$

81. $\frac{2}{3}x + 1 = \frac{1}{3}x - 6$

82. $\frac{4}{5}n + 2 = \frac{2}{5}n - 4$

83. $\frac{y}{5} + \frac{3}{4} = \frac{y}{2} + \frac{3}{4}$

84. $\frac{5n}{6} + \frac{1}{9} = \frac{3n}{2} + \frac{1}{9}$

85. $\frac{3}{8}\left(y - \frac{1}{2}\right) = \frac{1}{8}\left(y + \frac{1}{2}\right)$

86. $\frac{1}{2}\left(\frac{x}{2} + 1\right) = \frac{1}{3}\left(\frac{x}{2} - 1\right)$

87. $\frac{2x}{3} + \frac{x}{3} = -\frac{3}{4} + \frac{x}{2}$

88. $\frac{3}{4}x + \frac{1}{5}x = \frac{1}{2}x - \frac{3}{10}$

89. $x + \frac{2}{3}x - 2x = \frac{x}{6} - \frac{1}{8}$

90. $3x + \frac{1}{2}x - \frac{2}{5}x = \frac{x}{10} + \frac{7}{20}$

91. $3(1 + 9x) = 6(2 - 4x)$

92. $4(5 - x) = 8(3x + 10)$

93. $3(4x - 1) = 4(2x - 3) + 8$

94. $7(2x - 1) = 5(x + 6) - 13$

95. $5 - 3(2x + 1) = 4(x - 5) + 6$

96. $-2(y + 5) - 4 = 6(y - 2) + 2$

97. $8 + 4(2x - 3) = 5 - (x + 3)$

98. $8(3x + 5) - 9 = 9(x - 2) + 14$

99. $4.7 - 0.3x = 0.5x - 0.1$

100. $5.8 - 0.1x = 0.2x - 0.2$

101. $0.2(x + 3) = 0.1(x - 5)$

102. $0.4(x + 3) = 0.3(x - 6)$

103. $\frac{1}{2}(4 - 8x) = \frac{1}{3}(4x + 7) - 3$

104. $3 + \frac{1}{4}(x - 4) = \frac{2}{5}(2 + 3x)$

105. $0.6x - 22.9 = 1.5x - 18.4$

106. $0.1y + 3.8 = 5.72 - 0.3y$

107. $0.12n + 0.25n - 5.895 = 4.3n$

108. $0.15n + 32n - 21.0005 = 10.5n$

109. $0.7(x + 14.1) = 0.3(x + 32.9)$

110. $0.8(x - 6.21) = 0.2(x - 24.84)$

Determine whether each equation is a conditional equation, an identity, or a contradiction. See Example 5.

111. $2(3x - 1) + 5 = 3$

112. $-2x + 13 = -2(x - 7)$

113. $5x + 13 = -2(x - 7) + 3$

114. $3x + 9 = -3(x - 3) + 6x$

115. $7(x - 1) = -3(3 - x) + 4x$

116. $3(x - 2) + 4x = 6(x - 1) + x$

117. $5(x + 1) = 3(x + 1) + 2(x + 1)$

118. $8x - 20 + x = -3(5 - 2x) + 3(x - 4)$

119. $2x + 3x = 5.2(3 - x)$

120. $5.2x + 3.4x = 0.2(x - 0.42)$

WRITING & THINKING

121. Find the error(s) made in solving each equation and give the correct solution.

a.
$$\frac{1}{3}x+4=9$$
$$3\cdot\frac{1}{3}x+4=3\cdot 9$$
$$x+4=27$$
$$x+4-4=27-4$$
$$x=23$$

b.
$$5x+3=11$$
$$(5x-3)+(3-3)=11-3$$
$$2x+0=8$$
$$\frac{2x}{2}=\frac{8}{2}$$
$$x=4$$

122. Answer each question.

a. Simplify the expression $3(x+5)+2(x-7)$.

b. Solve the equation $3(x+5)+2(x-7)=31$.

c. How are the methods you used to answer parts **a.** and **b.** similar? How are they different?

123. Write an equation to represent each situation, using x to represent Ryan's current age. Determine whether each equation is a conditional equation, an identity, or a contradiction, and explain why that makes sense for the situation represented.

a. In 6 years, Ryan will be 20 years old.

b. In 6 years, Ryan will be 8 years older than he is now.

c. In 6 years, Ryan will be 3 years older than he will be 3 years from now.

Example 3: Calculating Average Interest Rate

Julie invested \$1500 in a risky high-tech stock on January 1st. On July 1st, her stock is worth \$2100. She knows that her investment does not earn interest at a constant rate, but she wants to determine her average annual rate of return at this point in the year. What is the average annual rate of return she has earned so far?

Solution

The interest that Julie has earned in half a year is \$600 (or \$2100 – \$1500). Replacing P with 1500, t with $\frac{1}{2}$, and I with 600 in the formula $I = Prt$, we have:

$$
\begin{aligned}
600 &= (1500)\left(\frac{1}{2}\right)r \\
\frac{1200}{1500} &= r \\
r &= 0.8 \\
r &= 80\% \text{ average rate of return per year}
\end{aligned}
$$

1.2 EXERCISES

PRACTICE

Solve each of the following equations for the indicated variable. See Example 1.

1. Circumference of a circle: $C = 2\pi r$; solve for r

2. Ideal Gas Law: $PV = nRT$; solve for T

3. Velocity: $v^2 = {v_0}^2 + 2ax$; solve for a

4. Area of a trapezoid: $A = \frac{1}{2}h(b+c)$; solve for h

5. Temperature conversions: $C = \frac{5}{9}(F-32)$; solve for F

6. Volume of a right circular cone: $V = \frac{1}{3}\pi r^2 h$; solve for h

7. Surface area of a rectangular prism: $A = 2lw + 2wh + 2hl$; solve for h

8. Distance: $d = rt_1 + rt_2$; solve for r

9. Kinetic energy of protons: $K = \frac{1}{2}mv^2$; solve for m

10. Finance: $A = P(1+rt)$; solve for t

APPLICATIONS

11. A riverboat leaves port and proceeds to travel downstream at an average speed of 15 miles per hour. How long will it take for the boat to arrive at the next port, 95 miles downstream?

12. Two trucks leave a warehouse at the same time. One travels due east at an average speed of 45 miles per hour, and the other travels due west at an average speed of 55 miles per hour. After how many hours will they be 450 miles apart?

13. Two cars leave a rest stop at the same time and proceed to travel down the highway in the same direction. One travels at an average rate of 62 miles per hour, and the other at an average rate of 59 miles per hour. How far apart are the two cars after four and a half hours?

14. Two trains are 630 miles apart, heading directly toward each other on parallel tracks. The first train is traveling at 95 mph, and the second train is traveling at 85 mph. How long will it be before the trains pass each other?

15. Two brothers, Rick and Tom, each inherit $10,000. Rick invests his inheritance in a savings account with an annual return of 2.25%, while Tom invests his in a CD paying 6.15% annually. How much more money does Tom have than Rick after 1 year?

16. Sarah, sister to Rick and Tom in the previous problem, also inherits $10,000, but she invests her inheritance in a global technology mutual fund. At the end of 1 year, her investment is worth $12,800. What has her effective annual rate of return been?

17. An industrial acid-etching procedure calls for 3 gallons of a 46% hydrofluoric acid solution, but the supplier currently only has 44% solution and 50% solution. How many gallons of each should be mixed for the procedure?

18. An agricultural stress test calls for soaking seeds in 8% saline solution. The scientist running the test wants to make use of 1 liter of 20% saline solution that is already made up. How much pure water should she add to the 20% solution to obtain an 8% solution?

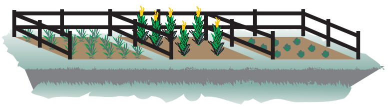
19. A total of 39 tickets were sold for a puppet show, with child tickets selling for $7.50 and adult tickets selling for $10.00. The ticket sales raised $330.00 in all. How many child tickets and how many adult tickets were sold?

20. Joe's Java Joint wants to make a blend of two coffees that can be sold for $15 per pound. The first of the two types of coffee costs $18 per pound, while the second costs $13 per pound. How many pounds of each should be mixed to get 10 pounds of the desired blend?

21. Bob buys a large screen digital TV priced at $9500, but pays $10,212.50 with tax. What is the rate of tax where Bob lives?

22. Will and Matt are brothers. Will is 6 feet, 4 inches tall, and Matt is 6 feet, 7 inches tall. How tall is Will as a percentage of Matt's height? How tall is Matt as a percentage of Will's height?

23. A farmer wants to fence in three square garden plots situated along a road, as shown, and he decides not to install fencing along the edge of the road. If he has 182 feet of fencing material total, what dimensions should he make each square plot?



24. Find three consecutive integers whose sum is 288. (**Hint:** If n represents the smallest of the three, then $n+1$ and $n+2$ represent the other two numbers.)

25. Find three consecutive odd integers whose sum is 165. (**Hint:** If n represents the smallest of the three, then $n+2$ and $n+4$ represent the other two numbers.)

26. Kathy buys last year's best-selling novel, in hardcover, for \$15.05. This is a 30% discount from the original price. What was the original price?

27. The highest point on Earth is the peak of Mount Everest. If you climbed to the top, you would be approximately 29,035 feet above sea level. Remembering that a mile is 5280 feet, what percentage of the height of the mountain would you have to climb to reach a point two miles above sea level?

TECHNOLOGY

Use a graphing utility to solve the following equations. Round your answers to two decimal places if necessary.

28. $453x = 95(34x + 291)$

29. $-0.23 = 0.79x - 0.47(x + 0.98)$

30. $254 + 0.98(x - 124) = 0$

31. $323x - 1745 = 531(68x - 887)$

We then proceed to solve the inequality.

$$\begin{aligned}\frac{72+x+77}{3} &> 75\\ 149+x &> 225\\ x &> 76\end{aligned}$$

Thus, the high temperature on the second day exceeded 76 degrees.

b. The phrase "must not detect more than 5 defective wafers per batch on average" means the average number must be less than or equal to 5. Let x denote the maximum number of defective wafers in the last batch.

$$\begin{aligned}\frac{(9)(4.78)+x}{10} &\le 5\\ 43.02+x &\le 50\\ x &\le 6.98\end{aligned}$$

The number of defective wafers found in the first 9 batches is $(9)(4.78) = 43.02$.

Since it is not possible to have a fractional number of wafers, there must have been 43 defective wafers in the first 9 batches, so the maximum allowable number of defective wafers in the final batch is 7.

1.3 EXERCISES

PRACTICE

Determine which elements of $S = \{12, -9, 3.14, -2.83, 1, 5.24, 8, -3, 4\}$ satisfy each inequality below.

1. $7y - 33.6 < -8.6 + 2y$

2. $-2.2y - 18.8 \ge 5.2(1-y)$

3. $-40 < 4y - 8 \le 4$

4. $-4 < -2(z-2) \le 2$

Solve the following linear inequalities. Describe the solution set using interval notation and by graphing. See Examples 2 and 3.

5. $4 + 3t \le t - 2$

6. $x - 7 \ge 5 + 3x$

7. $5y - 24 < -9.6 + 2y$

8. $-\frac{v+2}{3} > \frac{5-v}{2}$

9. $4.2x - 5.6 < 1.6 + x$

10. $8.5y - 3.5 \ge 2.5(3-y)$

11. $-2(3-x) < -2x$

12. $\frac{1-x}{5} > \frac{-x}{10}$

13. $4w + 7 \le -7w + 4$

14. $-5(p-3) > 19.8 - p$

15. $\frac{6f-2}{5} < \frac{5f-3}{4}$

16. $\frac{u-6}{7} \ge \frac{2u-1}{3}$

17. $0.04n + 1.7 < 0.13n - 1.45$

18. $2k + \frac{3}{2} < 5k - \frac{7}{3}$

19. $\dfrac{4x+4}{5} > \dfrac{3x+2.6}{4}$

20. $-1.4z - 19.6 \ge 4.4(1-z)$

21. $6m + \dfrac{7}{4} > \dfrac{4m+5.8}{5}$

22. $-3.9n - 5.4 \ge 6.2(2-3n)$

Solve the following compound inequalities. Describe the solution set using interval notation and by graphing. See Examples 4 and 5.

23. $-4 < 3x - 7 \le 8$

24. $5 \le 2m - 3 \le 13$

25. $-36 < 3x - 6 \le 12$

26. $2 < 3(x+2) \le 21$

27. $-8 \le \dfrac{z}{2} - 4 < -5$

28. $6(x-1) < 2(3x+5) \le 6x + 10$

29. $3 < \dfrac{w+3}{8} \le 9$

30. $4 \le \dfrac{p+7}{-2} < 9$

31. $\dfrac{1}{3} < \dfrac{7}{6}(l-3) < \dfrac{2}{3}$

32. $-10 < -2(4+y) \le 9$

33. $\dfrac{1}{4} \le \dfrac{g}{2} - 3 < 5$

34. $-1.2 \le \dfrac{x+3}{-5} \le 0.2$

35. $0.08 < 0.03c + 0.13 \le 0.16$

Solve the following absolute value inequalities. Describe the solution set using interval notation and by graphing. See Example 6.

36. $|x-2| \ge 5$

37. $|4-2x| > 11$

38. $4 + |3-2y| \le 6$

39. $4 + |3-2y| > 6$

40. $2|z+5| < 12$

41. $7 - \left|\dfrac{q}{2} + 3\right| \ge 12$

42. $4|z+3| \le 28$

43. $-3|4-t| < -6$

44. $-3|4-t| > -6$

45. $3|4-t| < -6$

46. $7 - |4-2y| \le -5$

47. $11 - \left|\dfrac{w}{4} + 1\right| \ge 12$

48. $5.5 + |x-7.2| \le 3.5$

49. $6 - 5|x+2| \ge -4$

50. $|2x-1| < x + 4$

51. $|3t+4| > -8$

52. $2 < |6w-2| + 7$

The words "and" and "or" can appear explicitly between two inequalities, and their meaning in such cases is the same as in absolute value inequalities. If two inequalities are joined by the word "and," the solution set consists of all those real numbers that satisfy both inequalities; that is, the solution set overall is the intersection of the two individual solution sets. If the word "or" appears between two inequalities, the solution set consists of all those real numbers that satisfy at least one of the two inequalities; in other words, the solution set overall is the union of the two individual solution sets.

Guided by the above paragraph, solve the following inequality problems. Describe the solution set using interval notation and by graphing.

53. $t < 2t - 3$ and $-3(t+4) > -57$

54. $7 - \frac{3x}{5} < \frac{2}{5}$ or $2 - 3x \geq 5$

55. $-2(a-1) < 4$ and $6 + a \leq 9$

56. $-2(a-1) < 4$ and $6 - a \leq 9$

57. $-2(a-1) < 4$ or $6 + a \leq 9$

58. $\frac{5n+6}{3} < -10$ and $-3(n-1) < -6$

59. $\frac{23x-3}{-7} \leq 7$ and $-x < -(4x-9)$

60. $7 - \frac{x}{3} \leq 14 + \frac{x}{2}$ or $-3x < 15$

APPLICATIONS

61. In a class in which the final course grade depends entirely on the average of four equally weighted 100-point tests, Cindy has scored 96, 94, and 97 on the first three. The professor has announced that there will be a 15-point bonus problem on the fourth test, and anyone who finishes the semester with an average of more than 100 will receive an A+. What interval of scores on the fourth test will give Cindy an A for the semester (an average between 90 and 100, inclusive), and what interval will give Cindy an A+?

62. In a series of 30 racquetball games played to date, Larry has won 10, giving him a winning average so far of 33.3% (to the nearest tenth of a percent). If he continues to play, what interval describes the number of games he must now win in a row to have an overall winning average greater than 50%?

63. Assume that the national average SAT score for high school seniors is 1020 out of 1600. A group of seven students receive their scores in the mail, and six of them look at their scores. Two students scored 1090, one got an 1120, two others each got a 910, and the sixth student received an 880. What interval of scores can the seventh student receive to pull the group's average above the national average?

64. The central bank of a certain country tries to keep the inflation rate below 5.0% on an annual basis. Assume that inflation rates for the first three quarters of a given year are as follows: 5.2%, 4.3%, and 4.7%. What interval of inflation rates for the final quarter would satisfy the government's goal?

1.4 EXERCISES

PRACTICE

Solve the following quadratic equations by factoring. See Example 1.

1. $2x^2 - x = 3$
2. $3x^2 - 7x = 0$
3. $x^2 - 14x + 49 = 0$
4. $9x - 5x^2 = -2$
5. $y(2y+9) = -9$
6. $2x^2 - 3x = x^2 + 18$
7. $(3x+2)(x-1) = 7 - 7x$
8. $3x^2 + 33 = 2x^2 + 14x$
9. $5x^2 + 2x + 3 = 4x^2 + 6x - 1$
10. $15x^2 + x = 2$
11. $(x-7)^2 = 16$
12. $4x^2 - 9 = 0$

Solve the following quadratic equations by taking square roots. See Example 2.

13. $(x-3)^2 = 9$
14. $(8t-3)^2 = 0$
15. $(2x+1)^2 - 7 = 0$
16. $(y-18)^2 - 1 = 0$
17. $9 = (3s+2)^2$
18. $(2x-1)^2 = 8$
19. $x^2 - 4x + 4 = 49$
20. $-3(n+7)^2 = -27$
21. $(3x-6)^2 = 4x^2$

Solve the following quadratic equations by completing the square. See Example 3.

22. $x^2 + 8x + 7 = -8$
23. $2x^2 + 6x - 10 = 10$
24. $2x^2 + 7x - 15 = 0$
25. $4x^2 - 4x - 63 = 0$
26. $u^2 + 10u + 9 = 0$
27. $4x^2 - 56x + 195 = 0$
28. $4x^2 + 32x - 260 = 0$
29. $z^2 + 26z + 2 = -23$
30. $y^2 + 22y + 96 = 0$

Solve the following quadratic equations using the quadratic formula. See Example 4.

31. $3x^2 - 4 = -x$
32. $2.1y^2 - 3.5y = 4$
33. $a(a+2) = -1$
34. $3x^2 - 2x = 0$
35. $6x^2 + 5x - 4 = 3x - 2$
36. $7x^2 - 4x = 51$
37. $4x^2 - 14x - 27 = 3$

Solve the following quadratic equations using any appropriate method.

38. $(z-11)^2 = 9$
39. $x^2 + 20x + 36 = -48$
40. $256t^2 - 324 = 0$
41. $(y-8)^2 = 36$
42. $(9y-6)^2 = 121y^2$
43. $2x^2 + 8x - 3 = 6x$
44. $x^2 - 6x = 27$
45. $3a^2 + 12a - 576 = 0$
46. $-3(b+5)^2 = -768$
47. $y^2 + 13y + 42 = 0$

48. $3x^2 - 6x = 0$

49. $7x^2 - 42x = 0$

50. $y^2 + 24y + 23 = 0$

51. $5x^2 - 5x - 10 = 0$

52. $4w^2 + 10w + 5 = 3w^2 + 18w - 10$

53. $|x^2 - 3x| = 2$ (**Hint:** Replace $|x^2 - 3x|$ first with $x^2 - 3x$ and solve the resulting equation, then replace it with $-(x^2 - 3x)$ and solve the resulting equation.)

54. $|x^2 - 8| = 1$

WRITING & THINKING

55. Factor the quadratic $9x^2 - 6x - 4$.

56. Factor the quadratic $4x^2 + 12x + 1$.

57. Determine b and c so that the equation $x^2 + bx + c = 0$ has the solution set $\{-3, 8\}$.

TECHNOLOGY

Use a graphing utility to solve the following quadratic equations.

58. $5x^2 - 3x = 17$

59. $(a + 4)(4a - 3) = 5$

60. $10\pi r + \pi r^2 = 107$

61. $4.8x^2 + 3.5x - 9.2 = 0$

$$(x-1)^{-\frac{1}{2}}=0 \quad \text{or} \quad x-2=0$$
$$\frac{1}{(x-1)^{\frac{1}{2}}}=0 \qquad\qquad x=2$$

The original equation has only one solution: $x=2$

1.5 EXERCISES

PRACTICE

Solve the following quadratic-like equations. See Example 1.

1. $(x-1)^2+(x-1)-12=0$

2. $(z-8)^2-7(z-8)+12=0$

3. $(y-5)^2-11(y-5)+24=0$

4. $(x^2-1)^2+(x^2-1)-12=0$

5. $(x^2+1)^2+(x^2+1)-12=0$

6. $(x^2-13)^2+(x^2-13)-12=0$

7. $(x^2-2x+1)^2+(x^2-2x+1)-12=0$

8. $2y^{\frac{2}{3}}+y^{\frac{1}{3}}-1=0$

9. $2x^{\frac{2}{3}}-7x^{\frac{1}{3}}+3=0$

10. $(x^2-6x)^2+4(x^2-6x)-5=0$

11. $(y^2-5)^2+5(y^2-5)-36=0$

12. $(x^2+7)^2+8(x^2+7)+12=0$

13. $(t^2-t)^2-8(t^2-t)+12=0$

14. $2x^{\frac{1}{2}}-5x^{\frac{1}{4}}+2=0$

15. $3x^{\frac{2}{3}}-x^{\frac{1}{3}}-2=0$

16. $y^{\frac{1}{2}}-5y^{\frac{1}{4}}+6=0$

17. $(z^2+4z)^2+7(z^2+4z)+12=0$

18. $5y^{\frac{2}{3}}+33y^{\frac{1}{3}}+18=0$

Solve the following polynomial equations by factoring. See Example 2.

19. $a^3-3a^2=a-3$

20. $2x^3+x^2+2x+1=0$

21. $2x^3-x^2=15x$

22. $x^4+5x^2-36=0$

23. $y^4+21y^2-100=0$

24. $y^3+8=0$

25. $5s^3+6s^2-20s=24$

26. $8a^3-27=0$

27. $16a^4=81$

28. $6x^3+8x^2=14x$

29. $14x^3+27x^2-20x=0$

30. $5z^3+28z^2=49z$

31. $27x^3+64=0$

32. $x^3-4x^2+x=4$

33. $x^3+27=0$

Solve the following equations by factoring. See Example 3.

34. $3x^{\frac{11}{3}}+2x^{\frac{8}{3}}-5x^{\frac{5}{3}}=0$

35. $(x-3)^{\frac{-1}{2}}+2(x-3)^{\frac{1}{2}}=0$

36. $(y-6)^{\frac{-5}{2}}+7(y-6)^{\frac{-3}{2}}=0$

37. $y^{-2}-2y^{-1}+1=0$

38. $2x^{\frac{13}{5}} - 5x^{\frac{8}{5}} + 2x^{\frac{3}{5}} = 0$

39. $(2x-5)^{\frac{1}{3}} - 3(2x-5)^{\frac{-2}{3}} = 0$

40. $x^{-4} - 13x^{-2} + 36 = 0$

41. $y^{\frac{7}{2}} - 5y^{\frac{5}{2}} + 6y^{\frac{3}{2}} = 0$

42. $(t+4)^{\frac{2}{3}} + 2(t+4)^{\frac{8}{3}} = 0$

43. $y^{-2} - 2y^{-1} - 35 = 0$

44. $x^{\frac{11}{2}} - 6x^{\frac{9}{2}} + 9x^{\frac{7}{2}} = 0$

45. $5y^{\frac{11}{3}} + 3y^{\frac{8}{3}} - 2y^{\frac{5}{3}} = 0$

46. $5y^{\frac{12}{5}} - 43y^{\frac{7}{5}} + 24y^{\frac{2}{5}} = 0$

47. $(3x-3)^{\frac{-1}{3}} - 5(3x-3)^{\frac{-4}{3}} = 0$

48. $x^{-2} + 8x^{-1} + 15 = 0$

49. $(y+3)^{\frac{2}{5}} + 4(y+3)^{\frac{7}{5}} = 0$

WRITING & THINKING

50. Find b, c, and d so the equation $x^3 + bx^2 + cx + d = 0$ has solutions of -3, -1, and 5.

51. Find b, c, and d so the equation $x^3 + bx^2 + cx + d = 0$ has solutions of -2, 0, and 6.

52. Find b and c so the equation $x^3 + bx^2 + cx = 0$ has solutions of 0, 1, and -7.

53. Find a, c, and d so the equation $ax^3 + 4x^2 + cx + d = 0$ has solutions of -4, 6, and -6.

54. Find a, b, and d so the equation $ax^3 + bx^2 + 3x + d = 0$ has solutions of -3, $-\frac{1}{2}$, and 0.

55. Find a, b, and c so the equation $ax^3 + bx^2 + cx + 6 = 0$ has solutions of $-\frac{3}{5}$, $\frac{2}{3}$, and 1.

Example 4: Equations with Positive Rational Exponents

Solve the following equations with rational exponents.

a. $x^{\frac{3}{4}} - 8 = 0$

Solution

$$
\begin{aligned}
x^{\frac{3}{4}} - 8 &= 0 \\
x^{\frac{3}{4}} &= 8 \\
\sqrt[4]{x^3} &= 8 \\
x^3 &= 8^4 \\
x &= 8^{\frac{4}{3}} \\
x &= \left(8^{\frac{1}{3}}\right)^4 \\
x &= (2)^4 \\
x &= 16
\end{aligned}
$$

Since the term containing the rational exponent can be rewritten as a radical expression, we will begin by isolating that term.

Raising both sides to the fourth power eliminates the fourth root.

Raising both sides to the $\frac{1}{3}$ power solves the equation for x, but we can evaluate the expression on the right-hand side.

Verify that this number solves the original equation.

b. $\left(18x^2 - 54x - 8\right)^{\frac{1}{6}} = 2$

Solution

$$
\begin{aligned}
\left(18x^2 - 54x - 8\right)^{\frac{1}{6}} &= 2 \\
\sqrt[6]{18x^2 - 54x - 8} &= 2 \\
18x^2 - 54x - 8 &= 2^6 \\
18x^2 - 54x - 8 &= 64 \\
18x^2 - 54x - 72 &= 0 \\
18\left(x^2 - 3x - 4\right) &= 0 \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x &= 4, -1
\end{aligned}
$$

The exponent of $\frac{1}{6}$ indicates we should raise both sides to the sixth power in order to eliminate the radical.

We are left with a second-degree polynomial equation that can be solved by factoring.

Note that both solutions solve the original equation.

1.6 EXERCISES

PRACTICE

State any restrictions on x, then solve the proportions. See Example 1.

1. $\frac{4x}{7} = \frac{x+5}{3}$

2. $\frac{3x+1}{4} = \frac{2x+1}{3}$

3. $\frac{10}{x} = \frac{5}{x-2}$

4. $\frac{8}{x-3} = \frac{12}{2x-3}$

5. $\dfrac{4}{x-4}=\dfrac{2}{x+3}$

6. $\dfrac{3}{x+5}=\dfrac{6}{x-2}$

7. $\dfrac{x+2}{5x}=\dfrac{x-6}{3x}$

8. $\dfrac{x-4}{3x}=\dfrac{x-2}{5x}$

9. $\dfrac{5x+2}{x-6}=\dfrac{11}{4}$

10. $\dfrac{x+9}{3x+2}=\dfrac{5}{8}$

State any restrictions on *x*, and then solve the equations. See Example 2.

11. $\dfrac{5x}{4}-\dfrac{1}{2}=-\dfrac{3}{16}$

12. $\dfrac{x}{6}-\dfrac{1}{42}=\dfrac{1}{7}$

13. $\dfrac{3x-1}{6}-\dfrac{x+3}{4}=\dfrac{7}{12}$

14. $\dfrac{x-2}{3}-\dfrac{x-3}{5}=\dfrac{13}{15}$

15. $\dfrac{2+x}{4}-\dfrac{5x-2}{12}=\dfrac{8-2x}{5}$

16. $\dfrac{4x+1}{5}=\dfrac{2x+3}{2}-\dfrac{x+2}{4}$

17. $\dfrac{2}{3x}=\dfrac{1}{4}-\dfrac{1}{6x}$

18. $\dfrac{1}{x}-\dfrac{8}{21}=\dfrac{3}{7x}$

19. $\dfrac{3}{5x}-\dfrac{1}{5}=\dfrac{3}{4x}$

20. $\dfrac{3}{8x}-\dfrac{7}{10}=\dfrac{1}{5x}$

21. $\dfrac{3}{4x}-\dfrac{1}{2}=\dfrac{7}{8x}+\dfrac{1}{6}$

22. $\dfrac{5}{3x}+\dfrac{1}{2}=\dfrac{7}{9x}-\dfrac{5}{6}$

23. $\dfrac{2}{4x+1}=\dfrac{4}{x^2+9x}$

24. $\dfrac{3}{4x-1}=\dfrac{4}{x^2+x}$

25. $\dfrac{9}{x^2-6x}=\dfrac{5}{2x-3}$

26. $\dfrac{-9}{x^2+5x}=\dfrac{8}{4-9x}$

27. $\dfrac{x}{x-4}-\dfrac{4}{2x-1}=1$

28. $\dfrac{x}{x+3}+\dfrac{1}{x+2}=1$

29. $\dfrac{x+2}{x+1}+\dfrac{x+2}{x+4}=2$

30. $\dfrac{3x-2}{x+4}+\dfrac{2x+5}{x-1}=5$

31. $\dfrac{2}{4x-1}+\dfrac{1}{x+1}=\dfrac{3}{x+1}$

32. $\dfrac{x-2}{x+4}-\dfrac{3}{2x+1}=\dfrac{x-7}{x+4}$

33. $\dfrac{x-2}{x-3}+\dfrac{x-3}{x-2}=\dfrac{2x^2}{x^2-5x+6}$

34. $\dfrac{x}{x-4}-\dfrac{12x}{x^2+x-20}=\dfrac{x-1}{x+5}$

35. $\dfrac{3x+5}{3x+2}+\dfrac{8x+16}{3x^2-4x-4}=\dfrac{x+2}{x-2}$

36. $\dfrac{3x+5}{3x+2}-\dfrac{4-2x}{3x^2+8x+4}=\dfrac{x+4}{x+2}$

37. $\dfrac{3}{3x-1}+\dfrac{1}{x+1}=\dfrac{4}{2x-1}$

38. $\dfrac{2}{x+1}+\dfrac{4}{2x-3}=\dfrac{4}{x-5}$

Solve the following radical equations. See Example 3.

39. $\sqrt{4-x}-x=2$

40. $\sqrt{x^2-4x+5}-x+2=0$

41. $\sqrt{x^2-4x+4}+2=3x$

42. $\sqrt{50+7s}-s=8$

43. $\sqrt[4]{2x+3}=-1$

44. $\sqrt{11x+3}+4x=18$

45. $\sqrt{x+10}+1=x-1$

46. $\sqrt{x+1}+10=x-1$

47. $\sqrt{x^2-10}-1=x+1$

48. $\sqrt[3]{5x^2-14x}=-2$

49. $\sqrt{4z+41}+3=z+2$

50. $\sqrt[3]{3-2x}-\sqrt[3]{x+1}=0$

51. $\sqrt[4]{x^2-x}=\sqrt[4]{x-1}$

52. $\sqrt[5]{7t^2+2t}=\sqrt[5]{5t^2+4}$

53. $\sqrt[3]{y^3-7y+2}=\sqrt[3]{2-3y}$

54. $\sqrt{3y+4}+\sqrt{5y+6}=2$

55. $\sqrt{3-3x}-3=\sqrt{3x+2}$

56. $\sqrt{2b-1}+3=\sqrt{10b-6}$

57. $\sqrt{5x+5}=\sqrt{4x-7}+2$

58. $\sqrt{14y^2-18y+4}+2=2y$

59. $\sqrt{9x+4}=\sqrt{7x+1}+1$

Solve the following equations. See Example 4.

60. $(x+3)^{\frac{1}{4}}+2=0$

61. $(2x-5)^{\frac{1}{4}}=(x-1)^{\frac{1}{4}}$

62. $(2x-1)^{\frac{2}{3}}=x^{\frac{1}{3}}$

63. $(3y^2+9y-5)^{\frac{1}{2}}=y+3$

64. $(3x-5)^{\frac{1}{5}}=(x+1)^{\frac{1}{5}}$

65. $w^{\frac{3}{5}}+8=0$

66. $z^{\frac{4}{3}}-\frac{16}{81}=0$

67. $x^{\frac{2}{3}}-\frac{25}{49}=0$

68. $(x-2)^{\frac{2}{3}}=(14-x)^{\frac{1}{3}}$

69. $(y-2)^{\frac{2}{3}}=(13y-66)^{\frac{1}{3}}$

70. $(x^2+21)^{\frac{-3}{2}}=\frac{1}{125}$

71. $(x^2+7)^{\frac{-3}{2}}=\frac{1}{64}$

APPLICATIONS

72. Computers: Making a statistical analysis, Ana found 3 defective computers in a sample of 20 computers. If this ratio is consistent, how many defective computers does she expect to find in a batch of 2400 computers?

73. Manufacturing: At the Bright-As-Day light bulb plant, 3 out of each 100 bulbs produced are defective. If the daily production is 4800 bulbs, how many are defective?

74. Education: The University of Arizona has a ratio of 1 professor for every 23 students. If there are 1600 faculty members at the university, how many students are enrolled there?

75. **Baseball:** New York Yankees player Didi Gregorius has a recorded batting average of 15 hits for every 50 times at bat. If he maintains this average, how many at bats will he need to achieve 111 hits? (Round to the nearest whole number.)

76. **Cartography:** On a map of Maryland, one inch represents 4 miles. If there are 8.5 inches between Baltimore, MD and Washington, DC, how far are the two cities from each other?

77. **Architecture:** A floor plan is drawn to scale in which 1 inch represents 4 feet. What size will the drawing be for a room that is 30 feet by 40 feet? (**Hint:** Set up two proportions.)

78. **Baking:** The recipe for Nestle Tollhouse Chocolate Chip Cookies calls for 2 cups of chocolate chips to make 5 dozen cookies. If you want to bake 17 dozen cookies, how many cups of chocolate chips do you need?

79. **Car maintenance:** In the instructions for Never-Ice Antifreeze it states that 4 quarts of antifreeze are needed for every 10 quarts of radiator capacity. If Sal's car has a 22-quart radiator, how many quarts of antifreeze will it need?

80. **Landscape architecture:** An architect is to draw plans for a city park. He intends to use a scale of $\frac{1}{2}$ inch to represent 25 feet. How many inches will be needed to use for the length and width of a rectangular playing field that is 50 yards by 125 yards? (1 yard = 3 feet)

81. **Testing cars:** A test driver wants to increase the speed of the car he is driving by 3 miles per hour every 2 seconds. But he can only check his speed every 5 seconds because he is busy with other items during the test drive.
    a. By how much should he increase his speed in 5 seconds?
    b. If he starts checking his speed at 40 miles per hour, how fast should he be going after 10 seconds?

82. **Decorating:** Jack and Diane are decorating a nursery room for their baby, which will be born in a few months. In one hour, Jack can get $\frac{1}{6}$ of the nursery done and Diane can get $\frac{1}{12}$ of the nursery done. If they work together, they can get $\frac{1}{x}$ of the nursery done in one hour. Determine how many hours it will take Jack and Diane to decorate the nursery if they work together by solving the equation $\frac{1}{6}+\frac{1}{12}=\frac{1}{x}$ for x.

83. **Printing:** A local print shop has a big order of pamphlets to print, so they decide to use two of their printers for the one job. The newer printer can print the pamphlets four times as fast as the older printer. That means in one hour, the newer printer can complete $\frac{1}{x}$ of the print job and the older printer can complete $\frac{1}{4x}$ of the print job. Working together, the printers can complete the job in 4 hours. Determine how many hours it would take the newer printer to print all of the pamphlets by itself by solving the equation $\frac{1}{x}+\frac{1}{4x}=\frac{1}{4}$ for x.

84. **Construction:** Two groups of civil engineers are surveying an area to prepare for the construction of a shopping center. The first group is full of new college graduates and it will take them four more hours than it takes the second group, which is full of seasoned professionals. The second group can complete the job in x hours. This means that in one hour, the first group can complete $\frac{1}{x+4}$ of the job and the second group can complete $\frac{1}{x}$ of the job. Working together, they can complete the surveying job in $\frac{15}{4}$ hours. Determine how many hours it would take each team to complete the job individually by solving the equation $\frac{1}{x+4}+\frac{1}{x}=\frac{4}{15}$ for x.

85. **Running:** Terrence and Alicia are competing in a marathon where the average running speed is x kilometers per hour. Terrence is running 2 kilometers per hour slower than the average running speed. Alicia is running 2 kilometers per hour faster than the average running speed. After a certain amount of time, Terrence ran 4 kilometers and Alicia ran 6 kilometers.
   a. Determine the speed of the average runner by solving the equation $\frac{4}{x-2}=\frac{6}{x+2}$ for x.
   b. What was Terrence's average running speed?
   c. What was Alicia's average running speed?
   d. How long did it take Terrence to run 4 kilometers and Alicia to run 6 kilometers?

WRITING & THINKING

In simplifying rational expressions, the result is a rational or polynomial expression. However, in solving equations with rational expressions, the goal is to find a value (or values) for the variable that will make the equation a true statement. Many students confuse these two ideas. To avoid confusing the techniques for adding and subtracting rational expressions with the techniques for solving equations, simplify the expression in part **a**. and solve the equation in part **b**. Explain, in your own words, the differences in your procedures. Assume no denominator has a value of 0.

86. **a.** $\frac{10}{x}+\frac{31}{x-1}+\frac{4x}{x-1}$
   b. $\frac{10}{x}+\frac{31}{x-1}=\frac{4x}{x-1}$

87. **a.** $\frac{-4}{x^2-16}+\frac{x}{2x+8}-\frac{1}{4}$
   b. $\frac{-4}{x^2-16}+\frac{x}{2x+8}=\frac{1}{4}$

88. **a.** $\frac{3x}{x^2-4}+\frac{5}{x+2}+\frac{2}{x-2}$
   b. $\frac{3x}{x^2-4}+\frac{5}{x+2}=\frac{2}{x-2}$

89. **a.** $\frac{7}{5x}+\frac{2}{x-4}-\frac{3}{5x}$
   b. $\frac{7}{5x}+\frac{2}{x-4}=\frac{3}{5x}$

90. **a.** $\frac{2}{x+9}-\frac{2}{x-9}+\frac{1}{2}$
   b. $\frac{2}{x+9}-\frac{2}{x-9}=\frac{1}{2}$

2.1 EXERCISES

PRACTICE

Plot the following sets of points in the Cartesian plane. See Example 1.

1. $\{(-3,2),(5,-1),(0,-2),(3,0)\}$

2. $\{(-4,0),(0,-4),(-3,-3),(3,-3)\}$

3. $\{(3,4),(-2,-1),(-1,-3),(-3,0)\}$

4. $\{(2,2),(0,3),(4,-5),(-1,3)\}$

5. $\{(0,5),(-3,2),(2,4),(1,1)\}$

6. $\{(8,3),(-3,4),(-4,-6),(3,-4)\}$

7. $\{(-5,-4),(3,2),(4,5),(-2,-1),(-4,-4),(1,1)\}$

8. $\{(-2,5),(0,1),(1,-1),(1,-3),(0,0),(-1,2),(0,-2)\}$

Identify the quadrant in which each point lies, if possible. If a point lies on an axis, specify which part (positive or negative) of which axis (*x* or *y*). See Example 1.

9. $(-2,-4)$

10. $(0,-12)$

11. $(4,-7)$

12. $(-2,0)$

13. $(9,0)$

14. $(3,26)$

15. $(-4,-7)$

16. $(0,1)$

17. $(17,-2)$

18. $(-\sqrt{2},4)$

19. $(-1,1)$

20. $(-4,0)$

21. $(3,-9)$

22. $(0,0)$

23. $(4,3)$

24. $(-3,-11)$

25. $(0,-97)$

26. $\left(\frac{1}{3},0\right)$

For each of the following equations, determine the value of the missing entries in the accompanying table of ordered pairs. Then plot the ordered pairs and sketch your guess of the complete graph of the equation. See Example 2.

27. $6x-4y=12$

x	y
0	?
?	0
3	?
?	3

28. $y=x^2+2x+1$

x	y
?	0
1	?
?	1
2	?
–3	?

29. $x=y^2$

x	y
0	?
1	?
4	?
9	?
?	$-\sqrt{2}$

30. $5x-2=-y$

x	y
?	0
0	?
1	?
?	7
–2	?

31. $x^2+y^2=9$

x	y
0	?
?	0
–1	?
1	?
?	2

32. $y=-x^2$

x	y
0	?
–1	?
1	?
–2	?
2	?

Determine **a.** the distance between the following pairs of points, and **b.** the midpoint of the line segment joining each pair of points. See Examples 3 and 4.

33. $(-2, 3)$ and $(-5, -2)$

34. $(-1, -2)$ and $(2, 2)$

35. $(0, 7)$ and $(3, 0)$

36. $\left(-\frac{1}{2}, 5\right)$ and $\left(\frac{9}{2}, -7\right)$

37. $(-2, 0)$ and $(0, -2)$

38. $(5, 6)$ and $(-3, -2)$

39. $(13, -14)$ and $(-7, -2)$

40. $(-8, 3)$ and $(2, 11)$

41. $(-3, -3)$ and $(5, -9)$

42. $(7, -7)$ and $(-7, -6)$

43. $(5, -4)$ and $(-1, 5)$

44. $(4, 6)$ and $(2, -7)$

45. $(8, 8)$ and $(-2, -2)$

46. $\left(3, \frac{26}{5}\right)$ and $\left(9, -\frac{14}{5}\right)$

47. Given $(10, 4)$ and $(x, -2)$, find x such that the distance between these two points is 10.

48. Given $(1, y)$ and $(13, -3)$, find y such that the distance between these two points is 15.

49. Given $(x, 3)$ and $(-6, y)$, find x and y such that the midpoint between these two points is $(2, 2)$.

Find the perimeter of the triangle whose vertices are the specified points in the plane.

50. $(-2, 3)$, $(-2, 1)$, and $(-5, -2)$

51. $(-1, -2)$, $(2, -2)$, and $(2, 2)$

52. $(6, -1)$, $(-6, 4)$, and $(9, 3)$

53. $(3, -4)$, $(-7, 0)$, and $(-2, -5)$

54. $(-3, 7)$, $(5, 1)$, and $(-3, -14)$

55. $(-12, -3)$, $(-7, 9)$, and $(9, -3)$

APPLICATIONS

56. Two college friends are taking a weekend road trip. Friday they leave home and drive 87 miles north for a night of dinner and dancing in the city. The next morning they drive 116 miles east to spend a day at the beach. If they drive straight home from the beach the next day, how far do they have to travel on Sunday?

57. Your backpacker's guide contains a grid map of Paris, with each unit on the grid representing 0.25 kilometers. If the Eiffel Tower is located at $(-8, -1)$ and the Arc de Triomphe is located at $(-8, 4)$, what is the direct distance (not walking distance, which would have to account for bridges and roadways) between the two monuments in kilometers?

58. Your hotel, located at $(-1, -2)$ on the map from Exercise 57, is advertised as exactly halfway between the Eiffel Tower and Notre Dame. What are the grid coordinates of Notre Dame on your map? Find the direct distance from the Eiffel Tower to Notre Dame, rounded to the nearest hundredth of a kilometer.

59. The navigator of a submarine plots the position of the submarine and surrounding objects using a Cartesian coordinate system, where each block is one square meter.

a. If his submarine is located at $(50, 231)$ and the mobile base to which he is heading is located at $(83, 478)$, how far is he from the mobile base?

b. Suppose there is another submarine located halfway between the first submarine and the mobile base. What is the position of the second sub?

60. At the entrance to Paradise Island Theme Park you are given a map of the park that is in the form of a grid, with the park entrance located at $(-5, -5)$. After walking past three rides and the restrooms, you arrive at the Tsunami Water Ride, which is located at $(-3, -1)$ on the grid. If you have traveled halfway along a straight line to your favorite ride, Thundering Tower, where on the grid is your favorite ride located? How far is Thundering Tower from the park entrance on the map?

WRITING & THINKING

61. Use the distance formula to prove that the triangle with vertices at the points $(1,1)$, $(-2,-5)$, and $(3,0)$ is a right triangle. Then determine the area of the triangle.

62. Use the distance formula to prove that the triangle with vertices at the points $(-2,2)$, $(1,-2)$, and $(2,5)$ is isosceles. Then determine the area of the triangle. (**Hint:** Make use of the midpoint formula.)

63. Use the distance formula to prove that the triangle with vertices at the points $(5,1)$, $(-3,7)$, and $(8,5)$ is a right triangle. Then determine the area of the triangle.

64. Use the distance formula to prove that the triangle with vertices at the points $(1,2)$, $(-2,0)$, and $(3,5)$ is isosceles. Then determine the area of the triangle. (**Hint:** Make use of the midpoint formula.)

65. Use the distance formula to prove that the triangle with vertices at the points $(2,2)$, $(6,3)$, and $(4,11)$ is a right triangle. Then determine the area of the triangle.

66. Use the distance formula to prove that the triangle with vertices at the points $(2,-1)$, $(4,3)$, and $(-2,-3)$ is isosceles. Then determine the area of the triangle. (**Hint:** Make use of the midpoint formula.)

67. Use the distance formula to prove that the polygon with vertices at the points $(-2,-1)$, $(6,5)$, $(-2,5)$, and $(6,-1)$ is a rectangle. Then determine the area of the rectangle. (**Hint:** It may help to plot the points before you begin.)

68. Plot the points $(-3,3)$, $(-5,-2)$, $(3,-2)$, and $(1,3)$ to demonstrate they are the vertices of a trapezoid. Then determine the area of the trapezoid.

TECHNOLOGY

Determine appropriate settings on a graphing utility so that each of the given points will lie within the viewing window. Answers will vary slightly.

69. $\{(-4, 1), (2, 8), (5, 7)\}$

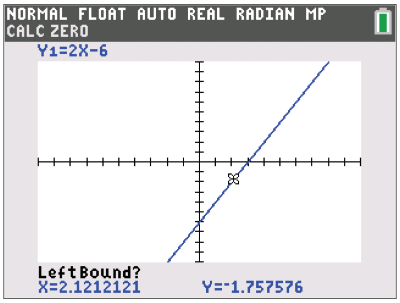
70. $\{(12, 3), (5, -11), (-9, 6)\}$

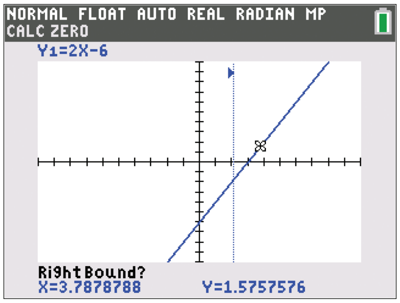
71. $\{(3, 2), (-2, 4), (5, -3)\}$

72. $\{(30, 55), (40, 25), (-80, -10)\}$

73. $\{(3.75, -8.5), (-5.25, 6.0), (7.5, -2.25)\}$

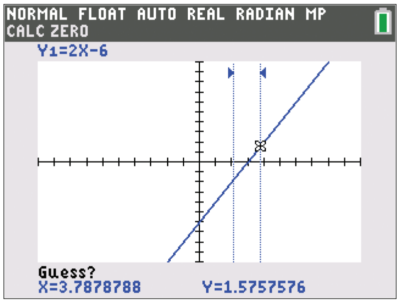
74. $\{(63, 99), (-87, 34), (45, -22)\}$

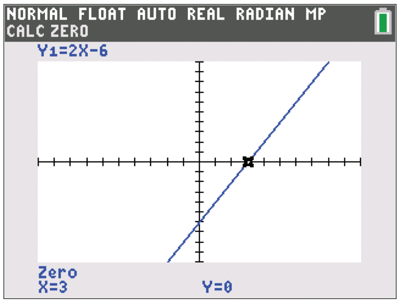




Use the right arrow to move the cursor to the right of where the line crosses the x-axis and press ENTER again. The text should now read "Guess?".

Press ENTER a third time and the x- and y-values of the x-intercept will appear at the bottom of the screen.





So the x-intercept is (3, 0). Both of these techniques can be used to find the x- and y-intercepts of any equation graphed with a calculator, not just linear equations.

2.2 EXERCISES

PRACTICE

Determine if the following equations are linear. See Example 1.

1. $3x+2(x-4y)=2x-y$

2. $9x+4(y-x)=3$

3. $9x^2-(x+1)^2=y-3$

4. $3x+xy=2y$

5. $8-4xy=x-2y$

6. $\dfrac{x-y}{2}+\dfrac{7y}{3}=5$

7. $\dfrac{6}{x}-\dfrac{5}{y}=2$

8. $3x-3(x-2y)=y+1$

9. $2y-(x+y)=y+1$

10. $(3-y)^2-y^2=x+2$

11. $x^2-(x-1)^2=y$

12. $(x+y)^2-(x-y)^2=1$

13. $x(y+1)=16-y(1-x)$

14. $\dfrac{x-3}{2}=\dfrac{4+y}{5}$

15. $x-2x^2+3=\dfrac{x-7}{2}$

16. $x-3=\dfrac{4x+17}{5}$

17. $13x-17y=y(7-2x)$

18. $y^2-3y=(1+y)^2-2x$

19. $x-1=\dfrac{2y}{x}-x$

20. $3x-4=89(x-y)-y$

21. $x-x(1+x)=y-3x$

22. $x^2-2x=3-x^2+y$

23. $\dfrac{2y-5}{14}=\dfrac{x-3}{9}$

24. $16x=y(4+(x-3))-xy$

Find the *x*- and *y*-intercepts of the given equations, if possible, and then sketch their graphs. See Examples 2 and 3.

25. $4x-3y=12$

26. $y-3x=9$

27. $5-y=10x$

28. $y-2x=y-4$

29. $3y=9$

30. $2x-(x+y)=x+1$

31. $x+2y=7$

32. $y-x=x-y$

33. $y=-x$

34. $2x-3=1-4y$

35. $3y+7x=7(3+x)$

36. $4-2y=-2-6x$

37. $x+y=1+2y$

38. $3y+x=2x+3y+4$

39. $3(x+y)+1=x-5$

Match each equation to the correct graph.

40. $y=2x+3$

41. $2x+3y=4$

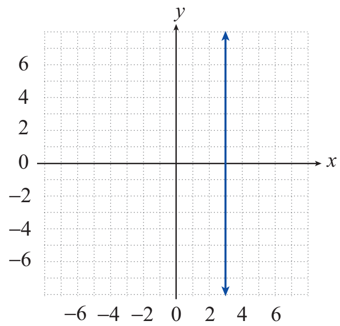
42. $2x-1=5$

43. $y+3-x=3$

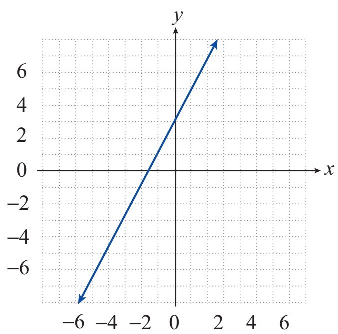
44. $4y+3=11$

45. $5y-x-1=4y+3x+5$

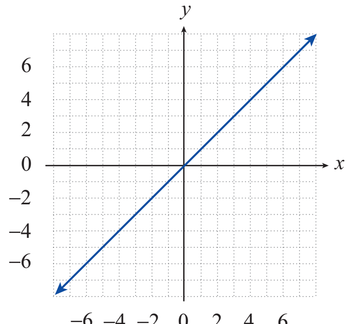
a.



b.



c.



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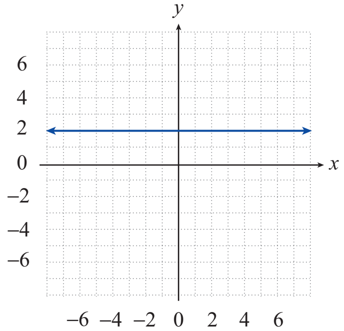
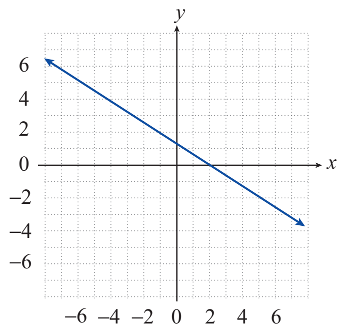
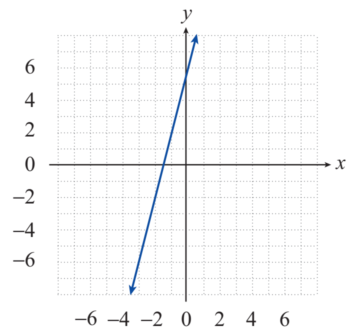


e.



f.



Solve each equation for the specified variable.

46. Standard form of a line: $ax + by = c$; solve for y

47. Perimeter of a triangle: $P = a + b + c$; solve for a

48. Surface area of a rectangular solid: $S = 2lw + 2wh + 2lh$; solve for w

APPLICATIONS

49. In your history class, you were told that the current population of Jamaica is approximately 24,000 more than 9 times the population of the Bahamas. Using j to represent the population of Jamaica and b to represent the population of the Bahamas, write this in the form of an equation. Then solve your equation for b to find an equation representing the population of the Bahamas. Are these equations linear?

50. The lowest point in the ocean, the bottom of the Mariana Trench, is about 1100 feet deeper than 26 times the depth of the lowest point on land, the Dead Sea. Find an equation to express the depth of the Mariana Trench, m, in terms of the depth of the Dead Sea, d. Then solve your equation for d to find the depth of the Dead Sea in terms of the depth of the Mariana Trench. Are these equations linear?

Solution

We already have a point (actually, two) on the line, but we still need the slope to use the point-slope form. We can calculate this using the two points and the slope formula.

$$m = \frac{-2-6}{-3-1} = \frac{-8}{-4} = 2$$

Now we can substitute into the point-slope form, then solve for y to obtain the desired slope-intercept equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 6 &= 2(x - 1) \\ y - 6 &= 2x - 2 \\ y &= 2x + 4 \end{aligned}$$

Note that no matter which point we substitute into the point-slope form, the resulting slope-intercept equation is the same.

We close this section with a summary of the different forms of linear equations, what information we need to write them, and what they are each most useful for.

Standard Form:	$ax + by = c$
Information Required:	Typically, we arrive at the standard form when given a linear equation in another form.
Potential Uses:	The standard form is most useful for easily calculating the x- and y-intercepts.
Slope-Intercept Form:	$y = mx + b$
Information Required:	The slope m and the y-intercept $(0, b)$.
Potential Uses:	The slope-intercept form makes it very easy to find the y-intercept and slope, and therefore to graph the line.
Point-Slope Form:	$y - y_1 = m(x - x_1)$
Information Required:	The slope m and a point on the line (x_1, y_1) or two points on the line (x_1, y_1) and (x_2, y_2).
Potential Uses:	The point-slope form allows us to find the equation for a line when the y-intercept is unknown.

2.3 EXERCISES

PRACTICE

Determine the slope of the line passing through the specified points. See Example 1.

1. $(0, -3)$ and $(-2, 5)$

2. $(-3, 2)$ and $(7, -10)$

3. $(4, 5)$ and $(-1, 5)$

4. $(3, -1)$ and $(-7, -1)$

5. $(3,-5)$ and $(3,2)$

6. $(0,0)$ and $(-2,5)$

7. $(-2,1)$ and $(-5,-1)$

8. $\left(\frac{1}{2},-7\right)$ and $\left(\frac{3}{4},-5\right)$

9. $\left(10,\frac{1}{5}\right)$ and $\left(4,-\frac{4}{5}\right)$

10. $(-2,4)$ and $(6,9)$

11. $(0,-21)$ and $(-3,0)$

12. $(-3,-5)$ and $(-2,8)$

13. $\left(\frac{1}{3},9\right)$ and $(2,4)$

14. $(29,-17)$ and $(31,-29)$

15. $(7,4)$ and $(-6,13)$

Determine the slopes of the lines defined by the following equations. See Example 2.

16. $8x-2y=11$

17. $2x+8y=11$

18. $12x-4y=-9$

19. $4y=13$

20. $\frac{x-y}{3}+2=4$

21. $7x=2$

22. $3y-2=\frac{x}{5}$

23. $3-y=2(5-x)$

24. $3(2y-1)=5(2-x)$

25. $\frac{x+2}{3}+2(1-y)=-2x$

26. $2y-7x=4y+5x$

27. $x-7=\frac{2y-1}{-5}$

Use the slope-intercept form to graph the equations. See Example 3.

28. $6x-2y=4$

29. $3y+2x-9=0$

30. $5y-15=0$

31. $x+4y=20$

32. $\frac{x-y}{2}=-1$

33. $3x+7y=8y-x$

34. $-4x-4y=8$

35. $-5x+3y+16=0$

36. $3x=3y-21$

Find the equation, in slope-intercept form, of the line with the given *y*-intercept and slope. See Example 4.

37. y-intercept $(0,-3)$; slope of $\frac{3}{4}$

38. y-intercept $(0,5)$; slope of -3

39. y-intercept $(0,-7)$; slope of $-\frac{5}{2}$

40. y-intercept $(0,6)$; slope of 4

41. y-intercept $(0,-9)$; slope of -5

42. y-intercept $(0,2)$; slope of $\frac{1}{2}$

Find the equation, in standard form, of the line passing through the given point with the given slope.

43. point $(-1, -3)$; slope of $\frac{3}{2}$

44. point $(6, 0)$; slope of $\frac{5}{4}$

45. point $(-3, 5)$; slope of 0

46. point $(-2, -13)$; undefined slope

47. point $(3, -1)$; slope of 10

48. point $(-1, 3)$; slope of $-\frac{2}{7}$

49. point $(5, 11)$; slope of -3

50. point $(5, -9)$; slope of $-\frac{1}{2}$

Find the equation, in standard form, of the line passing through the specified points.

51. $(-1, 3)$ and $(2, -1)$

52. $(1, 3)$ and $(-2, 3)$

53. $(2, -2)$ and $(2, 17)$

54. $(-9, 2)$ and $(1, 5)$

55. $(3, -1)$ and $(8, -1)$

56. $\left(\frac{4}{3}, 1\right)$ and $\left(\frac{2}{5}, \frac{3}{7}\right)$

57. $(-2, 8)$ and $(5, 6)$

58. $(8, -10)$ and $(8, 0)$

59. $(7, 5)$ and $(-9, 5)$

60. $(7, 7)$ and $(9, -8)$

61. $\left(\frac{2}{3}, \frac{5}{4}\right)$ and $\left(\frac{3}{5}, \frac{9}{8}\right)$

62. $(-5, -5)$ and $(10, -11)$

Match each equation or description to the correct graph.

63. $-3x - 2y = 17$

64. $-4y + 10 = -4x$

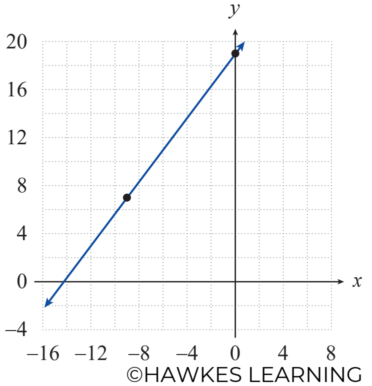
65. $-6y + 9 = \frac{x}{-2}$

66. point $(-9, 7)$; slope $\frac{4}{3}$

67. point $(-2, 4)$; slope -2

68. point $(0, -5)$; slope -9

a.



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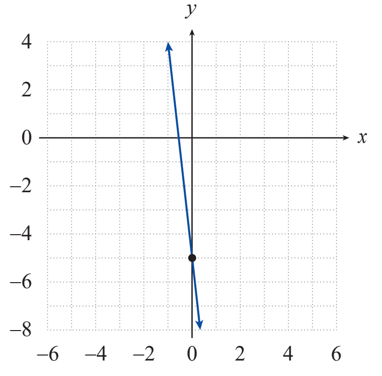

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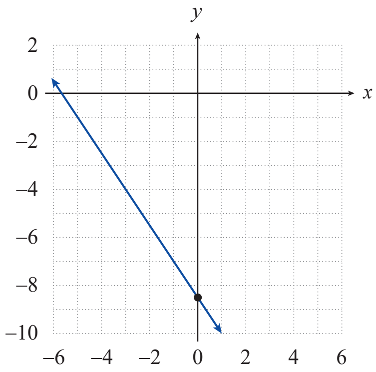

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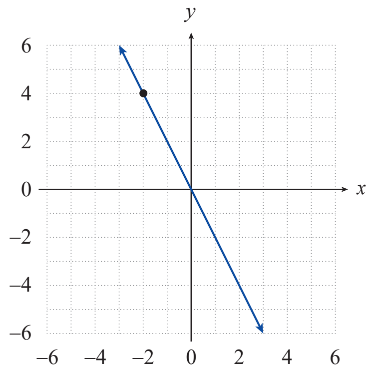
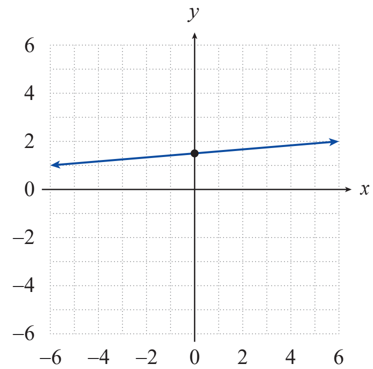


e.



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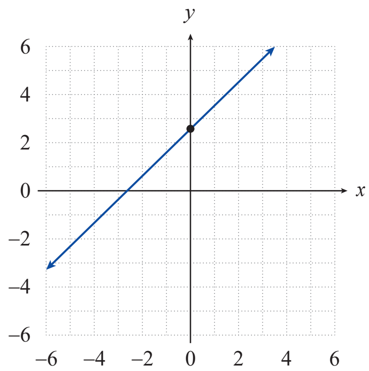


APPLICATIONS

69. A bottle manufacturer has determined that the total cost (C) in dollars of producing x bottles is $C = 0.25x + 2100$.

a. What is the cost of producing 500 bottles?

b. What are the fixed costs (costs incurred even when 0 bottles are produced)?

c. What is the increase in cost for each bottle produced?

70. Sales at Glover's Golf Emporium have been increasing linearly for the past couple of years. Last year, sales were \$163,000. This year, sales were \$215,000. If sales continue to increase at this linear rate, predict the sales for next year.

71. Amy owns stock in a company. If the stock had a value of \$2500 in 2018 when she purchased it, what has been the average change in value per year if in 2020 the stock was worth \$3150?

72. For tax and accounting purposes, businesses often have to depreciate equipment values over time. One method of depreciation is the straight-line method. Three years ago Hilde Construction purchased a bulldozer for \$51,500. Using the straight-line method, the bulldozer has now depreciated to a value of \$43,200. If V equals the value at the end of year t, write a linear equation expressing the value of the bulldozer over time. How many years from the purchase date will the value equal \$0? Round your answer to two decimal places.

Are the lines parallel?

No, the slopes are not equal.

Are the lines perpendicular?

Yes, the slopes are negative reciprocals of each other.

Thus, the lines are perpendicular.

b. One line is in point-slope form, so we can see its slope is 9. We calculate the slope of the other line using the two points given.

$$m=\frac{22-4}{2-0}=\frac{18}{2}=9$$

Are the lines parallel?

Yes, the slopes are equal. Thus, the lines are parallel. Note that we didn't need to find the equation of the second line.

c. Both lines are in slope-intercept form, so we can read off the slopes: $\frac{3}{4}$ and $\frac{4}{3}$.

Are the lines parallel?

No, the slopes are not equal.

Are the lines perpendicular?

No, the slopes are reciprocals, not *negative* reciprocals.

Thus, the lines are neither parallel nor perpendicular.

2.4 EXERCISES

PRACTICE

Find the equation, in slope-intercept form, for the line parallel to the given line and passing through the indicated point. See Examples 1 and 2.

1. $y-4x=7;\quad (-1,5)$

2. $6x+2y=19;\quad (-6,-13)$

3. $3x+2y=3y-7;\quad (3,-2)$

4. $2-\frac{y-3x}{3}=5;\quad (0,-2)$

5. $y-4x=7-4x;\quad (23,-9)$

6. $2(y-1)+\frac{x+3}{5}=-7;\quad (-5,0)$

7. $6y-4=-3(1-2x);\quad (-2,-2)$

8. $5-\frac{7y+5x}{2}=1;\quad (4,1)$

9. $2(y-1)-\frac{7x+1}{3}=-3;\quad (1,10)$

10. $8y-6=-3(4-x);\quad (11,-5)$

Each set of four ordered pairs defines the vertices, in counterclockwise order, of a quadrilateral. Determine if the quadrilateral is a parallelogram. See Example 3.

11. $\{(-2,2),(-5,-2),(2,-3),(5,1)\}$

12. $\{(-1,6),(-4,7),(-2,3),(1,1)\}$

13. $\{(-3,3),(-2,-2),(3,-1),(2,4)\}$

14. $\{(-2,-3),(-3,-6),(1,-2),(2,1)\}$

15. $\{(-6,-2),(-1,0),(-3,4),(-8,2)\}$

16. $\{(-3,-2),(3,-3),(5,2),(-1,3)\}$

17. $\{(-1,-1),(5,1),(3,5),(-2,3)\}$

18. $\{(0,1),(6,0),(7,4),(1,6)\}$

Determine if the two lines are parallel. See Example 6.

19. $y = 8x + 7$ and $y = -8x + 7$

20. $x - 5y = 2$ and $5x - y = 2$

21. $2x - 3y = (x-1) - (y-x)$ and $-2y - x = 9$

22. $3 - (2y + x) = 7(x - y)$ and $\dfrac{5y+1}{4} = 3 + 2x$

23. $6 = -12(x - y) + y$ and $13y = -12x + 3$

24. $\dfrac{2x-3y}{3} = \dfrac{x-1}{6}$ and $2y - x = 3$

25. $\dfrac{x-y}{2} = \dfrac{x+y}{3}$ and $\dfrac{2x+3}{5} - 4y = 1 + 2y$

26. $5 - (4y + 3x) = 5(x - y)$ and $y + 4 = 5 + 8x$

27. $7x - 2(x + 3) = 5y - x$ and $-6x = 1 - 5y$

28. $\dfrac{2y+11x}{3} = x + 1$ and $7x - 8y = 9x + 7$

29. $\dfrac{x-y}{5} = \dfrac{x+y}{3} - 1$ and $7 = -2(x - y) + 6y$

30. $2x + 5y = 14$ and the line passing through the points $(8,-5)$ and $(3,-3)$

Find the equation, in slope-intercept form, for the line perpendicular to the given line and passing through the indicated point. See Examples 4 and 5.

31. $3x + 2y = 3y - 7;\quad (3,-2)$

32. $6y + 2x = 1;\quad (-4,-12)$

33. $-y + 3x = 5 - y;\quad (-2,7)$

34. $x + y = 5;$ origin

35. $x = \dfrac{1}{4}y - 3;\quad (1,-1)$

36. $2(y + x) - 3(x - y) = -9;\quad (2,5)$

37. $4x + 8y = 4y - 3;\quad (-2,1)$

38. $\dfrac{3x-y}{4} = \dfrac{4x-5}{2};\quad (8,5)$

39. $4(y + x) - 8(x - y) = -1;\quad (6,10)$

40. $\dfrac{3x+4}{3} - 3y = 1 - 4y;\quad (2,-8)$

Determine if the two lines are perpendicular. See Example 6.

41. $x-5y=2$ and $5x-y=2$

42. $y=5x+4$ and $y=-\frac{1}{5}x-9$

43. $3x+y=2$ and $x+3y=2$

44. $\frac{3x-y}{3}=x+2$ and $x=9$

45. $5x-6(x+1)=2y-x$ and $2y-(x+y)=4y+x$

46. $-6y+3x=7$ and $8x-3(x+1)=3y-x$

47. $-x=-\frac{2}{5}y+2$ and $5y=2x$

48. $\frac{7x-5y}{4}=x+2$ and $-3y-3x=2x+4$

49. $3(4-x)=6y+3$ and $-3y-2x=3-8x$

50. $\frac{x-1}{2}+\frac{3y+2}{3}=-9$ and $3y-5x=x+5$

51. $1-\frac{2y-5x}{2}=7x+4$ and $9x-2y=11$

52. $y-\frac{2}{3}=4\left(x+\frac{7}{11}\right)$ and the line passing through the points $(-2,4)$ and $(7,-14)$

Each set of four ordered pairs defines the vertices, in counterclockwise order, of a quadrilateral. Use the ideas in this section to determine if the quadrilateral is a rectangle.

53. $\{(-2,2),(-5,-2),(2,-3),(5,1)\}$

54. $\{(2,-1),(-2,1),(-3,-1),(1,-3)\}$

55. $\{(1,2),(3,-3),(9,-1),(7,4)\}$

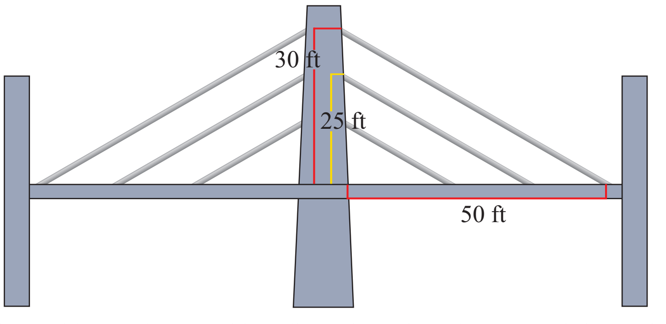
56. $\{(5,-7),(1,-13),(28,-31),(32,-25)\}$

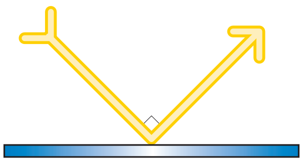
57. $\{(-5,-1),(0,-6),(5,-1),(0,4)\}$

58. $\{(-3,-3),(3,-2),(1,2),(-5,1)\}$

APPLICATIONS

59. A construction company is building a new suspension bridge that has support cables attached to a center tower at various heights. One cable is attached at a height of 30 feet and connects to the roadbed 50 feet from the base of the tower. If the support cables should run parallel to each other, how far from the base should the company attach a cable whose other end is connected to the tower at a height of 25 feet?





60. A light beam hits a mirror and is reflected off the mirror at a right angle. If the line formed by the original beam of light can be described by an equation of the form $y=-3.2x+b$ (for some constant b), write the form of an equation that describes the line of the reflected beam (use an arbitrary constant c in your answer).

$$\begin{aligned}\hat{y} &= -0.084x + 8.722\\ &= -0.084(64.1) + 8.722\\ &= -5.3844 + 8.722\\ &= 3.3376\end{aligned}$$

So, when a child's weight is 64.1 pounds, we can predict that his or her self-esteem score would be around 3.3.

d. The weight of 45.0 pounds is outside of the range of the original data since it is smaller than any of the other data pieces, so it is not appropriate to use the regression line for prediction.

e. Once again it is not appropriate to use the regression line for predictions in this case. The study included only children between the ages of 9 and 11. A 13-year-old is not in the same population as the study and cannot be assumed to have the same characteristics.

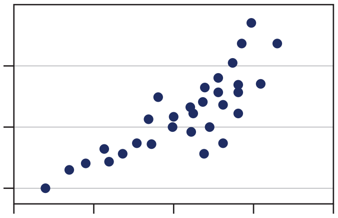
Skill Check Answers

1. Answers will vary. Examples may include: The number of candy bars consumed daily and weight gain or the distance between two locations and the length of time it takes to drive between the two.

2.5 EXERCISES

PRACTICE

In each scatter plot, determine whether there appears to be a positive linear correlation, a negative linear correlation, or no linear correlation.

1.


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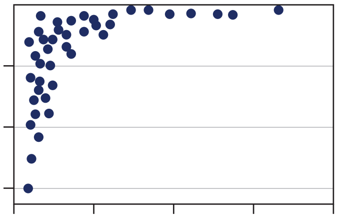

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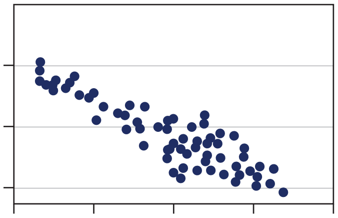

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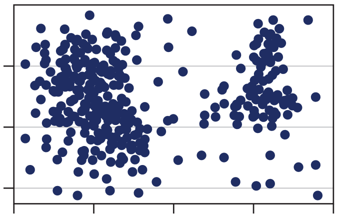


Consider each set of variables and predict whether the variables would have a weak negative relationship, a strong negative relationship, a weak positive relationship, a strong positive relationship, or no relationship at all.

5. Body weight and hours of exercise per week

6. A person's height and their self-esteem

7. Vision ability and IQ

8. Number of hours spent studying for a test and the grade on the test

Determine whether each correlation coefficient is statistically significant at the specified level of significance for the given sample size.

9. $r = 0.703, \quad \alpha = 0.01, \quad n = 12$

10. $r = 0.403, \quad \alpha = 0.05, \quad n = 25$

11. $r = 0.378, \quad \alpha = 0.05, \quad n = 29$

12. $r = 0.809, \quad \alpha = 0.01, \quad n = 8$

Use the linear regression model $\hat{y} = ax + b$, to predict the *y*-value for each value of *x*.

13. $\hat{y} = 28.01x + 17.83$

a. $x = 21$

b. $x = 31$

c. $x = 40$

14. $\hat{y} = -16.5x + 230.55$

a. $x = 5$

b. $x = 13$

c. $x = 35$

APPLICATIONS

For each data set, find the following.

a. Estimate the correlation in words as positive, negative, or no correlation.

b. Calculate the correlation coefficient *r*. Round your answer to the nearest thousandth.

c. Determine whether *r* is statistically significant at the 0.01 level of significance.

15. The following table gives the number of hours a student watches TV per week and his or her overall GPA.

Hours of TV per Week and Overall GPA									
TV Hours	20	10	25	15	14	13	21	9	5
GPA	2.0	2.46	2.3	2.9	3.0	3.2	3.5	3.3	3.7

16. The following table gives a sample of annual income and number of years of education.

Annual Income and Years of Education						
Annual Income	\$21,000	\$39,000	\$40,000	\$39,500	\$42,000	\$55,500
Years of Education	12	12	14	16	16	16
Annual Income	\$61,000	\$45,000	\$100,000	\$142,000	\$240,000	\$205,000
Years of Education	17	16	16	20	22	21

17. The following table shows the diastolic blood pressure reading and the stress test score for 20 adults.

Diastolic Blood Pressure Reading and Stress Test Score			
Stress Test Score	**Diastolic Blood Pressure Reading**	**Stress Test Score**	**Diastolic Blood Pressure Reading**
51	67	78	79
59	66	79	83
62	71	83	81
63	76	84	83
64	73	88	85
68	77	87	90
71	77	89	82
70	76	91	80
72	80	90	86
82	82	90	88

18. The following table shows the heights of identical twins in centimeters.

Heights of Identical Twins	
Sibling 1	**Sibling 2**
110.5	109.5
116.6	115.6
122.6	121.6
128.2	127.4
133.5	133.5
138.8	140.2
145.0	146.7
152.3	151.9
159.6	155.0
165.1	156.6
168.3	157.1
169.9	157.6
170.7	158.0

Solve each problem.

19. The following table gives the data for the number of cigarettes women smoked in their third trimester of pregnancy and the number of nonviolent crime arrests for their male babies.

Number of Cigarettes and Number of Arrests for Sons										
# of Cigarettes	0	5	3	10	22	19	30	15	8	12
# of Arrests	1	4	0	5	9	12	10	0	4	9

a. Determine the regression line $\hat{y} = ax + b$. Round the slope and y-intercept to the nearest thousandth.
b. Determine if the regression equation is appropriate, at the 0.05 level of significance, to use for making predictions. If so, answer part **c.**
c. If a mother smokes eight cigarettes in her third trimester, make a prediction for the number of times her son will be arrested for a nonviolent crime, if appropriate.

20. The following table shows students' test grades on the first two tests in an introductory literature class.

Test Grades in Introductory Literature Class												
Test 1 (x)	61	45	71	81	89	55	84	91	95	59	77	88
Test 2 (y)	67	79	68	80	87	68	87	90	97	71	77	74

a. Determine the regression line $\hat{y} = ax + b$. Round the slope and y-intercept to the nearest thousandth.
b. Determine if the regression equation is appropriate, at the 0.05 level of significance, to use for making predictions. If so, answer part **c.**
c. If a student scored a 70 on his first test, make a prediction for his score on the second test, if appropriate.

21. The following table shows the results on evaluations measuring self-esteem and perceived family support from 10 adolescents.

Self-Esteem and Perceived Family Support Evaluation Results										
Self-Esteem	30	31	31	28	27	26	15	32	27	33
Family Support	13	13	19	21	8	4	10	12	7	17

a. Determine the regression line $\hat{y} = ax + b$. Round the slope and y-intercept to the nearest thousandth.
b. Determine if the regression equation is appropriate, at the 0.05 level of significance, to use for making predictions. If so, answer part **c.**
c. If an adolescent had a self-esteem score of 22, make a prediction for his perceived family support score, if appropriate.

22. A medical equipment company wishes to show that a new device works with the same degree of accuracy and precision as an earlier model to perform an electrocardiogram. One of the measurements tested was the change in radio electric waves during a cardiac cycle. The following results were collected from both healthy adults and those with cardiovascular problems.

Change in Radio Electric Waves during Cardiac Cycle	
# of 5 mm Squares between R Waves	
Old	**New**
2	2
3	3
4	4.5
3	3
6	6
4	4.5
3	3
5	5
3	3.5
2	2
6	6
4	4
6	6
5	5
3	3
2	2

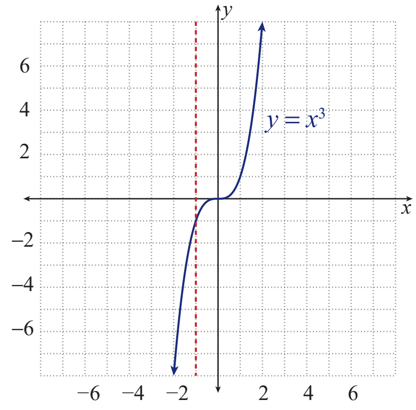
a. Determine the regression line $\hat{y} = ax + b$. Round the slope and y-intercept to the nearest thousandth.

b. Determine if the regression equation is appropriate, at the 0.01 level of significance, to use for making predictions. If so, answer part **c.**

c. If the old machine had a reading of 5.5, make a prediction for the new machine reading, if appropriate.

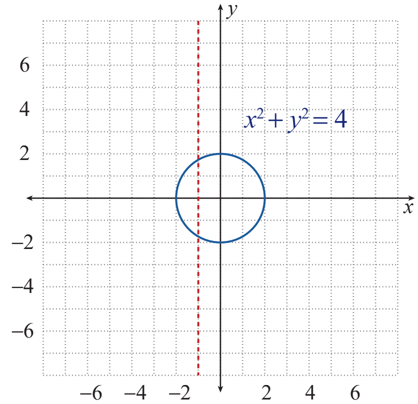
Solution

a.



The equation $y = x^3$ represents a function (which can also be written as $f(x) = x^3$). No vertical line can intersect the graph in more than one point.

b.



The graph of $x^2 + y^2 = 4$ is a circle. The graph shows that the equation does not represent a function. Vertical lines can be drawn that intersect the graph in more than one point.

3.1 EXERCISES

PRACTICE

In Exercises 1–9, evaluate the given function for parts **a.–d.**

1. $f(x) = 2x - 7$
   a. $f(5)$
   b. $f(-2)$
   c. $f(a + 1)$
   d. $f(a) + 1$

2. $f(x) = 3x + 5$
   a. $f(2)$
   b. $f(-1)$
   c. $f(a + 1)$
   d. $f(a) + 1$

3. $f(x) = x^2 - 2x + 1$
   a. $f(-2)$
   b. $f(3)$
   c. $f(a + 1)$
   d. $f(a) + 1$

4. $f(x) = 3x^2 - x + 2$
   a. $f(-3)$
   b. $f(2)$
   c. $f(a + 1)$
   d. $f(a) + 1$

5. $f(x) = x^3 + x^2 - 3x + 1$
a. $f(-1)$
b. $f(-3)$
c. $f(a + 1)$
d. $f(a) + 1$

6. $f(x) = 2x^3 - 4x^2 + x - 6$
a. $f(-2)$
b. $f(4)$
c. $f(a + 1)$
d. $f(a) + 1$

7. $f(x) = 4x^2 - 1$
a. $f(3)$
b. $f(a + 2)$
c. $f(x + h)$
d. $f(-2) - f(-1)$

8. $f(x) = 2 - 3x^2$
a. $f(5)$
b. $f(a - 3)$
c. $f(x + h)$
d. $f(3) - f(2)$

9. $f(x) = \sqrt{x+5}$
a. $f(-1)$
b. $f(a + 2)$, where $a \geq -7$
c. $f(x + h)$
d. $f(4) - f(1)$

10. Let $f(x) = \sqrt{x^2+1}$. Find **a.** $f(\sqrt{3})$, and **b.** $f(a+1)$.

11. Let $f(x) = \begin{cases} x-4 & \text{if } x \leq 2 \\ x^2-6 & \text{if } x > 2 \end{cases}$. Find **a.** $f(-1)$, **b.** $f(2)$, **c.** $f(2.5)$, and **d.** $f(3)$.

12. Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 3x-2 & \text{if } x \geq 0 \end{cases}$. Find **a.** $f(0)$, **b.** $f(-2)$, **c.** $f(1.5)$, and **d.** $f(3)$.

In Exercises 13–27, find $f(x+h) - f(x)$.

13. $f(x) = 3x - 1$

14. $f(x) = 5x - 2$

15. $f(x) = x^2 + 4$

16. $f(x) = x^2 - 3$

17. $f(x) = 2x^2 + 1$

18. $f(x) = 5 + 3x^2$

19. $f(x) = x^2 - x$

20. $f(x) = x^2 + 2x$

21. $f(x) = 3x - x^2$

22. $f(x) = 4x^2 - x$

23. $f(x) = 2x^2 - 3x$

24. $f(x) = x^3$

25. $f(x) = x^3 - 1$

26. $f(x) = x^3 + 7$

27. $f(x) = x^3 + 5$

In Exercises 28–39, determine the domain of each function.

28. $f(x) = \dfrac{3x+1}{(x-5)(x-6)}$

29. $f(x) = \sqrt{2x+10}$

30. $f(x) = \sqrt{x^2+2}$

31. $f(x) = \dfrac{5}{\sqrt{x+10}}$

32. $f(x) = \dfrac{2x}{x-2}$

33. $f(x) = \dfrac{x-3}{x+1}$

34. $f(x) = \dfrac{4}{x^2-x-12}$

35. $f(x) = x - 3$

36. $f(x) = 4 - 3x$

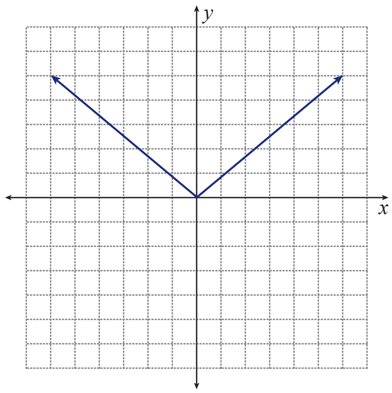
37. $f(x) = \dfrac{1}{\sqrt{2x+5}}$

38. $f(x) = \begin{cases} 3x+1 & \text{if } 0 \le x < 4 \\ 5x-2 & \text{if } x \ge 4 \end{cases}$

39. $f(x) = \begin{cases} 2-x^2 & \text{if } x \le 2 \\ x-4 & \text{if } x > 2 \end{cases}$

In Exercises 40–51, use the vertical line test to determine whether or not each graph represents a function.

40.



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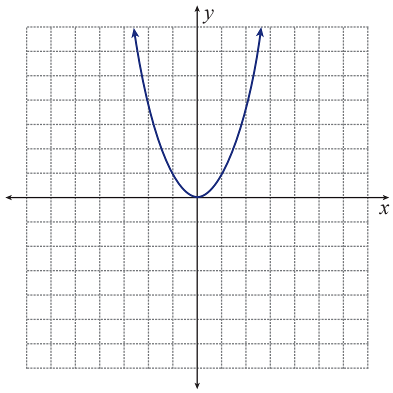

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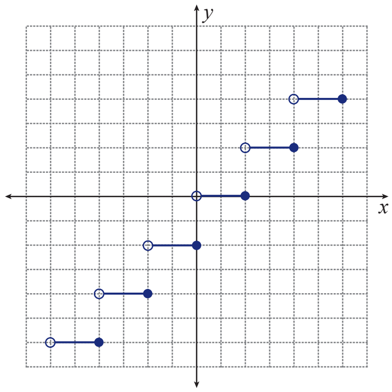

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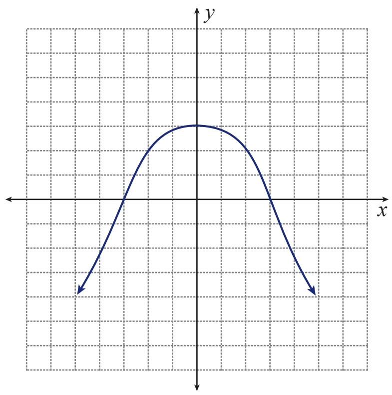

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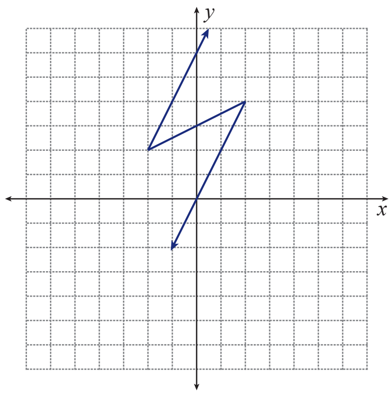

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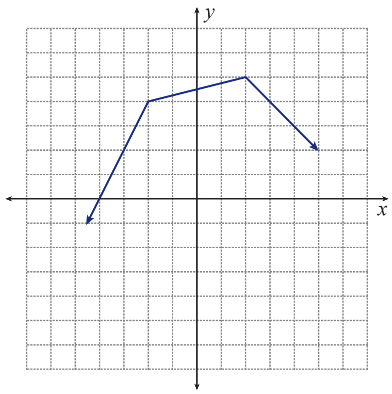

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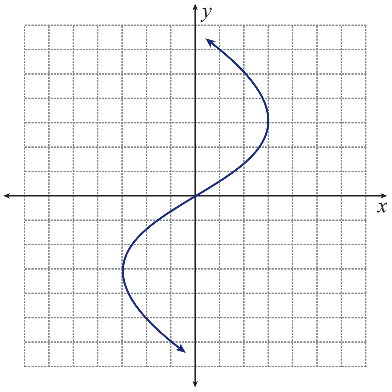

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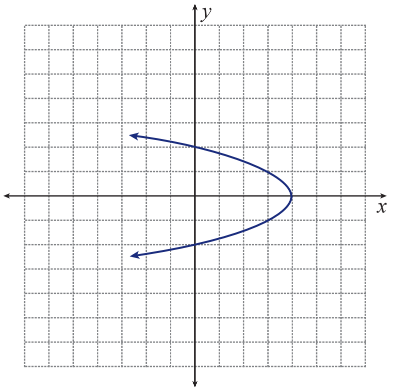
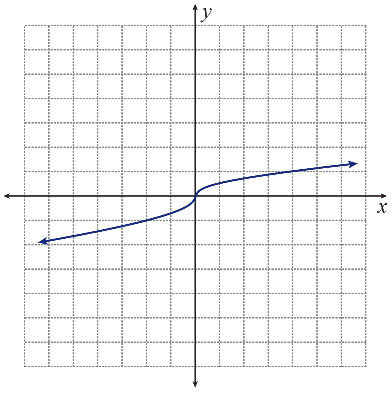


48.



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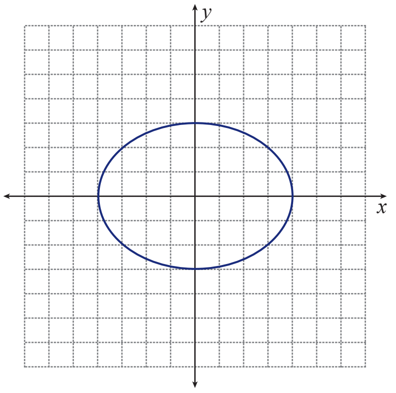

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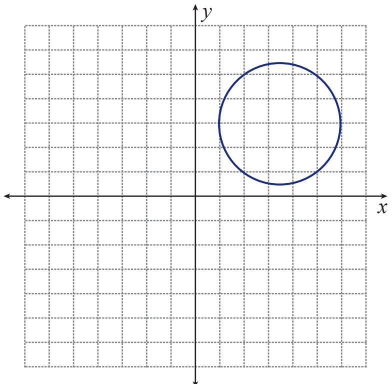

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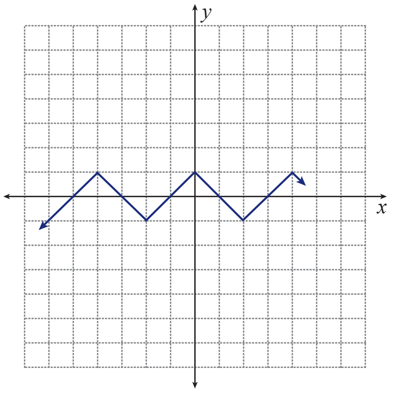


3.2 EXERCISES

PRACTICE

Find the break-even point given the revenue and cost functions in Exercises 1–4.

1. $R(x)=15x$
$C(x)=5x+30$

2. $R(x)=27x$
$C(x)=14x+442$

3. $R(x)=24x-0.2x^2$
$C(x)=8x+300$

4. $R(x)=9x-0.7x^2$
$C(x)=2x+11.2$

Find the equilibrium point given the supply and demand functions in Exercises 5–8.

5. $S(x)=2x+3$
$D(x)=15-x$

6. $S(x)=4x+7$
$D(x)=33-1.2x$

7. $S(x)=x^2+x$
$D(x)=35-x$

8. $S(x)=x^2+3$
$D(x)=51-2x$

In Exercises 9–18, the quantity represented by *y* is related to the quantity represented by *x* in one of the following ways:

(a) *y* is directly proportional to *x*
(b) *y* is inversely proportional to *x*
(c) *y* is directly proportional to the square of *x*
(d) *y* is inversely proportional to the square of *x*
(e) other

Match the given formula to the best choice of (a) through (e).

9. $y=32x$

10. $y=\frac{32}{x}$

11. $y=16x^2$

12. $y=\frac{-32}{x^2}$

13. $y=-x$

14. $y=2x+3$

15. $y=\pi x$

16. $y=(x-1)^2$

17. $y=\frac{x}{10}$

18. $y=\frac{wxv}{stu}$

APPLICATIONS

For Exercises 19–23, use the following situation:

Two students create a downloadable app which connects dots on a grid so that two players can play "Chase the Rabbit." They have developed a website on which to promote and sell the app for \$5.00 and they pay an outside vendor \$0.50 per purchase to manage the payments. They also pay another student \$27/day (5 days a week) to monitor a customer service email box to answer questions and relay orders.

Let x denote the number of copies of the app sold, let $C(x)$ be the weekly total cost function (linear), and let $R(x)$ be the revenue function.

19. **a.** Write the expression for $C(x)$.
b. Determine the weekly cost of selling 500 copies of the app.

20. **a.** Write the revenue function.
b. How much revenue is produced by the sale of 500 copies of the app.

21. **a.** Write the profit function.
b. Determine the profit from the sale of 500 copies of the app.

22. How many copies of the app must be sold in order to break even?

23. A business professor estimates that the campus craze for the game could become national, and, therefore, the game could be marketed nationally. If 100 colleges were to become market sites, find a profit function for all 100 colleges together. Assume total profits and the number of colleges involved are directly proportional.

For Exercises 24–28, use the following information:

An ideal gas satisfies a law which may be stated as $\frac{PV}{T} = 0.821n$ where P is the pressure in atmospheres (atm), V is the volume in liters (L), T is the temperature in kelvins (K) ($K = 273 + C$, where C is the temperature in degrees Celsius), and n is the number of moles (gram molecular weights). Thus, for one mole of gas, pressure and volume are indirectly proportional for a constant temperature, pressure and temperature are directly proportional for a fixed volume, and volume and temperature are directly proportional for a fixed pressure.

Assume, if necessary, that there is 1 mole (6.023×10^{23} molecules) present.

24. What volume is occupied by 1 mole of gas at a temperature of 300 K and a pressure of 2 atm?

25. A fixed volume of gas is heated from a temperature of 200 K and 0.5 atm of pressure to 300 K. What is the new pressure?

26. A gas at a pressure of 2 atm expands, at constant temperature, from 10 L to 15 L. What is the new pressure?

27. One mole of gas occupies a volume of 2 L and has a temperature of 136.5 K. What is the pressure in atmospheres?

28. A gas has a volume of 2 liters, a temperature of 30°C and a pressure of 1 atm. When the gas is heated to 60°C and its volume is compressed to a volume of 1.25 liters, what is its new pressure? (**Hint:** For this problem, $n \neq 1$.)

29. Modeling in business: The manager of a pie shop sells his pies for \$6.50. The overhead is \$378 per day and each pie costs \$1.10 to make.
a. Write the revenue function.
b. Write the cost function.
c. Write the profit function.
d. Find the break-even point.

30. Modeling in business: A certain style of athletic shoe costs \$11.80 per pair to produce. The fixed costs are \$864 per week. The shoes can be sold for \$19.00 per pair.
a. Write the revenue function.
b. Write the cost function.
c. Write the profit function.
d. Find the break-even point.

31. Modeling in manufacturing: A manufacturer of golf clubs finds that the fixed costs are \$5780 per week and the cost of producing each set of clubs is \$73.00. Each set of clubs can be sold for \$243.00.
a. Write the revenue function.
b. Write the cost function.
c. Write the profit function.
d. Find the break-even point.

32. Modeling in business: A soft drink company has fixed costs of \$4000 per day. The variable costs are \$2.75 per case of soda. Each case sells for \$5.25.
a. Write the revenue function.
b. Write the cost function.
c. Write the profit function.
d. Find the break-even point.

33. Modeling in production: The cost of producing 200 pens is \$290. Producing 250 pens would cost \$297.50.
a. Find the average cost per pen for additional 50 pens over 200.
b. Assuming the total cost function is linear, write an equation for the cost of producing x pens.
c. What are the fixed costs?

34. Modeling in production: The Blue Umbrella Company can produce 500 umbrellas per week at a cost of \$1800. It would cost \$1950 to produce 600 umbrellas.
a. Find the average cost of each of the additional 100 umbrellas over 500.
b. Assuming the total cost is a linear function, write an equation for the cost of producing x umbrellas.
c. What are the fixed costs?

35. Revenue-profit: It has been determined that the cost of producing x units of a certain item is $11x + 500$. The demand function is given by $p = D(x) = 31 - 0.5x$.
a. Write the revenue function.
b. Write the profit function.

36. **Modeling in sales:** The manager of a men's store knows he can sell 60 pairs of a certain style of sock when the price is \$1.20 per pair. If the price is \$1.50, he can sell only 48 pairs of socks. The total cost function for x pairs of socks is $C(x) = 0.70x + 15$ dollars.
   a. Assuming the demand function is linear, write an equation for $D(x)$.
   b. Write the revenue function.
   c. Write the profit function.

37. **Modeling in manufacturing:** A manufacturer of TVs can sell 800 TVs to his dealers at \$384 each. If the price is \$380, he can sell 1000 TVs. The total cost of producing x TVs is $C(x) = 3600 + 250x - 0.01x^2$ dollars.
   a. Assuming the demand function is linear, write an equation for $D(x)$.
   b. Write the revenue function.
   c. Write the profit function.

38. **Revenue:** Suppose the revenue R from the sale of a product is directly proportional to the number of units x of the product that are sold. Suppose also that the revenue from the sale of 65 units of the product is \$1820.
   a. Write a function for R in terms of x.
   b. Find the revenue if 75 units are sold.

39. **Interest:** Suppose the annual interest I earned on an investment is directly proportional to the amount of money invested P. Suppose also that an investment of \$8200 earns an annual interest of \$512.50.
   a. Write a function for I in terms of P.
   b. Find the annual interest earned by \$6000.

40. **Interest:** What will \$6000 accumulate to if it is deposited in a bank for three years and earns 5% a year with annual compounding?

41. **Price:** Suppose that for a certain product, the price per item p is inversely proportional to the number of items sold x. Suppose also that the price per item is \$8.50 when 40 items are sold.
   a. Write a function for p in terms of x.
   b. Find the price if 34 items are sold.

42. **Demand:** Pat has decided to produce a limited number of prints from one of her paintings. She plans to issue x prints, where $0 < x \leq 50$. If she wants her revenue to be \$5000, write a function for the demand $D(x)$.

43. **Number of orders:** The owner of a camera shop expects to sell 800 cameras of a particular style during the year. How many orders will the dealer need to place with his distributor if each order is for x cameras?

44. **Salary:** A salesperson's weekly salary depends on the amount of her sales. Her salary is \$250 per week plus a commission of 8% of her weekly sales in excess of \$2500. Write a function for her salary if her sales were x dollars.

45. **International calls:** For an international call, the telephone company charges 65 cents for the first 3 minutes or less, plus 15 cents for each additional minute. Write a cost function for a call x minutes long.

46. Car rental: The rate for renting a car at a local agency is \$22.50 per day plus \$0.10 for each mile driven in excess of 100. If a car is rented for one day, write a function for the cost in terms of the number of miles driven.

47. Agriculture: A farmer cultivates bananas. They cost him 38 cents per bunch to produce. He is able to sell only 85% of those he produces. If he sells his bananas at 75 cents per bunch, find a function for his profit in terms of the number of bunches he produces.

48. Retail profit: A grocery store bought bags of frozen corn for 59 cents per bag and stored it in two freezers. During the night, one freezer defrosted and ruined 14 bags. If the remaining frozen corn was sold for 98 cents per bag, find a function for the profit in terms of the number of bags bought.

49. Retail profit: It costs Liz \$12 to build a picture frame. She estimates that, if she charges x dollars per frame, she can sell $60 - x$ frames per week. Write a function for her weekly profit.

50. Retail profit: A toy retailer pays \$3 each for a particular doll. He estimates that, if he charges x dollars for each doll, he will be able to sell $300 - 20x$ dolls. Write a function for his profit.

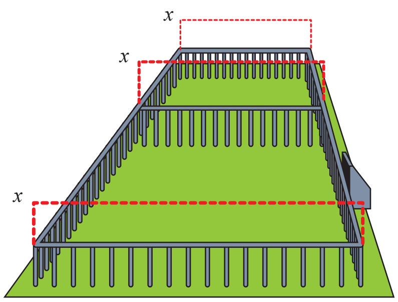
51. Area: The perimeter of a rectangle is 276 feet. If the rectangle is x feet long, write a function for the area $A(x)$.

52. Perimeter: The area of a rectangle is 426 cm^2. If the length of the rectangle is x centimeters, write a function for the perimeter $P(x)$.

53. Perimeter: The area of a rectangle is 288 ft^2. If the length of the rectangle is x feet, write a function for the perimeter $P(x)$.

54. Area: The perimeter of a rectangle is 197 inches. If the rectangle is x inches wide, write a function for the area $A(x)$.

55. Construction: The maintenance department at the city zoo wants to build a pen and divide it as shown in the diagram. If the department has a total of 720 feet of fencing, write a function for the area in terms of x.



3.3 EXERCISES

PRACTICE

Graph the following linear functions. See Example 1.

1. $f(x) = -5x + 2$

2. $g(x) = \dfrac{3x - 2}{4}$

3. $h(x) = -x + 2$

4. $p(x) = -2$

5. $g(x) = 3 - 2x$

6. $r(x) = 2 - \dfrac{x}{5}$

7. $f(x) = -2(1 - x)$

8. $a(x) = 3\left(1 - \dfrac{1}{3}x\right) + x$

9. $f(x) = 2 - 4x$

10. $g(x) = \dfrac{2x - 8}{4}$

11. $h(x) = 5x - 10$

12. $k(x) = 3x - \dfrac{2 + 6x}{2}$

13. $m(x) = \dfrac{-x + 25}{10}$

14. $q(x) = 1.5x - 1$

15. $w(x) = (x - 2) - (2 + x)$

Match the following functions with their graphs.

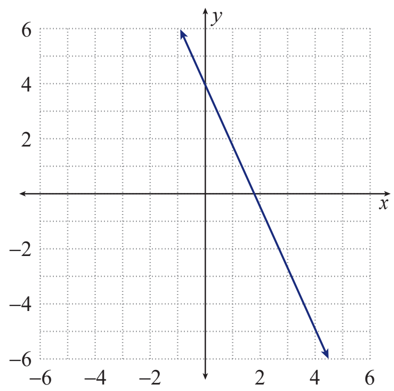
16. $f(x) = (8x - 14) - (-17 + 2x)$

17. $f(x) = 3x - \dfrac{7 + 8x}{3}$

18. $f(x) = \dfrac{6}{2} - \dfrac{2}{8}x$

19. $f(x) = 2\left(2 - \dfrac{8}{5}x\right) + x$

a.



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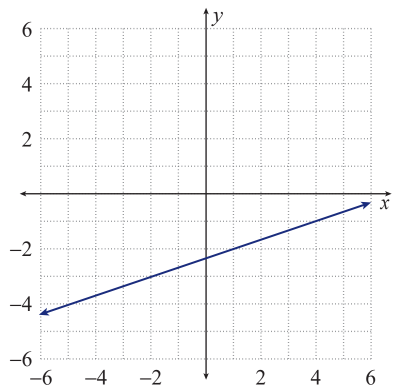

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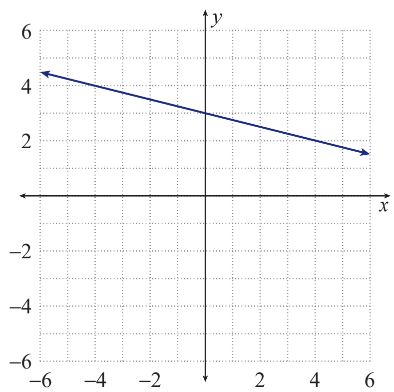
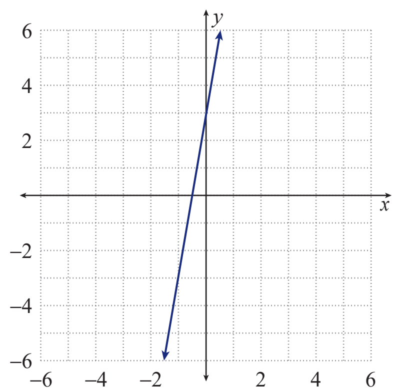


d.



Graph the following quadratic functions, accurately locating the vertices and *x*-intercepts (if any). See Example 2.

20. $f(x)=(x-2)^2+3$

21. $g(x)=-(x+2)^2-1$

22. $h(x)=x^2+6x+7$

23. $F(x)=3x^2+2$

24. $G(x)=x^2-x-6$

25. $p(x)=-2x^2+2x+12$

26. $q(x)=2x^2+4x+3$

27. $r(x)=-3x^2-1$

28. $s(x)=\dfrac{(x-1)^2}{4}$

29. $m(x)=x^2+2x+4$

30. $n(x)=(x+2)(2-x)$

31. $p(x)=-x^2+2x-5$

32. $f(x)=4x^2-6$

33. $k(x)=2x^2-4x$

34. $q(x)=(x+10)(x-2)+36$

Match the following functions with their graphs.

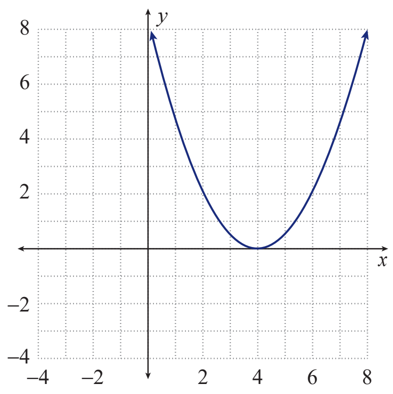
35. $f(x)=-x^2+2x$

36. $f(x)=x^2+7x+6$

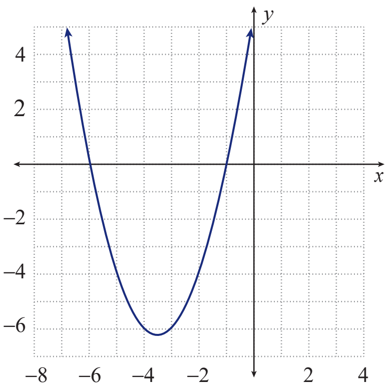
37. $f(x)=\dfrac{x^2-8x+16}{2}$

38. $f(x)=(x-5)(x+3)+16$

a.



b.



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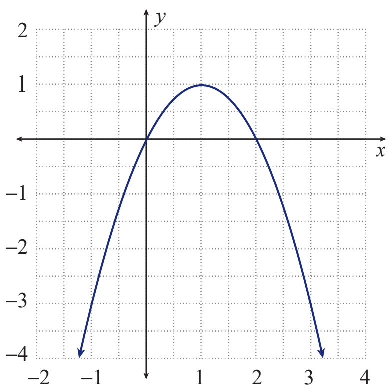

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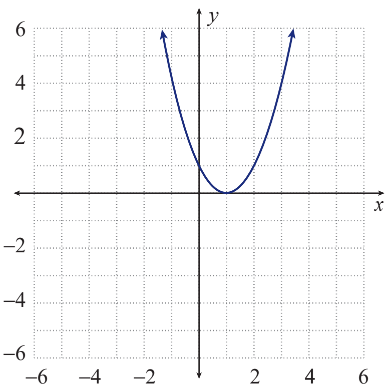


WRITING & THINKING

39. Without graphing, state the number of x-intercepts for each of the following functions and describe the location of the vertex in relation to the x-axis.

a. $y=(x-2)^2$

b. $y=(x-2)(x+2)$

c. $y=-(x-3)(x-1)$

d. $y=-(x-\sqrt{3})(x+\sqrt{3})$

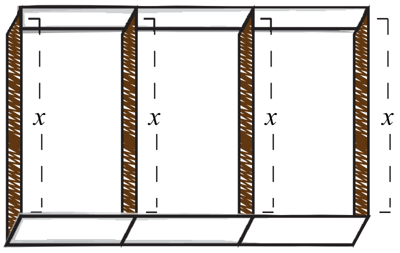
e. $y=x(x+1)$

f. $y=-(x^2+1)$

3.4 EXERCISES

APPLICATIONS

1. Cindy wants to construct three rectangular dog-training arenas side by side, as shown, using a total of 400 feet of fencing. What should the overall length and width be in order to maximize the area of the three combined arenas? (**Hint:** Let x represent the width, as shown, and find an expression for the overall length in terms of x.)



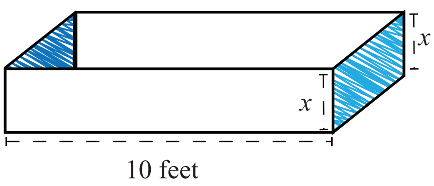
2. Among all the pairs of numbers with a sum of 10, find the pair whose product is maximum.

3. Among all rectangles that have a perimeter of 20, find the dimensions of the one whose area is largest.

4. Find the point on the line $2x + y = 5$ that is closest to the origin. (**Hint:** Instead of trying to minimize the distance between the origin and points on the line, minimize the square of the distance.)

5. Among all the pairs of numbers (x, y) such that $2x + y = 20$, find the pair for which the sum of the squares is minimum.

6. A rancher has a rectangular piece of sheet metal that is 20 inches wide by 10 feet long. He plans to fold the metal to create a narrow three-sided channel and weld two other sheets of metal to the ends to form a watering trough 10 feet long, as shown. How should he fold the metal in order to maximize the volume of the resulting trough?



7. Find a pair of numbers whose product is maximum if the pair must have a sum of 16.

8. Search the Seas cruise ship has a conference room onboard that can hold up to 60 people. Companies can reserve the room for groups of 38 or more. If the group contains 38 people, the company pays \$60 per person. The cost per person is reduced by \$1 for each person in excess of 38. Find the size of the group that maximizes the income for the owners of the ship and find this income.

9. The back of George's property is a creek. George would like to enclose a rectangular area, using the creek as one side and fencing for the other three sides, to create a vegetable garden. If he has 300 feet of material, what is the maximum possible area of the garden?

10. Find a pair of numbers whose product is maximum if two times the first number plus the second number is 48.

11. The total revenue for Thompson's Studio Apartments is given by the function $R(x) = 100x - 0.1x^2$, where x is the number of rooms rented. What number of rooms rented produces the maximum revenue?

12. The total revenue of Tran's Machinery Rental is given by the function $R(x) = 300x - 0.4x^2$, where x is the number of units rented. What number of units rented produces the maximum revenue?

13. The total cost of producing a type of small car is given by $C(x) = 9000 - 135x + 0.045x^2$, where x is the number of cars produced. How many cars should be produced to incur minimum cost?

14. The total cost of manufacturing a set of golf clubs is given by $C(x) = 800 - 10x + 0.20x^2$, where x is the number of sets of golf clubs produced. How many sets of golf clubs should be manufactured to incur minimum cost?

15. The owner of a parking lot is going to enclose a rectangular area with fencing, using an existing fence as one of the sides. The owner has 220 feet of new fencing material (which is much less than the length of the existing fence). What is the maximum possible area that the owner can enclose?

In Exercises 16–18, use the formula $h(t) = -16t^2 + v_0 t + h_0$ for the height at time t of an object thrown vertically upward with velocity v_0 (in feet per second) from an initial height of h_0 (in feet).

16. Sitting in a tree, 48 feet above ground level, Sue shoots a pebble straight up with a velocity of 64 feet per second. What is the maximum height attained by the pebble?

17. A ball is thrown upward with a velocity of 48 feet per second from the top of a 144-foot building. What is the maximum height of the ball?

18. A rock is thrown upward with a velocity of 80 feet per second from the top of a 64-foot-high cliff. What is the maximum height of the rock?

TECHNOLOGY

Use a graphing utility to graph each of the following quadratic functions. Then determine the vertex and x-intercepts.

19. $f(x) = 2x^2 - 16x + 31$

20. $f(x) = -x^2 - 2x + 3$

21. $f(x) = x^2 - 8x - 20$

22. $f(x) = x^2 - 4x$

23. $f(x) = 25 - x^2$

24. $f(x) = 3x^2 + 18x$

25. $f(x) = x^2 + 2x + 1$

26. $f(x) = 3x^2 - 8x + 2$

27. $f(x) = -x^2 + 10x - 4$

28. $f(x) = \frac{1}{2}x^2 + x - 1$

3.5 EXERCISES

PRACTICE

Sketch the graphs of the following functions. Pay particular attention to intercepts, if any, and locate these accurately. See Examples 1 through 4.

1. $f(x)=-\frac{x}{2}$
2. $g(x)=2x^2$
3. $F(x)=x^{\frac{1}{2}}$
4. $h(x)=x^{-1}$
5. $p(x)=-\frac{2}{x}$
6. $q(x)=-\sqrt[3]{x}$
7. $G(x)=-|x|$
8. $k(x)=\frac{1}{x^3}$
9. $G(x)=\frac{\sqrt{x}}{2}$
10. $H(x)=0.5x^{\frac{1}{3}}$
11. $r(x)=3|x|$
12. $p(x)=-\frac{1}{x^2}$
13. $W(x)=\frac{x^4}{16}$
14. $k(x)=\frac{x^3}{9}$
15. $h(x)=2\sqrt[3]{x}$
16. $d(x)=2x^5$
17. $S(x)=4x^{-2}$
18. $f(x)=-x^2$
19. $r(x)=\frac{\sqrt[3]{x}}{3}$
20. $s(x)=\frac{|x|}{3}$
21. $t(x)=\frac{x^6}{4}$
22. $f(x)=2[\![x]\!]$
23. $P(x)=-[\![x]\!]$
24. $m(x)=\left[\!\!\left[\frac{x}{2}\right]\!\!\right]$
25. $f(x)=\begin{cases}3-x & \text{if } x<-2\\ x^{\frac{1}{3}} & \text{if } x\geq -2\end{cases}$
26. $g(x)=\begin{cases}-x^2 & \text{if } x\leq 1\\ x^2 & \text{if } x>1\end{cases}$
27. $r(x)=\begin{cases}\frac{1}{x} & \text{if } x<1\\ -x & \text{if } x>1\end{cases}$
28. $p(x)=\begin{cases}x+1 & \text{if } x<-2\\ x^3 & \text{if } -2\leq x<3\\ -1-x & \text{if } x\geq 3\end{cases}$
29. $q(x)=\begin{cases}-1 & \text{if } x\in\mathbb{Z}\\ 1 & \text{if } x\notin\mathbb{Z}\end{cases}$
30. $s(x)=\begin{cases}\frac{x^2}{3} & \text{if } x<0\\ -\frac{x^2}{3} & \text{if } x\geq 0\end{cases}$
31. $v(x)=\begin{cases}x^2 & \text{if } -1\leq x\leq 1\\ |x| & \text{if } x<-1 \text{ or } x>1\end{cases}$
32. $M(x)=\begin{cases}x & \text{if } x\in\mathbb{Z}\\ -x & \text{if } x\notin\mathbb{Z}\end{cases}$
33. $t(x)=\begin{cases}x^4 & \text{if } x\leq 1\\ [\![x]\!] & \text{if } x>1\end{cases}$
34. $N(x)=\begin{cases}x^2 & \text{if } x\in\mathbb{Z}\\ [\![x]\!] & \text{if } x\notin\mathbb{Z}\end{cases}$

35. $h(x)=\begin{cases}-|x| & \text{if } x<2\\ [\![x]\!] & \text{if } x\ge 2\end{cases}$

36. $u(x)=\begin{cases}[\![x]\!] & \text{if } x\le 1\\ 2x-2 & \text{if } x>1\end{cases}$

Match the following functions to their graphs.

37. $f(x)=-2x^4$

38. $f(x)=-\dfrac{7}{9x^4}$

39. $f(x)=-4\left[\!\!\left[\dfrac{x}{4}\right]\!\!\right]$

40. $f(x)=-\dfrac{7\sqrt[3]{x}}{3}$

41. $f(x)=-\dfrac{8}{9}|x|$

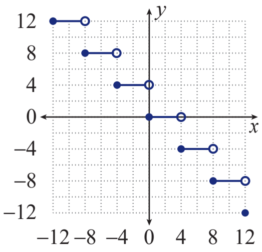
42. $f(x)=-4\sqrt{x}$

43. $f(x)=\dfrac{3}{7}|x|$

44. $f(x)=\begin{cases}-4x-12 & \text{if } x\le -3\\ \dfrac{5}{10}x^2 & \text{if } x>-3\end{cases}$

45. $f(x)=\begin{cases}\dfrac{-1}{3}|x| & \text{if } x<2\\ \left[\!\!\left[\dfrac{x}{2}\right]\!\!\right] & \text{if } x\ge 2\end{cases}$

46. $f(x)=\begin{cases}-\dfrac{1}{3}|x| & \text{if } x<2\\ \dfrac{x}{2} & \text{if } x\ge 2\end{cases}$

a.


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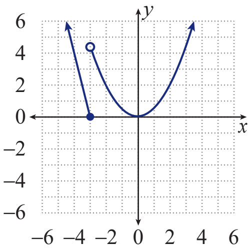

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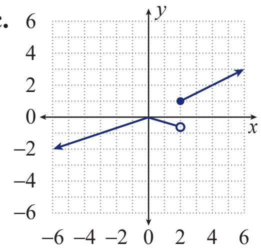

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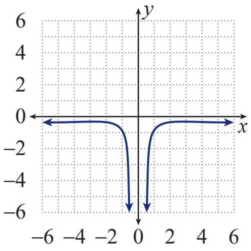

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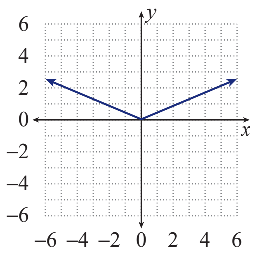

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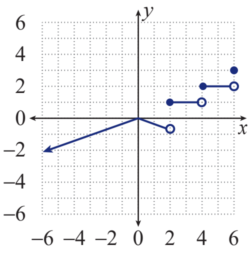

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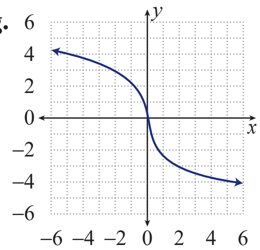

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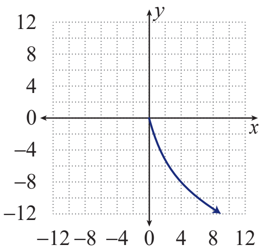

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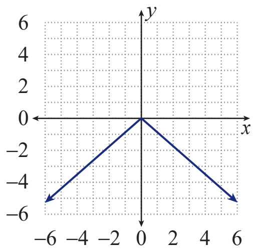

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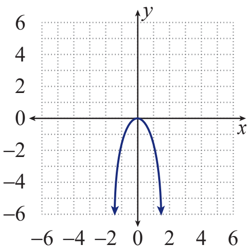


TECHNOLOGY

Use a graphing utility to graph the following functions. Experiment with different viewing windows until you obtain a sketch that seems to capture the meaningful parts of the graph.

47. $f(x) = 10x^5 - x^3$

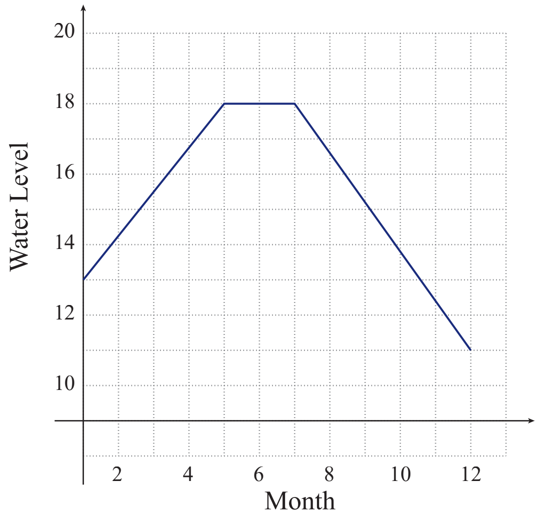
48. $g(x) = x^5 + x^2$

49. $f(x) = x^3 - 5x^2 + x$

50. $g(x) = \sqrt{x} - x^2$

51. $f(x) = \sqrt{x} + 3x - 1$

52. $g(x) = x^4 - 3x^3 + 2$



3.6 EXERCISES

PRACTICE

For each function or graph, determine the basic function that has been shifted, reflected, stretched, or compressed.

1. $f(x) = -(1-x)^2 + 2$

2. $f(x) = \dfrac{1}{x-4} + 5$

3. $f(x) = \sqrt[3]{x+6} - 2$

4. $f(x) = -2 + 2|x-3|$

5. $f(x) = \sqrt{x+2} - 5$

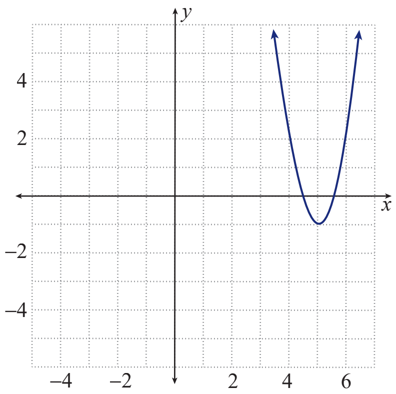
6. $f(x) = [\![-2-x]\!]$

7. $f(x) = \dfrac{1}{(x+2)^2} + 1$

8. $f(x) = \dfrac{\sqrt{-x}}{2} + 4$

9. $f(x) = (x+6)^3$

10.



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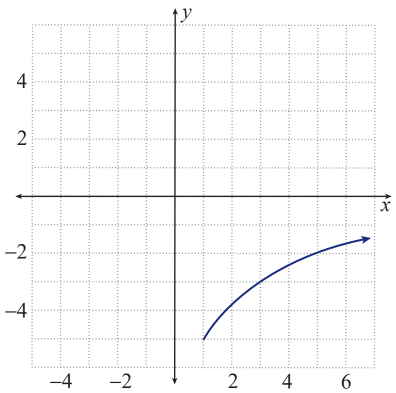

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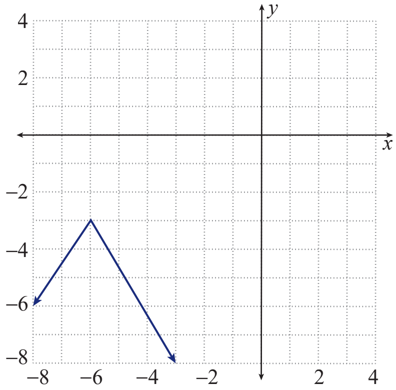

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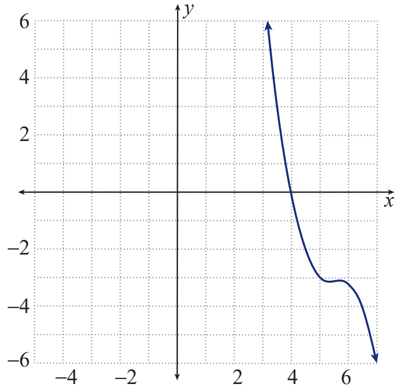


Sketch the graphs of the following functions by first identifying the more basic functions that have been shifted, reflected, stretched, or compressed. Then determine the domain and range of each function. See Examples 1 through 6.

14. $f(x) = (x+2)^3$

15. $G(x) = |x-4|$

16. $p(x) = -(x+1)^2 + 2$

17. $g(x) = \sqrt{x+3} - 1$

18. $q(x) = (1-x)^2$

19. $r(x) = -\sqrt[3]{x}$

20. $s(x) = \sqrt{2-x}$

21. $F(x) = \dfrac{|x+2|}{3} + 3$

22. $w(x) = \dfrac{1}{(x-3)^2}$

23. $v(x) = \dfrac{1}{3x} - 2$

24. $f(x) = \dfrac{1}{2-x}$

25. $k(x) = \sqrt{-x} + 2$

26. $b(x) = \sqrt[3]{x+2} - 5$

27. $b(x) = [\![x-4]\!] + 4$

28. $R(x) = 4 - 2|x|$

29. $S(x) = (3-x)^3$

30. $g(x) = -\dfrac{1}{x+1}$

31. $h(x) = \dfrac{x^2}{2} - 3$

32. $W(x) = 1 - |4-x|$

33. $W(x) = -\dfrac{|x-1|}{4}$

34. $S(x) = \dfrac{1}{x^2} + 3$

35. $V(x) = -3\sqrt{x-1} + 2$

36. $g(x) = x^2 - 6x + 9$ (**Hint:** Find a better way to write the function.)

37. $h(x) = \dfrac{|x|}{x}$ (**Hint:** Evaluate h at a few points to understand its behavior.)

38. $W(x) = \dfrac{x-1}{|x-1|}$

39. $s(x) = [\![x-2]\!]$

Write a formula for each of the functions described.

40. Use the function $g(x) = x^2$. Move the function 3 units to the left and 4 units down.

41. Use the function $g(x)=x^2$. Move the function 4 units to the right and 2 units up.

42. Use the function $g(x)=x^2$. Reflect the function across the x-axis and move it 6 units up.

43. Use the function $g(x)=x^2$. Move the function 2 units to the right and reflect across the y-axis.

44. Use the function $g(x)=x^3$. Move the function 1 unit to the left and reflect across the y-axis.

45. Use the function $g(x)=x^3$. Move the function 10 units to the right and 4 units up.

46. Use the function $g(x)=\sqrt{x}$. Move the function 5 units to the left and reflect across the x-axis.

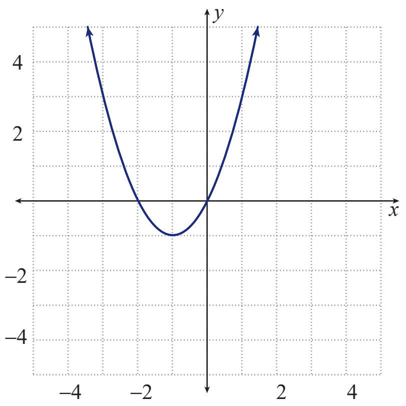
47. Use the function $g(x)=\sqrt{x}$. Reflect the function across the y-axis and move it 3 units down.

48. Use the function $g(x)=|x|$. Move the function 7 units to the left, reflect across the x-axis, and reflect across the y-axis.

49. Use the function $g(x)=|x|$. Move the function 8 units to the right, 2 units up, and reflect across the x-axis.

Use your knowledge about transformations to find a possible formula for the function $f(x)$ given its graph.

50.



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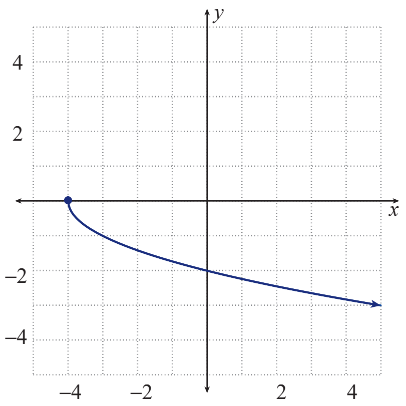

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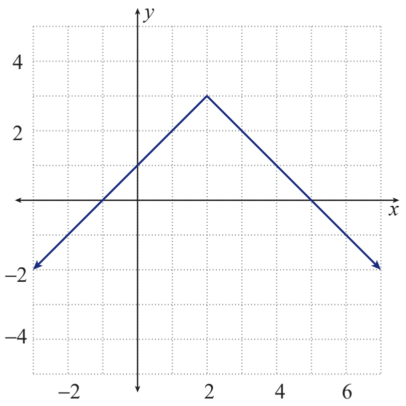

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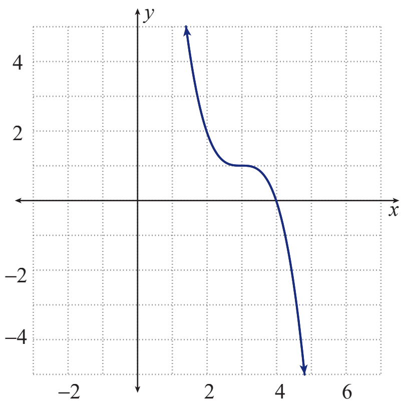


Determine if each of the following relations is a function. If so, determine whether it is even, odd, or neither. Also determine if it has *y*-axis symmetry, *x*-axis symmetry, origin symmetry, or none of these symmetries, and then sketch the graph of the relation. See Example 7.

54. $f(x)=|x|+3$ **55.** $g(x)=x^3$ **56.** $h(x)=x^3-1$

57. $w(x)=\sqrt[3]{x}$ **58.** $x=-y^2$ **59.** $3y-2x=1$

60. $x+y=1$ **61.** $F(x)=(x-1)^2$ **62.** $x=y^2+1$

63. $x=2|y|$ **64.** $g(x)=\dfrac{x^2}{5}-5$ **65.** $s(x)=\left[\!\left[x+\dfrac{1}{2}\right]\!\right]$

66. $m(x)=\sqrt[3]{x}-1$ **67.** $xy=2$ **68.** $x+y^2=3$

For each of the following functions, find the open intervals of monotonicity where the function is increasing, decreasing, or constant. See Examples 8 and 9.

69. $f(x)=(x+3)^2$ **70.** $g(x)=-|x-2|$ **71.** $h(x)=\dfrac{1}{x-1}$

72. $H(x)=\dfrac{1}{(x+3)^2}$ **73.** $G(x)=\sqrt{x+1}$ **74.** $F(x)=-2$

75. $p(x)=-30|x-1|$ **76.** $q(x)=(4-x)^2+1$

77. $r(x)=\dfrac{(x-7)^4}{-2}+4$ **78.** $P(x)=\begin{cases}(x+3)^2 & \text{if } x<-1\\ 1 & \text{if } x\geq -1\end{cases}$

79. $Q(x)=\begin{cases}|x-1| & \text{if } x\leq 3\\ 5-x & \text{if } x>3\end{cases}$

APPLICATIONS

80. During the summer months, the water level of a garden pool varies as water is added and as it evaporates. On May 1st the pool was 3.4 feet deep. After a steady and linear increase due to rain, the depth had increased to 4.9 feet on June 1st. By July 1st the water level had decreased linearly to 4.2 feet. Knowing that the pool would be covered for the winter, the owner filled the pool (in an essentially linear fashion) until it reached 5 feet on August 1st. Graph the water level as a function of time and determine the open intervals of monotonicity.

81. The profit made by a hot dog vendor is given by the function

$$P(x)=\begin{cases}2x-3 & \text{if } x\geq 0 \text{ and } x<7\\ \dfrac{1}{4}x^2 & \text{if } x\geq 7\end{cases}$$

where x is the number of hot dogs sold. Graph the profit function and determine the open intervals of monotonicity.

82. The cost incurred by a newspaper stand is given by the function

$$C(x)=\begin{cases}-2\sqrt{x}+8 & \text{if } x\geq 0 \text{ and } x<3\\ -x+8 & \text{if } x\geq 3\end{cases}$$

where x is the number of newspapers sold. Graph the cost function and determine the open intervals of monotonicity.

TECHNOLOGY

Mentally sketch the graph of the given function by identifying the basic shape that has been shifted, reflected, stretched, or compressed. Then use a graphing utility to graph the function and check your reasoning.

83. $f(x) = -2(3-x)^3 + 5$

84. $f(x) = \dfrac{3}{x+5} - 1$

85. $f(x) = \dfrac{-1}{(x-2)^2} - 3$

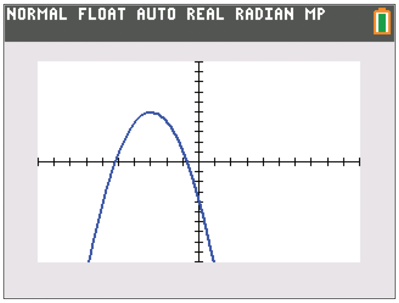
86. $f(x) = -3|x+2| - 4$

87. $f(x) = -\sqrt{1-x} + 2$

88. $f(x) = \sqrt[3]{2+x} - 1$

Write a possible equation for the function depicted on the graphing utility. The function is shown in a $[-10,10]$ by $[-10,10]$ viewing window.

89.



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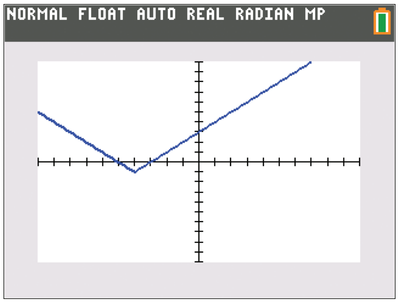

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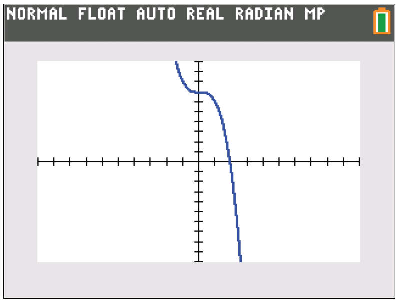

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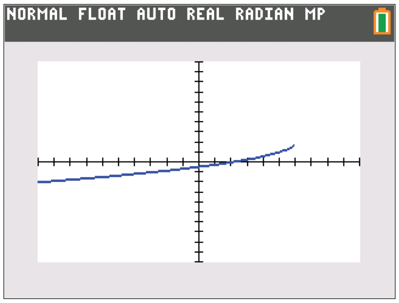

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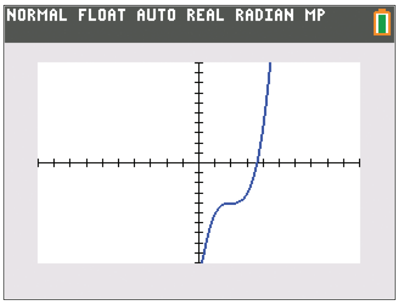

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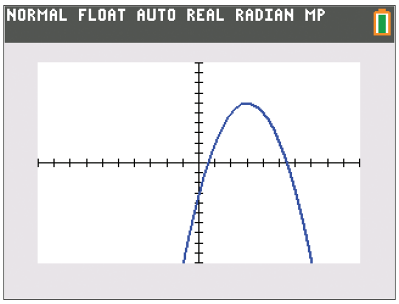


The leftmost zero occurs at approximately $x \approx -5.162$. Note that sometimes, as in this example, the display will read a very small number rather than exactly zero. To find the other zero, repeat the process, this time focusing on the rightmost zero. We find that it occurs at $x \approx 1.162$.

3.7 EXERCISES

PRACTICE

Verify that the given values of *x* solve the corresponding polynomial equations. See Example 1.

1. $9x^2 - 4x = 2x^3 + 15;\ x = -1$

2. $x^2 - 4x = -13;\ x = 2 - 3i$

3. $x^2 + 13 = 4x;\ x = 2 + 3i$

4. $3x^3 + (5 - 3i)x^2 = (2 + 5i)x - 2i;\ x = i$

5. $9x^2 - 4x = 2x^3 + 15;\ x = 3$

6. $9x^2 - 4x = 2x^3 + 15;\ x = \frac{5}{2}$

7. $3x^3 + (5 - 3i)x^2 = (2 + 5i)x - 2i;\ x = -2$

8. $x^5 - 10x^4 - 80x^2 = 32 - 80x - 40x^3;\ x = 2$

9. $4x^5 - 8x^4 - 12x^3 = 16x^2 - 25x - 69;\ x = 3$

10. $x^2 - 4x - 12 = 0;\ x = 6$

11. $23x^7 - 12x^5 = 63x^4 - 3x^2;\ x = 0$

12. $x^2 + 74 = 10x;\ x = 5 + 7i$

13. $4x^2 + 32x + (8 + i)x^3 = -8;\ x = 2i$

14. $8x - 17 = x^2;\ x = 4 - i$

15. $(5 - 3i)x - 3x = 4 - 6i;\ x = 2$

16. $x^6 - x^5 + 7x^4 + x^3 - 9x = -1;\ x = 1$

17. $6x^7 - 3x^5 = 3x^4 - 6x^2;\ x = -1$

Determine if the given values of *x* are solutions of the corresponding polynomial equations. See Example 1.

18. $16x = x^3 + x^2 + 20;\ x = -5$

19. $x^4 - 13x^2 + 12 = -x^3 + x;\ x = -1$

20. $x^4 - 3x^3 - 10x^2 = 0;\ x = 2$

21. $4x^5 - 216x^2 = 36x^3 - 24x^4;\ x = -6$

22. $x^3 - 8ix + 30 = 15x + 2x^2 + 16i;\ x = -i$

23. $x^3 - 7x^2 + 4x - 28 = 0;\ x = 2i$

Solve the following polynomial equations by factoring and/or using the quadratic formula, making sure to identify all the solutions.

24. $x^3 - x^2 - 6x = 0$

25. $x^2 - 2x + 5 = 0$

26. $x^4 + x^2 - 2 = 0$

27. $2x^2 + 5x = 3$

28. $9x^2 = 6x - 1$

29. $x^4 - 8x^2 + 15 = 0$

30. $x^3 - x^2 = 72x$

31. $x^2 + 5x = -\frac{25}{4}$

32. $2x^2+5=11x$

33. $x^4-8x^3+25x^2=0$

34. $x^4-13x^2+36=0$

35. $x^4+7x^2=8$

For each of the following polynomials, determine the degree and the leading coefficient; then determine the behavior of the graph as $x\to\pm\infty$.

36. $p(x)=2x^4-3x^3-6x^2-x-23$

37. $j(x)=4x^7+5x^5+12$

38. $r(x)=(3x+5)(x-2)(2x-1)(4x-7)$

39. $h(x)=-6x^5+2x^3-7x$

40. $g(x)=(x-5)^3(2x+1)(-x-1)$

41. $f(x)=-2(x+4)(x-4)(x^2)$

For each of the following polynomial functions, determine the behavior of its graph as $x\to\pm\infty$ and identify the x- and y-intercepts. Use this information to sketch the graph of each polynomial. See Example 2.

42. $f(x)=(x-3)(x+2)(x+4)$

43. $g(x)=(3-x)(x+2)(x+4)$

44. $f(x)=(x-2)^2(x+5)$

45. $h(x)=-(x+2)^3$

46. $r(x)=x^2-2x-3$

47. $s(x)=x^3+3x^2+2x$

48. $f(x)=-(x-2)(x+1)^2(x+3)$

49. $g(x)=(x-3)^5$

In Exercises 50–55, use the behavior as $x\to\pm\infty$ and the intercepts to match each polynomial with its graph.

50. $g(x)=(x+1)^2(x-3)^2$

51. $h(x)=1-(x+2)^2$

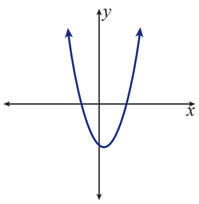
52. $f(x)=(x-1)(x+2)(3-x)$

53. $r(x)=x^2-x-6$

54. $s(x)=(x-1)^3-2$

55. $f(x)=(x-3)^2(4x+1)(x+2)(x-2)$

a.



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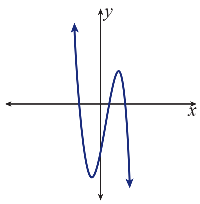

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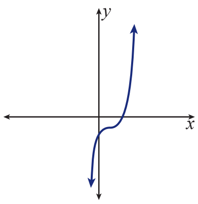

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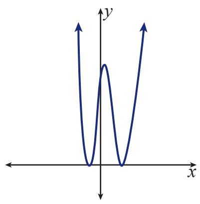

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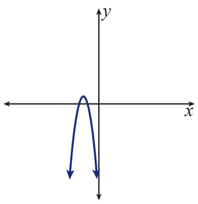

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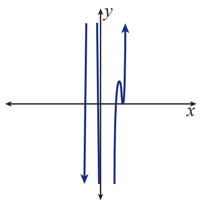


Match each of the following functions to the appropriate description.

56. $z(x)=(x-1)(x+2)(4-x)$

57. $r(x)=x^2-6x-7$

58. $s(x)=x^3+3x^2+2x$

59. $g(x)=(x-1)(x+4)(3-x)$

60. $s(x)=x^3+5x^2+4x$

61. $s(x)=x^2-5x-6$

a. cubic curve increasing as $x\to\infty$, has x-intercepts of 0, -1, and -2, and crosses the y-axis at 0

b. parabola that opens up, has x-intercepts at 6 and -1, crosses the y-axis at -6

c. cubic curve increasing as $x\to\infty$, has x-intercepts of 0, -1, and -4, and crosses the y-axis at 0

d. parabola that opens up, has x-intercepts at 7 and -1, crosses the y-axis at -7

e. cubic curve decreasing as $x\to\infty$, has x-intercepts at 1, 4, and -2, crosses the y-axis at -8

f. cubic curve decreasing as $x\to\infty$, has x-intercepts of 1, 3, and -4, and crosses the y-axis at -12

Solve the following polynomial inequalities. See Example 4.

62. $x^2-x-6\le 0$

63. $x^2 > x+6$

64. $(x+2)^2(x-1)^2>0$

65. $x^3+3x^2+2x<0$

66. $(x-2)(x+1)(x+3)\ge 0$

67. $(x-1)(x+2)(3-x)\le 0$

68. $-x^3-x^2+30x>0$

69. $(x^2-1)(x-4)(x+5)\le 0$

70. $x^4+x^2>0$

71. $4x^2<6x+4$

72. $x^2(x+4)(x-3)>0$

73. $(x-3)(x+4)(2-x)>0$

APPLICATIONS

For Exercises 74–78, use the fact that profit is equal to revenue minus cost.

74. A small start-up skateboard company projects that the cost per month of manufacturing x skateboards will be $C(x)=10x+300$, and the revenue per month from selling x skateboards will be $R(x)=-x^2+50x$. For what value(s) of x will the company break even or make a profit?

75. A manufacturer has determined that the revenue from the sale of x cameras is given by $R(x)=-x^2+15x$. The cost of producing x cameras is $C(x)=135-17x$. For what value(s) of x will the company break even or make a profit?

76. The revenue from the sale of x fire extinguishers is estimated to be $R(x) = 9 - x^2$. The total cost of producing x fire extinguishers is $C(x) = 209 - 33x$. For what value(s) of x will the company break even or make a profit?

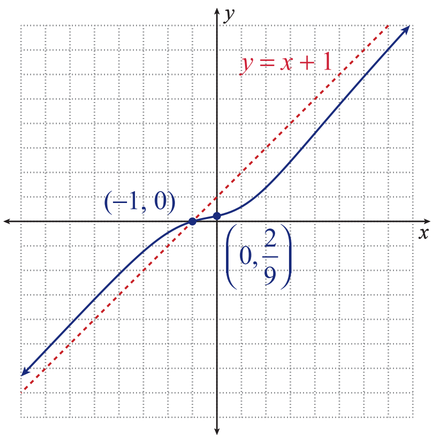
77. A manufacturer has determined that the cost and revenue of producing and selling x telescopes are $C(x) = 253 - 7x$ and $R(x) = 27x - x^2$, respectively. For what value(s) of x will the company break even or make a profit?

78. A company that produces and sells compact refrigerators has found that the revenue from the sale of x compact refrigerators is $R(x) = -x^2 + 30x - 370$. The cost function is given by $C(x) = 6 - 25x$. For what value(s) of x will the company break even or make a profit?

79. An electronics company is deciding whether or not to begin producing phones. The company must determine if a profit can be made on the phones. The profit function is modeled by the equation $P(x) = x + 0.27x^2 - 0.0015x^3 - 300$, where x is the number of phones produced in hundreds. Given this equation, how many phones must the company produce to make a profit?

80. The population of sea lions on an island is represented by the function $L(m) = 110m^2 - 0.35m^4 + 750$, where m is the number of months the sea lions have been observed on the island. Given this information, how many more months will there be sea lions on the island?

81. The population of mosquitoes in a city in Florida is modeled by the function $M(w) = 200w^2 - 0.01w^4 + 1200$, where w is the number of weeks since the town began spraying for mosquitoes. How many weeks will it take for all the mosquitoes to die?



c. The denominator of this function cannot be factored, so there are no restrictions on the domain of h. Further, we saw in Example 2b that this function has an oblique asymptote of $y = x + 1$.

As usual, we calculate the y-intercept by substituting $x = 0$.

$$h(0) = \frac{0^3 + 0^2 + 2(0) + 2}{0^2 + 9} = \frac{2}{9}$$

There are different approaches to finding the x-intercepts. Looking at the numerator, we might guess that -1 is a zero of the numerator. A quick calculation confirms this: $(-1)^3 + (-1)^2 + 2(-1) + 2 = 0$. This means we can factor the numerator. Using division, we find $x^3 + x^2 + 2x + 2 = (x + 1)(x^2 + 2)$. Thus, $(-1, 0)$ is the only x-intercept.

With the intercepts and a few other plotted points, we obtain the graph of h.

3.8 EXERCISES

PRACTICE

Find equations for the vertical asymptotes, if any, for each of the following rational functions. See Example 1.

1. $f(x) = \dfrac{5}{x-1}$

2. $f(x) = \dfrac{x^2+3}{x+3}$

3. $f(x) = \dfrac{x^2-4}{x+2}$

4. $f(x) = \dfrac{-3x+5}{x-2}$

5. $f(x) = \dfrac{3x^2+1}{x-2}$

6. $f(x) = \dfrac{x^2+2x}{x+1}$

7. $f(x) = \dfrac{x^2-4}{2x-x^2}$

8. $f(x) = \dfrac{x+2}{x^2-9}$

9. $f(x) = \dfrac{x^2-2x-3}{2x^2-5x-3}$

10. $f(x) = \dfrac{2x^2+2x-4}{x^2+2x+1}$

11. $f(x) = \dfrac{x^3-27}{x^2+5}$

12. $f(x) = \dfrac{x^2+5}{x^3-27}$

13. $f(x) = \dfrac{x^2-1}{x^2-8x+7}$

14. $f(x) = \dfrac{2x^2+7x-14}{2x^2+7x-15}$

15. $f(x) = \dfrac{x^3-6x^2+11x-6}{x^3+8}$

16. $f(x) = \dfrac{x^2-2x-15}{x-5}$

17. $f(x) = \dfrac{x^2-16}{x^2-4}$

18. $f(x) = \dfrac{x^2+4x+4}{x^2+x-2}$

Find equations for the horizontal or oblique asymptotes, if any, for each of the following rational functions. See Example 2.

19. $f(x)=\frac{5}{x-1}$

20. $f(x)=\frac{x^2+3}{x+3}$

21. $f(x)=\frac{x^4-4}{x^2+2}$

22. $f(x)=\frac{x^2-4}{2x-x^2}$

23. $f(x)=\frac{x+2}{x^2-9}$

24. $f(x)=\frac{x^2-2x-3}{2x^2-5x-3}$

25. $f(x)=\frac{2x^2+2x-4}{x^2+2x+1}$

26. $f(x)=\frac{-3x+5}{x-2}$

27. $f(x)=\frac{3x^2+1}{x-2}$

28. $f(x)=\frac{x^3-27}{x^2+5}$

29. $f(x)=\frac{x^2+5}{x^3-27}$

30. $f(x)=\frac{x^2+2x}{x+1}$

31. $f(x)=\frac{x^2-81}{x^3+7x-12}$

32. $f(x)=\frac{x^3-3x^2+2x}{x-7}$

33. $f(x)=\frac{x^2-9x+4}{x+2}$

34. $f(x)=\frac{-x^5+2x^2}{5x^5+3x^3-7}$

35. $f(x)=\frac{5x^2-x+12}{x-1}$

36. $f(x)=\frac{2x^2-5x+6}{x-3}$

Sketch the graphs of the following rational functions, making use of your work in the previous exercises and additional information about intercepts and any other points that may be useful. See Example 3.

37. $f(x)=\frac{5}{x-1}$

38. $f(x)=\frac{x^2+3}{x+3}$

39. $f(x)=\frac{x^2-4}{x+2}$

40. $f(x)=\frac{x^2-4}{2x-x^2}$

41. $f(x)=\frac{x+2}{x^2-9}$

42. $f(x)=\frac{x^2-2x-3}{2x^2-5x-3}$

43. $f(x)=\frac{2x^2+2x-4}{x^2+2x+1}$

44. $f(x)=\frac{-3x+5}{x-2}$

45. $f(x)=\frac{3x^2+1}{x-2}$

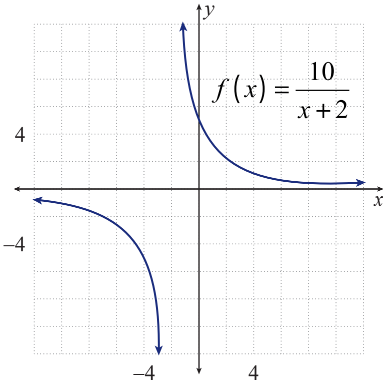
46. $f(x)=\frac{x^3-27}{x^2+5}$

47. $f(x)=\frac{x^2+5}{x^3-27}$

48. $f(x)=\frac{x^2+2x}{x+1}$

For each graph, find any **a.** vertical asymptotes, **b.** horizontal asymptotes, **c.** oblique asymptotes, **d.** visible *x*-intercepts, or **e.** visible *y*-intercepts.

49.



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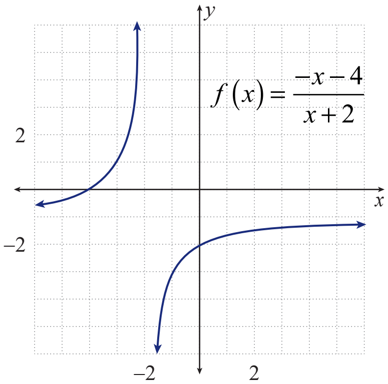

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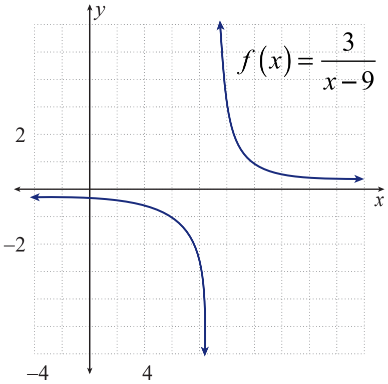

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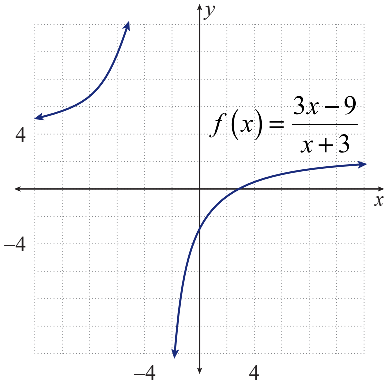

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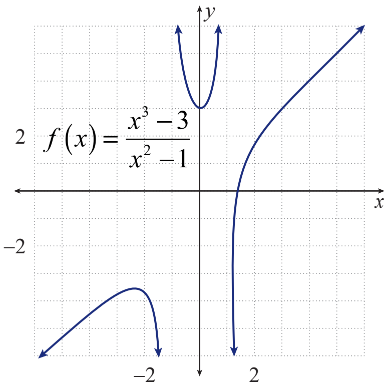

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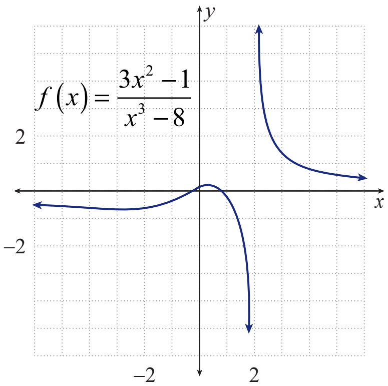

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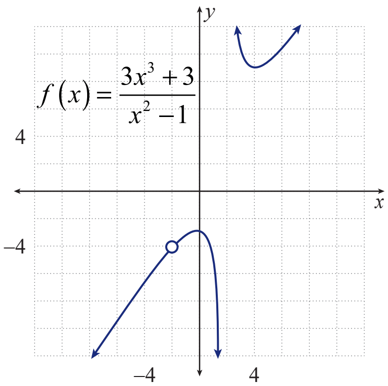

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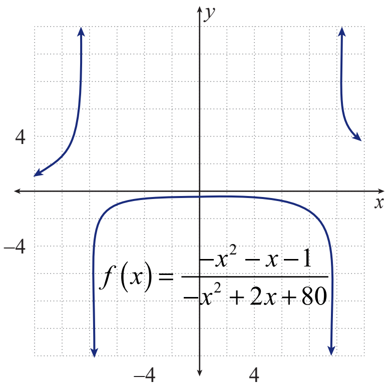


APPLICATIONS

57. April raises a species of aquarium fish, and the total number of fish she has follows the formula

$$p(t) = \frac{200t}{t+1},$$

where $t \geq 0$ represents the number of months since she began.

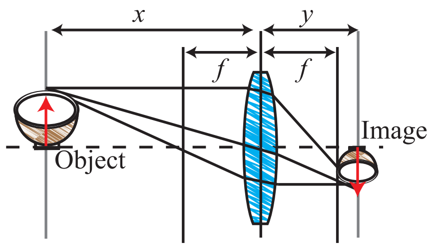
a. Sketch the graph of $p(t)$ for $t \geq 0$.

b. What happens to April's fish population in the long run?

58. If an object is placed a distance x from a lens with a focal length of f, the image of the object will appear a distance y on the opposite side of the lens, where x, f, and y are related by the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{f}$.

a. Express y as a function of x and f.

b. Graph your function for a lens with a focal length of 30 mm $(f = 30)$. What happens to y as the distance x increases?

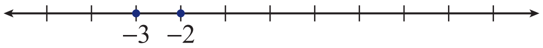


59. At t minutes after injection, the concentration (in mg/L) of a certain drug in the bloodstream of a patient is given by the formula

$$c(t) = \frac{20t}{t^2+1}.$$

a. Sketch the graph of $c(t)$ for $t \geq 0$.

b. What happens to the concentration of the drug in the long run?



we have the intervals $(-\infty, -3)$, $(-3, -2)$, and $(-2, \infty)$.

Interval	Test Point	Evaluate	Result
$(-\infty, -3)$	$x = -4$	$f(-4) = \dfrac{(-4)+3}{(-4)+2} = \dfrac{1}{2}$	$f(x) > 0$ on $(-\infty, -3)$ **Positive**
$(-3, -2)$	$x = -2.5$	$f(-2.5) = \dfrac{(-2.5)+3}{(-2.5)+2} = -1$	$f(x) < 0$ on $(-3, -2)$ **Negative**
$(-2, \infty)$	$x = 0$	$f(0) = \dfrac{(0)+3}{(0)+2} = \dfrac{3}{2}$	$f(x) > 0$ on $(-2, \infty)$ **Positive**

The final step is to evaluate which intervals satisfy each inequality. The solution to the first inequality is the union of the two intervals where f is positive.

$$(-\infty, -3) \cup (-2, \infty)$$

For the second inequality, we have to decide which endpoints to include. We include $x = -3$, since this is a zero of the rational function, but we do not include $x = -2$, since the value is not in the domain of f. Thus, the solution to the second inequality is

$$(-\infty, -3] \cup (-2, \infty).$$

3.9 EXERCISES

PRACTICE

Solve the following rational inequalities. See Examples 1 and 2.

1. $\dfrac{x+4}{2x} \geq 0$

2. $\dfrac{x}{x-4} \geq 0$

3. $\dfrac{x+6}{x^2} < 0$

4. $\dfrac{3x^2}{x+1} < 0$

5. $\dfrac{x+3}{x+9} > 0$

6. $\dfrac{2x+3}{x-4} < 0$

7. $\frac{3x-6}{2x-5}<0$

8. $\frac{4-3x}{2x+4}\leq 0$

9. $\frac{x+5}{x-7}\geq 1$

10. $\frac{2x+3}{x-1}>2$

11. $\frac{2x+5}{x-4}\leq -3$

12. $\frac{3x+2}{4x-1}<3$

13. $\frac{5-2x}{3x+4}<-1$

14. $\frac{8-x}{x+5}<-4$

15. $\frac{x(x+4)}{x-3}\leq 0$

16. $\frac{(x+3)(x-2)}{x+1}>0$

17. $\frac{x-5}{x(x+2)}\geq 0$

18. $\frac{-(x-3)^2}{(x-1)(x-4)}<0$

19. $2x<\frac{4}{x+1}$

20. $\frac{5}{x-2}\geq\frac{3x}{x-2}$

21. $\frac{5}{x-2}>\frac{3}{x+2}$

22. $\frac{x}{x^2-x-6}\leq\frac{-1}{x^2-x-6}$

23. $\frac{x}{x^2-x-6}\leq\frac{-2}{x^2-x-6}$

24. $x>\frac{1}{x}$

25. $\frac{4}{x-3}\leq\frac{4}{x}$

26. $\frac{x-7}{x-3}\geq\frac{x}{x-1}$

27. $\frac{x}{x^2+3x+2}>\frac{1}{x^2+3x+2}$

28. $\frac{1}{x-4}\geq\frac{1}{x+1}$

29. $\frac{x}{x+1}\geq\frac{x+1}{x}$

30. $\frac{x}{x^2-2x-3}>\frac{3}{x^2-2x-3}$

TECHNOLOGY

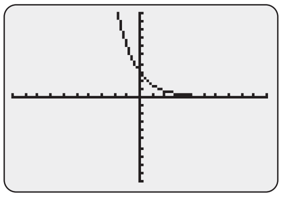
31. Use a graphing utility to graph the rational function $y=\frac{x^2+3x-4}{x}$.

a. Use the graph to find the solution set for $y\geq 0$.

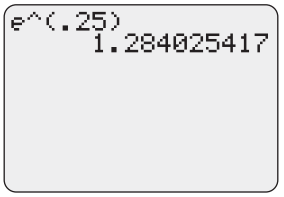
b. Use the graph to find the solution set for $y<0$.

c. Explain the effect of $x=0$ on the graph and why $x=0$ is not included in either parts **a.** or **b.**

Notice that only the fraction $\frac{1}{2}$ is being raised to the exponent, so we put it in parentheses. The graph looks like



Later, we will learn about the irrational number, e, which is used often in exponential equations. To calculate or input a value such as $e^{0.25}$ into a graphing calculator, type 2nd LN. Then, type in the exponent, 0.25, and close the parentheses by pressing). Then press ENTER:



4.1 EXERCISES

PRACTICE

Sketch the graphs of the following functions. State their domain and range. See Examples 1 and 2.

1. $f(x)=4^x$

2. $g(x)=(0.5)^x$

3. $s(x)=3^{x-2}$

4. $f(x)=\left(\frac{1}{3}\right)^{x+1}$

5. $r(x)=5^{x-2}+3$

6. $h(x)=1-2^{x+1}$

7. $f(x)=2^{-x}$

8. $r(x)=3^{2-x}$

9. $g(x)=3\left(2^{-x}\right)$

10. $h(x)=2^{2x}$

11. $s(x)=(0.2)^{-x}$

12. $f(x)=\frac{1}{2^x}+1$

13. $g(x)=3-2^{-x}$

14. $r(x)=\frac{1}{2^{3-x}}$

15. $h(x)=\left(\frac{1}{2}\right)^{5-x}$

16. $m(x)=3^{2x+1}$

17. $p(x)=2-4^{2-x}$

18. $q(x)=5^{3-2x}$

19. $r(x)=\left(\frac{9}{2}\right)^{-x}$

20. $p(x)=\left(\frac{1}{3}\right)^{2-x}$

21. $r(x)=1-\left(\frac{15}{4}\right)^{x}$

Solve the following exponential equations. See Example 3.

22. $5^x = 125$

23. $3^{2x-1} = 27$

24. $9^{2x-5} = 27^{x-2}$

25. $10^x = 0.01$

26. $4^{-x} = 16$

27. $2^x = \left(\frac{1}{2}\right)^{13}$

28. $2^{x+1} = 64^3$

29. $\left(\frac{2}{3}\right)^{x+3} = \left(\frac{9}{4}\right)^{-x}$

30. $\left(\frac{1}{5}\right)^{x-4} = 625^{\frac{1}{2}}$

31. $4^{3x+2} = \left(\frac{1}{4}\right)^{-2x}$

32. $5^x = 0.2$

33. $7^{x^2+3x} = \frac{1}{49}$

34. $3^{x^2+4x} = 81^{-1}$

35. $\left(\frac{1}{2}\right)^{x-3} = \left(\frac{1}{4}\right)^{x-5}$

36. $64^{x+\frac{7}{6}} = 2$

37. $6^{2x} = 36^{2x-3}$

38. $4^{2x-5} = 8^{\frac{x}{2}}$

39. $\left(\frac{2}{5}\right)^{2x+4} = \left(\frac{4}{25}\right)^{11}$

40. $4^{4x-7} = \frac{1}{64}$

41. $-10^x = -0.001$

42. $3^x = 27^{x+4}$

43. $1000^{-x} = 10^{x-8}$

44. $1^{3x-7} = 4^{2-x}$

45. $5^{3x-1} = 625^x$

46. $3^{2x-7} = 81^{\frac{x}{2}}$

Match the graphs of the following functions to the appropriate equation.

47. $f(x) = 2^{3x}$

48. $h(x) = 5^x - 1$

49. $g(x) = 2\left(4^{x-1}\right)$

50. $p(x) = 1 - 2^{-x}$

51. $f(x) = 6^{4-x}$

52. $r(x) = \frac{1}{3^x}$

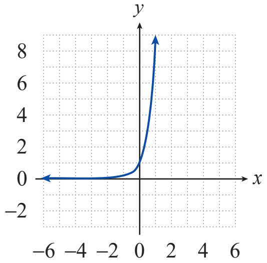
53. $m(x) = -2 + 2^{-3x}$

54. $g(x) = \left(\frac{1}{4}\right)^{1+x}$

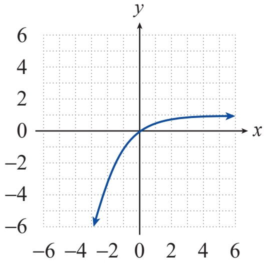
55. $h(x) = 3^{\frac{1}{2}x}$

56. $s(x) = 1^x - 4$

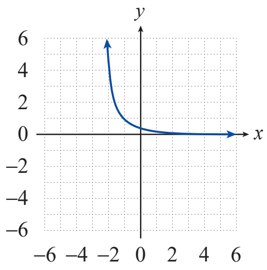
a.



b.



c.



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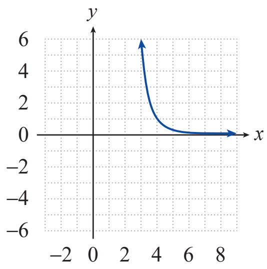
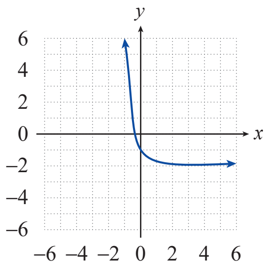
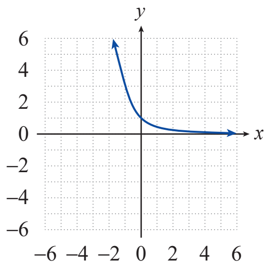


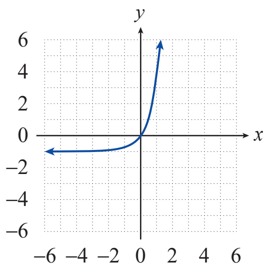
e.



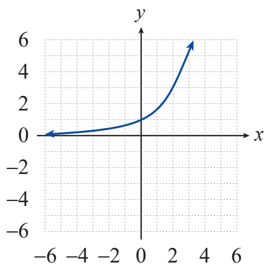
f.



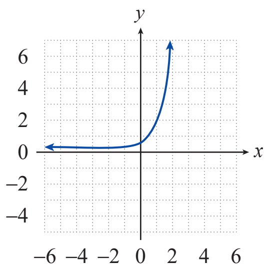
g.



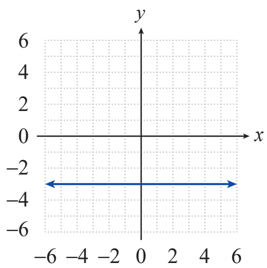
h.



i.



j.



Solution

Again, the solution boils down to substituting the correct values and evaluating the result. Here, $P = 10{,}000$, $r = 0.045$, and $t = 3.5$.

$$\begin{aligned} A(3.5) &= 10{,}000e^{(0.045)(3.5)} \\ &= 10{,}000e^{0.1575} \\ &\approx \$11{,}705.81 \end{aligned}$$

This account earns \$10.29 more than the quarterly compounded account in Example 3.

All exponential functions can be expressed with the base e (or any other base, for that matter). The base e is so commonly used for exponential functions that it is often called the *natural base*. For instance, the formula for the radioactive decay of carbon-14, using the base e, is

$$A(t) = A_0 e^{-0.000121t}.$$

You should verify that this version of the decay formula does indeed give the same values for $A(t)$ as the version derived in Example 2.

4.2 EXERCISES

APPLICATIONS

1. A new virus has broken out in isolated parts of Africa and is spreading exponentially through tribal villages. The growth of this new virus can be mapped using the following formula where V stands for the number of people in the village who are infected with the virus, P stands for the number of people in a village and d stands for the number of days since the virus first appeared. According to this equation, how many people in a village of 300 will be infected after 5 days?

$$V = P\left(1 - e^{-0.18d}\right)$$

2. A prototype for an electric motorcycle uses a battery whose energy capacity $C(d)$, in kilowatt-hours (kWh), is given by the formula $C(d) = 12e^{-0.02d}$, where d represents the number of days since receiving a full charge. What is the battery's energy capacity 30 days after being fully charged?

3. A young economics student has come across a very profitable investment scheme in which his money will accrue interest according to the equation listed below, where C represents the investment value after m months for an initial investment of I dollars. If this student invests \$1250 into this lucrative endeavor, how much money will he have after 24 months? I represents the investment and m represents the number of months the money has been invested for.

$$C = Ie^{0.08m}$$

4. A family releases a couple of pet rabbits into the wild. Upon being released the rabbits begin to reproduce at an exponential rate, as shown in the formula below. After 2 years how large is the rabbit population P, where n stands for the initial rabbit population (2) and m stands for the number of months?

$$P = ne^{0.5m}$$

5. Inside a business network, an email worm was downloaded by an employee. This worm goes through the infected computer's address book and sends itself to all the listed email addresses. This worm very rapidly works its way through the network following the equation below, where C is the number of computers in the network and W is the number of computers infected h hours after the worm is initially downloaded. After only 8 hours, how many computers has the worm infected if there are 150 computers in the network?

$$W = C\left(1 - e^{-0.12h}\right)$$

6. A construction crew has been assigned to build an apartment complex. The work of the crew can be modeled using the exponential formula below, where A is the total number of apartments to be built, w is the number of weeks, and F is the number of finished apartments. Out of a total of 100 apartments, how many apartments have been finished after 4 weeks of work?

$$F = A\left(1 - e^{-0.1w}\right)$$

7. The half-life of radium is approximately 1600 years.

a. Determine a so that $A(t) = A_0 a^t$ describes the amount of radium left after t years, where A_0 is the amount at time $t = 0$.

b. How much of a 1-gram sample of radium would remain after 100 years?

c. How much of a 1-gram sample of radium would remain after 1000 years?

8. The radioactive element polonium-210 has a relatively short half-life of 138 days, and one way to model the amount of polonium-210 remaining after t days is with the function $A(t) = A_0 e^{-0.005023t}$, where A_0 is the mass at time $t = 0$ (note that $A(138) = \frac{A_0}{2}$.) What percentage of the original mass of a sample of polonium-210 remains after 365 days?

9. A certain species of fish is to be introduced into a new man-made lake, and wildlife experts estimate the population will grow according to $P(t) = (1000)2^{\frac{t}{3}}$, where t represents the number of years from the time of introduction.

a. What is the doubling time for this population of fish?

b. How long will it take for the population to reach 8000 fish, according to this model?

10. The population of a certain inner-city area is estimated to be declining according to the model $P(t) = 237{,}000e^{-0.018t}$, where t is the number of years from the present. What does this model predict the population will be in ten years?

11. In an effort to control vegetation overgrowth, 100 rabbits are released in an isolated area that is free of predators. After one year, it is estimated that the rabbit population has increased to 500. Assuming exponential population growth, what will the population be after another six months?

12. Assuming a current world population of 7.75 billion people, an annual growth rate of 1.9% per year, and a worst-case scenario of exponential growth, what will the world population be in **a.** 10 years? **b.** 50 years?

13. Madiha has \$3500 that she wants to invest in a simple savings account for two and a half years, at which time she plans to close out the account and use the money as a down payment on a car. She finds one local bank offering an annual interest rate of 2.75% compounded monthly, and another bank offering an annual interest rate of 2.7% compounded daily (365 times per year). Which bank should she choose?

14. Madiha, from the last problem, does some more searching and finds an online bank offering an annual rate of 2.75% compounded continuously. How much more money will she earn over two and a half years if she chooses this bank rather than the local bank offering the same rate compounded monthly?

15. Tom hopes to earn \$1000 in interest in three years time from \$10,000 that he has available to invest. To decide if it's feasible to do this by investing in a simple monthly compounded savings account, he needs to determine the annual interest rate such an account would have to offer for him to meet his goal. What would the annual rate of interest have to be?

16. An investment firm claims that its clients usually double their principal in five years time. What annual rate of interest would a savings account, compounded monthly, have to offer in order to match this claim?

17. The function $C(t) = C_0(1 + r)^t$ models the rise in the cost of a product that has a cost of C_0 today, subject to an average yearly inflation rate of r for t years. If the average annual rate of inflation over the next decade is assumed to be 3%, what will the inflation-adjusted cost of a \$100,000 house be in 10 years? Round your answer to the nearest dollar.

18. Given the inflation model $C(t) = C_0(1 + r)^t$ (see Exercise 17), and given that a loaf of bread that currently sells for \$3.60 sold for \$3.10 six years ago, what has the average annual rate of inflation been for the past six years?

19. The function $N(t) = \dfrac{10{,}000}{1 + 999e^{-t}}$ models the number of people in a small town who have caught the flu t weeks after the initial outbreak.

a. How many people were ill initially?
b. How many people have caught the flu after eight weeks?
c. Determine what happens to the function $N(t)$ as $t \to \infty$.

20. The concentration $C(t)$, in milligrams per liter, of a certain drug in the bloodstream after t minutes is given by the formula $C(t) = 0.05(1 - e^{-0.2t})$. What is the concentration after 10 minutes?

21. Carbon-11 has a radioactive half-life of approximately 20 minutes, and is useful as a diagnostic tool in certain medical applications. Because of the relatively short half-life, time is a crucial factor when conducting experiments with this element.
a. Determine a so that $A(t) = A_0 a^t$ describes the amount of carbon-11 left after t minutes, where A_0 is the amount at time $t = 0$.
b. How much of a 2 kg sample of carbon-11 would be left after 30 minutes?
c. How many milligrams of a 2 kg sample of carbon-11 would be left after six hours?

22. Charles has recently inherited \$8000 that he wants to deposit into a savings account. He has determined that his two best bets are an account that compounds annually at a rate of 3.20% and an account that compounds continuously at an annual rate of 3.15%. Which account would pay Charles more interest?

23. Marshall invests \$1250 in a mutual fund which boasts a 5.7% annual return compounded semiannually (twice a year). After three and a half years, Marshall decides to withdraw his money.
   a. How much is in his account?
   b. How much has he made in interest from his investment?

24. Adam is working in a lab testing bacteria populations. After starting out with a population of 375 bacteria, he observes the change in population and notices that the population doubles every 27 minutes.
   a. Find the equation for the population P in terms of time t in minutes, rounding a to the nearest thousandth.
   b. Find the population after two hours.

25. Your credit union offers a special interest rate of 10% compounded monthly for the first year for a student savings account opened in August if the student deposits \$5000 or more. You received a total of \$9000 for graduation, and you decide to deposit all of it in this special account. Assuming you open your account in August and make no withdrawals for the first year, how much money will you have in your account at the end of February (after six months)? How much will you have at the end of the following July (after one full year)?

26. You have a savings account of \$3000 with an interest rate of 6.8%.
   a. How much interest would be earned in two years if the interest is compounded annually?
   b. How much interest would be earned in two years if the interest is compounded semiannually?
   c. In which case do you make more money on interest? Explain why this is so.

27. If \$2500 is invested in a continuously compounded certificate of deposit with an annual interest rate of 4.2%, what would be the account balance at the end of three years?

28. The new furniture store in town boasts a special in which you can buy any set of furniture in their store and make no monthly payments for the first year. However, the fine print says that the interest rate of 7.25% is compounded quarterly beginning when you buy the furniture. You are considering buying a set of living room furniture for \$4000 but know you cannot save up more than \$4500 in one year's time. Can you fully pay off your furniture on the one year anniversary of having bought the furniture? If so, how much money will you have left over? If not, how much more money will you need?

29. When Nicole was born, her grandmother was so excited about her birth that she opened a certificate of deposit in Nicole's honor to help send her to college. Now at age 18, Nicole's account has \$81,262.93. How much did her grandmother originally invest if the interest rate has been 8.1% compounded annually?

30. Inflation is a relative measure of your purchasing power over time. The formula for inflation is the same as the compound interest formula, but with $n = 1$. Given the current values below, what will the values of the following items be 10 years from now if inflation is at 6.4%?
   a. an SUV: \$38,000
   b. a loaf of bread: \$1.79
   c. a gallon of milk: \$3.40
   d. your salary: \$34,000

31. Depreciation is the decrease of an item's value and can be determined using a formula similar to that for compound interest: $V = P(1-r)^t$, where V is the new value. If the particular car you buy upon graduation from college costs \$17,500 and depreciates at a rate of 16% per year, what will the value of the car be in 5 years when you pay it off?

32. Assume the interest on your credit card is compounded continuously with an APR (annual percentage rate) of 19.8%. If you put your first term bill of \$3984 on your credit card, but do not have to make payments until you graduate (4 years later), how much will you owe when you start making payments?

33. Suppose you deposit \$5000 in an account for five years at an annual interest rate of 8.5%.

a. What would be the ending account balance if the interest is continuously compounded?

b. What would be the ending account balance if the interest is compounded daily?

c. Are these two answers similar? Why or why not?

TECHNOLOGY

Use a graphing utility to sketch the graphs of the following functions.

34. $m(x) = 1 - 3e^x$

35. $p(x) = e^{4x} - 2$

36. $b(x) = \dfrac{1}{e^{x-2}}$

37. $m(x) = e^{2x^2-3x+1}$

38. $g(x) = e^{x+3} - 3$

39. $m(x) = 6e^{2x} - 2$

4.3 EXERCISES

PRACTICE

Write the following equations in logarithmic terms.

1. $625 = 5^4$

2. $216 = 6^3$

3. $x^3 = 27$

4. $b^2 = 3.2$

5. $4.2^3 = C$

6. $1.3^2 = V$

7. $4^x = 31$

8. $16^{2x} = 215$

9. $(4x)^{\sqrt{3}} = 13$

10. $e^x = \pi$

11. $2^{e^x} = 11$

12. $4^e = N$

Write the following logarithmic equations as exponential equations.

13. $\log_3 81 = 4$

14. $\log_2\left(\frac{1}{8}\right) = -3$

15. $\log_b 4 = \frac{1}{2}$

16. $\log_y 9 = 2$

17. $\log_2 15 = b$

18. $\log_5 8 = d$

19. $\log_5 W = 12$

20. $\log_7 T = 6$

21. $\log_\pi (2x) = 4$

22. $\log_{\sqrt{3}} (2\pi) = x$

23. $\ln 2 = x$

24. $\ln(5x) = 3$

Sketch the graphs of the following functions. State their domain and range. See Examples 2 and 3.

25. $f(x) = \log_3(x - 1)$

26. $g(x) = \log_5(x + 2) - 1$

27. $r(x) = \log_{\frac{1}{2}}(x - 3)$

28. $p(x) = 3 - \log_2(x + 1)$

29. $q(x) = \log_3(2 - x)$

30. $s(x) = \log_{\frac{1}{3}}(5 - x)$

31. $h(x) = \log_7(x - 3) + 3$

32. $m(x) = \log_{\frac{1}{2}}(1 - x)$

33. $f(x) = \log_3(6 - x)$

34. $p(x) = 4 - \log(x + 3)$

35. $s(x) = -\log_{\frac{1}{3}}(-x)$

36. $g(x) = \log_5(2x) - 1$

Match the graph of the appropriate equation to the logarithmic function.

37. $f(x) = \log_2(x-1)$

38. $f(x) = \log_2(2-x)$

39. $f(x) = \log_2(-x)$

40. $f(x) = \log_2(x-3)$

41. $f(x) = 1 - \log_2 x$

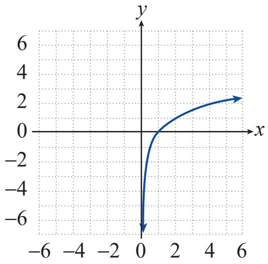
42. $f(x) = -\log_2 x$

43. $f(x) = -\log_2(-x)$

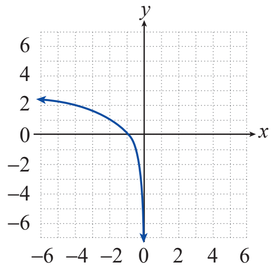
44. $f(x) = \log_2 x$

45. $f(x) = \log_2(x+3)$

a.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

b.


y
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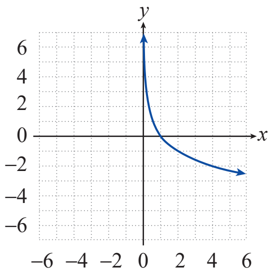
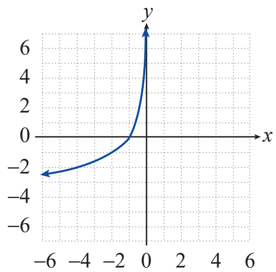
x
−6 −4 −2 0 2 4 6

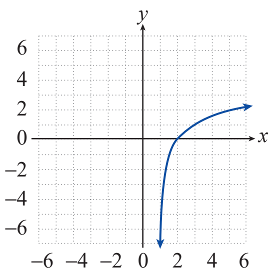
c.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

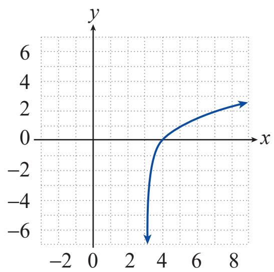
d.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

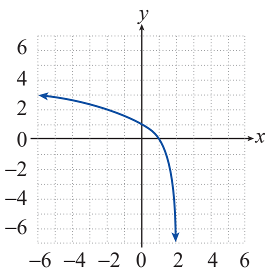
e.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

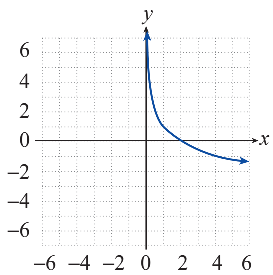
f.


y
6 4 2 0 −2 −4 −6
x
−2 0 2 4 6 8

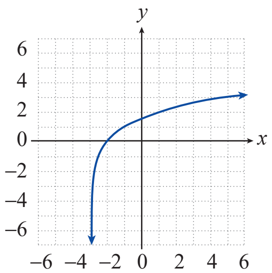
g.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

h.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

i.


y
6 4 2 0 −2 −4 −6
x
−6 −4 −2 0 2 4 6

Evaluate the following logarithmic expressions without the use of a calculator. See Examples 4 and 6.

46. $\log_7\left(\sqrt{7}\right)$

47. $\log_{\frac{1}{2}} 4$

48. $\log_9\left(\frac{1}{81}\right)$

49. $\log_3 27$

50. $\log_{27} 3$

51. $\log_9\left(\frac{1}{3}\right)$

52. $\log_{27} 9$

53. $\log_{\frac{1}{16}}\left(\frac{1}{8}\right)$

54. $\log_3\left(\log_{27} 3\right)$

55. $\ln e^{2.89}$

56. $\log(0.0001)$

57. $\log_a\left(a^{\frac{5}{3}}\right)$

58. $\ln\left(\frac{1}{e}\right)$

59. $\log\left(\log\left(10^{10}\right)\right)$

60. $\log_3 1$

61. $\ln\left(\sqrt[5]{e}\right)$

62. $\log_{\frac{1}{16}} 4$

63. $\log_8 4^{\log 1000}$

Use the elementary properties of logarithms to solve the following equations. See Example 5.

64. $\log_{16} x = \frac{3}{4}$

65. $\log_{16}\left(x^{\frac{1}{2}}\right) = \frac{3}{4}$

66. $\log_{16} x = -\frac{3}{4}$

67. $\log_5\left(5^{\log_3 x}\right) = 2$

68. $\log_a\left(a^{\log_b x}\right) = 0$

69. $\log_3\left(9^{2x}\right) = -2$

70. $\log_{\frac{1}{3}}\left(3^x\right) = 2$

71. $\log_7(3x) = -1$

72. $4^{\log_3 x} = 0$

73. $\log\left(x^{10}\right) = 10$

74. $\log_x\left(\log_{\frac{1}{2}}\left(\frac{1}{4}\right)\right) = 1$

75. $6^{\log_x\left(e^2\right)} = e$

Hint: Note that $\log_a b = \log_{a^2} b^2$. This follows from the fact that
$\log_a b = y \Leftrightarrow b = a^y \Leftrightarrow b^2 = a^{2y} = \left(a^2\right)^y \Leftrightarrow \log_{a^2} b^2 = y$.

Solve the following logarithmic equations, using a calculator if necessary to evaluate the logarithms. See Examples 5 and 6. Express your answer either as a fraction or a decimal rounded to two decimal places.

76. $\log(3x) = 2.1$

77. $\log\left(x^2\right) = -2$

78. $\ln(x+1) = 3$

79. $\ln(2x) = -1$

80. $\ln\left(e^x\right) = 5.6$

81. $\ln\left(\ln\left(x^2\right)\right) = 0$

82. $\log 19 = 3x$

83. $\log\left(e^x\right) = 5.6$

84. $\log_9(2x-1) = 2$

85. $\log\left(\log(x-2)\right) = 1$

86. $\log\left(300^{\log x}\right) = 9$

The Decibel Scale

In the **decibel scale**, I_0 is the intensity of a just-discernible sound, I is the intensity of the sound being analyzed, and D is its decibel level.

$$D = 10\log\left(\frac{I}{I_0}\right)$$

Decibel levels range from 0 for a barely discernible sound, to 60 for the level of normal conversation, to 80 for heavy traffic, to 120 for a loud rock concert, and finally (as far as humans are concerned) to around 160, at which point the eardrum is likely to rupture.

Example 8: The Decibel Scale

Given that $I_0 = 10^{-12}$ watts/meter^2, what is the decibel level of a jet airliner's engines at a distance of 45 meters, for which the sound intensity is 50 watts/meter^2?

Solution

$$\begin{aligned} D &= 10\log\left(\frac{50}{10^{-12}}\right) \\ &= 10\log\left(5\times 10^{13}\right) \\ &= 10\left(\log 5 + 13\right) \\ &\approx 137 \end{aligned}$$

In other words, the sound level would probably not be literally earsplitting, but it would be very painful.

4.4 EXERCISES

PRACTICE

Use the properties of logarithms to expand the following expressions as much as possible. Simplify any numerical expressions that can be evaluated without a calculator. See Example 3.

1. $\log_5\left(125x^3\right)$

2. $\ln\left(\frac{x^2y}{3}\right)$

3. $\ln\left(\frac{e^2p}{q^3}\right)$

4. $\log\left(100x\right)$

5. $\log_9\left(9xy^{-3}\right)$

6. $\log_6\left(\sqrt[3]{\frac{p^2}{q}}\right)$

7. $\ln\left(\frac{\sqrt{x^3}\,pq^5}{e^7}\right)$

8. $\log_a\sqrt[5]{\frac{a^4b}{c^2}}$

9. $\log\left(\log\left(100x^3\right)\right)$

10. $\log_3\left(9x + 27y\right)$

11. $\log\left(\frac{10}{\sqrt{x+y}}\right)$

12. $\ln\left(\ln\left(e^{ex}\right)\right)$

13. $\log_2\left(\frac{y^2+z}{16x^4}\right)$ **14.** $\log\left(\log\left(100{,}000^{2x}\right)\right)$ **15.** $\log_b\left(\sqrt{\frac{x^4y}{z^2}}\right)$

16. $\ln\left(7x^2-42x+63\right)$ **17.** $\log_b\left(ab^2c^b\right)$ **18.** $\ln\left(\ln\left(e^{e^x}\right)\right)$

Use the properties of logarithms to condense the following expressions as much as possible, writing each answer as a single term with a coefficient of 1. See Example 4.

19. $\log x-\log y$ **20.** $\log_5 x-2\log_5 y$

21. $\log_5\left(x^2-25\right)-\log_5\left(x-5\right)$ **22.** $\ln\left(x^2y\right)-\ln y-\ln x$

23. $\frac{1}{3}\log_2 x+\log_2\left(x+3\right)$ **24.** $\frac{1}{5}\left(\log_7\left(x^2\right)-\log_7\left(pq\right)\right)$

25. $\ln 3+\ln p-2\ln q$ **26.** $2\left(\log_5\left(\sqrt{x}\right)-\log_5 y\right)$

27. $\log\left(x-10\right)-\log x$ **28.** $2\log a^2b-\log\left(\frac{1}{b}\right)+\log\left(\frac{1}{a}\right)$

29. $3\left(\ln\left(\sqrt[3]{z^2}\right)-\ln\left(xy\right)\right)$ **30.** $\log_2\left(4x\right)-\log_2 x$

31. $\log_5 20-\log_5 5$ **32.** $\log 30-\log 2-\log 5$

33. $\ln 15+\ln 3$ **34.** $\ln 8-\ln 4+\ln 3$

35. $0.5\log_3 16-\log_3 4$ **36.** $3\log_7 2-2\log_7 4$

37. $0.25\ln 81+\ln 4$ **38.** $2\left(\log 4-\log 1+\log 2\right)$

39. $\log 11+0.5\log 9-\log 3$ **40.** $3\log_4\left(x^2\right)+\log_4\left(x^6\right)$

41. $\log_8\left(2x^2-2y\right)-0.25\log_8 16$ **42.** $\log_{3x} x^2+\log_{3x} 18-\log_{3x} 6$

Use the properties of logarithms to write each of the following as a single term that does not contain a logarithm.

43. $5^{2\log_5 x}$ **44.** $10^{\log y^2-3\log x}$ **45.** $e^{2-\ln x+\ln p}$

46. $e^{5\left(\ln\sqrt[5]{3}+\ln x\right)}$ **47.** $10^{\log x^3-4\log y}$ **48.** $a^{\log_a b+4\log_a\sqrt{a}}$

49. $10^{2\log x}$ **50.** $10^{4\log x-2\log x}$ **51.** $\log_4 16\cdot\log_x x^2$

52. $e^{\ln x+2+\ln x^2}$ **53.** $4^{\log_4(3x)+0.5\log_4\left(16x^2\right)}$ **54.** $4^{2\log_2 6-\log_2 9}$

Evaluate the following logarithmic expressions. See Example 5.

55. $\log_4 17$ **56.** $2\log_{\frac{1}{3}} 5$ **57.** $\log_9 8$

58. $\log_2 0.01$ **59.** $\log_{12} 10.5$ **60.** $\log\left(\ln 2\right)$

61. $\log_6 3^4$ **62.** $\log_7 14.3$ **63.** $\log_{\frac{1}{2}}\pi^{-2}$

64. $\log_{\frac{1}{5}} 626$ **65.** $\ln\left(\log 123\right)$ **66.** $\log_{17} 0.041$

67. $\log 16$

68. $\log_3 9$

69. $\log_5 20$

70. $\log_8 26$

71. $\log_4 0.25$

72. $\log_{1.8} 9$

73. $\log_{2.5} 34$

74. $\log_{0.5} 10$

75. $\log_4 2.9$

76. $\log_{0.4} 14$

77. $\log_{0.2} 17$

78. $\log_{0.16} 2.8$

Without using a calculator, evaluate the following expressions.

79. $\log_4 16$

80. $\log_5 25^3$

81. $\ln e^4 + \ln e^3$

82. $\log_4 \dfrac{1}{64}$

83. $\ln e^{1.5} - \log_4 2$

84. $\log_2 8^{(2\log_2 4 - \log_2 4)}$

Find the value of *x* in each of the following equations. Express your answer in exact form, or rounded to two decimal places.

85. $\log_x 1024 = 4$

86. $\log_6 729 = x$

87. $\log_2 529 = x$

88. $\log_4 625 = x$

89. $\log_x 729 = 9$

90. $\log_4 x = 8$

91. $\log_{12} x = 1$

92. $\log_x 16{,}807 = 7$

93. $\log_4 x = 10$

APPLICATIONS

94. A certain brand of tomato juice has a $\left[H_3O^+\right]$ concentration of 6.31×10^{-5} moles/liter. What is the pH of this brand?

95. One type of detergent, when added to neutral water with a pH of 7, results in a solution with a $\left[H_3O^+\right]$ concentration that is 5.62×10^{-4} times weaker than that of the water. What is the pH of the solution?

96. What is the concentration of $\left[H_3O^+\right]$ in lemon juice with a pH of 2.1?

97. An earthquake in Chile in 2019 measured 6.7 on the Richter scale. What was the intensity, relative to a 0-level earthquake, of this event?

98. How much stronger was the 2001 Gujarat earthquake (6.9 on the Richter scale) than the 2019 earthquake described in Exercise 97?

99. A construction worker operating a jackhammer would experience noise with an intensity of 20 watts/meter^2 if it weren't for ear protection. Given that $I_0 = 10^{-12} \text{ watts/meter}^2$, what is the decibel level for such noise?

100. A microphone picks up the sound of a thunderclap and measures its decibel level as 105. Given that $I_0 = 10^{-12} \text{ watts/meter}^2$, with what sound intensity did the thunderclap reach the microphone?

101. Matt, a lifeguard, has to make sure that the pH of the swimming pool stays between 7.2 and 7.6. If the pH is out of this range, he has to add chemicals that alter the pH level of the pool. If Matt measures the $\left[H_3O^+\right]$ concentration in the swimming pool to be 2.40×10^{-8} moles/liter, what is the pH? Does he need to change the pH by adding chemicals to the water?

102. The intensity of a cat's soft purring is measured to be 2.19×10^{-11}. Given that $I_0=10^{-12}$ watts/meter^2, what is the decibel level of this noise?

103. Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference between the temperature of the object and the surrounding temperature. If C denotes the surrounding temperature and T_0 denotes the temperature at time $t = 0$, the temperature of an object at time t is given by $T(t)=C+(T_0-C)e^{-kt}$, where k is a constant that depends on the particular object under discussion.

a. You are having friends over for tea and want to know how long after boiling the water it will be drinkable. If the temperature of your kitchen stays around 74°F and you found online that the constant k for tea is approximately 0.049, how many minutes after boiling the water will the tea be drinkable (you prefer your tea no warmer than 140°F)? Recall that water boils at 212°F.

b. As you intern for your local crime scene investigation department, you are asked to determine at what time a victim died. If you are told k is approximately 0.1947 for a human body and the body's temperature was 72°F at 1:00 a.m., and the body has been in a storage building at a constant 60°F, approximately what time did the victim die? Recall the average temperature for a human body is 98.6°F. Note in this situation, t is measured in hours.

c. When helping your father cook a turkey, you were told to remove the turkey when the thickest part had reached 180°F. If you remove the turkey and place it on the table in a room that is 72°F, and it cools to 155°F in 20 minutes, what will the temperature of the turkey be at lunch time (an hour and 15 minutes after the turkey is removed from the oven)? Should you warm the turkey before eating?

HELPFUL HINT

The list price of a product in a store is based on two factors.

1. The cost of the product for the store—usually called the wholesale price
2. The percentage the product is marked up from the wholesale price

Skill Check 4

Use the list price of $630 for the bike in Example 8 to find the sale price if the bike was reduced 40%. Compare the 40% reduction in price to the wholesale price in Example 8.

Example 8: Computing Percentage Increase

Pear Bike Company uses a 40% profit margin to determine the list price of their products. If the company buys a bike at a wholesale price of $450, what would be the list price for the bike?

Solution

To determine a percentage increase, we need to recognize that the final price will be equal to the original price of $450 increased by 40%. So, the amount of the increase is

$$\$450(0.40) = \$180.$$

This means the total price for the bike would be

$$\$450 + \$180 = \$630.$$

Skill Check Answers

1. $1079.33
2. 65%
3. −32.36%, or a 32.36% decrease
4. $378; It is less than the wholesale price in Example 8.

5.1 EXERCISES

APPLICATIONS

1. Jeff was recently hired for a job with an annual income of $48,000. Using the federal income tax rate of 15%, what amount should Jeff expect to pay in federal taxes?

2. Megan has a job where she has a take-home salary each month of $1800. If Megan wants to spend no more than 25% of her income on rent, how much rent can Megan afford?

3. Erica earns $670 weekly at her job at a local newspaper. Calculate the weekly taxes for Erica's salary. If Erica wishes to purchase a car where her payments are no more than 15% of her take home pay for a month, what is the maximum monthly car payment she can afford? Assume that the tax rate is 27.2% and that a month has 4 weeks.

4. Jack rents an apartment for $650 per month, pays his car payment of $470 per month, has utilities that cost $420 per month, and spends $850 per month on food and entertainment. Determine Jack's monthly expenses.

5. Using the information from Exercise 4, if Jack has $1500 remaining after his monthly expenses, what is his take-home pay?

6. The given table represents the estimated cost for an on-campus student to attend the University of California, Los Angeles (UCLA) in the 2013–2014 academic year (10 months). Create a monthly expense budget for attending UCLA and determine a monthly income that would allow a student to attend UCLA without incurring any debt.

Estimated Cost of Attending UCLA for the 2013–2014 Academic Year

Budget Category	On-Campus Student
University Fees	$12,685
Room & Board	$14,454
Books & Supplies	$1536
Transportation	$807
Personal	$1395
Health Insurance	$1323
Loan Fees	$156
Total	$32,356

Source: Financial Aid Office, "2013-2014 Undergraduate and Graduate Budgets", UCLA. Accessed April 2014. http://www.fao.ucla.edu/publications/2013-2014/Budget_Figures.pdf

7. A car with a list price of $18,000 will be discounted 35% at the time of purchase. What is the sale price?

8. The discount on a TV amounts to $240. If the sale price is $1575, what was the list price of the TV?

9. You purchase a pair of jeans with a list price of $175. If the sales tax is 8.5%, what is the total cost of the jeans? Round your answer to the nearest cent.

10. At a restaurant, your total bill is $48.55. You wish to give a tip of 18% of the total bill. What is the amount of the tip? Round your answer to the nearest cent.

11. Jamie found a receipt for a wireless speaker for $127.59, tax included. If the sales tax rate was 9%, what was the selling price of the wireless speaker before taxes? Round your answer to the nearest cent.

12. The average cost of a car in 1990 was $12,500. In 2010, the average price of a car was $21,800. What is the percentage increase in the average price of a car? Round your answer to the nearest tenth of a percent.

13. During the housing price decline of 2009, the value of a house decreased by 30% in one year. If the 2010 value of the house was $115,000, what was the house worth prior to the decline? Round your answer to the nearest cent.

14. The wholesale price of a coat is 50% less than the retail price. If the retail price is $199, what is the wholesale price?

15. A store is having a 60% off sale. The sale price of an item is $125. What is the list price?

16. The local sales tax is 8%. If a pair of shoes sells for $115, what is the total cost after tax?

17. The value of your house is $245,000. If your local property tax rate is 2.5% of the value, how much are your property taxes?

WRITING & THINKING

18. You receive a 10% decrease in your pay. What percentage increase in pay would you have to receive in order to gain your original pay rate again? Round your answer to the nearest whole percent.

5.2 EXERCISES

PRACTICE

Calculate the simple interest for each situation. Round your answer to the nearest cent, if necessary.

1. Determine the interest owed on \$800 for 5 years at a rate of 8.5%.

2. Determine the interest owed on \$5000 for 2 years at a rate of 10%.

3. Determine the interest owed on \$1200 for 30 months at a rate of 19.5%.

4. Determine the interest owed on \$550 for 5 years at a rate of 8.5%.

Use the compound interest formula to find the following for each situation.

a. Calculate the total amount in the account after the given time period.

b. Determine the amount of interest earned for each.

5. $P = \$2500$, $r = 6.5\%$ compounded weekly, $t = 10$ years

6. $P = \$3500$, $r = 4.5\%$ compounded monthly, $t = 10$ years

7. $P = \$2500$, $r = 6.5\%$ compounded daily, $t = 10$ years

8. $P = \$5650$, $r = 8\%$ compounded biannually, $t = 15$ years

9. $P = \$2500$, $r = 6.5\%$ compounded yearly, $t = 10$ years

10. $P = \$15{,}000$, $r = 6\%$ compounded semiannually, $t = 25$ years

11. $P = \$12{,}500$, $r = 8\%$ compounded biweekly, $t = 15$ years

12. $P = \$7300$, $r = 19.9\%$ compounded weekly, $t = 20$ years

An account is compounded monthly for five years with an APR of 9%. For each principal amount, calculate the following.

a. The total amount paid

b. The amount of interest paid

13. \$15,000

14. \$30,000

15. \$60,000

16. \$120,000

APPLICATIONS

17. You are purchasing a new computer using the store's "90 days same as cash" deal. If the cost of the computer is \$1575 (tax included) with an annual interest rate of 19.99%, how much would you owe on the 91st day if you make no payments during the first 90 days?

18. Assume you wish to borrow \$500 for two weeks and the amount of interest you must pay is \$20 per \$100 borrowed. What is the APR at which you are borrowing money? Round your answer to the nearest hundredth.

19. A couple deposits \$25,000 into an account earning 6% annual interest for 25 years.
a. Calculate the future value of the investment if interest is compounded monthly.
b. Calculate the future value if the interest on the investment is compounded weekly.

20. Suppose your salary in 2016 was \$65,000. If the annual inflation rate is 4%, what salary do you need to make in 2024 in order for it to keep up with inflation?

21. Suppose that \$15,000 is deposited for eight years at 5% APR.
a. Calculate the simple interest earned.
b. Calculate the interest earned if interest is compounded weekly.
c. Calculate the interest earned if interest is compounded monthly.

22. A payday loan is made for six weeks, where the amount of interest owed per \$100 borrowed is \$20. If you borrow \$500 for six weeks, answer the following questions.
a. How much do you owe at the end of six weeks?
b. What is the APR for this transaction?

23. Angela deposits \$2500 into an account with an APR of 5.5% for 10 years. Find the amount of interest earned for each of the following situations.
a. Interest is compounded annually.
b. Interest is compounded monthly.
c. Interest is compounded weekly.
d. Interest is compounded daily.
e. Interest is compounded continuously.

24. David deposits \$4000. Determine the APY for each of the following.
a. APR of 5% compounded monthly
b. APR of 5% compounded weekly
c. APR of 5% compounded daily
d. APR of 7.5% compounded monthly
e. APR of 7.5% compounded weekly
f. APR of 7.5% compounded daily

25. Determine the simple interest earned on \$10,000 after 10 years if the APR is each of the following rates.
   a. 3%
   b. 6%
   c. 12%
   d. 24%

26. Suppose the First Bank of Lending offers a CD (Certificate of Deposit) that has a 6.45% interest rate and is compounded quarterly for three years. You decide to invest \$5500 into this CD.
   a. Determine how much money you will have at the end of the three years?
   b. Find the APY.

27. The First Bank of Lending lists the following APR for loans. Determine the APY, or effective interest rate for each category.

First Bank of Lending Loan APR

Loan Amount	APR*
< \$20,000	11.25%
\$20,000–\$99,999	8.99%
≥ \$100,000	5.75%

*interest is compounded quarterly

$$\begin{aligned} PMT &= FV \cdot \frac{\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]} \\ &= 750{,}000 \cdot \frac{\left(\frac{0.08}{12}\right)}{\left[\left(1+\frac{0.08}{12}\right)^{12 \cdot 45}-1\right]} \\ &\approx 142.19 \end{aligned}$$

So, if you wish to retire and make \$60,000 per year without affecting the principal in your retirement fund, you need to start at age 22 depositing approximately \$143 per month. With a fixed APR of 8% over the years of your retirement, you will never outlive your retirement savings.

Skill Check Answers

1. \$34,931.95
2. \$16,359.26
3. \$1,242,475.47

5.3 EXERCISES

PRACTICE

For each situation, use the formula for present value of money to calculate the amount you need to invest now in one lump sum to reach the amount given. Round your answer to the nearest cent, if necessary.

1. \$25,000 after 10 years with an APR of 8% compounded monthly

2. \$25,000 after 10 years with an APR of 12% compounded monthly

3. \$100,000 after 18 years with an APR of 6% compounded quarterly

4. \$1,000,000 after 40 years with an APR of 10% compounded monthly

APPLICATIONS

5. Alexis and Will are purchasing a home. They wish to save money for 10 years and purchase a house that has a value of \$180,000 with cash. If they deposit money into an account paying 12% interest, how much do they need to deposit each month in order to make the purchase?

6. Marilyn wishes to retire at age 65 with \$2,000,000 in the bank. At the age of 21, she decides to begin depositing money into an account with an APR of 11%. What is the monthly payment Marilyn must make in order to make this happen?

7. Repeat Exercise 6 with an APR of 6%.

8. Repeat Exercise 6 with a desired retirement amount of $1,500,000.

9. Suppose you wish to retire at the age of 65 with $80,000 in savings. Determine your monthly payment into an IRA (Individual Retirement Account) if the APR is 7.5% and you begin making payments at
  a. 20 years old.
  b. 30 years old.
  c. 40 years old.

10. Revere College predicts that in 18 years it will take $200,000 to attend the college for four years. Debbie wishes to save money for her child's college fund. How much should Debbie put aside in an account with an APR of 9% compounded monthly in order to have $200,000 in the account in 18 years?

11. Repeat Exercise 10 with an interest rate earned of 5%.

12. Repeat Exercise 10 with an interest rate earned of 3.5%.

13. Suppose you'd like to save enough money to pay cash for your next car. The goal is to save an extra $26,000 over the next 6 years. What amount of quarterly payments must you make into an account that earns 5.5% interest in order to reach your goal?

14. Repeat Exercise 13 if the interest rate earned is 6.5%.

15. Repeat Exercise 13 if the interest rate earned is 3.5%.

16. Willie deposits a fixed monthly amount into an annuity account for his child's college fund. He wishes to accumulate a future value of $75,000 in 15 years.
  a. Assuming an APR of 3.5%, how much money should Willie deposit monthly in order to reach his goal?
  b. How much of the $75,000 will Willie ultimately deposit in the account, and how much is interest earned?

17. Repeat Exercise 16 with an APR of 6%.

18. Repeat Exercise 16 with an accumulated amount of $125,000.

19. Blake starts an IRA (Individual Retirement Account) to save for retirement at the age of 22. He deposits $450 each month. The IRA has an average annual interest rate of 7%.
  a. How much money will he have saved upon retirement at the age of 65?
  b. Determine the amount of money Blake deposited over the length of the investment and how much he made in interest.

20. Jimmie has a job at an advertising agency earning $54,000 per year. Jimmie is currently 26 years old and wishes to retire at age 67 with a retirement income of $75,000. How much money would Jimmie need to invest each month into a growth stock mutual fund with an interest rate of 6.5% in order to withdraw $75,000 per year without reducing the principal?

Monitor your accounts and pay your bills on time. Use the online resources that most banks provide free of charge and check your account balance often. One of the best ways to get good interest rates on loans is to have a healthy credit score. Although we don't go into detail in the course about the components that make up a personal credit score, paying your bills on time, every time, will give you a good start to a good score.

Ask questions. Finally, don't assume you can't ask for help with financial matters. Ask questions and shop around to look for the best possibilities for you. Often, there isn't just one right answer that fits everyone's needs. There are plenty of resources available to the public, even online resources. Use them!

Skill Check Answers

1. $875.11

5.4 EXERCISES

PRACTICE

Consider a credit card with a balance of $7000. You wish to pay off the credit card in each scenario. Round your answer to the nearest cent, if necessary.

a. Calculate the amount of a monthly payment within the time frame given.

b. Calculate the total amount paid over the time period.

1. APR of 17.99% paid off within 1 year

2. APR of 12.5% paid off within 2 years

3. APR of 24% paid off within 3 years

Consider a credit card with a balance of $5560. You wish to pay off the credit card in each scenario. Calculate the following. Round your answer to the nearest cent, if necessary.

a. Calculate the amount of a monthly payment within the time frame given.

b. Calculate the total amount paid over the time period.

4. APR of 14.99% paid off within 1 year

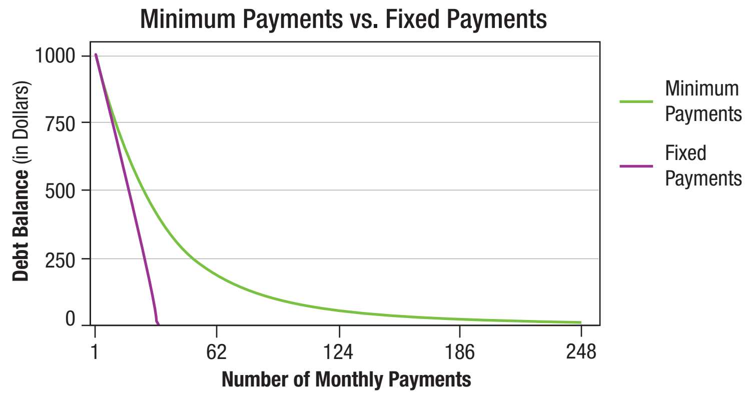
5. APR of 11.99% paid off within 2 years

6. APR of 5.9% paid off within 3 years

APPLICATIONS

Round your answer to the nearest cent, if necessary.

7. Given the chart below, solve the following problems.



a. Estimate the total amount paid when a debt balance was paid using a fixed monthly payment of $40.

b. Estimate the total amount paid when a debt balance was paid using the minimum monthly payment of $18.

8. Rachel is purchasing a new camera that costs $3800 for her photography business. Rachel uses a credit card that has an APR of 16.99%.

a. How long will it take her to pay off the camera if she makes monthly payments of $75?

b. How much will she pay in the long run for the camera if she makes monthly payments of $75?

c. How long will it take her to pay off the camera if she makes monthly payments of $150?

d. How much will she pay in the long run for the camera if she makes monthly payments of $150?

9. Tommy gets to choose from one of the new car incentives when he purchases his car next week. He can either choose 0.9% APR financing for 48 months or $1000 cash back with a 4.75% APR over 48 months. Compare the two incentives that Tommy has to choose from if the new car he wishes to buy is $32,457 and he has saved a down payment of $3500.

10. Mike bought a new car and financed $25,000 to make the purchase. He financed the car for 60 months with an APR of 6.5%. Determine each of the following.

a. Mike's monthly payment

b. Total cost of Mike's car

c. Total interest Mike pays over the life of the loan

11. Omar wants to purchase three vans for his delivery business. Each van costs $38,000. He wishes to finance the purchase for 48 months and has acquired an APR of 4.5%. Determine each of the following.

a. Omar's monthly payment

b. Total cost of Omar's vans

c. Total interest paid by Omar over the life of the loan

12. Jamal bought a new car for $32,000. He paid a 10% down payment and financed the remaining balance for 36 months with an APR of 4.5%. Determine each of the following.

a. Jamal's monthly payment

b. Total cost of Jamal's car

c. Total interest Jamal pays over the life of the loan

13. Susan wants to buy a new computer from Banana Computers. The company sells a laptop model for \$2650. Susan decides to finance the computer for 24 months at an APR of 12.5%. Determine each of the following.
  a. Susan's monthly payment
  b. Total cost of the computer
  c. Total interest paid over the 24 months

14. Amanda and Ferobee are buying a house on a 30-year mortgage. They can only pay \$800 per month for a mortgage. If they have an APR of 3.75%, what is the maximum price of a mortgage that they can take out?

15. Brad decides to purchase a \$250,000 house. He wants to finance the entire balance. He has received an APR of 4.5% for a 30-year mortgage.
  a. What is Brad's monthly payment?
  b. Over the course of the loan, how much interest will Brad pay?
  c. What is Brad's total cost if he takes all 30 years to pay off the house?
  d. If he changed the term to 15 years instead of 30 years, what would his monthly payment be?
  e. With a 15-year mortgage, how much interest will Brad pay?
  f. With a 15-year mortgage, what is the total cost of the house?

16. The city of Nettleton recently completed a new school building. The entire cost of the project was \$19,000,000. The city has put the project on a 20-year loan with an APR of 2.4%. There are 15,000 families that will be responsible for paying the loan.
  a. Determine the amount of the monthly payment for the loan.
  b. Determine the amount that each family should be required to pay each year to cover the cost of the school.
  c. Determine the total cost of the school.

17. You want to buy a car and finance \$20,000 to do so. You can afford a payment of up to \$450 per month. The bank offers three choices for the loan: a four-year loan with an APR of 7%, a five-year loan with an APR of 7.5%, and a six-year loan with an APR of 8%. Which option best meets your needs, assuming you want to pay the least amount of interest?

18. A credit card has a balance of \$5000 at an APR of 9.99%. You plan to pay \$500 each month in an effort to clear the debt quickly. How long will it take you to pay off the balance?

19. A credit card has a balance of \$11,500 at an APR of 14.99%. You plan to pay \$650 each month in an effort to clear the debt quickly. How long will it take you to pay off the balance?

20. Suppose you have a student loan of \$80,000 with an APR of 4.5% for 25 years.
  a. What is your monthly payment?
  b. If you decide you want to pay off the loan in 15 years instead of 25, what is your monthly payment?
  c. What is your savings for paying the loan off in 15 years instead of 25?

21. Suppose you have graduated from college and want to purchase a house. Your take-home pay is \$4560 per month and you wish to stay within the recommended guidelines for mortgage amounts by only spending $\frac{1}{4}$ of your take-home pay on a house payment. You have \$18,500 saved for a down payment. With your good credit and the down payment you can get an APR from your bank of 4.35%, compounded monthly.
  a. What is the total cost of a house you could afford with a 15-year mortgage?
  b. What is the most that you could afford with a traditional 30-year mortgage instead of a 15-year?

WRITING & THINKING

22. What if you won the Powerball Lottery with a jackpot of \$150 million? Calculate the amount of money you would receive over a 25-year period with each of the following two options. Which option gives you the most money over the 25 years?

Option 1: Taking all the money at once with a 40% penalty, and pay the income tax of 38% on the lump sum, and investing the remaining amount into an account earning 6% interest for 25 years.

Option 2: Acquire the money as part of an annuity to be paid out in 25 equal payments over a 25-year period, paying the income tax of 38% on the income from the winnings each year.

6.1 EXERCISES

PRACTICE

Use the method of substitution to solve the following systems of equations. If a system is dependent, express the solution set in terms of one of the variables. See Examples 1 and 2.

1. $\begin{cases} 2x - y = -12 \\ 3x + y = -13 \end{cases}$

2. $\begin{cases} 2x - 4y = -6 \\ 3x - y = -4 \end{cases}$

3. $\begin{cases} 3y = 9 \\ x + 2y = 11 \end{cases}$

4. $\begin{cases} -3x - y = 2 \\ 9x + 3y = -6 \end{cases}$

5. $\begin{cases} 2x + y = -2 \\ -4x - 2y = 5 \end{cases}$

6. $\begin{cases} 5x - y = -21 \\ 9x + 2y = -34 \end{cases}$

7. $\begin{cases} 2x - y = -3 \\ -4x + 2y = 6 \end{cases}$

8. $\begin{cases} 3x + 6y = -12 \\ 2x + 4y = -8 \end{cases}$

9. $\begin{cases} 2x + 5y = 33 \\ 3x = -3 \end{cases}$

10. $\begin{cases} 5x + 2y = 8 \\ 2x + y = 6 \end{cases}$

11. $\begin{cases} -2x + y = 5 \\ 9x - 2y = 5 \end{cases}$

12. $\begin{cases} 3x + y = 4 \\ -2x + 3y = 1 \end{cases}$

13. $\begin{cases} 4x - y = -1 \\ -8x + 2y = 2 \end{cases}$

14. $\begin{cases} 4x - 2y = 3 \\ -2x + y = -7 \end{cases}$

15. $\begin{cases} 9x - y = -1 \\ 3x + 2y = 44 \end{cases}$

Use the method of elimination to solve the following systems of equations. If a system is dependent, express the solution set in terms of one of the variables. See Examples 3 and 4.

16. $\begin{cases} 2x - 3y = 8 \\ 8x + 5y = -2 \end{cases}$

17. $\begin{cases} -2x + 3y = 13 \\ 4x + 2y = -18 \end{cases}$

18. $\begin{cases} 5x + 7y = 1 \\ -2x + 3y = -12 \end{cases}$

19. $\begin{cases} x + 2y = 17 \\ 3x + 4y = 39 \end{cases}$

20. $\begin{cases} 5x - 10y = 9 \\ -x + 2y = -3 \end{cases}$

21. $\begin{cases} -2x - 2y = 4 \\ 3x + 3y = -6 \end{cases}$

22. $\begin{cases} 4x + y = 11 \\ 3x - 2y = 0 \end{cases}$

23. $\begin{cases} 7x + 8y = -3 \\ -5x - 4y = 9 \end{cases}$

24. $\begin{cases} -2x - y = 9 \\ 4x + 2y = 1 \end{cases}$

25. $\begin{cases} -2x + 4y = 6 \\ 3x - y = -4 \end{cases}$

26. $\begin{cases} 5x - 6y = -1 \\ -4x + 3y = -10 \end{cases}$

27. $\begin{cases} \frac{2}{3}x + y = -3 \\ 3x + \frac{5}{2}y = -\frac{7}{2} \end{cases}$

28. $\begin{cases} \frac{x}{5} - y = -\frac{11}{5} \\ \frac{x}{4} + y = 4 \end{cases}$

29. $\begin{cases} \frac{2}{3}x + 2y = 1 \\ x + 3y = 0 \end{cases}$

30. $\begin{cases} -x - 5y = -6 \\ \frac{3}{5}x + 3y = 1 \end{cases}$

Use any convenient method to solve the following systems of equations. If a system is dependent, express the solution set in terms of one or more of the variables, as appropriate. See Examples 5 and 6.

31. $\begin{cases} x-y+4z=-4 \\ 4x+y-2z=-1 \\ -y+2z=-3 \end{cases}$

32. $\begin{cases} x+2y=-1 \\ y+3z=7 \\ 2x+5z=21 \end{cases}$

33. $\begin{cases} x+y=4 \\ y+3z=-1 \\ 2x-2y+5z=-5 \end{cases}$

34. $\begin{cases} 2x-y=0 \\ 5x-3y-3z=5 \\ 2x+6z=-10 \end{cases}$

35. $\begin{cases} 3x-y+z=2 \\ -6x+2y-2z=-4 \\ -3x+y-z=-2 \end{cases}$

36. $\begin{cases} 2x-3y=-2 \\ x-4y+3z=0 \\ -2x+7y-5z=0 \end{cases}$

37. $\begin{cases} 3x-y+z=2 \\ -6x+2y-2z=1 \\ 5x+2y-3z=2 \end{cases}$

38. $\begin{cases} 4x-y+5z=6 \\ 4x-3y-5z=-14 \\ -2x-5z=-8 \end{cases}$

39. $\begin{cases} 3x+8z=3 \\ -3x+y-7z=-2 \\ x+2y+3z=3 \end{cases}$

40. $\begin{cases} x+2y+z=8 \\ 2x-3y-4z=-16 \\ x-5y+5z=6 \end{cases}$

41. $\begin{cases} 2x-7y-4z=7 \\ -x+4y+2z=-3 \\ 3y-4z=-1 \end{cases}$

42. $\begin{cases} 4x+4y-2z=6 \\ x-5y+3z=-2 \\ -2x-2y+z=3 \end{cases}$

43. $\begin{cases} 2x+3y+4z=1 \\ 3x-4y+5z=-5 \\ 4x+5y+6z=5 \end{cases}$

44. $\begin{cases} x-4y+2z=-1 \\ 2x+y-3z=10 \\ -3x+12y-6z=3 \end{cases}$

45. $\begin{cases} x+2y+3z=29 \\ 2x-y-z=-2 \\ 3x+2y-6z=-8 \end{cases}$

46. $\begin{cases} 5x-2y+z=14 \\ 8x+4y=12 \\ 9x=18 \end{cases}$

47. $\begin{cases} 2x+5y=6 \\ 3y+8z=-6 \\ x+4y=-5 \end{cases}$

48. $\begin{cases} 4x+3y+4z=5 \\ 5x-6y-2z=-12 \\ 5z=20 \end{cases}$

49. $\begin{cases} 9x+4y-8z=-4 \\ -6x+3y-9z=-9 \\ 8y-3z=18 \end{cases}$

50. $\begin{cases} 21x-7y+51z=141 \\ 13x+9y-5z=-19 \\ 19x-8y+23z=30 \end{cases}$

APPLICATIONS

51. Karen empties out her purse and finds 45 loose coins, consisting entirely of nickels and pennies. If the total value of the coins is $1.37, how many nickels and how many pennies does she have?

52. What choice of a, b, and c will force the graph of the polynomial $f(x) = ax^2 + bx + c$ to have a y-intercept of 5 and to pass through the points $(1,3)$ and $(2,0)$?

53. A tour organizer is planning on taking a group of 40 people to a musical. Balcony tickets cost $29.95 and regular tickets cost $19.95. The organizer collects a total of $1048.00 from her group to buy the tickets. How many people chose to sit in the balcony?

54. How many ounces each of a 12% alcohol solution and a 30% alcohol solution must be combined to obtain 60 ounces of an 18% solution?

55. Eliza's mother is 20 years older than Eliza, but 3 years younger than Eliza's father. Eliza's father is 7 years younger than three times Eliza's age. How old is Eliza?

56. An investor decides at the beginning of the year to invest some of his cash in an account paying 8% annual interest, and to put the rest in a stock fund that ends up earning 15% over the course of the year. He puts $2000 more in the first account than in the stock fund, and at the end of the year he finds he has earned $1310 in interest. How much money was invested at each of the two rates?

57. Jack and Tyler went shopping for summer clothes. Shirts were $12.47 each, including tax, and shorts were $17.23 per pair, including tax. Jack and Tyler spent a total of $156.21 on 11 items. How many shirts and pairs of shorts did they buy?

58. Three years ago, Bob was twice as old as Marla. Fifteen years ago, Bob was three times as old as Marla. How old is Bob?

59. Deyanira empties her pockets and finds 42 coins consisting of quarters, dimes, and pennies. There are twice as many pennies as dimes and quarters total. If the total value of the coins is $2.13, how many coins of each denomination does she have?

60. If an investor has invested $1000 in stocks and bonds, how much has he invested in stocks if he invested four times more in stocks than in bonds?

61. Twelve years ago, Jim was twice as old as Kristin. Sixteen years ago, Jim was three times older than Kristin. How old is Jim?

62. A movie brought in $740 in ticket sales in one day. Tickets during the day cost $5 and tickets at night cost $7. If 120 tickets were sold, how many were sold during the day?

63. A computer has 24 screws in its case. If there are 7 times more slotted screws than thumb screws, how many thumb screws are in the computer?

64. Jael has $10,000 she would like to invest. She has narrowed her options down to a certificate of deposit paying 5% annually, bonds paying 4% annually, and stocks with an expected annual rate of return of 13.5%. If she wants to invest twice as much in the stocks as in the certificate of deposit and she wants to earn $1000 in interest by the end of the year, how much should she invest in each type of investment?

65. Lea ordered fruit baskets for three of her coworkers. One contained 5 apples, 2 oranges, and 1 mango and cost \$6.81. Another contained 2 mangos, 8 oranges, and 3 apples and cost \$11.88. The third contained 4 apples, 4 oranges, and 4 mangos and cost \$11.04. How much did each type of fruit cost?

TECHNOLOGY

Solve each of the following systems of equations using a graphing utility.

66. $\begin{cases} 98x + 43y - 82z = -784 \\ -65x + 34y = 3032 \\ 28y - 13z = 966 \end{cases}$

67. $\begin{cases} 7.5x + 5.2y - 9.3z = -23.971 \\ -6.8x + 4.4y = 2.708 \\ 0.9x - 1.88y = -2.0194 \end{cases}$

68. $\begin{cases} -5x + 2y - 20z = 14 \\ 2x - 3y + 10z = -19 \\ 7x + 4y - 7z = -7 \end{cases}$

69. $\begin{cases} -5.5x + 2.2y - 5.1z = 11.29 \\ 1.8x + 4.9y - 0.5z = 7.066 \\ 3.9x - 2.6y + 6.3z = -3.698 \end{cases}$

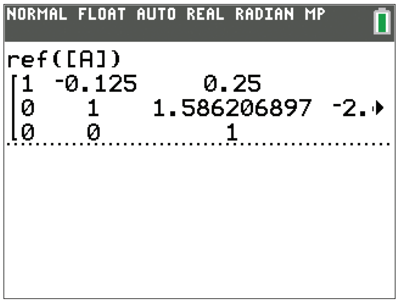
70. $\begin{cases} 5x - 10y + 11z = 19 \\ 27x + 9y + 7z = -44 \\ 2x + 19y - 4z = -3 \end{cases}$

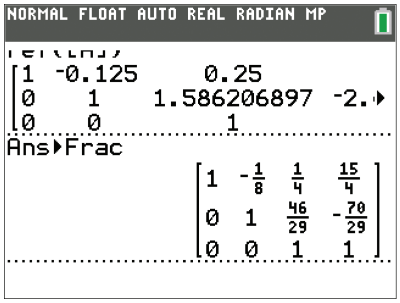
71. $\begin{cases} -23x + 17y - 7z = -51 \\ -13x + 25y - 11z = 45 \\ 51x - 21y - 28z = -58 \end{cases}$

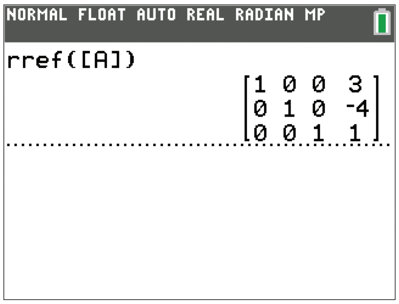
Return to the main screen by pressing 2nd mode. Now that the matrix is stored in the calculator as matrix A, we can perform operations. Press 2nd x^{-1} to access the MATRIX menu again, but this time select MATH. Scroll down to select A:ref(. When you press enter, ref(will appear on the main screen.

Now we need to select which matrix we want to put in row echelon form, matrix A. To do this, press 2nd x^{-1} once more. NAMES and 1:[A] should already be highlighted, so press enter. Add the right-hand parenthesis and press enter.

To view this matrix with fractional entries, press math and enter, since 1:Frac is already highlighted. Press enter again on the main screen.







Since the entire matrix does not fit on the screen, use the arrow keys to view the right side of the matrix.

Finding the reduced row echelon form of this matrix is similar. Press 2nd x^{-1} and select MATH but this time arrow down farther to highlight B:rref(and press enter. Select matrix A again, add the right-hand parenthesis and press enter.

6.2 EXERCISES

PRACTICE

1. Let $A=\begin{bmatrix}4 & -1\\ 0 & 3\\ 9 & -5\end{bmatrix}$. Determine the following, if possible:

a. The order of A **b.** The value of a_{12} **c.** The value of a_{23}

2. Let $B=\begin{bmatrix}-7 & 2 & 11\end{bmatrix}$. Determine the following, if possible:

a. The order of B **b.** The value of b_{12} **c.** The value of b_{31}

3. Let $C = \begin{bmatrix} 1 & 0 \\ 5 & -3 \\ 2 & 9 \\ \pi & e \\ 10 & -7 \end{bmatrix}$. Determine the following, if possible:

a. The order of C **b.** The value of c_{23} **c.** The value of c_{51}

4. Let $D = \begin{bmatrix} -8 & 13 & -1 \\ 0 & 6 & 3 \\ 0 & -9 & 0 \end{bmatrix}$. Determine the following, if possible:

a. The order of D **b.** The value of d_{23} **c.** The value of d_{33}

5. Let $E = \begin{bmatrix} -443 & 951 & 165 & 274 \\ 286 & -653 & 812 & -330 \\ 909 & 377 & 429 & -298 \end{bmatrix}$. Determine the following, if possible:

a. The order of E **b.** The value of e_{42} **c.** The value of e_{21}

6. Let $A = \begin{bmatrix} 9 & 5 & 0 \\ 7 & 4 & 2 \end{bmatrix}$. Determine the following, if possible:

a. The order of A **b.** The value of a_{22} **c.** The value of a_{13}

7. Let $B = \begin{bmatrix} 8 & 1 \\ 3 & 0 \\ 6 & 7 \end{bmatrix}$. Determine the following, if possible:

a. The order of B **b.** The value of b_{12} **c.** The value of b_{13}

8. Let $C = \begin{bmatrix} 65 & 32 & 91 & 45 \\ 23 & 18 & 75 & 47 \\ 8 & 63 & 28 & 31 \end{bmatrix}$. Determine the following, if possible:

a. The order of C **b.** The value of c_{43} **c.** The value of c_{23}

9. Let $D = \begin{bmatrix} 4 & 9 & 7 & 1 & 8 \\ 5 & 3 & 0 & 2 & 6 \end{bmatrix}$. Determine the following, if possible:

a. The order of D **b.** The value of d_{21} **c.** The value of d_{24}

Construct the augmented matrix that corresponds to each of the following systems of equations. See Example 2. (Answers may appear in slightly different, but equivalent, form.)

10. $\begin{cases} 4x+5y-3z=8 \\ 7x-2y+9=3 \\ 5x-6y+3z=0 \end{cases}$

11. $\begin{cases} y-2z+4=3x \\ \dfrac{x}{2}-4y-1=z \\ 3(-y+z)-1=0 \end{cases}$

12. $\begin{cases} 5x+\dfrac{y-z}{2}=3 \\ 7(z-x)+y-2=0 \\ x-(4-z)=y \end{cases}$

13. $\begin{cases} \dfrac{2-3x}{2}=y \\ 3z+2(x+y)=0 \\ 2x-y=2(x-3z) \end{cases}$

14. $\begin{cases} 2(z+3)-x+y=z \\ -3(x-2y)-1=5z \\ \dfrac{x}{3}-(y-2z)=x \end{cases}$

15. $\begin{cases} \dfrac{12x-1}{5}+\dfrac{y}{2}=\dfrac{3z}{2} \\ y-(x+3z)=-(1-y) \\ 2x-2-z-2y=7x \end{cases}$

16. $\begin{cases} \dfrac{3x+4y}{2}-3z=6 \\ 3(x-2y+9z)=0 \\ 2x+6y=3-z \end{cases}$

17. $\begin{cases} \dfrac{2x-4y}{3}=2z \\ 8x=2(y-3z)+7 \\ 3x=2y \end{cases}$

18. $\begin{cases} \dfrac{2(2x-y)}{3}+z=7 \\ 4=\dfrac{3}{-x+y+3z} \\ 4x-8y+4=9x \end{cases}$

19. $\begin{cases} 0.5x-14y=\dfrac{z}{4}-8 \\ \dfrac{x}{5}-y+\dfrac{z}{4}=\dfrac{y}{6}-3 \\ \dfrac{2}{3}\left(\dfrac{4}{y-x-1}\right)=\dfrac{5}{z} \end{cases}$

Construct the system of equations that corresponds to each of the following matrices.

20. $\left[\begin{array}{cc|c} 5 & 3 & 9 \\ 1 & 4 & 12 \end{array}\right]$

21. $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 3 \end{array}\right]$

22. $\left[\begin{array}{ccc|c} 14 & 0 & 1 & 16 \\ 3 & 6 & 4 & 0 \\ 8 & 2 & 5 & 21 \end{array}\right]$

23. $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 16 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 4 \end{array}\right]$

24. $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 22 \\ 1 & 3 & 1 & 17 \\ 1 & 1 & 4 & 8 \end{array}\right]$

25. $\left[\begin{array}{ccc|c} 0 & 9 & 13 & 27 \\ 2 & 0 & 21 & 19 \\ 7 & 18 & 0 & 32 \end{array}\right]$

Fill in the blanks by performing the indicated row operations. See Example 4.

26. $\left[\begin{array}{cc|c} 3 & 2 & -7 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{-3R_2+R_1} \underline{\ ?\ }$

27. $\left[\begin{array}{cc|c} 2 & -5 & 3 \\ -4 & 3 & -1 \end{array}\right] \xrightarrow{2R_1+R_2} \underline{\ ?\ }$

28. $\left[\begin{array}{cc|c} 4 & 2 & -8 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{\frac{1}{2}R_1} \underline{\ ?\ }$

29. $\left[\begin{array}{cc|c} 9 & -2 & 7 \\ 1 & 3 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \underline{\ ?\ }$

30. $\left[\begin{array}{cc|c} 4 & 1 & 5 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{2R_1} \underline{\ ?\ }$

31. $\left[\begin{array}{cc|c} 8 & -2 & -4 \\ 3 & -1 & 7 \end{array}\right] \xrightarrow{-2R_2} \underline{\ ?\ }$

32. $\left[\begin{array}{cc|c} 9 & 12 & -6 \\ 15 & -3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \underline{\ ?\ }$

33. $\left[\begin{array}{cc|c} 4 & 12 & -6 \\ 7 & 3 & 9 \end{array}\right] \xrightarrow{\frac{1}{2}R_1+R_2} \underline{\ ?\ }$

34. $\left[\begin{array}{cc|c} 3 & 0 & 1 \\ 5 & 7 & -2 \end{array}\right] \xrightarrow{3R_1+R_2} \underline{\ ?\ }$

35. $\left[\begin{array}{cc|c} 8 & -2 & 10 \\ 9 & -3 & 0 \end{array}\right] \xrightarrow[-\frac{2}{3}R_2]{\frac{1}{2}R_1} \underline{\ ?\ }$

36. $\left[\begin{array}{ccc|c} 5 & 2 & 9 & 7 \\ 1 & 3 & -5 & 0 \\ 2 & -4 & 1 & 8 \end{array}\right] \xrightarrow[-R_1+R_3]{2R_2} \underline{\ ?\ }$

37. $\left[\begin{array}{ccc|c} 6 & -2 & 5 & 14 \\ -7 & 19 & 2 & 3 \\ -9 & 11 & -4 & 7 \end{array}\right] \xrightarrow[0.5R_3]{3R_1} \underline{\ ?\ }$

38. $\left[\begin{array}{ccc|c} 5 & 3 & 13 & 15 \\ 17 & 9 & -8 & -14 \\ 4 & -11 & 19 & 8 \end{array}\right] \xrightarrow{-2R_2+R_3} \underline{\ ?\ }$

39. $\left[\begin{array}{ccc|c} 8 & 11 & 18 & 2 \\ 14 & 33 & -3 & -5 \\ -9 & 21 & 12 & 9 \end{array}\right] \xrightarrow[-2R_3+R_2]{\frac{1}{3}R_3+R_1} \underline{\ ?\ }$

40. $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 3 & -1 & 8 & 2 \\ -5 & 0 & 2 & 7 \end{array}\right] \xrightarrow[5R_1+R_3]{-3R_1+R_2} \underline{\ ?\ }$

41. $\left[\begin{array}{ccc|c} 2 & 3 & -3 & 5 \\ 1 & 1 & 3 & 4 \\ 3 & 3 & 9 & 12 \end{array}\right] \xrightarrow[-3R_2+R_3]{-2R_2+R_1} \underline{\ ?\ }$

42. $\left[\begin{array}{cc|c} -3 & 2 & 2 \\ 5 & -4 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \underline{\ ?\ }$

43. $\left[\begin{array}{cc|c} -5 & 20 & -15 \\ 2 & -12 & 5 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{5}R_1} \underline{\ ?\ }$

44. $\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ -3 & 2 & 8 & -2 \\ 1 & 5 & 2 & 6 \end{array}\right] \xrightarrow[3R_3+R_2]{-2R_3+R_1} \underline{\ ?\ }$

45. $\left[\begin{array}{ccc|c} 1 & 5 & -9 & 11 \\ 1 & 4 & -1 & 4 \\ 4 & 3 & 5 & 45 \end{array}\right] \xrightarrow[-4R_1+R_3]{-R_1+R_2} \underline{\ ?\ }$

For each matrix, determine if it is in row echelon form, reduced row echelon form, or neither.

46. $\left[\begin{array}{cc|c} 1 & 5 & 4 \\ 0 & 1 & 3 \end{array}\right]$

47. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 12 \end{array}\right]$

48. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 8 \end{array}\right]$

49. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 5 & 1 & 0 & 14 \\ 3 & 4 & 1 & -16 \end{array}\right]$

50. $\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 9 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

51. $\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & 6 \end{array}\right]$

Use Gaussian elimination and back-substitution to solve the following systems of equations. See Example 5.

52. $\begin{cases} 2x-4y=-6 \\ 3x-y=-4 \end{cases}$

53. $\begin{cases} 2x-5y=11 \\ 3x+2y=7 \end{cases}$

54. $\begin{cases} 5x-y=-21 \\ 9x+2y=-34 \end{cases}$

55. $\begin{cases} x-4y=-11 \\ 7x-y=4 \end{cases}$

56. $\begin{cases} x+2y=17 \\ 3x+4y=39 \end{cases}$

57. $\begin{cases} 2x+6y=4 \\ -4x-7y=7 \end{cases}$

58. $\begin{cases} 3x-2y=5 \\ -5x+4y=-3 \end{cases}$

59. $\begin{cases} 2x+y=-2 \\ -4x-2y=5 \end{cases}$

60. $\begin{cases} 6x-16y=10 \\ -3x+8y=4 \end{cases}$

61. $\begin{cases} 2x-3y=0 \\ 5x+y=17 \end{cases}$

62. $\begin{cases} 6x+3y=3 \\ x+y=3 \end{cases}$

63. $\begin{cases} 3x+6y=-12 \\ 2x+4y=-8 \end{cases}$

64. $\begin{cases} 4x+5y=9 \\ 8x+3y=-17 \end{cases}$

65. $\begin{cases} \frac{2}{3}x+2y=1 \\ x+3y=0 \end{cases}$

66. $\begin{cases} 13x-17y=-3 \\ -19x+15y=-35 \end{cases}$

67. $\begin{cases} 3x-9y-7z=-9 \\ 5x+11y-z=17 \\ -4x-8y+7z=5 \end{cases}$ **68.** $\begin{cases} 8x-y+5z=-8 \\ 11x-2y+9z=-9 \\ 7x-3y+13z=4 \end{cases}$ **69.** $\begin{cases} 17x+13y+8z=46 \\ -12x+3y+28z=-19 \\ 14x+5y-15z=-15 \end{cases}$

Use Gauss-Jordan elimination to solve the following systems of equations. See Example 6.

70. $\begin{cases} 2x-3y=8 \\ 8x+5y=-2 \end{cases}$ **71.** $\begin{cases} \frac{2}{3}x+y=-3 \\ 3x+\frac{5}{2}y=-\frac{7}{2} \end{cases}$ **72.** $\begin{cases} 3y=9 \\ x+2y=11 \end{cases}$

73. $\begin{cases} 6x+2y=-4 \\ -9x-3y=6 \end{cases}$ **74.** $\begin{cases} 3y=6 \\ 5x+2y=4 \end{cases}$ **75.** $\begin{cases} 3x+8y=-4 \\ x+2y=-2 \end{cases}$

76. $\begin{cases} -3x+2y=5 \\ 5x-2y=1 \end{cases}$ **77.** $\begin{cases} 9x-11y=10 \\ -4x+3y=-12 \end{cases}$ **78.** $\begin{cases} 9x-15y=-6 \\ -3x+11y=-10 \end{cases}$

79. $\begin{cases} 3x-8y=7 \\ 18x-35y=-23 \end{cases}$ **80.** $\begin{cases} 4x+y-3z=-9 \\ 2x-3z=-19 \\ 7x-y-4z=-29 \end{cases}$ **81.** $\begin{cases} -5x+9y+3z=1 \\ 3x+2y-6z=9 \\ x+4y-z=16 \end{cases}$

82. $\begin{cases} 2x-y=0 \\ 5x-3y-3z=5 \\ 2x+6z=-10 \end{cases}$ **83.** $\begin{cases} x+y=4 \\ y+3z=-1 \\ 2x-2y+5z=-5 \end{cases}$ **84.** $\begin{cases} 2x-3y=-2 \\ x-4y+3z=0 \\ -2x+7y-5z=0 \end{cases}$

85. $\begin{cases} 3x+8z=3 \\ -3x-7z=-3 \\ x+3z=1 \end{cases}$ **86.** $\begin{cases} 3x-y+z=2 \\ -6x+2y-2z=1 \\ 5x+2y-3z=2 \end{cases}$ **87.** $\begin{cases} x+2y=-1 \\ y+3z=7 \\ 2x+5z=21 \end{cases}$

88. $\begin{cases} 2x+8y-z=-5 \\ -5x+3y+4z=-6 \\ x-4y-5z=-8 \end{cases}$ **89.** $\begin{cases} 7x-8y+2z=-2 \\ 5x-3y-z=-3 \\ 8x+y-3z=7 \end{cases}$

90. $\begin{cases} 8x+14y-3z=3 \\ -6x+2y+7z=-13 \\ 8x+19y+3z=11 \end{cases}$ **91.** $\begin{cases} 8x+5y+3z=-2 \\ 12x-y-18z=1 \\ 7x+6y+10z=19 \end{cases}$

92. $\begin{cases} 4x+8y+7z=27 \\ -2x+9y-8z=-15 \\ 9x+13y+7z=-33 \end{cases}$ **93.** $\begin{cases} w-x+2z=9 \\ 2w+3y=-1 \\ -2w-5y-z=0 \\ x+2y=-4 \end{cases}$

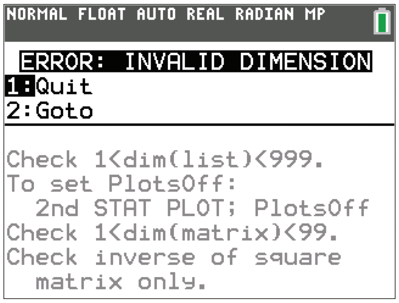
94. $\begin{cases} 3w-x+5y+3z=2 \\ -4w-10y-2z=10 \\ w-x+2z=7 \\ 4w-2x+5y+5z=9 \end{cases}$

APPLICATIONS

95. The sum of three integers is 155. The first integer is sixteen more than the second. The third integer is seven less than the sum of the first integer and twice the second. What are the three integers?

96. Mario bought a pound of bacon, a dozen eggs, and a loaf of bread to make breakfast for his family. The total cost was \$7.42. The bacon cost \$0.03 more than twice the price of the bread and the eggs cost \$0.03 less than half the price of the bread. Find the price of each item.

97. The Pizza House sells three sizes of pizzas: small, medium, large. The prices of the pizzas are \$9.00, \$12.00, and \$15.00, respectively. In one day, they sold 82 pizzas for a total of \$1098.00. If the number of large pizzas sold was twice the number of medium pizzas sold, how many of each size pizza did the Pizza House sell?



Remember that the definition of determinant only applies to square matrices; matrices that are not square matrices do not have determinants.

6.3 EXERCISES

PRACTICE

Evaluate each of the following determinants. See Example 1.

1. $\begin{vmatrix} 4 & -3 \\ 1 & 2 \end{vmatrix}$ **2.** $\begin{vmatrix} 5 & -2 \\ 5 & -2 \end{vmatrix}$ **3.** $\begin{vmatrix} 0 & 3 \\ -5 & 2 \end{vmatrix}$ **4.** $\begin{vmatrix} 34 & -2 \\ 17 & -1 \end{vmatrix}$

5. $\begin{vmatrix} a & x \\ x & b \end{vmatrix}$ **6.** $\begin{vmatrix} 5x & 2 \\ -x & 1 \end{vmatrix}$ **7.** $\begin{vmatrix} -2 & 2 \\ -2 & -2 \end{vmatrix}$ **8.** $\begin{vmatrix} ac & 2ad \\ bc & db \end{vmatrix}$

9. $\begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix}$ **10.** $\begin{vmatrix} w & x \\ y & z \end{vmatrix}$ **11.** $\begin{vmatrix} -2 & 9 \\ 5 & -3 \end{vmatrix}$ **12.** $\begin{vmatrix} 2y & 3x \\ y-1 & x^2 \end{vmatrix}$

Solve for *x* by calculating the determinant.

13. $\begin{vmatrix} x-2 & 2 \\ 2 & x+1 \end{vmatrix} = 0$ **14.** $\begin{vmatrix} x+7 & -2 \\ 9 & x-2 \end{vmatrix} = 0$ **15.** $\begin{vmatrix} x+1 & 8 \\ 1 & x+3 \end{vmatrix} = 0$

16. $\begin{vmatrix} x-8 & 11 \\ -2 & x+5 \end{vmatrix} = 0$ **17.** $\begin{vmatrix} x+6 & 2 \\ -1 & x+3 \end{vmatrix} = 0$ **18.** $\begin{vmatrix} x-4 & -4 \\ 3 & x+9 \end{vmatrix} = 0$

19. $\begin{vmatrix} x+5 & 3 \\ 3 & x-3 \end{vmatrix} = 0$ **20.** $\begin{vmatrix} x+3 & 6 \\ 5 & x+7 \end{vmatrix} = 0$ **21.** $\begin{vmatrix} x-3 & 2 \\ 1 & x-4 \end{vmatrix} = 0$

Use the matrix $A = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$ to evaluate the following. See Example 2.

22. The minor of a_{12}

23. The cofactor of a_{12}

24. The minor of a_{22}

25. The cofactor of a_{22}

26. The cofactor of a_{32}

27. The cofactor of a_{33}

28. The minor of a_{13}

29. The cofactor of a_{21}

30. The cofactor of a_{31}

Find the determinant by the method of expansion by cofactors along the given row or column. See Example 3.

31. $\begin{vmatrix} 4 & 5 & 3 \\ -1 & 2 & 7 \\ 11 & 6 & 2 \end{vmatrix}$ Expand along Row 3

32. $\begin{vmatrix} 8 & 2 & 0 \\ 3 & 4 & 7 \\ 1 & 0 & 2 \end{vmatrix}$ Expand along Column 3

33. $\begin{vmatrix} 5 & 8 & 5 \\ 0 & -6 & 3 \\ 2 & 4 & -1 \end{vmatrix}$ Expand along Row 2

34. $\begin{vmatrix} -4 & 2 & 1 \\ 9 & 12 & 8 \\ 0 & 6 & -3 \end{vmatrix}$ Expand along Column 1

35. $\begin{vmatrix} 13 & 0 & -7 \\ 4 & 2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$ Expand along Row 2

36. $\begin{vmatrix} 7 & 0 & 1 \\ 2 & 5 & 3 \\ 8 & 6 & 2 \end{vmatrix}$ Expand along Column 3

37. $\begin{vmatrix} 8 & 0 & -7 & 5 \\ 4 & -2 & 3 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & 6 & 0 \end{vmatrix}$ Expand along Row 4

38. $\begin{vmatrix} 4 & -2 & 9 & 2 \\ 7 & 0 & 1 & 7 \\ -6 & 3 & 0 & 1 \\ 3 & 1 & 2 & 0 \end{vmatrix}$ Expand along Column 2

Evaluate each of the following determinants. In each case, minimize the required number of computations by carefully choosing a row or column to expand along, and use the properties of determinants to simplify the process. See Examples 3 and 4.

39. $\begin{vmatrix} 2 & 0 & 1 \\ -5 & 1 & 0 \\ 3 & -1 & 1 \end{vmatrix}$

40. $\begin{vmatrix} 12 & 3 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{vmatrix}$

41. $\begin{vmatrix} 12 & 3 & 6 \\ 2 & 2 & -4 \\ 0 & 2 & 0 \end{vmatrix}$

42. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

43. $\begin{vmatrix} 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix}$

44. $\begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}$

45. $\begin{vmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{vmatrix}$

46. $\begin{vmatrix} 0 & 2 & 0 & 0 \\ -2 & -4 & 5 & 9 \\ 1 & 3 & -1 & 1 \\ 0 & 7 & 0 & 2 \end{vmatrix}$

47. $\begin{vmatrix} x & x & 0 & 0 \\ yz & x^3 & z & x^4 \\ z & xy & x & 0 \\ x^2 & 0 & 0 & 0 \end{vmatrix}$

Use Cramer's Rule to solve each system of equations. See Examples 5 and 6.

48. $\begin{cases} 2x-3y=8 \\ 8x+5y=-2 \end{cases}$

49. $\begin{cases} 5x+7y=9 \\ 2x+3y=-7 \end{cases}$

50. $\begin{cases} 5x-10y=9 \\ -x+2y=-3 \end{cases}$

51. $\begin{cases} -2x-2y=4 \\ 3x+3y=-6 \end{cases}$

52. $\begin{cases} \frac{2}{3}x+y=-3 \\ 3x+\frac{5}{2}y=-\frac{7}{2} \end{cases}$

53. $\begin{cases} \frac{2}{3}x+2y=1 \\ x+3y=0 \end{cases}$

54. $\begin{cases} x+2y=-1 \\ y+3z=7 \\ 2x+5z=21 \end{cases}$

55. $\begin{cases} 2x-y=0 \\ 5x-3y-3z=5 \\ 2x+6z=-10 \end{cases}$

56. $\begin{cases} 3x+8z=3 \\ -3x-7z=-3 \\ x+3z=1 \end{cases}$

57. $\begin{cases} 3w-x+5y+3z=2 \\ -4w-10y-2z=10 \\ w-x+2z=7 \\ 4w-2x+5y+5z=9 \end{cases}$

58. $\begin{cases} 2w+x-3y=3 \\ w-2x+y=1 \\ x+z=-2 \\ y+z=0 \end{cases}$

59. $\begin{cases} 3w-2x+y-5z=-1 \\ w+x-y+4z=2 \\ 4w-x-z=1 \\ 5w-x=9 \end{cases}$

60. $\begin{cases} -4x+y=1 \\ 7x+2y=407 \end{cases}$

61. $\begin{cases} 5x-4y=-49 \\ 24x-19y=179 \end{cases}$

62. $\begin{cases} 2w-3x+4y-z=21 \\ w+5x=2 \\ -2x+3y+z=12 \\ -3w+4z=-5 \end{cases}$

63. $\begin{cases} -5x+10y=3 \\ \frac{7}{2}x-7y=20 \end{cases}$

64. $\begin{cases} 23x+21y=-4 \\ x-3y=-8 \end{cases}$

65. $\begin{cases} w-x+y-z=2 \\ 2w-x+3y=-5 \\ x-2z=7 \\ 3w+4x=-13 \end{cases}$

APPLICATIONS

66. The three sides of a triangle are related as follows: the perimeter is 43 feet, the second side is 5 feet more than twice the first side, and the third side is 3 feet less than the sum of the other two sides. Find the lengths of the three sides of the triangle.

67. Eric's favorite candy bar and ice cream flavor have fat and calorie contents as follows: each candy bar has 5 grams of fat and 280 calories; each serving of ice cream has 10 grams of fat and 150 calories. How many candy bars and servings of ice cream did he eat during the weekend he consumed 85 grams of fat and 2300 calories from these two treats?

68. A farmer plants soybeans, corn, and wheat and rotates the planting each year on her 500-acre farm. In a particular year, the profits from her crops were $120 per acre of soybeans, $100 per acre of corn, and $80 per acre of wheat. She planted twice as many acres of corn as soybeans. How many acres did she plant with each crop that year if she made a total profit of $51,800?

TECHNOLOGY

Using a graphing utility, find the determinant of the matrix.

69. $\begin{vmatrix} 0.1 & 0.4 & -0.7 \\ 0.3 & -0.1 & 0.2 \\ 0.5 & -0.2 & 0.3 \end{vmatrix}$

70. $\begin{vmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & -0.2 & -0.1 \\ -0.1 & -0.4 & 0.5 \end{vmatrix}$

71. $\begin{vmatrix} 2.2 & 0.3 & -1.7 \\ 0.4 & -0.2 & 0.1 \\ 0.2 & 0.3 & -1.6 \end{vmatrix}$

72. $\begin{vmatrix} 3.1 & 0.6 & -1.1 \\ 1.2 & 5.2 & -7.3 \\ -0.1 & -4.1 & 6.5 \end{vmatrix}$

73. $\begin{vmatrix} 13 & 23 & -21 \\ 17 & -32 & 14 \\ 15 & 12 & -16 \end{vmatrix}$

74. $\begin{vmatrix} 25 & 32 & 17 \\ -13 & 14 & -24 \\ 16 & 26 & 36 \end{vmatrix}$

Use a graphing utility and Cramer's Rule to solve each system of equations.

75. $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$

76. $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

77. $\begin{cases} w + x + y + z = 6 \\ -w + 2x + 3y = 0 \\ 2w - 3x + 4y + z = 4 \\ w + x + 2y - z = 0 \end{cases}$

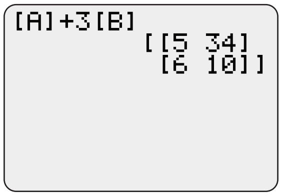
for large n (the value for n at which this happens will vary depending on the technology used). This means that, after a few months, store **A** can count on roughly 60% of the town's customers and store **B** can count on roughly 40% (the actual identities of the customers will keep changing from month to month, but the relative proportions will have stabilized). We can verify that the situation is stable by applying the transition matrix to an assumed 1000 customers split 60 : 40.

$$\begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}\begin{bmatrix} 600 \\ 400 \end{bmatrix} = \begin{bmatrix} 600 \\ 400 \end{bmatrix}$$

Basic Matrix Operations Using Technology

To perform algebra on matrices using the calculator, we must first create the matrices we wish to use by defining them in the MATRIX menu under EDIT. Once the matrices have been defined, we can select them in the MATRIX menu under NAMES and use them in the operations we wish to perform.

For instance, suppose we have two matrices, $A = \begin{bmatrix} -1 & 4 \\ 9 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 10 \\ -1 & 4 \end{bmatrix}$, and we want to find $A + 3B$. After creating matrices A and B, we would enter in the following.



Note that if we try to perform an operation that isn't possible due to dimension size, we will get an error.

6.4 EXERCISES

PRACTICE

Given $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -5 \\ 3 & 0 \\ -2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 \\ 6 & 10 \\ -3 & 7 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 2 & 5 \\ -2 & -4 & 1 \end{bmatrix}$,

determine the following, if possible. See Examples 1, 3, and 4.

1. $3A - B$ **2.** $B - 2D$ **3.** $3C$ **4.** $\frac{1}{2}D$

5. $3D + C$ **6.** $A + B + C$ **7.** $2A + 2B$ **8.** $\frac{3}{2}B + \frac{1}{2}C$

9. $C - 3A$ **10.** $3C - A$ **11.** $4A - 3D$ **12.** $2(A - 3B)$

Determine the values of the variables that will make each of the following equations true, if possible. See Examples 1–4.

13. $\begin{bmatrix} 2a & b & 3 \\ -5 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 3 \\ -5 & 9 & c-3 \end{bmatrix}$

14. $\begin{bmatrix} x \\ -9 \\ -1+z \end{bmatrix} = \begin{bmatrix} 8 \\ 3y \\ 5 \end{bmatrix}$

15. $\begin{bmatrix} a & 2b & c \end{bmatrix} + 3\begin{bmatrix} a & 2 & -c \end{bmatrix} = \begin{bmatrix} 8 & 2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} w & 5x \\ 2y & z \end{bmatrix} - 5\begin{bmatrix} w & x \\ y & -z \end{bmatrix} = \begin{bmatrix} w+5 & 0 \\ 6 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3x \\ 2y \end{bmatrix} + \begin{bmatrix} x \\ -y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

18. $\begin{bmatrix} 2a & 3b & c \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

19. $\begin{bmatrix} x \\ 3x \end{bmatrix} - \begin{bmatrix} y \\ 2y \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix}$

20. $7\begin{bmatrix} -1 \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 5x \end{bmatrix} + 3\begin{bmatrix} y \\ 1 \end{bmatrix}$

21. $2\begin{bmatrix} x \\ 2y \end{bmatrix} - 3\begin{bmatrix} 5y \\ -3x \end{bmatrix} = \begin{bmatrix} -9 \\ 31 \end{bmatrix}$

22. $2\begin{bmatrix} 3r & s & 2t \end{bmatrix} - \begin{bmatrix} r & s & t \end{bmatrix} = \begin{bmatrix} 15 & 3 & 9 \end{bmatrix}$

23. $2\begin{bmatrix} 2x^2 & x \\ 7x & 4 \end{bmatrix} - \begin{bmatrix} 5x \\ x-2 \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 6 & x^2 \end{bmatrix}$

24. $\begin{bmatrix} -x \\ 3 \end{bmatrix} - 5\begin{bmatrix} 2 \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ 3x \end{bmatrix}$

25. $3\begin{bmatrix} 2a \\ -a \end{bmatrix} - 3\begin{bmatrix} 3b \\ 2b \end{bmatrix} = \begin{bmatrix} 3 \\ -54 \end{bmatrix}$

26. $2\begin{bmatrix} -s \\ -7 \end{bmatrix} + 2\begin{bmatrix} -2r \\ r \end{bmatrix} = -2\begin{bmatrix} 8 \\ s \end{bmatrix}$

Evaluate the following matrix products, if possible. See Examples 5 and 6.

27. $\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 0 & 3 \\ 9 & 4 \end{bmatrix}$

28. $\begin{bmatrix} 0 & -8 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 3 & 7 \end{bmatrix}$

29. $\begin{bmatrix} 3 & 7 \end{bmatrix}\begin{bmatrix} 0 & -8 \\ 5 & 6 \end{bmatrix}$

30. $\begin{bmatrix} 5 & 0 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$

31. $\begin{bmatrix} 3 & 9 & -4 \\ 0 & 0 & 2 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}\begin{bmatrix} 5 & 0 & -3 \end{bmatrix}$

33. $\begin{bmatrix} -3 & -6 & -3 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ 6 & -8 \\ -8 & 8 \end{bmatrix}$

34. $\begin{bmatrix} 4 & -5 \\ 7 & -9 \end{bmatrix}\begin{bmatrix} -8 & 3 \end{bmatrix}$

35. $\begin{bmatrix} -3 \\ -5 \\ -6 \end{bmatrix}\begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

Given $A = \begin{bmatrix} -3 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 4 \\ 7 \\ -2 \end{bmatrix}$, and $D = \begin{bmatrix} -5 & 4 \\ -1 & -1 \end{bmatrix}$, determine the following, if possible. See Examples 5 and 6.

36. AB **37.** BA **38.** $BA + B$ **39.** A^2

40. C^2 **41.** CB **42.** D^2 **43.** $CD + C$

44. DA **45.** AD **46.** DB **47.** $(BD)A$

APPLICATIONS

48. Suppose that each month 20% of store B's customers switch to store A, and 10% of store A's customers switch back to store B. At the start of January, store A has 300 customers and store B has 700. How many customers can each store expect at the start of February? At the start of March?

49. Given the percentages stated in the last problem, what long-term proportion of the town's customers can each store expect? (A graphing utility may be used to compute high powers of the transition matrix, or you can use the method described in the following exercise.)

WRITING & THINKING

50. Suppose P is a 2×2 transition matrix, and we want to determine the effect of applying high powers of P to the matrix

$$\begin{bmatrix} x \\ y \end{bmatrix},$$

where $x + y$ is a fixed constant, say c. (In our competing store situation, $x + y = 1000$.) If the long-term behavior approaches a steady state, as in our two-store example, then there is some value for x and some value for y such that $x + y = c$ and

$$P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

In other words, once the steady state has been reached, applying the matrix P to it has no effect on the state.

We can use this fact to actually solve for x and y as follows. Given the matrix

$$P=\begin{bmatrix}0.7 & 0.45\\ 0.3 & 0.55\end{bmatrix},$$

write the equation

$$P\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$$

in system form. You should find that the two equations that result are actually identical. But if we now also use the fact that $x + y = 1000$, we can solve for the variables and find that $x = 600$ and $y = 400$. Verify that this is indeed the case.

51. Your friend Jared is having trouble with matrices, so you offer to study with him. Check his solution of the following problem. If the solution is incorrect, explain the error that has been made.

$$\begin{aligned}
&2\begin{bmatrix}-8 & -9 & -1\\ -8 & 1 & 5\end{bmatrix}-2\begin{bmatrix}-2 & 3\\ 5 & -7\\ -8 & -1\end{bmatrix}\\
&=\begin{bmatrix}-16 & -18 & -2\\ -16 & 2 & 10\end{bmatrix}+\begin{bmatrix}4 & -6\\ -10 & 14\\ 16 & 2\end{bmatrix}\\
&=\begin{bmatrix}-16+4-18-10-2+16 & -16-6-18+14-2+2\\ -16+4+2-10+10+2 & -16-6+2+14+10+2\end{bmatrix}\\
&=\begin{bmatrix}-26 & -26\\ -8 & 6\end{bmatrix}
\end{aligned}$$

TECHNOLOGY

Given $A=\begin{bmatrix}3.8 & -1.2 & 4.6\end{bmatrix}$, $B=\begin{bmatrix}-8.2 & -4.9\\ 7.4 & -1.3\\ 3.5 & -2.1\end{bmatrix}$, $C=\begin{bmatrix}6.3\\ 5.7\end{bmatrix}$, and $D=\begin{bmatrix}2.8 & -7.1\\ -5.4 & 6.6\end{bmatrix}$,

use a graphing utility to determine the following, if possible.

52. BD

53. CA

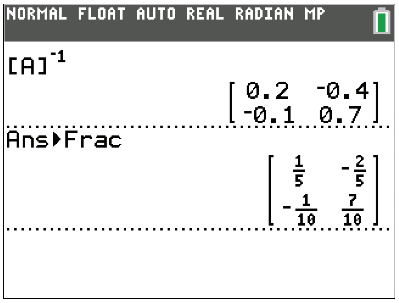
54. D^2

55. AB

56. DC

57. BC

If we defined matrix A to be $\begin{bmatrix} 7 & 4 \\ 1 & 2 \end{bmatrix}$, we would find the following to be its inverse.



6.5 EXERCISES

PRACTICE

Write each of the following systems of equations as a single matrix equation. See Example 1.

1. $\begin{cases} 14x - 5y = 7 \\ x + 9y = 2 \end{cases}$

2. $\begin{cases} x - 5 = 9y \\ 3y - 2x = 8 \end{cases}$

3. $\begin{cases} -6 - 2y = x \\ 9x + 14 = 3y \end{cases}$

4. $\begin{cases} x - y = 5 \\ 2 - z = x \\ z - 3y = 4 \end{cases}$

5. $\begin{cases} 3x_1 - 7x_2 + x_3 = -4 \\ x_1 - x_2 = 2 \\ 8x_2 + 5x_3 = -3 \end{cases}$

6. $\begin{cases} x_3 = x_2 \\ x_2 = x_1 \\ x_1 = x_3 \end{cases}$

7. $\begin{cases} \dfrac{3x - 8y}{5} = 2 \\ y - 2 = 0 \end{cases}$

8. $\begin{cases} x - 7y = 5 \\ \dfrac{6 + x}{2} = 3y - 2 \end{cases}$

9. $\begin{cases} 4x = 3y - 9 \\ 13 - 2x = -4y \end{cases}$

10. $\begin{cases} -\dfrac{7}{3}y = \dfrac{5 - x}{6} \\ x - 5(y - 3) = -2 \end{cases}$

11. $\begin{cases} 2x - y = -3z \\ y - x = 17 \\ 2 + z + 4x = 5y \end{cases}$

12. $\begin{cases} 2x_1 - 3x_3 = 7 \\ x_2 - 10x_3 = 0 \\ 2x_1 - x_2 + x_3 = 1 \end{cases}$

Find the inverse of each of the following matrices, if possible. See Examples 2 and 3.

13. $\begin{bmatrix} 0 & 4 \\ -5 & -1 \end{bmatrix}$

14. $\begin{bmatrix} -2 & -2 \\ -1 & 2 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$

16. $\begin{bmatrix} -1 & -1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$

17. $\begin{bmatrix} -\frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{2} \end{bmatrix}$

18. $\begin{bmatrix} -7 & 2 \\ 7 & -2 \end{bmatrix}$

19. $\begin{bmatrix} -2 & -4 & -2 \\ 1 & -4 & 1 \\ 4 & -3 & 4 \end{bmatrix}$

20. $\begin{bmatrix} -3 & 0 & -4 \\ 2 & 5 & 4 \\ 1 & -5 & -2 \end{bmatrix}$

21. $\begin{bmatrix} -\frac{5}{11} & -\frac{8}{11} & 1 \\ \frac{13}{11} & \frac{12}{11} & -2 \\ -\frac{2}{11} & -\frac{1}{11} & 0 \end{bmatrix}$

22. $-\frac{1}{31}\begin{bmatrix} 17 & -8 & -2 \\ 1 & 5 & 9 \\ -6 & 1 & 8 \end{bmatrix}$ **23.** $\begin{bmatrix} -1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 4 & -1 \end{bmatrix}$ **24.** $\begin{bmatrix} -1 & 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{4} \end{bmatrix}$

25. $\begin{bmatrix} -\frac{6}{5} & -\frac{2}{5} & -1 \\ \frac{3}{5} & \frac{1}{5} & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **26.** $\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & -3 \\ 1 & 0 & 2 \end{bmatrix}$ **27.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

28. $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ **29.** $\begin{bmatrix} \frac{2}{3} & \frac{8}{9} & \frac{1}{9} \\ -\frac{1}{3} & \frac{2}{9} & -\frac{2}{9} \\ -\frac{1}{3} & -\frac{7}{9} & -\frac{2}{9} \end{bmatrix}$ **30.** $\begin{bmatrix} -3 & -3 & -4 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 2 & 2 & 3 \end{bmatrix}$

For each pair of matrices, determine if either matrix is the inverse of the other.

31. $\begin{bmatrix} -5 & -2 \\ -7 & 4 \end{bmatrix}, \begin{bmatrix} 10 & 4 \\ 14 & -8 \end{bmatrix}$ **32.** $\begin{bmatrix} 9 & -18 \\ 3 & 12 \end{bmatrix}, \begin{bmatrix} -3 & -6 \\ -1 & -4 \end{bmatrix}$

33. $\begin{bmatrix} -6 & -1 & 1 \\ 4 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & 3 \\ 2 & 5 & -8 \\ -3 & -7 & 10 \end{bmatrix}$ **34.** $\begin{bmatrix} -1 & 4 & 5 \\ 3 & -11 & -17 \\ 4 & -17 & -19 \end{bmatrix}, \begin{bmatrix} -80 & -9 & -13 \\ -11 & -1 & -2 \\ -7 & -1 & -1 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 2 \\ 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 4 & 0 & -2 \\ 6 & 8 & 4 \\ 2 & 2 & -6 \end{bmatrix}$ **36.** $\begin{bmatrix} -7 & 0 & -2 \\ -10 & -1 & -2 \\ -7 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 4 & -7 & 6 \\ 3 & -7 & 7 \end{bmatrix}$

Solve the following systems by the inverse matrix method, if possible. If the inverse matrix method doesn't apply, use any other method to determine if the system is inconsistent or dependent. See Example 4.

37. $\begin{cases} -2x-2y=9 \\ -x+2y=-3 \end{cases}$ **38.** $\begin{cases} 3x+4y=-2 \\ -4x-5y=9 \end{cases}$ **39.** $\begin{cases} -2x+3y=1 \\ 4x-6y=-2 \end{cases}$

40. $\begin{cases} -2x+4y=5 \\ x-4y=-3 \end{cases}$ **41.** $\begin{cases} -5x=10 \\ 2x+2y=-4 \end{cases}$ **42.** $\begin{cases} -3x+y=2 \\ 9x-3y=5 \end{cases}$

43. $\begin{cases} 8x+2y=26 \\ -16x-2y=-90 \end{cases}$ **44.** $\begin{cases} 3x-7y=-2 \\ -6x+14y=4 \end{cases}$ **45.** $\begin{cases} 3y=15 \\ 8x+4y=20 \end{cases}$

46. $\begin{cases} 4y+3z=-254 \\ 2x-2y-z=100 \\ -x+y-2z=155 \end{cases}$ **47.** $\begin{cases} 2x-y-3z=-10 \\ 2y-z=11 \\ -x+4z=0 \end{cases}$ **48.** $\begin{cases} 3y-4z=15 \\ x+2y-3z=9 \\ -x-y+2z=-5 \end{cases}$

WRITING & THINKING

Solve the following sets of systems by the inverse matrix method. **Hint:** The coefficient matrix is the same for the three systems within a set.

49. $\begin{cases} x+2y-z=2 \\ 3x+3y-z=-5 \\ 4x+4y-z=1 \end{cases}$ $\begin{cases} x+2y-z=1 \\ 3x+3y-z=1 \\ 4x+4y-z=1 \end{cases}$ $\begin{cases} x+2y-z=0 \\ 3x+3y-z=1 \\ 4x+4y-z=1 \end{cases}$

50. $\begin{cases} -x-y-2z=4 \\ x+3y+3z=0 \\ -3y-2z=9 \end{cases}$ $\begin{cases} -x-y-2z=1 \\ x+3y+3z=0 \\ -3y-2z=0 \end{cases}$ $\begin{cases} -x-y-2z=-2 \\ x+3y+3z=-3 \\ -3y-2z=1 \end{cases}$

51. $\begin{cases} -x+z=6 \\ -x+3y+2z=-11 \\ 2x-4y-3z=13 \end{cases}$ $\begin{cases} -x+z=-2 \\ -x+3y+2z=2 \\ 2x-4y-3z=-1 \end{cases}$ $\begin{cases} -x+z=-4 \\ -x+3y+2z=2 \\ 2x-4y-3z=0 \end{cases}$

TECHNOLOGY

Using a graphing utility, find the inverse of each of the following matrices, if possible. Round your answers to three decimal places if necessary.

52. $\begin{bmatrix} -7 & 3 \\ -1 & 2 \end{bmatrix}$

53. $\begin{bmatrix} -6 & 2 \\ -5 & 5 \end{bmatrix}$

54. $\begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

55. $\begin{bmatrix} 2.3 & 7.8 \\ -3.4 & 1.6 \end{bmatrix}$

56. $\begin{bmatrix} 4.5 & -9.4 & 6.9 \\ 8.6 & -2.8 & 1.2 \\ 3.1 & 0.3 & -7.0 \end{bmatrix}$

57. $\begin{bmatrix} 38 & -44 & 72 \\ -93 & 16 & 29 \\ 65 & 23 & -19 \end{bmatrix}$

6.6 EXERCISES

PRACTICE

1. The input-output matrix A and corresponding demand matrix E for a local economy are given.

$$A = \begin{array}{cc} \text{A} & \text{M} \\ \left[\begin{array}{cc} 0.7 & 0.1 \\ 0.2 & 0.6 \end{array}\right] & \begin{array}{l} \text{Agriculture} \\ \text{Manufacturing} \end{array} \end{array}, \quad E = \begin{bmatrix} 8000 \\ 5000 \end{bmatrix}$$

a. Identify the inputs needed from each industry to produce one unit of output from manufacturing.
b. Identify the inputs needed from each industry to produce 3 units of output from agriculture.
c. Find $I - A$ and $(I - A)^{-1}$.
d. Find the production matrix X.

2. The input-output matrix A and corresponding demand matrix E for a local real estate market are given.

$$A = \begin{array}{cc} \text{A} & \text{S} \\ \left[\begin{array}{cc} 0.4 & 0.3 \\ 0.3 & 0.1 \end{array}\right] & \begin{array}{l} \text{Apartments} \\ \text{Single-Family Homes} \end{array} \end{array}, \quad E = \begin{bmatrix} 900 \\ 600 \end{bmatrix}$$

a. Identify the inputs needed from each industry to produce one unit of output from single-family homes.
b. Identify the inputs needed from each industry to produce 4 units of output from apartments.
c. Find $I - A$ and $(I - A)^{-1}$.
d. Find the production matrix X.

3. The input-output matrix A and corresponding demand matrix E for a local economy market are given.

$$A = \begin{array}{cc} \text{A} \quad \text{Mi} \quad \text{Ma} & \\ \left[\begin{array}{ccc} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.3 \end{array}\right] & \begin{array}{l} \text{Agriculture} \\ \text{Mining} \\ \text{Manufacturing} \end{array} \end{array}, \quad E = \begin{bmatrix} 400 \\ 1200 \\ 800 \end{bmatrix}$$

a. Identify the inputs needed from each industry to produce one unit of output from agriculture.
b. Identify the inputs needed from each industry to produce 2 units of output from mining.
c. Find $I - A$ and $(I - A)^{-1}$.
d. Find the production matrix X.

4. The input-output matrix A and corresponding demand matrix E for a local real estate market are given.

$$A = \begin{array}{c} \begin{array}{ccc} \text{S} & \text{M} & \text{R} \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Single-Family} \\ \text{Multi-Family} \\ \text{Rental} \end{array}, \quad E = \begin{bmatrix} 600 \\ 200 \\ 400 \end{bmatrix}$$

a. Identify the inputs needed from each industry to produce one unit of output from rentals.

b. Identify the inputs needed from each industry to produce 3 units of output from single family.

c. Find $I - A$ and $(I - A)^{-1}$.

d. Find the production matrix X.

Given the following input-output matrix A and corresponding production matrix X, find the external demand matrix E.

5. $$A = \begin{array}{c} \begin{array}{cc} \text{H} & \text{HC} \end{array} \\ \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Hospitality} \\ \text{Health Care} \end{array}, \quad X = \begin{bmatrix} 1700 \\ 2900 \end{bmatrix}$$

6. $$A = \begin{array}{c} \begin{array}{cc} \text{C} & \text{M} \end{array} \\ \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Construction} \\ \text{Manufacturing} \end{array}, \quad X = \begin{bmatrix} 2200 \\ 2600 \end{bmatrix}$$

7. $$A = \begin{array}{c} \begin{array}{ccc} \text{C} & \text{O} & \text{G} \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Coal} \\ \text{Oil} \\ \text{Natural Gas} \end{array}, \quad X = \begin{bmatrix} 800 \\ 1200 \\ 600 \end{bmatrix}$$

8. $$A = \begin{array}{c} \begin{array}{ccc} \text{C} & \text{Au} & \text{Ai} \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.1 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Computers} \\ \text{Automobiles} \\ \text{Aircraft} \end{array}, \quad X = \begin{bmatrix} 800 \\ 600 \\ 1000 \end{bmatrix}$$

APPLICATIONS

9. Suppose that in a certain local economy we have manufacturing and agriculture industries. To produce one dollar in output, each industry needs the following input.

- The manufacturing industry requires \$0.10 from itself and \$0.30 from agriculture.
- The agriculture industry requires \$0.30 from manufacturing and \$0.40 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$300 in manufacturing and \$600 in agriculture. Solve for the production necessary to meet these internal and surplus demands.

10. Suppose that in a certain local economy we have natural gas and coal industries. To produce one dollar in output, each industry needs the following input.

- The natural gas industry requires \$0.40 from itself and \$0.30 from coal.
- The coal industry requires \$0.20 from natural gas and \$0.40 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$600 in natural gas and \$900 in coal. Solve for the production necessary to meet these internal and surplus demands.

11. Suppose that in a certain local economy we have computer manufacturing and automobile manufacturing industries. To produce one dollar in output, each industry needs the following input.

- The computer manufacturing industry requires \$0.10 from itself and \$0.20 from automobile manufacturing.
- The automobile manufacturing industry requires \$0.20 from computer manufacturing and \$0.40 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$400 in computer manufacturing and \$800 in automobile manufacturing. Solve for the production necessary to meet these internal and surplus demands.

12. Suppose that in a certain local economy we have coal, oil, and natural gas industries. To produce one dollar in output, each industry needs the following input.

- The coal industry requires \$0.10 from itself, \$0.20 from oil, and \$0.30 from natural gas.
- The oil industry requires no input from coal, \$0.40 from itself, and \$0.20 from natural gas.
- The natural gas industry requires \$0.10 from coal, \$0.20 from oil, and \$0.30 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$10,000 in coal, \$10,000 in oil, and \$10,000 in natural gas. Solve for the production necessary to meet these internal and surplus demands.

13. Suppose that in a certain local entertainment economy we have museums, theaters, and sporting events industries. To produce one dollar in output, each industry needs the following input.

- The museum industry requires \$0.40 from itself, \$0.30 from theater, and \$0.20 from sporting events.

- The theater industry requires \$0.20 from museums, \$0.40 from itself, and \$0.30 from sporting events.

- The sporting events industry requires \$0.20 from museums, \$0.40 from theater, and \$0.30 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$500 in museums, \$750 in theaters, and \$400 in sporting events. Solve for the production necessary to meet these internal and surplus demands.

14. Suppose that in a certain local information economy we have media, data processing, and telephone company industries. To produce one dollar in output, each industry needs the following input.

- The media industry requires \$0.30 from itself, \$0.20 from data processing, and no input from telephone companies.

- The data processing industry requires \$0.40 from media, \$0.20 from itself, and \$0.10 from telephone companies.

- The telephone company industry requires \$0.10 from media, \$0.10 from data processing, and \$0.20 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$300 in media, \$300 in data processing, and \$300 in telephone companies. Solve for the production necessary to meet these internal and surplus demands.

15. Suppose that in a certain local health economy we have curative, rehabilitative, and preventative industries. To produce one dollar in output, each industry needs the following input.

- The curative industry requires \$0.20 from itself, \$0.10 from rehabilitative, and \$0.20 from preventative.

- The rehabilitative industry requires \$0.20 from curative, \$0.30 from itself, and \$0.20 from preventative.

- The preventative industry requires \$0.20 from curative, \$0.40 from rehabilitative, and \$0.20 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$500 in curative, \$1500 in rehabilitative, and \$1000 in preventative. Solve for the production necessary to meet these internal and surplus demands.

16. Suppose that in a certain local economy we have coal, gasoline, electric, and natural gas industries. To produce one dollar in output, each industry needs the following input.

- The coal industry requires \$0.10 from itself, \$0.20 from gasoline, no input from electric, and \$0.20 from natural gas.
- The gasoline industry requires \$0.40 from coal, \$0.10 from itself, no input from electric, and \$0.10 from natural gas.
- The electric industry requires no input from coal, \$0.10 from gasoline, \$0.20 from itself, and \$0.30 from natural gas.
- The natural gas industry requires \$0.20 from coal, \$0.10 from gasoline, no input from electric, and \$0.10 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$300 in coal, \$600 in gasoline, \$600 in electric, and \$900 in natural gas. Solve for the production necessary to meet these internal and surplus demands.

17. Suppose that in a certain local economy we have manufacturing, agricultural, health, and energy industries. To produce one dollar in output, each industry needs the following input.

- The manufacturing industry requires \$0.20 from itself, \$0.20 from agriculture, \$0.20 from health, and \$0.20 from energy.
- The agricultural industry requires \$0.20 from manufacturing, \$0.20 from itself, \$0.20 from health, and \$0.20 from energy.
- The health industry requires \$0.20 from manufacturing, \$0.20 from agriculture, \$0.20 from itself, and \$0.20 from energy.
- The energy industry requires \$0.10 from manufacturing, no input from agriculture, \$0.30 from health, and \$0.40 from itself.

Suppose further that in addition to the internal demand, there is a surplus demand from outside of the industries for \$1000 in manufacturing, \$500 in agriculture, \$500 in health, and \$500 in energy. Solve for the production necessary to meet these internal and surplus demands.

18. Suppose that in a certain local economy we have durable manufacturing and nondurable manufacturing industries. To produce one dollar in output, each industry needs the following input.

- The durable manufacturing industry requires \$0.10 from itself and \$0.30 from nondurable manufacturing.
- The nondurable manufacturing industry requires \$0.30 from durable manufacturing and \$0.60 from itself.

Suppose further that the total production capacity of durable manufacturing is \$500 and of nondurable manufacturing is \$800. Find the external demand.

19. Suppose that in a certain local economy we have agriculture, manufacturing, and energy industries. To produce one dollar in output, each industry needs the following input.

- The agriculture industry requires \$0.30 from itself, no input from manufacturing, and \$0.20 from energy.
- The manufacturing industry requires \$0.10 from agriculture, \$0.40 from itself, and \$0.20 from energy.
- The energy industry requires \$0.30 from agriculture, \$0.40 from manufacturing, and \$0.20 from itself.

Suppose further that the total production capacity of agriculture is \$800, of manufacturing is \$800, and of energy is \$1000. Find the external demand.

20. Suppose that in a certain local economy we have coal, gasoline, electric, and natural gas industries. To produce one dollar in output, each industry needs the following input.

- The coal industry requires \$0.30 from itself, \$0.20 from gasoline, \$0.10 from electric, and \$0.20 from natural gas.
- The gasoline industry requires \$0.20 from coal, \$0.10 from itself, \$0.10 from electric, and \$0.10 from natural gas.
- The electric industry requires no input from coal, \$0.10 from gasoline, \$0.20 from itself, and \$0.30 from natural gas.
- The natural gas industry requires \$0.20 from coal, \$0.10 from gasoline, no input from electric, and \$0.10 from natural gas.

Suppose further that the total production capacity of coal is \$800, of gasoline is \$400, of electric is \$300, and of natural gas is \$1000. Find the external demand.

21. Suppose a closed economy consists of two industries—with x_1 equal to the value of the agriculture output, and x_2 equal to the value of the manufacturing output—and has the following input-output matrix.

$$\begin{array}{cc} & \begin{array}{cc} \text{A} & \text{M} \end{array} \\ A = & \begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix} \begin{array}{l} \text{Agriculture} \\ \text{Manufacturing} \end{array} \end{array}$$

a. Find the productions (outputs of agriculture and manufacturing) for each industry x_1 and x_2.

b. For every \$8 produced by the manufacturing industry, how many dollars' worth of production is needed from the agricultural industry?

22. Suppose a closed economy consists of two industries—with x_1 equal to the value of the energy output, and x_2 equal to the value of the real estate output—and has the following input-output matrix.

$$A = \begin{array}{c} \begin{array}{cc} \text{E} & \text{R} \end{array} \\ \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Energy} \\ \text{Real Estate} \end{array}$$

a. Find the productions (outputs of energy and real estate) for each industry x_1 and x_2.

b. For every \$3 produced by the real estate industry, how many dollars' worth of production is needed from the energy industry?

23. Suppose a closed economy consists of three industries—with x_1 equal to the value of the agriculture output, x_2 equal to the value of the manufacturing output, and x_3 equal to the value of the energy output—and has the following input-output matrix.

$$A = \begin{array}{c} \begin{array}{ccc} \text{A} & \text{M} & \text{E} \end{array} \\ \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.5 & 0.4 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Agriculture} \\ \text{Manufacturing} \\ \text{Energy} \end{array}$$

a. Find the productions (outputs of agriculture, manufacturing, and energy) for each industry x_1, x_2, and x_3.

b. For every \$33 produced by the energy industry, how many dollars' worth of production is needed from the agricultural industry and how much is needed from the manufacturing industry?

WRITING & THINKING

24. Use the given input-output matrix to answer the questions that follow.

$$\begin{array}{c} \begin{array}{cccc} \text{C} & \text{G} & \text{E} & \text{N} \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.4 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Coal} \\ \text{Gasoline} \\ \text{Electric} \\ \text{Natural Gas} \end{array}$$

a. Which industry is most dependent on its own production for its operation?

b. Which industry is least dependent on its own production?

c. Which industry is most dependent on the natural gas industry?

d. Which industry would be most affected by a rise in the cost of coal?

7.1 EXERCISES

PRACTICE

Solve the following linear inequalities by graphing their solution sets. See Example 1.

1. $x-3y<6$

2. $y<2x-1$

3. $x>\frac{3}{4}y$

4. $x-3y\geq 6$

5. $3x-y\leq 2$

6. $\frac{2x-y}{4}>1$

7. $y<-2$

8. $x+1\geq 0$

9. $x+y<0$

10. $x+y>0$

11. $-(y-x)>-\frac{5}{2}-y$

12. $-2y\leq -x+4$

13. $5(y+1)\geq -x$

14. $3x-7y\geq 7(1-y)+2$

15. $x-y<2y+3$

Graph the solution sets that satisfy the following inequalities. See Example 2.

16. $y>-3x-6$ or $y\leq 2x-7$

17. $y\geq -2$ and $y>1$

18. $y\geq -2x-5$ and $y\leq -6x-9$

19. $y\leq 4x+4$ and $y>7x+7$

20. $x-3y\geq 6$ and $y>-4$

21. $x-3y\geq 6$ or $y>-4$

22. $3x-y\leq 2$ and $x+y>0$

23. $x>1$ and $y>2$

24. $x>1$ or $y>2$

25. $x+y>-2$ and $x+y<2$

26. $y>-2$ and $2y>-3x-4$

27. $3y>x+2$ or $4y\leq -x-2$

28. $y\leq -x$ and $2y+3x>-4$

29. $5x+6y<-30$ and $x\geq 2$

30. $6y-2x>-6$ or $y>6$

31. $x>-3$ or $y\geq 4$

32. $-2y<-3x-6$ or $-3y\geq -6x-18$

33. $x<6$ and $x\geq -5$

Graph the solution sets that satisfy the following linear absolute value inequalities. See Example 3.

34. $|x-3|<2$

35. $|x-3|>2$

36. $|3y-1|\leq 2$

37. $|2x-4|>2$

38. $1-|y+3|<-1$

39. $|x+1|<2$ and $|y-3|\leq 1$

40. $|x-3|\geq 1$ or $|y-2|\leq 1$

41. $|x-y|<1$

42. $|x+y|\geq 1$

43. $|4x-2y-3|\leq 5$

44. $|2x-3|\geq 1$ or $|2y+3|\geq 1$

45. $|y-3x|\leq 2$ and $|y|<2$

Match the following inequalities to the appropriate graph.

46. $-8y+5x \geq -8y+5$

47. $x < -2$ and $x \geq -5$

48. $|-7x-4y+23| \leq 16$

49. $y \leq 3x-6$ and $y > 2x-4$

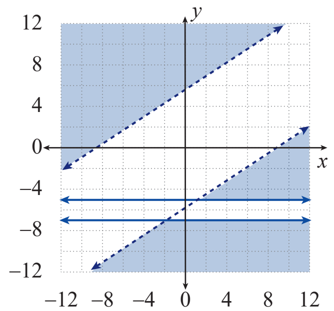
50. $|3y-2x| > 17$ and $|y+6| \geq 1$

51. $4(y+2) < -x$

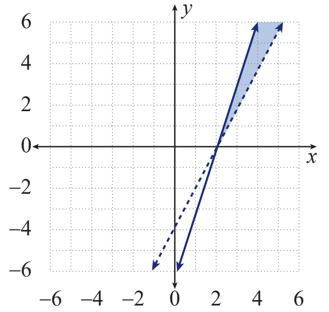
52. $-y < 6x+3$ or $4y \geq 3x-6$

53. $|7x+4| \leq 5$ or $|7y+4| \leq 5$

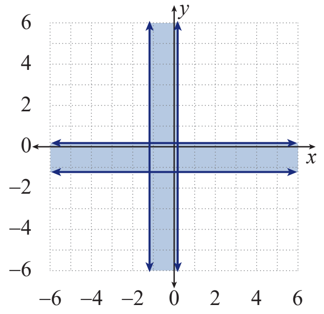
a.



b.



c.



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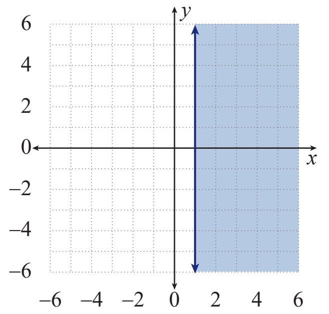

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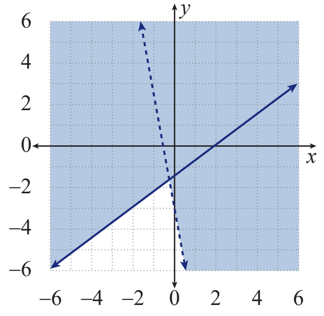
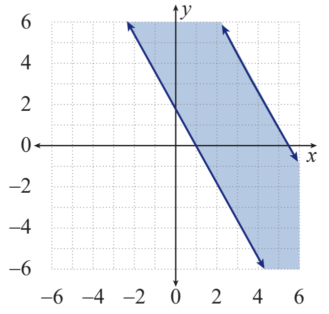
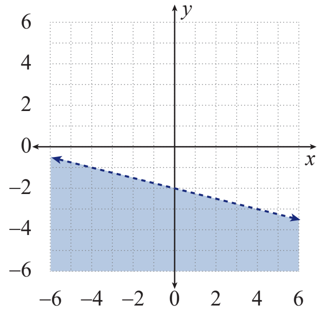


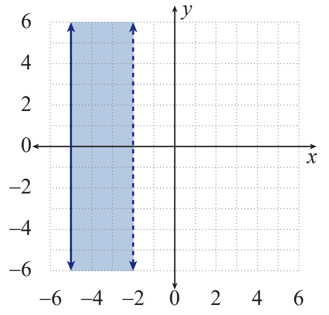
f.



g.



h.



Graph the solution set of each of the following systems of inequalities.

54. $\begin{cases} y \geq -2 \\ y > 1 \end{cases}$

55. $\begin{cases} y \geq -2x-5 \\ y \leq -6x-9 \end{cases}$

56. $\begin{cases} y \leq 4x+4 \\ y > 7x+7 \end{cases}$

57. $\begin{cases} x-3y \geq 6 \\ y > -4 \end{cases}$

58. $\begin{cases} 3x-y \leq 2 \\ x+y > 0 \end{cases}$

59. $\begin{cases} x > 1 \\ y > 2 \end{cases}$

60. $\begin{cases} x+y > -2 \\ x+y < 2 \end{cases}$

61. $\begin{cases} y > -2 \\ 2y > -3x-4 \end{cases}$

62. $\begin{cases} y \leq -x \\ 2y+3x > -4 \end{cases}$

63. $\begin{cases} 5x+6y < -30 \\ x \geq 2 \end{cases}$

64. $\begin{cases} x < 6 \\ x \geq -5 \end{cases}$

65. $\begin{cases} |x+1| < 2 \\ |y-3| \leq 1 \end{cases}$

66. $\begin{cases} |y-3x| \leq 2 \\ |y| < 2 \end{cases}$

APPLICATIONS

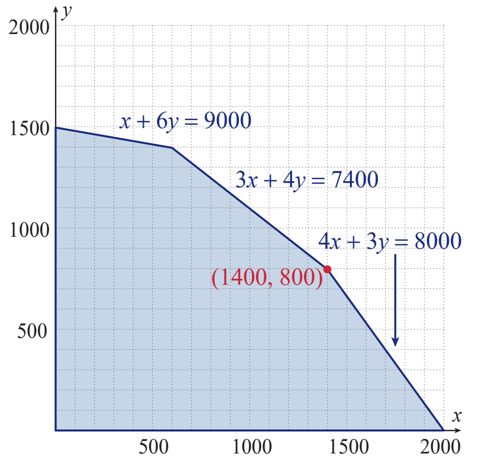
67. It costs Happy Land Toys \$5.50 in variable costs per doll produced. If total costs must remain less than \$200, write a linear inequality describing the relationship between cost and dolls produced.

68. Trish is having a garden party where she wants to have several arrangements of lilies and orchids for decoration. The lily arrangements cost \$12 each and the orchids cost \$22 each. If Trish wants to spend less than \$150 on flowers, write a linear inequality describing the number of each arrangement she can purchase. Graph the inequality.

69. Rob has 300 feet of fencing he can use to enclose a small rectangular area of his yard for a garden. Assuming Rob may or may not use all the fencing, write a linear inequality describing the possible dimensions of his garden. Graph the inequality.

70. Flowertown Canoes produces two types of canoes. The two-person model costs \$73 to produce and the one-person model costs \$46 to produce. Write a linear inequality describing the number of each canoe the company can produce and keep costs under \$1750. Graph the inequality.

Next we sketch the feasible region.



The profit function in this situation is $f(x, y) = 7x + 8y$, and our goal is to maximize this function. The vertices of the feasible region are (0, 1500), (600, 1400), (1400, 800), (2000, 0), and (0, 0). Of course, manufacturing 0 units of Model *A* and Model *B* is not going to be profitable, so we really only need to evaluate *f* at four vertices:

$$f(0,1500) = 12,000$$
$$f(600,1400) = 15,400$$
$$f(1400,800) = 16,200$$
$$f(2000,0) = 14,000$$

From these calculations, we conclude that the maximum profit is generated from making 1400 units of Model *A* and 800 units of Model *B*. The maximum profit would be $16,200.

7.2 EXERCISES

PRACTICE

Find the minimum and maximum values of the given functions, subject to the given constraints. See Examples 2 and 3.

1. Objective Function:
$f(x,y) = 2x + 3y$
Constraints:
$$\begin{cases} x \geq 0, y \geq 0 \\ x + y \leq 7 \end{cases}$$

2. Objective Function:
$f(x,y) = 4x + y$
Constraints:
$$\begin{cases} x \geq 0, y \geq 0 \\ x + y \leq 3 \end{cases}$$

3. Objective Function:
$f(x,y) = 2x + 5y$
Constraints:
$$\begin{cases} x \geq 0, y \geq 0 \\ x + y \leq 7 \end{cases}$$

4. Objective Function:
$f(x,y) = 7x + 4y$
Constraints:
$$\begin{cases} x \geq 0, y \geq 0 \\ 3x + y \leq 3 \end{cases}$$

5. Objective Function:
$f(x,y)=5x+6y$
Constraints:
$$\begin{cases} 0\le x\le 7 \\ 0\le y\le 10 \\ 8x+5y\ge 40 \end{cases}$$

6. Objective Function:
$f(x,y)=9x+7y$
Constraints:
$$\begin{cases} 0\le x\le 20 \\ 0\le y\le 10 \\ 6x+12y\ge 140 \end{cases}$$

7. Objective Function:
$f(x,y)=6x+4y$
Constraints:
$$\begin{cases} 0\le x\le 4 \\ 0\le y\le 5 \\ 4x+3y\ge 10 \end{cases}$$

8. Objective Function:
$f(x,y)=3x+7y$
Constraints:
$$\begin{cases} 0\le x\le 8 \\ 0\le y\le 6 \\ 7x+10y\ge 50 \end{cases}$$

9. Objective Function:
$f(x,y)=6x+8y$
Constraints:
$$\begin{cases} x\ge 0,\, y\ge 0 \\ 4x+y\le 16 \\ x+3y\le 15 \end{cases}$$

10. Objective Function:
$f(x,y)=x+2y$
Constraints:
$$\begin{cases} x\ge 0,\, y\ge 0 \\ 3x+y\le 45 \\ x+3y\le 24 \end{cases}$$

11. Objective Function:
$f(x,y)=6x+y$
Constraints:
$$\begin{cases} x\ge 0,\, y\ge 0 \\ 3x+4y\ge 24 \\ 3x+4y\le 48 \end{cases}$$

12. Objective Function:
$f(x,y)=15x+30y$
Constraints:
$$\begin{cases} x\ge 0,\, y\ge 0 \\ 5x+7y\ge 70 \\ 5x+7y\le 140 \end{cases}$$

13. Objective Function:
$f(x,y)=3x+10y$
Constraints:
$$\begin{cases} x\ge 0 \\ 2x+4y\ge 8 \\ 5x-y\le 10 \\ x+3y\le 40 \end{cases}$$

14. Objective Function:
$f(x,y)=20x+30y$
Constraints:
$$\begin{cases} x\ge 0 \\ 12x+6y\ge 120 \\ 9x-6y\le 144 \\ x+4y\le 12 \end{cases}$$

APPLICATIONS

15. A plane carrying relief food and water can carry a maximum of 50,000 pounds and is limited in space to carrying no more than 6000 cubic feet. Each container of water weighs 60 pounds and takes up 1 cubic foot, and each container of food weighs 50 pounds and takes up 10 cubic feet. What is the region of constraint for the numbers of containers of food and water that the plane can carry?

16. A furniture company makes two kinds of sofas, the Standard model and the Deluxe model. The Standard model requires 40 hours of labor to build, and the Deluxe model requires 60 hours of labor to build. The finish of the Deluxe model uses both teak and fabric, while the Standard uses only fabric, with the result that each Deluxe sofa requires 5 square yards of fabric and each Standard sofa requires 8 square yards of fabric. Given that the company can use 200 hours of labor and 25 square yards of fabric per week building sofas, what is the region of constraint for the numbers of Deluxe and Standard sofas the company can make per week?

17. Sarah is looking through a clothing catalog, and she is willing to spend up to \$80 on clothes and \$10 for shipping. Shirts cost \$12 each plus \$2 shipping, and a pair of pants costs \$32 plus \$3 shipping. What is the region of constraint for the numbers of shirts and pairs of pants Sarah can buy?

18. Suppose you inherit \$75,000 from a previously unknown (and highly eccentric) uncle and that the inheritance comes with certain stipulations regarding investments. First, the dollar amount invested in bonds must not exceed the dollar amount invested in stocks. Second, a minimum of \$10,000 must be invested in stocks, and a minimum of \$5000 must be invested in bonds. Finally, a maximum of \$40,000 can be invested in stocks. What is the region of constraint for the dollar amounts that can be invested in the two categories of stocks and bonds?

19. A manufacturer produces two models of computers. The times (in hours) required for assembling, testing, and packaging each model are listed in the following table.

Process	Model X	Model Y
Assemble	2.5	3
Test	2	1
Package	0.75	1.25

The total times available for assembling, testing, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively. The profits per unit are \$50 for Model X and \$52 for Model Y. How many of each type should be produced to maximize profit? What is the maximum profit?

20. A manufacturer produces two types of fans. The times (in minutes) required for assembling, packaging, and shipping each type are listed in the following table.

Process	Type X	Type Y
Assemble	20	25
Package	40	10
Ship	10	7.5

The total times available for assembling, packaging, and shipping are 4000 minutes, 4800 minutes, and 1500 minutes, respectively. The profits per unit are \$4.50 for Type X and \$3.75 for Type Y. How many of each type should be produced to maximize profit? What is the maximum profit?

21. Ashley is making a set of patchwork curtains for her apartment. She needs a minimum of 16 yards of the solid material, at least 5 yards of the striped material, and at least 20 yards of the flowered material. She can choose between two sets of precut bundles. The olive-based bundle costs \$10 per bundle and contains 8 yards of the solid material, 1 yard of the striped material, and 2 yards of the flowered material. The cranberry-based bundle costs \$20 per bundle and includes 2 yards of the solid material, 1 yard of the striped material, and 7 yards of the flowered material. How many of each bundle should Ashley buy to minimize her cost and yet buy enough material to complete the curtains? What is her minimum cost?

22. A volunteer has been asked to drop off some supplies at a facility housing victims of a hurricane evacuation. The volunteer would like to bring at least 60 bottles of water, 45 first aid kits, and 30 security blankets on his visit. The relief organization has a standing agreement with two companies that provide victim packages. Company A can provide packages of 5 water bottles, 3 first aid kits, and 4 security blankets at a cost of \$1.50. Company B can provide packages of 2 water bottles, 2 first aid kits, and 1 security blanket at a cost of \$1.00. How many of each package should the volunteer pick up to minimize the cost? What total amount does the relief organization pay?

23. On your birthday your grandmother gave you \$25,000, but told you she would like you to invest the money for 10 years before you use any of it. Since you wish to respect your grandmother's wishes, you seek out the advice of a financial adviser. She suggests you invest at least \$15,000 in municipal bonds yielding 6% and no more than \$5000 in Treasury bills yielding 9%. How much should be placed in each investment so that income is maximized?

24. A boutique cell phone manufacturer produces two models: a retro model flip phone and a smart phone. The manufacturer's quota per day is to produce at least 100 flip phones and 80 smart phones. No more than 200 flip phones and 170 smart phones can be produced per day due to limitations on production. A total of at least 200 phones must be shipped every day.

a. If the production costs are \$5 for a flip phone and \$7 for a smart phone, how many of each model should be produced on a daily basis to minimize cost and what would that cost be?
b. If each flip phone results in a \$2 loss but each smart phone results in a \$5 gain, how many of each model should be manufactured daily to maximize profit? What is the maximum profit if this number of phones is produced?

This time, column 2 has a negative entry in the bottom row. Thus, we choose column 2 to be the pivot column. However, this example is different than previous ones because now there are no positive entries in the second column. Thus, the method we used in earlier examples to identify a pivot row won't work here. What does this mean?

Note that the bottom row corresponds to the equation $x_1 - 5x_2 + 2s_1 + z = 4$, or equivalently $z = 4 - x_1 + 5x_2 - 2s_1$. Since increasing x_1 and s_1 will decrease z (and because x_1 and s_1 are nonbasic variables), z will be optimized if $x_1 = 0$ and $s_1 = 0$. The second row corresponds to the equation $-x_1 - 2s_1 + s_2 = 0$, and if we substitute $x_1 = 0$ and $s_1 = 0$ into this equation, we obtain $s_2 = 0$. The first row corresponds to the equation $x_1 - 2x_2 + x_3 + s_1 = 2$, and if we substitute $x_1 = 0$ and $s_1 = 0$ into this equation, we obtain $-2x_2 + x_3 = 2$; or equivalently $x_3 = 2 + x_2$. Going back to $z = 4 - x_1 + 5x_2 - 2s_1$, we see that increasing x_2 will increase the value of z. However, looking at $x_3 = 2 + x_2$, we see that increasing x_2 will increase the value of x_3.

Therefore, if we increase the size of x_2 without bound, we will obtain a value of x_3 that gets infinitely large, and thus $f(x_1, x_2, x_3) = x_1 + x_2 + 2x_3$ gets infinitely large.

So, we conclude that $f(x_1, x_2, x_3)$ has no maximum value because it is unbounded. In addition, we observe that when using the simplex method to solve maximization problems with no solution, there is some point in the process in which a pivot column in a simplex tableau will have no positive entries.

7.3 EXERCISES

PRACTICE

For each given simplex tableau:

- **a.** identify the basic and nonbasic variables,
- **b.** find a basic feasible solution corresponding to this tableau,
- **c.** find the pivot element, and
- **d.** perform one pivot operation.

1. $$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 1 & 1 & 0 & 0 & 6 \\ 1 & 2 & 0 & 1 & 0 & 4 \\ \hline -7 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

2. $$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 0 & 1 & 4 & 0 & 10 \\ 6 & 1 & 0 & 2 & 0 & 15 \\ \hline 9 & 0 & 0 & -8 & 1 & 30 \end{array}\right] \end{array}$$

3. $$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 5 \\ 3 & 0 & 3 & -1 & 1 & 0 & 3 \\ \hline -3 & 0 & -11 & 7 & 0 & 1 & 35 \end{array}\right] \end{array}$$

4. $$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 9 & 0 & 3 & 1 & -1 & 0 & 0 & 54 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 18 \\ 3 & 0 & 7 & 0 & -1 & 1 & 0 & 58 \\ \hline -14 & 0 & -6 & 0 & 6 & 0 & 1 & 108 \end{array}\right] \end{array}$$

For each given maximization problem, find an optimal solution if one exists. If there are multiple solutions, find at least two. If there are no solutions, explain why. See Examples 1, 4, and 5.

5. Maximize $f(x_1,x_2)=5x_1+6x_2$ subject to the following constraints.

$$\begin{cases} x_1+2x_2\le 10 \\ 3x_1+x_2\le 15 \end{cases}$$

6. Maximize $f(x_1,x_2)=3x_1+x_2$ subject to the following constraints.

$$\begin{cases} x_1+4x_2\le 5 \\ 2x_1+x_2\le 3 \\ 3x_1+2x_2\le 5 \end{cases}$$

7. Maximize $f(x_1,x_2)=4x_1+8x_2$ subject to the following constraints.

$$\begin{cases} 3x_1+x_2\le 6 \\ x_1+2x_2\le 4 \end{cases}$$

8. Maximize $f(x_1,x_2,x_3)=5x_1+7x_2+2x_3$ subject to the following constraints.

$$\begin{cases} x_1+2x_2-x_3\le 5 \\ 2x_1+x_2-2x_3\le 4 \end{cases}$$

9. Maximize $f(x_1,x_2,x_3)=5x_1+6x_2+3x_3$ subject to the following constraints.

$$\begin{cases} 3x_1+x_2+x_3\le 18 \\ x_1+4x_2+x_3\le 18 \\ x_1+x_2+2x_3\le 19 \end{cases}$$

10. Maximize $f(x_1,x_2)=7x_1-3x_2$ subject to the following constraints.

$$\begin{cases} x_1+x_2\le 15 \\ 4x_1+2x_2\le 21 \end{cases}$$

11. Maximize $f(x_1,x_2,x_3)=6x_1+5x_2+4x_3$ subject to the following constraints.

$$\begin{cases} 2x_1+x_2\le 46 \\ x_1+3x_3\le 54 \\ x_1+x_2+x_3\le 60 \end{cases}$$

12. Maximize $f(x_1,x_2,x_3)=7x_1+4x_2+x_3$ subject to the following constraints.

$$\begin{cases} 2x_1+x_2+x_3\le 40 \\ x_1+x_2+2x_3\le 30 \end{cases}$$

APPLICATIONS

Use the simplex method to solve each given application. See Examples 2 and 3.

13. A tool company manufactures drills and table saws. Suppose it costs the company \$30 to produce each drill, and the company makes a profit of \$6 for each drill sold. Moreover, it costs the company \$80 to produce each table saw, and the company makes a profit of \$11 for each table saw sold. In addition, suppose the company does not expect the total daily demand for the tools to exceed 100, and the company will halt production for the day once they spend a total of \$4500 in inventory. How many drills and table saws should the company produce and sell to optimize daily profit?

14. A company sells two kinds of fruit juices: Tropical Blend and Florida Sunshine. Suppose the Tropical Blend juice is 40% orange juice and 60% pineapple juice, and the Florida Sunshine juice is 60% orange juice and 40% pineapple juice. The company has a daily supply of 100 gallons of orange juice and 120 gallons of pineapple juice. In addition, suppose the company makes a profit of \$0.50 per gallon of Tropical Blend sold and \$0.40 per gallon of Florida Sunshine sold. How many gallons of Tropical Blend and Florida Sunshine juice should be sold in order for the company to make an optimal daily profit?

15. A company produces and sells two models of roller skates. The following table summarizes how long it takes (in minutes) to assemble and package one pair of each model.

	A	B
Assemble	45	60
Package	10	5

Suppose the company makes a \$3.50 profit for each pair of model A roller skates sold and a \$4 profit for each pair of model B roller skates sold. In addition, suppose the time available for assembling and packaging is 1200 hours and 200 hours, respectively. How many pairs of each model of roller skate should the company sell to maximize profit?

16. A candy store sells 2 kinds of trail mix. The standard trail mix consists of 50% peanuts and 50% M&M's, and the Chocolate Lover's trail mix consists of 37.5% peanuts and 62.5% M&M's. Suppose the candy store has a daily inventory of 100 pounds of peanuts and 150 pounds of M&M's. In addition, suppose the store makes \$0.25 in profit for each pound of standard trail mix sold and \$0.20 in profit for every pound of Chocolate Lover's sold. How many pounds of each kind of trail mix should the store sell to maximize daily profit?

17. A contractor builds two types of homes. Type A requires one lot, \$200,000 in capital, and 165 worker-days of labor. Type B requires one lot, \$250,000 of capital, and 150 worker-days of labor. The contractor owns 170 lots, has \$40,000,000 available in capital, and has 27,600 worker-days of labor. The profit for selling each Type A home is \$45,000, and the profit for each Type B home is \$56,000. How many of each type of home should the contractor build to maximize profit?

18. A shoe manufacturer makes running shoes and basketball shoes. Each pair of running shoes requires 2 units of leather, 5 units of cotton, and 1 unit of rubber. Each pair of basketball shoes requires 4 units of leather, 4 units of cotton, and 2 units of rubber. Shipments are such that leather is limited to 120 units per day, cotton is limited to 150 units per day, and rubber is limited to 80 units per day. If the profits of the shoe manufacturer are \$15 per pair of running shoes and \$25 per pair of basketball shoes, how many pairs of each kind of shoe should be produced to maximize profit? What is the maximum profit?

19. A landlord has a brand new, empty apartment complex with 110 units available for rent; 25 of them have one bedroom, 50 have two bedrooms, and the other 35 have three bedrooms. He has set the rent at \$550 per month for a one-bedroom unit, \$850 per month for two bedrooms, and \$1150 per month for three bedrooms. Assume that he must rent to one person per bedroom, and the laws restrict him to at most 200 occupants in the complex. How many of each type of apartment should the landlord rent to maximize the revenue? What is the maximum revenue?

20. A manufacturer produces 3 models of wooden rocking chairs. The time required for assembling, finishing, and packaging for each model are as follows.

	Model A	**Model B**	**Model C**
Assembling	2 hr 30 min	2 hr 30 min	3 hr 30 min
Finishing	1 hr 30 min	1 hr 20 min	1 hr 30 min
Packaging	30 min	45 min	1 hr

The total time for assembling, finishing, and packaging is 4500 hours, 2200 hours, and 1200 hours, respectively. The profit for each model is \$55 (Model A), \$60 (Model B), and \$77 (Model C). How many of each type should be produced to maximize profit? What is the maximum profit?

21. A furniture store manufactures bookshelves, bed frames, and recliners. Each bookshelf uses 9 units of wood, each bed frame uses 23 units of wood, and each recliner uses 5 units of wood. Each bookshelf uses 1 unit of steel, each bed frame uses 4.5 units of steel, and each recliner uses 7.5 units of steel. It takes 35 minutes to assemble a bookshelf, 80 minutes to assemble a bed frame, and 75 minutes to assemble a recliner. Suppose the store has 600 units of wood and 200 units of steel in its inventory. Also, suppose the total time available for assembly is 3025 minutes. Finally, the price the store sells each item is \$200 (bookshelf), \$550 (bed frame), and \$600 (recliner). How many of each kind of furniture should the store produce to maximize revenue? What is the maximum revenue?

22. A grower has 60 acres of land available for planting 3 crops. It costs \$150 to produce an acre of carrots and the profit is \$50 per acre. It costs \$120 to produce an acre of potatoes and the profit is \$42 per acre. Finally, it costs \$180 to produce an acre of beets and the profit is \$55 per acre. Assume the grower's cost cannot exceed \$8700, and the grower must plant at least 10 acres of beets. How many acres of each crop should the grower plant in order to maximize profit? What is the maximum profit?

23. A shoe company is advertising their latest running shoe. They have a budget of \$8000 per month for advertising. Newspaper ads cost \$120 each and can occur a maximum of 20 times per month. Radio ads cost \$400 each and can occur a maximum of 24 times a month. Suppose each newspaper ad reaches 8000 women under the age of 24 and each radio ad reaches 10,000 women of this age group. If the company wants to maximize its ad exposure to women under 24, how many of each ad should it purchase? What is the maximum number of exposures?

24. A candidate running for president wants to purchase radio and television ads to maximize his exposure. Suppose he has a $1 million budget for ads in his campaign, and election day is 6 months away. Each radio ad costs him $800 and can occur a maximum of 100 times over the next 6 months. Each television ad costs $4000, and can occur a maximum of 300 times over the next 6 months. Suppose each radio ad reaches 20 thousand voters and each television ad reaches 80 thousand voters. How many of each kind of ad should this presidential candidate purchase in order to maximize how many voters he reaches?

25. A researcher is studying how the rabbit population grows in two controlled environments. He wants each male rabbit to be in environment A for 30 minutes and each female rabbit to be in environment A for 20 minutes. He wants each male rabbit to be in environment B for 18 minutes and each female rabbit to be in environment B for 30 minutes. Suppose the researcher has the following time limits: 1040 minutes available for environment A and 750 minutes for environment B. How many male and female rabbits should the researcher use to maximize the total number of rabbits he uses in his study?

26. Biologists at a university have a research pond on their grounds that is stocked with bluegill and bass. Suppose each bluegill consumes 2 units of food per day while each bass consumes 8 units of food per day. In addition, suppose each bluegill produces 1 unit of waste per day while each bass produces 5 units of waste per day. Suppose the total daily supply of food for this pond is no more than 220 units. Moreover, in order to sustain life in the pond, the biologists want no more than a total of 120 units of waste excreted in the pond per day, and the bluegill population should be no more than 50. How many bluegill and bass should the biologists stock in this pond to maximize the number of fish they can study?

27. A botanist is studying how three different types of plant (onion, turnip, and radish) respond to three different controlled environments (A, B, and C). The following table describes how many days each plant is to be tested in each environment.

	Onion	**Turnip**	**Radish**
Environment A	4	1	3
Environment B	2	3	2
Environment C	1	2	1.5

There is limited time available in these environments: the researcher has 27 days for Environment A, 23 days for Environment B, and 15 days for Environment C. How many of each kind of plant should the researcher test in order to maximize the number of plants she is able to test?

28. A natural habitat in a zoo contains several feeding areas for lions, tigers, and bears. There are 3 foods (A, B, and C) available at each of these feeding areas. The following table describes how many pounds of each type of food is required to feed one animal.

	Food A	**Food B**	**Food C**
Lion	4	1	3
Tiger	5	3	2
Bear	3	6	2

Suppose the zoo has the following amounts of food available: 700 pounds (A), 490 pounds (B), and 455 pounds (C). How many of each type of animal can the zoo support so that the number of lions, tigers, and bears is a maximum?

WRITING & THINKING

29. Look once again at Example 4 in this section. Is the constraint $2x_1 + 5x_2 \le 80$ necessary? Explain your answer.

30. Consider a store that manufactures and sells two brands of clocks, say brand A and brand B. Suppose x_1 and x_2 denote the number of brand A and brand B clocks sold, respectively. Suppose the profit to be maximized (in dollars) is $f(x_1, x_2) = 5x_1 + 6x_2$. Moreover, suppose we have the following constraints.

$$\begin{cases} x_1 + 2x_2 \le 15 \\ 3x_1 + 2x_2 \le 26 \end{cases}$$

a. Use the simplex method to find the exact values of x_1 and x_2 that maximize f with the constraints. Do these values make sense in the context of this problem? Explain.

b. If your answer to part a. is no, can you find a solution that is optimal, feasible, and makes sense in the context of this problem? Explain.

31. Recall that it is important to always use the pivot row when doing row operations in any given iteration of the pivoting process. This exercise explores a possible pitfall of failing to do this. Look back at Example 3 of this section. At one stage in this problem, we arrived at the following simplex tableau.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} \frac{6}{7} & \frac{8}{7} & 1 & \frac{4}{7} & 0 & 0 & 0 & \frac{4800}{7} \\ \frac{6}{7} & \frac{1}{7} & 0 & -\frac{10}{7} & 1 & 0 & 0 & \frac{2700}{7} \\ \frac{18}{7} & -\frac{11}{7} & 0 & -\frac{16}{7} & 0 & 1 & 0 & \frac{6840}{7} \\ \hline -\frac{9}{70} & \frac{22}{35} & 0 & \frac{32}{35} & 0 & 0 & 1 & \frac{7680}{7} \end{array}\right]$$

a. Explain why $\frac{18}{7}$ is the pivot element.

b. Since $\frac{6}{7}$ appears in rows 1 and 2 of the first column, the row operation $-R_2 + R_1$ seems like an intuitive first step in introducing a 0 in the top left entry. Also note that it doesn't use the pivot row, R_3. Try the sequence of row operations: $-R_2 + R_1$, $-\frac{1}{3}R_3 + R_1$, $\frac{1}{20}R_3 + R_4$, then $\frac{7}{18}R_3$. Do you obtain the optimal feasible solution of $(x_1, x_2, x_3, s_1, s_2, s_3, z) = (380, 0, 360, 0, 60, 0, 1146)$ with this new tableau?

c. How many basic variables are in the tableau you obtained in part b.? Is the number of basic variables different than 3, which is the number of constraints in this problem?

d. Can you do any additional row operations on the tableau you obtained in part b. to introduce another basic variable and arrive at the optimal feasible solution? Explain your answer.

Simplex Method for Maximization vs. the Duality Principle for Minimization

The following table gives a summary and a comparison of the techniques for maximization and minimization for solving optimization problems.

Simplex Method: Maximization	Duality Principle: Minimization
1. Maximize $f(x_1, x_2, \ldots, x_n)$ subject to constraints of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \le c$.	**1.** Minimize $f(x_1, x_2, \ldots, x_n)$ subject to constraints of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \ge c$.
2. Introduce slack variables $s_1, s_2, \ldots, s_m$, and form the initial simplex tableau.	**2.** Write down the matrix for minimization with constraints, A.
3. Repeat the pivoting process until an optimal solution is reached.	**3.** Write down the matrix for the dual maximization problem, A^T.
4. Look at the final simplex tableau. The maximum value of f is in the lower-right corner. The nonzero coordinates at which the maximum is attained are in the last column in the rows corresponding to the 1 entry of each x_i that is basic, and each x_i that is nonbasic is 0.	**4.** Introduce slack variables $s_1, s_2, \ldots, s_m$, and form the initial simplex tableau.
	5. Repeat the pivoting process until an optimal solution to the dual problem is reached.
	6. Look at the final simplex tableau. The minimum value of f is in the lower-right corner. The coordinates at which the minimum is attained are in the last row of the columns corresponding to the slack variables.

7.4 EXERCISES

PRACTICE

For each given matrix A, find the transpose A^T.

1. $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 6 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 \\ 6 & -5 \\ 7 & 2 \\ -1 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 6 & 10 & -3 & 8 \\ 4 & -1 & 3 & 7 \end{bmatrix}$

For each given (primal) minimization problem, state the dual maximization problem.

5. Minimize $f(x_1, x_2) = 7x_1 + 2x_2$ subject to the following constraints.

$$\begin{cases} 3x_1 + 5x_2 \geq 18 \\ 2x_1 + 6x_2 \geq 15 \end{cases}$$

6. Minimize $f(x_1, x_2) = 3x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} x_1 \geq 3 \\ 2x_1 + x_2 \geq 5 \end{cases}$$

7. Minimize $f(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + 3x_2 \geq 16 \\ 5x_1 + 4x_3 \geq 20 \end{cases}$$

8. Minimize $f(x_1, x_2, x_3) = 3x_1 + 4x_2 + 5x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \geq 8 \\ 2x_1 + 2x_2 + x_3 \geq 14 \\ x_1 + 3x_2 + x_3 \geq 12 \end{cases}$$

Solve each given minimization problem. See Example 1.

9. Minimize $f(x_1, x_2) = 3x_1 + x_2$ subject to the following constraints.

$$\begin{cases} 4x_1 + 5x_2 \geq 13 \\ 2x_1 + x_2 \geq 5 \end{cases}$$

10. Minimize $f(x_1, x_2) = 2x_1 + 4x_2$ subject to the following constraints.

$$\begin{cases} 3x_1 + x_2 \geq 6 \\ x_1 + x_2 \geq 4 \end{cases}$$

11. Minimize $f(x_1, x_2, x_3) = x_1 + 2x_2 + x_3$ subject to the following constraints.

$$\begin{cases} x_1 + 3x_2 \geq 6 \\ 2x_1 + x_3 \geq 10 \end{cases}$$

12. Minimize $f(x_1, x_2, x_3) = 11x_1 + 8x_2 + 7x_3$ subject to the following constraints.

$$\begin{cases} x_1 + 2x_2 + 2x_3 \geq 8 \\ x_1 + 2x_2 + x_3 \geq 6 \\ 4x_1 + x_2 + x_3 \geq 4 \end{cases}$$

13. Minimize $f(x_1, x_2, x_3) = 7x_1 + 10x_2 + 7x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + 2x_3 \geq 5 \\ x_1 + 3x_2 \geq 6 \\ 2x_1 + x_2 + x_3 \geq 4 \end{cases}$$

14. Minimize $f(x_1, x_2) = 3x_1 + 4x_2$ subject to the following constraints.

$$\begin{cases} 2x_1 + 2x_2 \geq 7 \\ 2x_1 + 3x_2 \geq 10 \end{cases}$$

15. Minimize $f(x_1, x_2, x_3) = 8x_1 + 3x_2 + 7x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \geq 9 \\ 2x_1 + x_2 + 2x_3 \geq 11 \\ x_1 + 3x_2 + 3x_3 \geq 17 \end{cases}$$

16. Minimize $f(x_1, x_2) = 6x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 \geq 8 \\ 3x_2 \geq 20 \\ -x_1 + x_2 \geq 1 \end{cases}$$

APPLICATIONS

Use the simplex method to solve each given application. See Examples 2 and 3.

17. A tool company produces wrenches and screwdrivers at factories in two locations (plant A and plant B). It has orders to produce 12,000 wrenches and 10,000 screwdrivers. For each hour, plant A can produce 400 wrenches and 200 screwdrivers, and the operation costs are $1200 per hour. At plant B, 200 wrenches and 500 screwdrivers can be produced in an hour, and the operation costs are $1500 per hour. How many hours should the company run each plant to fill the orders at a minimum cost? What is the minimum cost?

18. An electronics factory produces televisions and DVD players on two assembly lines. Line 1 can produce 50 televisions and 75 DVD players at a cost of $225 per hour. Line 2 can produce 60 televisions and 60 DVD players at a cost of $210 per hour. The factory needs to produce at least 300 televisions and 375 DVD players to fill an order. How many hours should each line be run to fill the order at a minimum cost? What is the minimum cost?

19. An athlete drinks two kinds of dietary drinks to help maintain a healthy diet. Drink I has 3 units of protein, 3 units of carbohydrates, and 2 units of vitamin D per one-quart (32 oz) bottle. Drink II has 2 units of protein, 3 units of carbohydrates, and 5 units of vitamin D per bottle. Suppose each bottle of drink I costs $8 and each bottle of drink II costs $12. Finally, suppose the athlete wants to consume a minimum of 7 units of protein per day, 9 units of carbohydrates per day, and 12 units of vitamin D per day. Find the combination of drinks that the athlete should purchase and consume per day in order to minimize the cost while meeting the dietary requirements.

20. A swine farmer is considering two different types of feed for fattening his pigs. Brand 1 feed costs 40 cents per pound and brand 2 feed costs 35 cents per pound. Brand 1 feed contains 10 units of fiber and 4 units of protein per pound. Brand 2 feed has 12 units of fiber and 3 units of protein per pound. Suppose the farmer would like his pigs to consume at least 160 units of fiber and 55 units of protein for each feeding. How many pounds of each feed should the farmer buy per feeding to satisfy the nutritional requirements at a minimum cost?

21. A petroleum company owns 3 refineries (A, B, and C), and each one produces 3 grades of oil (5W30, 10W30, and 10W40). The following table summarizes how many barrels of each kind of oil per day each refinery produces for the petroleum company.

	5W30	10W30	10W40
Refinery A	300	300	200
Refinery B	200	400	250
Refinery C	250	300	450

Suppose the daily costs to operate the refineries are $25,000 (A), $30,000 (B), and $40,000 (C). Also, suppose the petroleum company has orders totaling 5300 barrels of 5W30, 6900 barrels of 10W30, and 6500 barrels of 10W40. How many days should the petroleum company run each of the three refineries so that the costs of operation are minimized and yet the orders are fulfilled?

22. Three factories dump waste water containing three kinds of pollutants into a lagoon. In order to reduce the pollution levels, the factories must treat the waste water they dump. The following table shows the possible percent reduction of each pollutant at each factory.

	Factory 1	Factory 2	Factory 3
Pollutant 1	40%	10%	30%
Pollutant 2	50%	15%	20%
Pollutant 3	30%	25%	35%

Suppose the costs to dump each ton of waste at each factory is as follows: \$100 (factory 1), \$40 (factory 2), and \$60 (factory 3). Moreover, suppose the state requires a reduction of at least 110 tons per day of pollutant 1, at least 120 tons per day of pollutant 2, and at least 122 tons per day of pollutant 3. Find the number of tons of waste that must be treated each day in order to minimize the cost of treatment. What is the minimum cost?

23. Biologists in a lab feed two different foods (I and II) to mice; each food contains three ingredients (A, B, and C). Food I has 6 units of ingredient A per gram, 2 units of ingredient B per gram, and 1 unit of ingredient C per gram. Food II has 4 units of ingredient A per gram, 8 units of ingredient B per gram, and 3 units of ingredient C per gram. Suppose that the lab requires the mice to have consumed a total of least 60 grams of ingredient A and 50 grams of ingredient B. Also suppose that ingredient C can be harmful to the mice. How many grams of each type of food should be given to the mice to meet the requirements for ingredients A and B, while at the same time minimizing the amount of ingredient C?

24. A nutritionist advises an individual who is suffering from iron and vitamin B deficiency to take at least 3000 milligrams (mg) of iron, 2000 mg of vitamin B1, and 1500 mg of vitamin B2 over a period of time. Two vitamin pills are suitable, brand A and brand B. Each brand A pill costs 10 cents and contains 50 mg of iron, 15 mg of vitamin B1, and 10 mg of vitamin B2. Each brand B pill costs 15 cents and contains 10 mg of iron, 28 mg of vitamin B1, and 22 mg of vitamin B2. What combination of pills should the individual purchase in order to meet the minimum iron and vitamin requirements at the lowest cost?

25. A candidate wishes to use a combination of radio and television advertisements in his campaign. Suppose that every one-minute spot on the radio reaches 10 thousand voters, and every one-minute spot on television reaches 70 thousand voters. The candidate feels that he needs to reach at least 3 million voters, and he must buy at least 90 minutes of advertisements. If a radio ad costs \$75 per minute, and a television ad costs \$450 per minute, how many minutes of each medium should the candidate use to minimize the cost of advertisements?

26. Biologists at a university have a research pond on their grounds that is stocked with bass and catfish. Suppose each bass consumes 8 units of food per day while each catfish consumes 20 units of food per day. In addition, suppose each bass produces 6 units of waste per day while each catfish produces 14 units of waste per day. Suppose the biologists require that the total amount of food the fish eat in this pond is at least 220 units. In addition, the biologists want to keep at least 20 fish in this pond. How many bass and catfish should the biologists stock in this pond to minimize the amount of waste produced?

27. A dieting company offers three types of packaged meals (I, II, and III), and it groups its customers into three groups (A, B, and C) depending on the diet. The following table gives the percent of the daily nutritional requirements that a serving of each meal provides and the number of ounces of detrimental substances in each serving of food.

	Meal I	Meal II	Meal III
Group A	20% per serving	30% per serving	10% per serving
Group B	30% per serving	10% per serving	10% per serving
Group C	10% per serving	20% per serving	20% per serving
Detrimental Substances	0.4 oz per serving	0.5 oz per serving	0.2 oz per serving

Determine the combination of food types that will provide at least 100% of the daily requirements for each group and will minimize the amount of detrimental substances.

WRITING & THINKING

28. Suppose we want to minimize $f(x_1, x_2) = 4x_1 + 6x_2$ subject to the following constraints.

$$\begin{cases} 2x_1 + 2x_2 \geq 5 \\ x_1 - 3x_2 \leq -6 \end{cases}$$

a. Notice that this optimization problem is different because one of the constraints has the $\leq$ inequality. Rewrite the second constraint so that it has the $\geq$ inequality.

b. Use the methods of this section to solve this problem.

29. Suppose we want to minimize $f(x_1, x_2) = 4x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} -3x_1 + 6x_2 \geq 1 \\ x_1 - 2x_2 \geq 3 \\ x_1 + x_2 \geq 2 \end{cases}$$

a. Try using the methods of this section to solve this problem. What do you find?

b. Since there are only two decision variables, draw a graph of the feasible region. Then use the graphical approach to solve this problem.

30. Suppose we want to minimize $f(x_1, x_2) = 6x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} 3x_1 + 4x_2 \geq 7 \\ 2x_1 + x_2 \geq 3 \end{cases}$$

Find two solutions to this problem. (**Hint:** At some point, you should obtain a tableau in which you can either choose row 1 or row 2 as the pivot row. Try each one.)

Simplex Method: Maximization	Duality Principle: Minimization
1. Maximize objective function f subject to constraints involving $\leq$.	**1.** Minimize objective function f subject to constraints involving $\geq$.
2. Form the initial simplex tableau.	**2.** Write down the matrix for minimization with constraints, A.
3. Apply the simplex method to obtain an optimal solution.	**3.** Write down the matrix for the dual maximization problem, A^T.
	4. Form the initial simplex tableau for the dual problem.
	5. Apply the simplex method to obtain an optimal solution. Use entries in the last row of columns corresponding to slack variables for the coordinates.

TABLE 2

7.5 EXERCISES

PRACTICE

For each given system of inequalities (constraints), rewrite as a system of equations by adding slack or subtracting surplus variables.

1. $\begin{cases} 5x_1 + x_2 \leq 15 \\ 3x_1 + 2x_2 \geq 7 \end{cases}$

2. $\begin{cases} 2x_1 + 3x_2 \leq 7 \\ 4x_1 - x_2 \geq 5 \\ x_1 + x_2 \leq 4 \end{cases}$

3. $\begin{cases} 8x_1 + 3x_2 - x_3 \geq 10 \\ 6x_2 - x_3 \geq 7 \\ x_1 + x_2 + x_3 \leq 8 \end{cases}$

4. $\begin{cases} x_1 + x_2 + 2x_3 \leq 12 \\ 2x_1 + x_2 + 3x_3 \leq 17 \\ x_1 - x_2 + x_3 \geq 7 \\ 2x_2 \leq 5 \end{cases}$

5. $\begin{cases} 4x_1 - x_3 \geq 9 \\ x_2 + 2x_3 \leq 13 \\ 5x_1 - x_2 + 4x_3 \geq 14 \\ 3x_1 + x_2 + x_3 \leq 17 \end{cases}$

6. Suppose you want to minimize $f(x_1, x_2) = x_1 + 3x_2$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 \leq 5 \\ 2x_1 - x_2 \geq 3 \end{cases}$$

Rewrite this as a maximization problem and rewrite the system of constraints using slack variables as in the simplex method.

7. Suppose you want to minimize $f(x_1, x_2, x_3) = 4x_1 + 7x_2 + 9x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \leq 9 \\ -x_1 + 3x_3 \geq 6 \\ 5x_1 - x_2 - x_3 \geq 7 \end{cases}$$

Rewrite this as a maximization problem and rewrite the system of constraints using slack variables as in the simplex method.

8. Consider again the optimization problem in Exercise 6. Rewrite this as a maximization problem, but this time rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

9. Consider again the optimization problem in Exercise 7. Rewrite this as a maximization problem, but this time rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

10. Suppose you want to minimize $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 6x_3$ subject to the following constraints.

$$\begin{cases} 3x_1 + 2x_2 + x_3 \le 8 \\ 2x_1 + x_3 = 5 \\ -x_1 - x_2 + 4x_3 \ge 7 \end{cases}$$

Rewrite this as a maximization problem, and rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

Use either the simplex method or the big M method to solve each given optimization problem. See Examples 1, 2, and 3.

11. Minimize $f(x_1, x_2) = 4x_1 + 6x_2$ subject to the following constraints.

$$\begin{cases} 2x_1 + 3x_2 \le 10 \\ 4x_1 - x_2 \ge 4 \end{cases}$$

12. Maximize $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 4x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 + x_3 \le 7 \\ 3x_1 - x_2 + 2x_3 \ge 4 \\ x_1 + x_3 \ge 5 \end{cases}$$

13. Maximize $f(x_1, x_2, x_3) = 2x_1 - 2x_2 + 6x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 \le 20 \\ x_1 + x_3 = 5 \\ x_2 + x_3 \ge 10 \end{cases}$$

14. Minimize $f(x_1, x_2) = 3x_1 + 4x_2$ subject to the following constraints.

$$\begin{cases} 4x_1 + 5x_2 \ge 18 \\ x_1 + x_2 = 4 \end{cases}$$

15. Minimize $f(x_1, x_2, x_3) = 7x_1 + 2x_2 + 3x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \ge 6 \\ 3x_1 - x_2 + 2x_3 = 7 \\ -x_1 + x_3 \le 2 \end{cases}$$

16. Minimize $f(x_1, x_2) = 3x_1 + 5x_2$ subject to the following constraints.

$$\begin{cases} x_1 + 3x_2 \le 8 \\ 3x_1 - x_2 = 5 \end{cases}$$

17. Maximize $f(x_1, x_2, x_3) = 6x_1 + 4x_2 + 5x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 + 3x_3 \le 18 \\ x_1 + 2x_2 + 2x_3 \le 16 \\ x_1 + x_2 + x_3 = 10 \end{cases}$$

18. Minimize $f(x_1, x_2, x_3) = 10x_1 + 8x_2 + 15x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 - x_3 \ge 6 \\ x_1 + x_2 + 2x_3 = 6 \\ -x_1 + 4x_2 - x_3 \ge 24 \end{cases}$$

APPLICATIONS

Use either the simplex method or the big M method to solve each given application. See Example 4.

19. A truck driver is tasked with delivering stereos for an electronics company. Suppose this company has stereos at two warehouses; the warehouse in Charlotte has 500 stereos, and the warehouse in Charleston has 200 of them. A store in Raleigh needs 350 stereos, and a store in Spartanburg needs 250 stereos. Suppose the truck driver earns \$5 in profit per stereo delivered to Raleigh from Charlotte and \$4 for each stereo delivered to Spartanburg from Charlotte. Moreover, he earns \$7 for each stereo delivered to Raleigh from Charleston and \$5 for each stereo delivered to Spartanburg from Charleston. How many stereos should the truck driver deliver to each store from each warehouse to maximize his earnings?

20. A Mexican food truck produces and sells beef tacos and beef burritos. Suppose each taco uses 8 ounces of beef and 1 ounce of cheese, and suppose each burrito uses 12 ounces of beef and 5 ounces of cheese. The supply of beef is limited to 1180 ounces, but the supply of cheese is plentiful, and the workers on the truck wish to use up at least 200 ounces of cheese to minimize waste. If each taco sells for \$6 and each burrito sells for \$7, how many of each kind of food should the truck produce to maximize revenue?

21. A car company has two factories. Factory A has 600 cars (of a particular model) in stock, and factory B has 400 cars (of that same model) in stock. Two dealerships order this car model. Dealership I needs 300 cars and dealership II needs 400 cars. The cost of shipping per car from the two factories to the dealerships is given below.

	Dealership I	**Dealership II**
Factory A	\$40	\$30
Factory B	\$35	\$45

How should the company ship the cars from the two factories to the dealerships in order to minimize the shipping cost? (**Hint:** Use an approach similar to the one in Exercise 19.)

22. An ice cream company sells ice cream bars and ice cream sandwiches. One of its plants makes each treat and packs the bars and sandwiches into 12-count packages. There are enough lines running at this plant so that each package of bars takes 2 seconds to produce, and each package of sandwiches takes 5 seconds. Suppose the plant has at most 15 hours (54,000 seconds) available each day for producing and packaging the ice cream products. Also, limitations and product demand indicate that the number of packages bars plus the number of packages of sandwiches should be at least 18,000, and the number of bars should be at least 1.5 times the number of sandwiches. If the plant's manufacturing costs are \$2.50 for each package of bars and \$2 for each package of sandwiches, how many packages of each ice cream product should the plant produce to satisfy the constraints at a minimum cost?

23. Three water purification facilities can handle at most 8 million gallons during a certain time period. Plant I leaves 11% of certain impurities and costs $40,000 per million gallons. Plant II leaves 8% of certain impurities and costs $50,000 per million gallons. Plant III leaves 5% of certain impurities and costs $70,000 per million gallons. The desired level of impurities in the water from all plants combined is at most 8%. If plants I and III must handle at least 5 million gallons, find the number of gallons each plant should handle so the water is treated at a minimum cost.

24. A silverware manufacturer produces forks, knives, and spoons. The manager feels that restricting the types of silverware produced could increase revenue. The following table gives the price of each piece of silverware, the raw materials cost for each piece, and the profit made.

	Price	Materials Cost	Profit
Knives	$4	$0.10	$0.60
Forks	$6	$0.10	$0.25
Spoons	$5	$0.15	$1.10

The monthly revenue must be at least $12,000 and the raw material costs should be no more than $280. Also, suppose there should be at least as many forks produced as spoons. How many of each type of silverware should be produced to maximize profit?

25. A candidate for president has budgeted a maximum of $900,000 for political advertisements during her presidential campaign. Suppose each minute of television time costs $30,000 and each one-page newspaper ad costs $12,000. Each television ad is expected to be viewed by 2 million voters, and each newspaper ad is expected to be seen by 200 thousand voters. The candidate is advised to use at least 8 television ads and at least 5 newspaper ads. How should the candidate allocate the advertising budget to reach as many voters as possible?

26. An aquarium contains several feeding areas for dolphins, sharks, and stingrays. There are 3 foods (A, B, and C) available at each of these feeding areas. The following table describes how many pounds of each type of food is required to feed one animal.

	Food A	Food B	Food C
Dolphin	3	1	2
Shark	2	6	6
Stingray	1	2	2

Suppose the aquarium has 300 pounds of food A available and 400 pounds of food B available. Also suppose the aquarium has a surplus of food C and would like to use at least 450 pounds of it. How many of each type of animal can the aquarium support so that the number of dolphins, sharks, and stingrays is a maximum?

27. A family has a side business of selling three kinds of snow cones from their trailer: Tropical Paradise, Arctic Chill, and Paradise Blast. Each Tropical Paradise snow cone uses 3 ounces of mango syrup, 1 ounce of blueberry syrup, and 8 ounces of shaved ice. Each Arctic Chill snow cone uses 3 ounces of blueberry syrup and 8 ounces of shaved ice. Each Paradise Blast snow cone uses 2 ounces of mango syrup, 2 ounces of blueberry syrup, and 8 ounces of shaved ice. Suppose 225 ounces of blueberry syrup and 52.5 pounds (840 ounces) of shaved ice are available, and suppose that all of the ice has to get used up. In addition, suppose there is plenty of mango syrup in stock, and the family would like to use at least 150 ounces of it. If the family makes a profit of \$0.40 for each Tropical Paradise snow cone, \$0.60 profit for each Arctic Chill snow cone, and \$0.50 profit for each Paradise Blast snow cone, how many of each kind of snow cone should the family sell in order to maximize their profit?

28. A company produces three kinds of 3-ring binders depending on their thickness: 1.5-inch, 2-inch, and 3-inch. The number of units of steel, cardboard, and plastic needed to make each kind of 3-ring binder are in the table below.

	1.5-inch	**2-inch**	**3-inch**
Steel	2	3	6
Cardboard	10	18	30
Plastic	8	12	16

Suppose the company wants to use no more than 280 units of steel, no less than 1520 units of cardboard, and exactly 1000 units of plastic per day. If it costs the company \$2 to produce each 1.5-inch binder, \$4 to produce each 2-inch binder, and \$6 to produce each 3-inch binder, then how many of each kind of binder should the company produce in order to minimize cost?

29. A plant that makes popsicles has a mixing tank. Currently, it contains 60 gallons of a mixture that is 15% orange syrup, 18% high fructose corn syrup, and 41% water. The foreman at the plant wants to add more mixture to the tank before the lines run out of fluid to make the popsicles. He has 3 different mixes available. Mix I contains 20% orange syrup, 40% high fructose corn syrup, and 35% water. Mix II contains 10% orange syrup, 30% high fructose corn syrup, and 40% water. Mix III contains 25% orange syrup, 40% high fructose corn syrup, and 15% water. The desired final mixture should have at most 18% orange syrup, exactly 30% high fructose corn syrup, and at least 35% water. If the costs of each mix per gallon are \$60 (I), \$45 (II), and \$30 (III), how many gallons of each mix should the foreman use to minimize the cost? **Hint:** If the final mix needs at least 35% water, that means

$$\begin{aligned}&41\%(\text{mix in tank})+35\%(\text{mix I})+40\%(\text{mix II})+15\%(\text{mix III})\\&\qquad\geq 35\%(\text{mix in tank}+\text{mix I}+\text{mix II}+\text{mix III}).\end{aligned}$$

30. An eccentric barbecue restaurant produces and sells three kinds of bratwursts well-known to the locals in the area: Plain Jane, Spicy Samantha, and Cheezy Chelsea. All of these bratwursts consist of pork, spices, and cheese. The following table summarizes how much of each ingredient are in each kind of bratwurst.

	Plain Jane	Spicy Samantha	Cheezy Chelsea
Pork	8 oz	6 oz	7 oz
Spices	1 oz	3 oz	1 oz
Cheese	1 oz	1 oz	2 oz

Suppose the restaurant has 200 pounds (3200 ounces) of pork available, and they want to use at least 800 ounces of spices. Suppose they also have 600 ounces of cheese and must use all of it due to an obscure health code regulation preventing the restaurant from having any traces of unused cheese. Finally, suppose each Plain Jane brings in a profit of \$0.70, each Spicy Samantha brings in a profit of \$0.50, and each Cheezy Chelsea brings in a profit of \$0.60. How many of each kind of sausage should the restaurant produce and sell to maximize its profit?

WRITING & THINKING

31. Use the big M method to try and maximize $f(x_1, x_2) = 3x_1 + 9x_2$ subject to the following constraints.

$$\begin{cases} x_1 + 3x_2 = 7 \\ 2x_1 + x_2 \le 5 \end{cases}$$

What do you find?

32. Use the big M method to try and maximize $f(x_1, x_2) = x_1 + 2x_2$ subject to the following constraints.

$$\begin{cases} x_1 - x_2 \le 5 \\ 2x_1 - x_2 = 20 \end{cases}$$

What do you find?

33. Consider the problem of minimizing $f(x_1, x_2) = 2x_1 + 4x_2$ subject to the following constraints.

$$\begin{cases} 3x_1 + x_2 = 6 \\ x_1 + x_2 \ge 4 \end{cases}$$

a. Use the big M method to solve.

b. According to the first constraint, $3x_1 + x_2 = 6$ holds, and therefore the inequality $3x_1 + x_2 \ge 6$ must also hold. It seems intuitive that we could use the Theorem of Duality (instead of the big M method) to solve this problem if we replace the first constraint with $3x_1 + x_2 \ge 6$. Try it. Do you obtain the same solution as in part a.? If you obtain a different solution, can you explain why?

8.1 EXERCISES

✔ CONCEPT CHECK

Determine whether each statement is true or false. If the statement is false, explain why.

1. It is always possible to list every element of a set using the roster method.

2. $\varnothing \sim \{0\}$

3. Let $U = \{\text{set of all students enrolled at Xavier University}\}$ and $A = \{x \mid x \in U \text{ and } x \text{ is a student with less than 30 earned credit hours}\}$. Then $A' = \{x \mid x \in U \text{ and } x \text{ is a student with at least 30 earned credit hours}\}$.

4. $0 \in \varnothing$

5. $x \in \{x, y, z\}$

6. $\{x\} \in \{x, y, z\}$

7. $|\varnothing| = 0$

8. Let $Y = \{\text{Tim, Emilia, Aleesa, Whit}\}$, then $|Y| = 4$.

9. Let $A = \{\text{Sounds, Angels, Ravens, Titans}\}$ and $B = \{201, 38, 46, 23\}$, then $A \sim B$.

PRACTICE

Write each set using the roster method.

10. A is the set of months of the year that have exactly 30 days.

11. B is the set of states whose names begin with the letter N.

12. C is the set of positive numbers smaller than 100 that have 2 digits that are the same.

13. D is the set of last names of people who teach this course at your school.

14. E is the set of planets in our solar system.

15. F is the set of weekdays.

Write each set using set-builder notation.

16. Let G be the set of whole numbers.

17. Let H be the set of natural numbers less than or equal to 50.

18. Let U be the set of all the states in the United States of America and J be the set of all states that border an ocean.

19. Let U be the set of all students enrolled at a school of higher education and K be the set of all collegiate athletes.

Write a description for each set. It is possible for more than one description to be correct.

20. $J = \{-11, -9, -7, -5, -3, -1\}$

21. $K = \{\$1, \$2, \$5, \$10, \$20, \$50, \$100\}$

22. $M = \{\text{Saturday, Sunday}\}$

23. $N = \{\text{January, June, July}\}$

Write each set using the roster method.

24. $A = \{x \mid x \text{ is an odd number and } 20 < x < 30\}$

25. $B = \{x \mid x \leq 15 \text{ and } x \text{ is a positive multiple of } 3\}$

26. $C = \{x \mid 2x = 4\}$

27. $D = \{x \mid x \text{ is a state that shares a common border with Colorado}\}$

Use the given sets to answer each question.

$P = \{\text{lasagna, rotini, orzo, tortellini, penne}\}$

$Q = \{x \mid x \text{ is a pasta shape}\}$

$R = \{\text{penne, tortellini, orzo, rotini, lasagna}\}$

$S = \{\text{marinara, pesto, alfredo, Bolognese, carbonara}\}$

28. Is $P = Q$? Why or why not?

29. Is $P = R$? Why or why not?

30. Is $P = S$? Why or why not?

31. Are any of P, Q, R, and S equivalent? Explain.

Use the given sets to answer each question.

$A = \{\text{Business, Physics, Psychology, Kinesiology, Graphic Design, History}\}$

$B = \{\text{History, Graphic Design, Kinesiology, Physics, Business}\}$

$C = \{\text{Art History, Education, Nursing, Biology, Statistics}\}$

$D = \{x \mid x \text{ is a university major}\}$

32. Is $A = B$? Why or why not?

33. Is $B = C$? Why or why not?

34. Is $A = D$? Why or why not?

35. Are any of A, B, C, and D equivalent? Explain.

Determine the cardinal number of each set.

36. $W = \{0, 3, 4, 5, 6, 7, 8, 9, 10\}$

37. $X = \{x \mid x \in \mathbb{Z}, x \text{ is even, and } |x| < 10\}$

38. The empty set

39. $Y = \{x \mid x \text{ is a United States president, past or present}\}$

Use the set $A = \{\text{b, a, s, k, e, t}\}$ to solve each problem.

40. Find $|A|$.

41. If $U = \{\text{a, b, c, d, …, x, y, z}\}$, find A'.

42. If $U = \{\text{a, b, c, d, …, x, y, z}\}$, find $|A'|$.

43. If $U = \{\text{a, b, c, d, …, x, y, z, A, B, C, D, …, X, Y, Z}\}$, find A'.

44. If $U = \{\text{a, b, c, d, …, x, y, z, A, B, C, D, …, X, Y, Z}\}$, find $|A'|$.

Use the set $B = \{1, 2, 3, 4\}$ to solve each problem.

45. Find $|B|$.

46. If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find B'.

47. If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find $|B'|$.

48. If $U = \{-9, -8, -7, -6, \ldots, 6, 7, 8, 9\}$, find B'.

49. If $U = \{-9, -8, -7, -6, \ldots, 6, 7, 8, 9\}$, find $|B'|$.

APPLICATIONS

The table shows the total number of official Olympic medals for the time period from the first modern Olympics in 1896 through the Winter Games of 2014 for the top 10 countries, some of which are no longer countries. Let the universal set consist of the 10 countries listed. Solve each problem.

Top 10 All-Time Olympic Medal Winning Countries

Team	Gold	Silver	Bronze	Combined Total
United States (USA)	1072	860	749	2681
Soviet Union (URS)	473	376	355	1204
Great Britain (GBR)	246	276	284	806
Germany (GER)	252	260	270	782
France (FRA)	233	254	293	780
Italy (ITA)	235	200	228	663
Sweden (SWE)	193	204	230	627
China (CHN)	213	166	147	526
Russia (RUS)	182	162	177	521
East Germany (GDR)	192	165	162	519

Source: Wikipedia, s.v. "All-time Olympic Games medal table," accessed July 2014, http://en.wikipedia.org/wiki/All-time_Olympic_Games_medal_table

50. Let X equal the set of countries who have won more than 1000 medals overall. Write the set X using the roster method.

51. Let Y equal the set of countries who have won between 500 and 1000 medals overall. Write the set Y using the roster method.

52. Let Z equal the set of countries who have won less than 500 medals overall. Write the set Z using the roster method.

53. Let G equal the set of countries who have won more than 200 Gold medals. Write the set G using the roster method.

54. Is $X = G$? Explain.

55. Is $X \sim G$? Explain.

WRITING & THINKING

56. Give an example of a set that cannot be represented using the roster method.

57. Describe two sets of which you are a member.

58. Let A be a set in which one of the elements is President Barack Obama. List at least three different universal sets for A.

59. Let X be a set in which one of the elements is π. List at least three different universal sets for X.

Example 8: Applying the Inclusion-Exclusion Principle

A standard deck of playing cards has 52 cards (26 of which are red and 26 of which are black) divided into 4 suits (clubs, spades, diamonds, and hearts), where there are 13 of each suit (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king). Of these cards, 12 are considered face cards (4 kings, 4 queens, and 4 jacks). Find the number of cards in a standard deck that are either clubs or face cards.

Solution

We start the solution by writing what we are looking for using set notation.

$$|\text{clubs} \cup \text{face cards}| = |\text{clubs}| + |\text{face cards}| - |\text{clubs} \cap \text{face cards}|$$

If the set A consists of clubs and the set B consists of face cards, then this is equivalent to

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

There are 13 clubs in the deck and there are 12 face cards. However, there are 3 face cards that are also clubs (king of clubs, queen of clubs, and jack of clubs). Therefore,

$$|A \cup B| = 13 + 12 - 3 = 22.$$

So, the number of cards that are either clubs or face cards is 22.

Skill Check 3

Find the number of playing cards that are either even (2, 4, 6, 8, 10) or are diamonds.

Skill Check Answers

1. $\{n, e, t, o\}$

2. $A \cap B = \{o\}$, so $(A \cap B)' = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q, r, s, t, u, v, w, x, y, z\}$.
   $A' = \{a, b, c, e, f, g, i, j, k, l, m, p, q, r, s, t, v, w, x, y, z\}$ and
   $B' = \{a, b, d, e, f, g, h, i, j, l, m, n, p, q, s, t, u, v, w, x, y, z\}$, so
   $A' \cup B' = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q, r, s, t, u, v, w, x, y, z\}$.
   Thus, $(A \cap B)' = A' \cup B'$.

3. 28

8.2 EXERCISES

PRACTICE

Use the given sets to solve each problem.

$U = \{1, 2, 3, \ldots, 20\}$ $\quad$ $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$B = \{2, 4, 6, 8, 10, 12\}$ $\quad$ $C = \{5, 7, 9, 11, 13, 15\}$

1. Find $A \cup B$.

2. Find $A \cup C$.

3. Find $B \cap C$.

4. Find $A \cap C$.

5. Verify $(A \cup B)' = A' \cap B'$.

6. Verify $(A \cap B)' = A' \cup B'$.

Use the given sets to solve each problem.

$U = \{\text{a, b, c, d, …, z}\}$ $\quad A = \{\text{n, u, m, b, e, r, s}\}$ $\quad B = \{\text{r, u, l, e}\}$

7. Find $A \cup B$.

8. Find $A \cap B$.

9. Find $|A \cap B|$.

10. Verify $(A \cup B)' = A' \cap B'$.

11. Verify $(A \cap B)' = A' \cup B'$.

Use the given sets to solve each problem.

$U = \{\text{A, B, C, D, …, Z}\}$ $\quad A = \{\text{I, C, E}\}$ $\quad B = \{\text{C, U, B, E}\}$

12. Find $A \cup B$.

13. Find $A \cap B$.

14. Find $|A \cap B|$.

15. Verify $(A \cup B)' = A' \cap B'$.

16. Verify $(A \cap B)' = A' \cup B'$.

Use the given sets to solve each problem.

$U = \{\text{A, B, C, D, …, Z}\}$ $\quad A = \{\text{F, A, C, T, O, R}\}$ $\quad B = \{\text{P, R, O, D, U, C, T}\}$

17. Find $A \cup B$.

18. Find $A \cap B$.

19. Find $|A \cap B|$.

20. Verify $(A \cup B)' = A' \cap B'$.

21. Verify $(A \cap B)' = A' \cup B'$.

Use the given sets to solve each problem.

$U = \{\text{a, b, c, d, …, z}\}$ $\quad M = \{\text{b, r, i, d, g, e}\}$

$N = \{\text{g, a, t, o, r}\}$ $\quad K = \{\text{b, a, l, i, s, t, e, r}\}$

22. Find $N \cap (M \cup K)$.

23. Find $N \cup (M \cap K)$.

24. Find $K \cup (N \cap M)$.

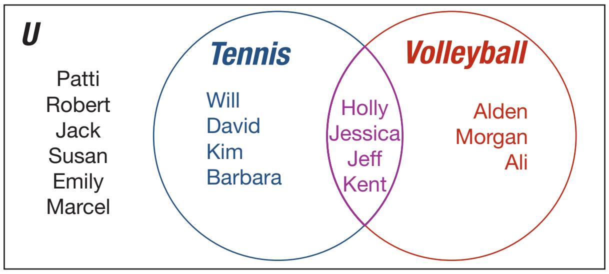
25. Verify $(M \cup K)' = M' \cap K'$.

26. Verify $(M \cap N)' = M' \cup N'$.

APPLICATIONS

27. A grocery store found that 275 of its customers use push carts to shop, 185 used a carry basket to shop, and that 145 used both a push cart and a carry basket. How many customers use only a push cart or a carry basket? Draw the Venn diagram.

Use the Venn diagram to solve each problem.



28. Which students played only tennis?

29. Determine which students played tennis or volleyball.

30. Determine which students played tennis and volleyball.

31. Find the number of students that play tennis or volleyball.

Solve each problem.

32. Determine the number of playing cards in a standard deck that are red cards or face cards.

33. Determine the number of playing cards in a standard deck that are odd numbered cards or black cards.

WRITING & THINKING

Show that each pair of sets is equal by drawing a Venn diagram of each set.

34. $A \cap B$ and $B \cap A$

35. $A \cup B$ and $B \cup A$

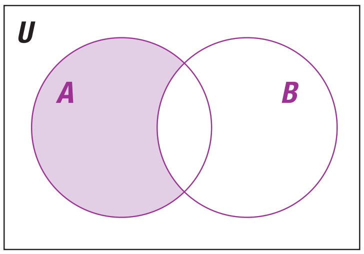
36. $(A \cap B) \cap C$ and $A \cap (B \cap C)$

37. $(A \cup B) \cup C$ and $A \cup (B \cup C)$

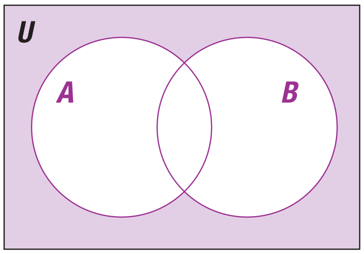
38. $A \cup \varnothing$ and A

Use set notation to represent each shaded region.

39.



40.



8.3 EXERCISES

PRACTICE

Identify the sample space for each experiment.

1. A coin is flipped and then a single die is rolled.

2. Choosing a number from all positive two-digit integers where the digits are repeated (for example, 11).

3. Five marbles are in a bag, one of each of the following colors: blue (B), clear (C), green (G), yellow (Y), and red (R). Two marbles are drawn consecutively. Assume that the first marble is not put back in the bag before the second marble is drawn. Order of the selection matters. In other words, BG and GB are two different selections.

4. When choosing an outfit, you have a choice of three shirts: white (W), black (B), or patterned (P); a choice of two types of jeans: faded (F) or dark wash (D); and a choice of four pairs of shoes: sandals (S), running shoes (R), climbing boots (C), or mules (M). List the outcomes in the sample space in regard to the outfits (combination of shirt, jeans, and shoes) you could pick from.

Determine whether each probability is empirical or classical.

5. In order to find the percentage of bass that Troy had in his pond this spring, he decided to spend three days catching a total of 15 fish each day and counting how many of those were bass.

6. Jason wants to know how likely he is to win a raffle if he bought 3 of the 1000 tickets that were sold.

7. Virginia wants to know how likely it is for her to win a backgammon game if she only needs to roll a double six to win.

8. Emre wants to see how many students drive a hybrid car. He surveys 100 college freshmen at a Winter Welcome event and asks what kind of car they drive.

APPLICATIONS

Calculate each empirical probability. Round your answer to the nearest millionth when necessary.

9. A news organization asked a selection of voters exiting a polling place their age bracket in order to paint a picture of turnout on election day. Here's the record of the results collected so far.

Voting Age			
17–29	**30–44**	**45–64**	**65 and Older**
9	8	32	15

a. What is the probability that the next voter to exit will be between 30 and 44?
b. What is the probability that the next voter to exit will be in either of the youngest two age groups?
c. What is the probability that the next voter to exit will be under 65?

10. The blood types of 200 people are collected at a doctor's office. The table shows the breakdown of patients per blood type. Based on the data in the table, what is the probability that a new patient will have type O blood?

Blood Type Survey Results	
Blood Type	**Number of Patients**
A	50
B	65
O	70
AB	15

11. As students were exiting the student center on campus, Vicki took note whether they were listening to headphones. The table shows results that she collected. Based on Vicki's data, what is the probability that a randomly selected student will have headphones on when exiting the student center?

Data for Students Exiting the Student Center	
	Number of Students
Headphones	33
No Headphones	51

12. A sample of 500 active-duty military showed that 82 of them suffered from post-traumatic stress disorder (PTSD). However, 95 of those studied reported that they would be too embarrassed to seek mental health services.
a. Based on this sample, if a soldier is chosen at random, what is the probability that he/she suffers from PTSD?
b. Based on this sample, if a soldier is chosen at random, what is the probability that he/she would be willing to seek mental health services?

Use classical probability to calculate each probability. Assume individual outcomes are equally likely. Round your answer to the nearest millionth when necessary.

13. What is the probability that, out of 235 attendees (including yourself) at a conference, you are selected to win the door prize at the opening session?

14. A standard die is rolled.
   a. Find the probability that the roll produces a number less than 3.
   b. Find the probability that the number rolled is an even number.
   c. Find the probability that the number rolled is greater than 0.

15. On an American roulette wheel, there are 18 red pockets, 18 black pockets, and 2 green pockets. What is the probability of landing on a red pocket?

16. The table shows a breakdown for all employees on nonfarm payrolls in the United States during March 2014 (the values are not seasonally adjusted).

Employees on Nonfarm Payrolls (in Thousands), March 2014		
	Area of Employment	**Number of Employees (in Thousands)**
Private Sector	Goods-Producing	18,558.2
	Wholesale Trade	5803.7
	Retail Trade	15,004.0
	Transportation and Warehousing	4524.8
	Utilities	550.3
	Information	2653.0
	Financial Activities	7870.0
	Professional and Business Services	18,832.0
	Education and Health Services	21,481.0
	Leisure and Hospitality	14,143.0
	Other Private Service-Providing Services	5464.0
Public Sector	Federal Government	2705.0
	State Government	5217.0
	Local Government	14,341.0
	Total Nonfarm Employees	**137,147.0**
	Source: Bureau of Labor Statistics. "Table B-1. Employees on nonfarm payrolls by industry sector and selected industry detail." Accessed June 2014. http://www.bls.gov/news.release/empsit.t17.htm	

   a. Find the probability that a random nonfarm employee was employed in the leisure and hospitality sector during March 2014.
   b. Find the probability that a random nonfarm employee was in the public sector during March 2014.

17. What is the probability that a card drawn randomly from a standard deck of cards will be an ace?

18. A book contains 321 pages numbered 1, 2, 3, …, 321. If a student randomly opens the book, what is the probability that the page has a two- or three-digit number and all of its digits are the same?

19. Consider parents with four biological children. Find the probability that all four siblings are boys.

20. Mason has 213 songs on his iPod. He's categorized them in the following manner: 20 from sound tracks, 8 spiritual, 31 jazz, 16 Latin American, 27 R&B, 47 rock, and 64 pop. If Mason puts his iPod on shuffle, what is the probability that the first song played is a pop song?

21. Landon decided to play a joke on his friends at work. When he stocked the vending machine with sodas, he randomly put the drinks into the different slots. If he put in 15 Diet Coke, 15 Coke, 12 Sprite, 17 Dr. Pepper, and 10 Fanta cans, what is the probability that the next person will get a Fanta drink when they put their money into the machine?

8.4 EXERCISES

PRACTICE

Create a tree diagram to list the outcomes in each sample space.

1. Find the sample space for the gender of each child in regard to birth order for a family with three children.

2. In picking out a new car, there are several choices to make. The color can be red, white, or silver. The seats can be cloth or leather. Finally, it can have a sunroof, a moonroof, or neither. Find the sample space for the possible new car combinations using a tree diagram.

3. Use a tree diagram to find the sample space for tossing a coin three times.

4. Four students are randomly selected from an algebra class and asked whether they suffer from math anxiety. Find the sample space for the possible outcomes of the survey using a tree diagram.

Evaluate each factorial expression.

5. $5!$

6. $9!$

7. $\frac{8!}{6!}$

8. $1!$

9. $0!$

10. $\frac{10!}{2!4!}$

11. $\frac{5!}{3!2!}$

12. $\frac{12!}{8!(3-1)!}$

13. $\frac{23!}{11(25-4)!}$

Evaluate each permutation or combination.

14. ${}_8P_2$

15. ${}_7P_4$

16. ${}_5P_1$

17. ${}_4P_4$

18. ${}_3C_2$

19. ${}_{30}C_1$

20. ${}_5C_5$

21. $\frac{{}_3C_2}{{}_3P_2}$

22. $\frac{{}_5P_3}{{}_5C_3}$

23. ${}_7C_4 + {}_7C_3 + {}_7C_2 + {}_7C_1$

24. ${}_7P_4 + {}_7P_3 + {}_7P_2 + {}_7P_1$

APPLICATIONS

25. How many three-digit area codes can be made from the digits 0 through 9? Assume that the digits may repeat and that area codes beginning with 0 are allowed.

26. How many three-digit area codes can be made from the digits 0 through 9 if the first digit is not allowed to be a 0 and the digits are allowed to repeat?

27. How many six-character password codes are possible if you are allowed both lowercase letters and the digits 0 through 9, and repeating characters are allowed?

28. When ordering a new e-reader, you have several choices to make. You can choose from five price ranges, decide between Wi-Fi and 5G, choose to have a one-year or three-year warranty, and pick a black, white, or silver casing. How many possible e-readers are there for you to choose from?

29. The Youngs are planning their next family night. They always have dinner out somewhere and then do something fun together. There are two adults and four boys in the family. Each family member is allowed two meal suggestions, and each boy is allowed three activity suggestions. Assuming no family members choose the same thing, how many different family night possibilities are there?

30. If there are seven children lining up for recess, how many different ways can they line up?

31. The college's soccer team will play 11 games next fall. Each game can result in one of three outcomes: a win, a loss, or a tie. Find the total number of possible outcomes for the season record.

32. Harper is deciding on her schedule for next semester. She must take each of the following classes: English 102, College Algebra, History 102, and Biology 101. If there are 16 sections of English 102, 14 sections of College Algebra, 7 sections of History 102, and 12 sections of Biology 101, how many different possible schedules are there for Harper to choose from? Assume there are no time conflicts between the different classes.

33. If there were no restrictions on the use of the number zero, in theory, how many seven-digit telephone numbers are possible?

34. How many four-digit even whole numbers exist?

35. In the new ice creamery, several choices need to be made before tasting "a little bit of heaven," as the advertisement suggests. First, there are three cup sizes to choose from, then 39 different ice cream flavors to decide from, and finally several mix-ins to choose from: 10 candy bars, 8 fruits, 8 nuts, and 10 cookies or cakes. If you want a medium cup with one flavor of ice cream and two mix-ins, one candy bar and one nut, how many possible choices are there for you to have your "taste of heaven"?

Determine whether to use a permutation or combination to answer each question, and then determine the total number of outcomes.

36. There are 10 board members on the Community Arts Council. In how many ways can a president and treasurer be chosen? Assume that no member can hold both positions at the same time.

37. In how many ways can a committee of five people be chosen from a pool of 120 employees?

38. If there are 84 runners in a race, in how many ways can 1^{st}, 2^{nd}, and 3^{rd} place ribbons be given out?

39. Elliot has to submit three photographs for the school art show. This semester he has taken 29 photographs that he thinks are show-worthy. In how many ways can he choose the photographs to submit?

40. Avery was born on 10/15/1995. How many eight-digit codes could she make using the digits in her birthday?

41. There are 18 tenured faculty in the biology department on campus. The department needs one tenured faculty member to facilitate undergraduate research, one member to supervise graduate advising, and one to coordinate grant proposals. In how many ways can these tasks be assigned, if a member may be appointed to only one duty?

42. A service organization on campus needs a group of six students from their organization's membership of 123 students to serve as program attendants at graduation. In how many ways can the attendants be chosen?

43. In how many ways can the letters in the word STATISTICS be arranged?

44. In how many ways can the letters in the word TENNESSEE be arranged?

WRITING & THINKING

45. Without calculating the permutations, decide which of the following words would produce the greatest number of four-letter arrangements.

a. PASS
b. TEST
c. FAIR
d. FREE

46. Determine how many ways a group of three students from your class could be selected.

47. Count the number of possible outfits you have in your closet based on the number of pants, shirts, and pairs of shoes you have. Assume that all match well together or that you would be making your own fashion statement some days.

48. Calculate the number of ways to arrange the letters in

a. your first name,
b. your last name,
c. both your first and last names.

Solution

a. The probability that imported walnuts purchased in the United States during this period were from Austria is found by dividing the weight of walnuts imported from Austria by the total weight of walnuts imported during this time period. The first number is given in the table as 1938 pounds. To find the total weight of walnuts imported, we need to add together all of the weights of walnuts imported.

$$\text{total walnut weight} = 4373 + 1268 + 5239 + 1533 + 1938 + 4157 = 18{,}508$$

The probability that the walnuts were from Austria is then found by

$$P(\text{walnuts from Austria}) = \frac{1938}{18{,}508} \approx 0.104711.$$

b. We could find the probability that the walnuts came from somewhere other than Austria by combining all the remaining places together. However, given that we just calculated the probability that the walnuts were from Austria, it is easier for us to just use the complement.

$$P(\text{not from Austria}) = 1 - P(\text{walnuts from Austria}) \approx 1 - 0.104711 = 0.895289$$

c. Again, it will be easier for us to use the complement here. First calculate the probability that the pistachios you purchased came from either Italy or Switzerland.

$$\text{total pistachio weight} = 2012 + 2030 + 262 + 64 + 115 + 323 = 4806$$

$$P(\text{from Italy or Switzerland}) = \frac{\text{Italy} + \text{Switzerland}}{\text{total pistachio imports}} = \frac{115 + 64}{4806} \approx 0.037245$$

Now, to calculate the probability that the pistachios came from somewhere other than Switzerland or Italy, we'll find the probability of the comple1ment.

$$\begin{aligned} P(\text{not from Italy or Switzerland}) &= 1 - P(\text{from Italy or Switzerland}) \\ &\approx 1 - 0.037245 \\ &= 0.962755 \end{aligned}$$

Skill Check Answers

1. e. All of the above

8.5 EXERCISES

PRACTICE

1. Describe the complement of the set of odd numbers greater than 0 within the set of positive integers.

2. Let the event E be the sum of a pair of dice that is divisible by 3. List the events in E^c.

3. The following is a table of the ages of boys on a soccer team.

Let $A = \{\text{soccer players older than 9}\}$. How many players are in the complement of A?

Ages of Boys on Soccer Team	
Age	**Number of Boys**
8	3
9	6
10	7
11	2

4. Describe the complement of the set of face cards in a standard deck of cards.

5. In a company, all employees who have worked there for more than five years receive a gift. Describe the complement of this group of employees.

APPLICATIONS

Find the classical probability for each scenario. Round your answer to the nearest millionth when necessary.

6. Find the probability of obtaining exactly one head when flipping four coins.

7. There are two sets of balls numbered 1 through 5 placed in a bowl. If two balls are randomly chosen without replacement, find the probability that the balls have the same number.

8. William and Gavin are going to play video games after work. Together they have 48 games. If they decide to randomly choose two games to play, what is the probability that the two games they choose consist of William's favorite game and Gavin's favorite game? Assume they have different favorites.

9. A local pizza parlor has the following list of toppings available for selection. The parlor is running a special to encourage patrons to try new combinations of toppings. They list all possible three-topping pizzas (three distinct toppings) on individual cards and give away a free pizza every hour to a lucky winner.

Pizza Toppings			
Green Peppers	Onions	Pepperoni	Sausage
Baby Portabello Mushrooms	Black Olives	Ham	Spicy Italian Sausage
Roma Tomatoes	Pineapple	Beef	Grilled Chicken
Jalapeño Peppers	Banana Peppers	Bacon	Extra Cheese

a. How many three-topping pizza cards are there?

b. Find the probability that the first winner randomly selects the card with the pizza containing green peppers, ham, and bacon on it.

10. A combination padlock is a lock in which a sequence of numbers is used as the "key" to open the lock. Suppose a combination padlock has 10 digits to choose from for each of the four sections of the lock.

a. Does the "key" for a combination padlock involve permutations or combinations?

b. How many possible "keys" are there for the combination padlock?

c. What is the probability that you randomly buy one of these locks whose "key" is made of 4 of the same digit?

11. Four students, three girls and a boy, have arranged to meet on the first day of class and sit in the front row. Suppose they agree to sit in the first four seats in the order that they arrive.

a. How many possible seating arrangements are there for the four friends?

b. What is the probability that all three girls end up sitting next to one another?

12. A committee of four is being formed randomly from the employees at a school: 5 administrators, 37 teachers, and 4 staff.

a. How many ways can the committee be formed?

b. What is the probability that all four members are staff?

c. What is the probability that no member is an administrator?

13. A hand of poker is made up of five cards from a standard deck of cards.

a. How many possible hands of poker are there in a standard deck of 52 cards?

b. A royal flush consists of the cards ace, king, queen, jack, and ten, all in the same suit. What is the probability of being dealt a royal flush?

14. Find the probability that three people randomly line up to buy tickets in order of their height (tallest, middle, shortest). Assume that no two people in the line are of the exact same height.

15. Matthew needs to set the passcode on his smartphone. It must be a four-digit number and repeated digits are allowed.

a. How many possible passcodes are there for Matthew to choose from?

b. How many possible passcodes are there for Matthew if he decides to choose four distinct numbers?

c. A spy sneaks a look at Matthew's phone and sees his fingerprints on the screen over four numbers. What is the probability that the spy is able to unlock the phone on his first try?

d. The spy knows the fingerprint trick and so on his phone he uses a repeated digit in his code. If you could see the three fingerprints on the spy's phone, what is the probability that you could unlock the phone on your first attempt?

e. Based on parts **c.** and **d.**, is it better to repeat a digit or have four distinct digits in the code on your phone for security purposes?

16. A hand of blackjack consists of two cards. The dealer deals you a hand from a fresh deck.

a. What is the probability that the two cards have the same face value, for instance, both cards are kings or both cards are 5s?

b. If aces count 1 or 11, picture cards count 10, and card numbers 2 through 10 are equal to their face value, what is the probability that the two cards sum to 21?

17. One option to play the lottery is called "3-way any order." In order to play this method, you select three digits, from 0 to 9, such that precisely two of the digits are the same (for example, 1, 1, 2). You're a winner if your three digits show up in any order in the lottery's three randomly chosen digits. Digits may be repeated when the lottery chooses the winning number. Find the probability of winning with the "3-way any order" method.

18. Another option of playing the lottery is to choose three numbers (allowing repetition) in the exact order they will appear. Find the probability of winning the lottery with one ticket.

19. Ian is playing Scrabble. What is the probability that the next three letters he draws from the bag spell out his name in the order that he draws them? Assume there is one of each letter in the alphabet left in the bag.

20. For a pickup game of basketball, jerseys are in a box and people start grabbing them. The box contains three extra-large, seven large, and four medium jerseys. If you are first to the box and grab two jerseys, what is the probability that you randomly grab two extra-large jerseys?

21. A junk drawer at home contains a half dozen pens, two of which work. What is the probability that you randomly grab two pens from the drawer and don't end up with a pen that works?

Find each probability using complements.

22. A bag contains each letter of the alphabet. Find the probability that a randomly selected letter from the bag will not be one of the five vowels.

23. Find the probability of randomly choosing a letter other than the letter O from a bag that contains the eighteen letters of the Italian city GUIDONIA MONTECELIO.

24. Using the table containing the breakdown of all employees on nonfarm payrolls in the United States during March 2014, find the probability that a randomly selected US nonfarm worker was not in either retail trade or wholesale trade.

Employees on Nonfarm Payrolls (in Thousands), March 2014		
	Area of Employment	**Number of Employees (in Thousands)**
Private Sector	Goods-Producing	18,558.2
	Wholesale Trade	5803.7
	Retail Trade	15,004.0
	Transportation and Warehousing	4524.8
	Utilities	550.3
	Information	2653.0
	Financial Activities	7870.0
	Professional and Business Services	18,832.0
	Education and Health Services	21,481.0
	Leisure and Hospitality	14,143.0
	Other Private Service-Providing Services	5464.0
Public Sector	Federal Government	2705.0
	State Government	5217.0
	Local Government	14,341.0
	Total Nonfarm Employees	**137,147.0**
	Source: Bureau of Labor Statistics. "Table B-1. Employees on nonfarm payrolls by industry sector and selected industry detail." Accessed June 2014. http://www.bls.gov/news.release/empsit.t17.htm	

25. In June 2011, the week of the final mission of the US space shuttle program, a Pew Research poll asked 1502 US adults whether the United States must continue to be a world leader in space exploration. The following table gives a breakdown of their opinions.

The United States Continuing to be a World Leader in Space Exploration Is. . .		
Essential	**Not Essential**	**Don't Know**
871	571	60
Source: Pew Research Center. "Majority Sees U.S. Leadership in Space Essential." July 5, 2011. http://www.people-press.org/2011/07/05/majority-sees-u-s-leadership-in-space-as-essential/		

a. Find the probability that someone responded "essential."

b. Find the probability that someone did not respond "essential."

26. Find the probability of rolling two dice and not getting the same number on both dice.

27. Suppose a family has five pets. Find the probability that at least one of the pets is male.

Skill Check Answers

1. **a.** Mutually exclusive **b.** Not mutually exclusive

8.6 EXERCISES

PRACTICE

Determine whether each situation contains independent events.

1. The color of car driven by three randomly chosen classmates.

2. A password must be six characters long with no repeated characters. Are the choices of consecutive characters independent?

3. There are 15 board members, of which seven are men and eight are women. Two randomly chosen members will serve on the United Way campaign committee. If you wish to find the probability that both members chosen are the same sex, do you treat these selections as independent events?

4. Are receiving a bill in Monday's mail and receiving a letter from your grandparents in Monday's mail independent events?

5. Naomi and Amelia both put two business cards into the basket at a coffee shop. The shop owner selects three cards from the basket. Are the two events that Naomi's card is chosen and Amelia's card is chosen independent?

APPLICATIONS

Calculate the probability of each set of mutually exclusive events. Round your answer to the nearest millionth when necessary.

6. Suppose that the probability of obtaining zero defective items in a sample of 50 items off the assembly line is 0.34 while the probability of obtaining 1 defective item in the sample is 0.46. What is the probability of the following?
   a. Obtaining no more than one defective item in a sample.
   b. Obtaining more than one defective item in a sample.

7. A pair of dice is rolled. What is the probability that the sum of the numbers is either 7 or 11?

8. A single letter from the word MISSISSIPPI is chosen. What is the probability of choosing an S or an I?

9. What is the probability that a card selected from a deck will be either an ace or a queen?

10. A reporter for an international newspaper is given an assignment that is randomly chosen from the following destinations worldwide: 13 continental United States assignments, 7 South American assignments, 21 European Union assignments, and 5 Asian assignments. Find the probability that he gets an assignment in Asia or South America.

11. The following table shows the breakdown of opinions for both faculty and students in a recent survey about the new restructuring of the campus to be a walking campus.

Survey Results on Restructuring Campus to a Walking Campus				
	Favor	Oppose	Neutral	**Total**
Faculty	12	4	3	**19**
Student	33	57	28	**118**
Total	**45**	**61**	**31**	**137**

a. Find the probability that a randomly selected person is either a faculty member in favor of the change or a student who has an opinion either for or against.

b. Find the probability that a randomly selected person is either neutral or in favor of the restructuring.

12. The probability of the stoplight being green at the intersection of Meeting Street and Main Street is 0.55, while the probability of it being yellow is 0.15. Find the probability that the light is red when you get to the intersection of Meeting Street and Main Street. Assume that the light will be working and will be a solid color: red, yellow, or green.

13. In a box of pens and pencils, the probability of randomly choosing a sharpened pencil is 0.54 and the probability of randomly choosing a pen from the box is 0.39. Find the probability of randomly selecting either an unsharpened pencil or a pen from the box.

Calculate the probability of each set of events that are not mutually exclusive. Round your answer to the nearest millionth when necessary.

14. A pair of dice is rolled. What is the probability that the sum of the numbers is an even number or a multiple of 3?

15. A bag of eleven marbles contains five marbles with red on them, three with green on them, seven with black on them, and four with black and red on them. What is the probability that a randomly chosen marble has either black or red on it?

16. What is the probability that a card selected from a deck will be either an ace or a spade?

17. The following is a table showing the results of a poll taken on campus.

Will You Vote in the Upcoming Election?		
	Male	**Female**
Yes	16	24
No	19	11
Not decided	21	22

a. What is the probability that a randomly selected student from this poll would be a male who has not decided whether he will vote in the upcoming election?
b. What is the probability that a randomly selected student from this poll is female or will not vote in the upcoming election?
c. What is the probability that a randomly selected student from this poll has decided to vote in the upcoming election?

18. Out of a class of 30 students, there are 16 students who study Latin, 21 who study German, and 7 who study both. What is the probability that a randomly selected student from the class will study only Latin?

19. Of the 11 instructors in the English department, four are new to the department and three are female. However, there is only one who fits all of the descriptions. Find the probability that if you randomly choose a course taught by these instructors, you get either a new instructor or a female instructor.

20. The following is a table representing the students who are on the Student Government Board.

Students on the Student Government Board		
	On-Campus Housing	**Off-Campus Housing**
Freshman	3	1
Sophomore	3	2
Junior	2	3
Senior	0	3
Graduate Student	0	2

Find the probability that a randomly chosen member of the Student Government Board is either a sophomore or lives in on-campus housing.

Calculate the probability of each set of independent events. Round your answer to the nearest millionth when necessary.

21. Suppose the probability that my pet will be alive in five years is 0.65 and the probability that my cousin's pet will be alive in five years is 0.48. Find the probability that both of these pets will be alive in five years assuming that they are independent events.

22. Two dice are thrown. Find the probability of getting an even number on the first die and an odd number on the second die.

23. Find the probability of choosing a heart and then an ace from a standard deck of cards with replacement.

24. On any given day at the beach, there is a 49% chance of precipitation. What is the probability that you will get precipitation for three days in a row on your beach vacation? Assume that the weather on a particular day at the beach is independent of the weather the day before.

Calculate each conditional probability. Round your answer to the nearest millionth when necessary.

25. A swim team consists of four boys and three girls. A relay team of four swimmers is chosen at random from the team members. What is the probability that there are two boys on the relay team given that there are two girls on the relay team?

26. Emma is playing Monopoly, a game played with two dice. What is the probability that the sum of the two dice she rolls is less than 4 given that she rolls an odd number?

27. Hunter bets his friend that he can draw two aces in a row from a standard deck of cards. What is the probability that Hunter draws a second ace given that his first card was an ace?

28. The probability that a student passes Intermediate Algebra is 0.55. The probability that a student passes College Algebra given that they pass Intermediate Algebra is 0.70. What is the probability that a student passes both College Algebra and Intermediate Algebra?

29. On each point in racquetball, a player is allowed two serves. Suppose while playing racquetball, Tim gets his first serve in about 75% of the time. He gets his first serve in and wins the point about 50% of the time. What is the probability that he wins the point, given that he gets his first serve in?

Calculate each probability. Round your answer to the nearest millionth when necessary.

30. The following table shows the student demographics for a sociology class.

Sociology 101 Student Demographics

	Male	**Female**
Freshman	3	11
Sophomore	4	9
Junior	0	3
Senior	1	0

a. Find the probability that a randomly selected student from the class is a male.

b. Find the probability that if two students are randomly selected, without replacement, the first is a female junior and the second is a male sophomore.

31. Arianna likes chicken and apple sausage, but not chicken and asiago cheese sausage. There are 18 pieces of each kind of sausage on a sausage and cheese plate. What is the probability that Arianna randomly skewers three pieces of sausage that she likes given that the first two are to her liking?

32. A swim team consists of four boys and three girls. A relay team of four swimmers is chosen at random.

- **a.** What is the probability that two boys and two girls are chosen for the relay team?
- **b.** What is the probability that Jim is one of the two boys and Jane is one of the two girls?

33. James has 20 applications on the home screen of his smartphone. His nephew accidently deletes five of the apps on his home screen. What is the probability that the app originally in the top right corner and the app originally in the bottom left corner have not been deleted?

34. The probability that an e-mail is spam is 0.05, the probability that the word "offer" is in an e-mail is 0.02, and the probability that the word "bank" is in an e-mail is 0.1. The probability that the word "offer" appears given that the e-mail is spam is 0.2, and the probability that the word "bank" appears given that the e-mail is spam is 0.4.

- **a.** Find the probability that an e-mail contains the word "bank" and is spam.
- **b.** If the words are assumed to appear independently, find the probability that an e-mail that contains "offer" and "bank" is spam.

Skill Check 2

If the odds on a bet are 4 : 1 against, what is the probability of winning?

In fact, rather than being an expression for the probability of winning, the term "odds against" represents the ratio of the chance of losing to the chance of winning. In our example, the phrase "odds against of 3 : 1" means that 3 out of 4 times you're likely to lose. Since odds are ratios, you can also express odds as fractions.

$$\text{Odds against} = \frac{P(\text{losing})}{P(\text{winning})}$$

In this particular case, the fraction is

$$\text{Odds against winning} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{1}{4}\right)} = \frac{3}{1} = 3.$$

Skill Check Answers

1.

$x_i P(x_i)$
1.35
1.80
2.25

$E(X) = 5.4$

2. $\frac{1}{5}$ or 20%

We should note that in the betting world, "odds against" are most often quoted because it is the most convenient way to understand the payout if the bet is a successful one for the player. So odds of 20 : 1 mean that for every $1.00 you bet, the bookie will pay you $20.00 if you win. The bookie is out to make money; in reality, the odds represent how much the bookie is willing to pay out rather than the true ratio.

Finally, note that odds may also be expressed as "odds for" (also called "odds in favor"), in which case the ratio is

$$\text{Odds for} = \frac{P(\text{winning})}{P(\text{losing})}.$$

8.7 EXERCISES

PRACTICE

Calculate the expected value of each scenario. Round your answer to the nearest hundredth when necessary.

1.

x_i	$P(x_i)$
1	0.21
2	0.58
3	0.06
4	0.15
5	0

2.

x_i	$P(x_i)$
–$1.50	0.3
$0.00	0.5
$2.75	0.1
$5.00	0.1

3.

x_i	$P(x_i)$
25	$\frac{1}{3}$
15	$\frac{2}{5}$
10	$\frac{1}{15}$
5	$\frac{1}{5}$

4. Let x_i represent the number of even numbers showing when a pair of standard six-sided dice are rolled.

x_i	$P(x_i)$
0	$\frac{9}{36}=\frac{1}{4}=0.25$
1	$\frac{18}{36}=\frac{1}{2}=0.5$
2	$\frac{9}{36}=\frac{1}{4}=0.25$

APPLICATIONS

5. Suppose Piper eats out twice a week 15% of the time, she eats out once a week 35% of the time, and she doesn't eat out anytime during the week 50% of the time. What is the expected value for the number of times Piper eats out during a week?

6. Suppose that you and a friend are playing cards and decide to make a bet. If you draw two aces in succession from a standard deck of cards without replacing the first card, you win \$50.00. Otherwise, you pay your friend \$10.00.
   a. What is the expected value of your bet?
   b. If the same bet was made 25 times, how much would you expect to win or lose?

7. A European roulette wheel has only one green slot instead of two. Using Example 2 from this section as a guide, calculate the expected winnings on a European roulette wheel if a player bets \$1.00 on red to play the game.

8. Jim likes to day-trade on the Internet. On a good day, he averages a \$1100 gain. On a bad day, he averages a \$900 loss. Suppose that he has good days 25% of the time, bad days 35% of the time, and the rest of the time he breaks even.
   a. What is the expected value for one day of Jim's day-trading hobby?
   b. If Jim day-trades every weekday for three weeks, how much money should he expect to win or lose?

9. A university in town is raffling off \$20,000 for student scholarships. You can buy one ticket for \$10, three tickets for \$25, or five tickets for \$40. Assume that the university sells 10,000 tickets.
   a. Find the expected value for each of the three ticket options: purchasing just one ticket, purchasing three tickets, or purchasing five tickets.
   b. Should you buy one, three, or five tickets in order to maximize the money you expect to have at the end of the raffle?

10. You need to borrow money from your sister. She's feeling quirky on the day you ask and says she wants you to flip a coin. Heads, you get \$15, tails you get \$5. Thinking this is weird, you ask your mother for money instead. She says she'll let you roll a die and she'll give you \$2 times the number that appears on the die. Before agreeing to either of these unique offers from the "mathy" folk in your family, you decide to see which is the better offer by calculating the expected value for each method (realizing that you too fit the bill of a "mathy" member of your family). Which offer should you take? Explain your reasoning.

11. Assume that stock in Degree Compass, a predictive analytics company in higher education, returns the percentages shown in the table.

Degree Compass Stock Returns	
Annual Return Rate	**Probability**
15%	0.17
30%	0.51
45%	0.32

Calculate the expected value of the return rate for stock in Degree Compass.

12. During the NCAA basketball tournament season, affectionately called *March Madness*, part of one team's strategy is to always foul their opponent's tall forward. Because he is so tall, he makes 57% of shots he takes close to the basket. However, when he is fouled, his free throw shooting percentage is only 51.5%. The shots he makes close to the basket are worth two points and each of the two free throw shots after being fouled are worth one point.

a. Calculate the expected value of the number of points the forward makes when he takes a shot close to the basket.

b. Calculate the expected value of the number of points the forward makes when he shoots two foul shots.

c. Based on these expected values, is fouling the tall forward a good strategy? Explain your answer.

13. On your next multiple-choice test, each question has four incorrect answers and one correct answer to choose from. Your professor tells you that each correct answer you make, you receive 1 point, but you lose $\frac{1}{4}$ point for each incorrect answer.

a. What is your expected gain or loss on a question if you have no idea of the correct answer and end up simply guessing?

b. What is your expected gain or loss if you guess on all 25 questions?

14. The stock prices of Web Movies on the 1st of the month for the first 6 months of 2014 were \$177.41, \$207.90, \$214.63, \$239.09, \$242.19, and \$259.99 respectively.

a. Calculate the average change in the stock prices of Web Movies in a month.

b. Use linearity of expectation to estimate the stock price on January 1st, 2015.

15. If the odds on a bet are 6 : 1 against, what is the probability of winning?

16. Suppose the probability of a football team winning a playoff game is 0.25. What are the odds of winning?

17. Suppose the odds of a teenage male having an accident are 2 : 3. What is the probability of a teenage male having an accident?

18. An insurance company claims the probability of surviving a certain type of cancer is 95%. What are the odds of surviving?

19. The UVest investment company publishes that the odds of increasing your wealth with their company is 5 : 2. What is the probability of UVest increasing your investment?

20. Odds against being struck by lightning in one year are 1,000,000 to 1.[1]

a. If you live to be 80, what are the odds against being struck by lightning over your lifetime? Assume each year has the same probability.

b. The National Weather Service gives the odds against being struck by lightning over an 80-year lifetime as 10,000 to 1. Why do you think this is different from the answer you got in part **a.**?

21. Overall odds in favor of winning in a state lottery game are 4.63 : 1.

a. Find the probability of winning in the lottery game.

b. The prize for this lottery game is \$100. If the cost to play the game is \$2.00, what is the expected value for playing this game?

22. Suppose the odds on a bet are 10 : 1 against. Your friend tells you he thinks the odds are too generous. Odds are considered less generous if the probability of losing is greater. Write down some less generous odds.

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1 NWS Lightning Safety, http://www.lightningsafety.noaa.gov

d. Because of the ease of choosing shoppers right in their own store, this is convenience sampling. In this case, convenience sampling is a viable method for gaining a representative sample since the store would be interested in knowing the thoughts of their customers.

e. Cluster sampling was used here because the counties are the natural clusters and all of the households in some of the counties received the surveys.

Skill Check Answers

1. Answers will vary. For example: size of engine, manufacturer, make, safety rating, number of doors.

2. No, since cluster sampling is an "all from one group" method, comparing mpg from cars in only certain price ranges would not produce a representative sample.

3. The location of the entrance ramp might lend itself to having cars only on one end of the price scale depending on the businesses located in the area.

4. No, although students in a given class represent a variety of different backgrounds, it is unlikely that everyone would drive a wide range of different types of cars, which would be representative of the population of cars driven in the United States.

9.1 EXERCISES

✔ CONCEPT CHECK

Fill in each blank with the correct term.

1. A ______ involves gathering data from every member of a population.

2. A subset of a population is called a ______. ______________ are numerical descriptors of characteristics of this subset.

3. A _________ is a particular group of interest in a statistical study. _________ _________ describe numerical characteristics of this entire group.

4. A __________________ does not favor one subgroup over another from a population.

PRACTICE

Decide if each numerical value from a statistical study is a population parameter or sample statistic.

5. After the presentation by the United Way representative, 67 of the night's 300+ attendees were interviewed about their reaction. Of the 67 people interviewed, 79% said they were motivated to donate more of their time and/or money to helping nonprofit organizations.

6. The Nashville Country Music Marathon reported an average finish time of 4:46:28 for the 4082 finishers.

7. A medical study showed that the heart-rate profile during exercise and recovery is a predictor of sudden death. Of the 5713 men studied, the mean maximum heart rate during exercise (expressed as the percentage of the predicted maximum heart rate) was found to be 96% in subjects who died suddenly from cardiac causes during the 23 years of the follow-up period of the study.[1]

8. An estimated 65% of drivers in rural or town areas view traffic conditions in a good light versus 39% of drivers in the city or suburbs based on a recent news report that aired nationally.

In each scenario, identify the population, the sample, and any population parameters or sample statistics that are given.

9. A medical study followed 675 cancer patients who received either the new drug *Lafent* or a standard chemotherapy drug. 76% of those who received *Lafent* were still living after eight months, compared to only 63% of those who received the standard chemotherapy drug.

10. US realtors across the country are encouraged by the latest reports on the real estate market. Taking into account seasonal factors, sales rose by nearly 9% in the West, 3.5% in the South, 3.4% in the Northeast, and 1% in the Midwest.

11. The online version of *Health* magazine, Health.com, reported the following: "Data from the Women's Health Study . . . examined the medical records of more than 36,000 women who had no history of depression at the start of the study. Roughly 18% of the women were experiencing some form of migraine or had suffered from the headaches in the past. Over the next 14 years, 11% of the study participants received a depression diagnosis." One of the conclusions of the study was that middle-aged women who experience migraines are 40% more likely to become depressed.[2]

12. In a recent study, US graduates of for-profit higher education institutions were between 4.8 and 6.7 percentage points more likely to be unemployed than graduates of nonprofit institutions and community colleges.[3]

1 Xavier Jouven et al., "Heart-Rate Profile During Exercise as a Predictor of Sudden Death."
2 Matt McMillen, *Health*, http://www.health.com
3 Dan Berrett, *The Chronicle of Higher Education*, http://www.chronicle.com

Identify the sampling technique used in each scenario.

13. To ensure the quality of its product, a company tests every 15^{th} item off the assembly line.

14. A student committee in the biology department was formed by randomly choosing three biology majors from every level of student, that is, 1^{st} year, 2^{nd} year, etc.

15. In order to choose the winners of free tickets to the campus concert, student housing printed out all 60 pages of student IDs and then chose three of the pages randomly.

16. A candidate for the local school board surveyed parents picking up their children from the public school on three school days in the last month.

17. Surveying moviegoers as they exited the late movie, researchers determined that only young adults under the age of 22 see movies in the theater anymore.

18. In order to sample the production yield of milk from the farm's herd of cows, random samples are taken from each of the five different types of cows.

19. Using Excel to generate an arbitrary list of customers, the marketing department sent promotional materials to the top 250 names on the list.

APPLICATIONS

Determine an appropriate sampling method for each scenario. Give your reasons for choosing the method. Describe any potential biases that the researcher might need to consider.

20. A state senator wishes to survey his constituents on issues in the upcoming legislature session.

21. A pharmaceutical company wishing to test the outcomes of a new drug for a particular skin disease finds 1200 patients willing to participate in the study. They can only choose 100 of them for the actual study.

22. Medical scientists wish to evaluate the effects of vitamin E and low-dose aspirin in primary prevention of cardiovascular disease and cancer in apparently healthy women.

23. Researchers wish to study the effects of watching too much television at a young age for children diagnosed with autism.

WRITING & THINKING

24. Explain how cluster sampling is different from stratified sampling.

25. *Self-selected* samples are a type of convenience sampling in which the participants volunteer to participate in the study rather than being chosen by the researcher. One issue with this type of sampling is that people who "self-select" to be in a survey often either have very strong opinions about an issue or desire monetary rewards for their participation. For instance, consider online polls regarding political views that are conducted by popular news outlets. Describe some of the potential responses from those who take the time to log on and respond. Is it reasonable to generalize the results from a study like this to include all of the American public? Why?

26. Consider how you would collect data for a study of illegal immigrants working in the United States. How would you go about collecting the data? Is it possible to conduct a census on this population of people? How reliable do you think your results would be? Discuss any issues you might encounter.

27. Often words like "best" and "worst" are used in reporting data. However, descriptive words like these are hard to measure in a concrete way. Consider how you could measure small towns to compile a list of "America's 50 Best Small Towns to Live In." Name at least five distinct measurements.

28. Researchers try to eliminate as many biases in studies as they can in order to get a true picture of the population they are studying. Describe as many potential sources of bias as you can if you were asked to study the effects of alcohol on college campuses.

Misleading Graphs

Spirit Week Participation

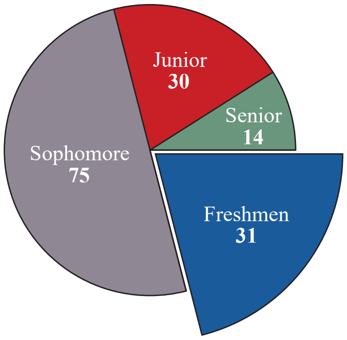


FIGURE 10

As we said at the beginning of the section, it's important to display data so that it is organized clearly and it effectively conveys the intended message. Ideally, graphs should be able to stand alone without the need for additional information in order to be understood. However, sometimes graphs either intentionally or unintentionally convey the wrong message about data or are not quite clear enough to get their message across. It's important to be aware that there are visually misleading and/or ambiguous graphs out there. For example, making the bars different widths on a bar chart might imply that one category is somehow larger than another. Similarly, a distorted piece of a pie graph might inaccurately lead the reader to assume that one section of data is larger than another. An example of this is the graph in Figure 10.

In their graph, the freshmen wanted to emphasize the fact that they came in second place for spirit week—although they just grabbed second place by a tiny margin. If the exact numbers were not on the graph, the emphasis on the freshmen piece of pie might visually imply that the freshman wedge is considerably larger than the junior wedge. Watch out for these visual manipulations when interpreting graphs.

Skill Check Answers

1. 0 pets **2.** 6,488,203 people

9.2 EXERCISES

✔ CONCEPT CHECK

Fill in each blank with the correct term.

1. A _________ is a graph that represents a frequency distribution.

2. When comparing parts of data to the whole, a _______ visually shows this using sections of a circle.

3. A ___ graph is best to use when showing data over a time period.

4. When comparing multiple categories from different populations, a __________ ___ graph or a _________ graph can be used.

5. A __________________ is a literal count of each member of a data set and how often it occurs.

APPLICATIONS

6. The grades on the first statistics test for Ms. Seago's class are listed in the following table. Construct a frequency distribution for the grades.

Grades on Statistics Test 1

A	C	F	C	C	D	F
B	D	F	B	A	A	F
B	C	C	A	B	F	D

7. The following table gives the grouped frequency distribution of weights of 194 babies in kilograms. Answer the questions that follow based on the distribution.

Weights of Babies

Birth Weight (kg)	0.00–0.99	1.00–1.99	2.00–2.99	3.00–3.99	4.00–4.99	5.00–5.99
Frequency	2	17	39	89	46	1

a. How many classes are in the grouped frequency distribution?
b. What is the class width?
c. What is the value of the lower class limit of the 3rd class?
d. What is the value of the upper class limit of the 5th class?
e. What is the relative frequency of the 4th class? Give your answer as a percentage rounded to the nearest tenth.

8. The following table represents a grouped frequency distribution of the number of hours spent on the computer per week for 55 students.

Hours	Number of Students
0.0–4.4	9
4.5–8.9	14
9.0–13.4	21
13.5–17.9	11

a. Calculate the relative frequencies (as percentages rounded to the nearest tenth) for each class.
b. What percentage of the students used the computer between 9 and 13.4 hours per week?
c. What percentage of the students used the computer less than 9 hours per week?

9. Sisscon is a phone answering service. The following data are the numbers of calls per day reported by the company for the last month.

10	72	64	32	78	62	11
37	45	32	52	38	70	66
13	21	14	13	39	73	62
41	63	44	23	27	22	21
55	24	53	43	20	16	22

a. Create a grouped frequency distribution for the data using 8 classes and then use it to answer the following questions. Let the first lower class limit be 0 and the class width equal 10.

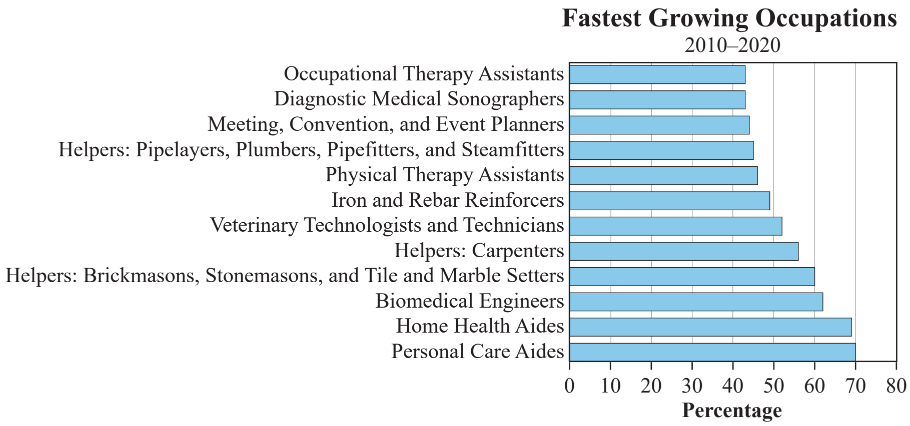
b. Calculate the relative frequencies for each class. Give your answers as percentages.

c. For what percentage of the days is the number of calls between 40 and 49?

d. For what percentage of the days is the number of calls in the single digits?

e. What is the most common range for the number of calls per day?

10. Consider the bar graph, constructed from a 2012 study, of the predicted fastest growing occupations between 2010 and 2020.[1]

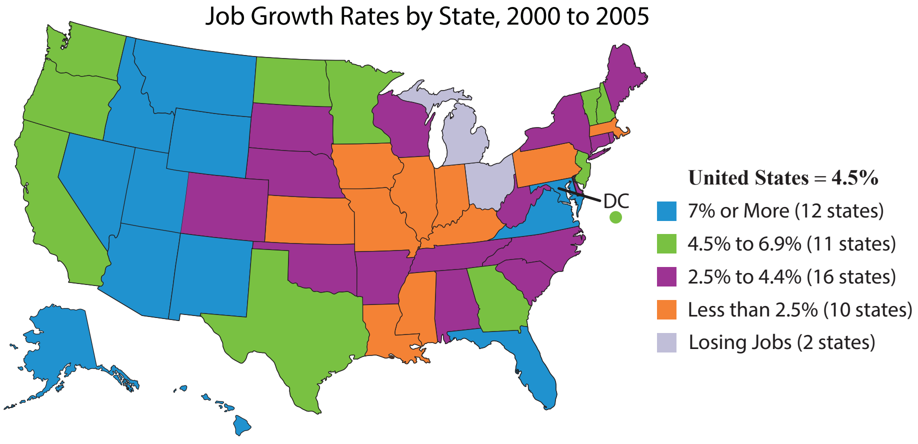


a. At a quick glance, which occupation was predicted to grow the most between 2010 and 2020? What was the predicted amount of growth?

b. How many new jobs were predicted to be available in the fastest-growing occupation in 2020?

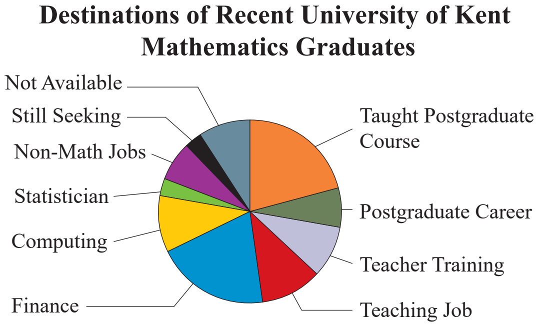
1 BLS Occupational Outlook Handbook, http://www.bls.gov/ooh

11. Answer the following questions about the Job Growth graph.[2]



a. How many states lost jobs between 2000 and 2005?
b. In what part of the country were jobs growing the most in this time period?

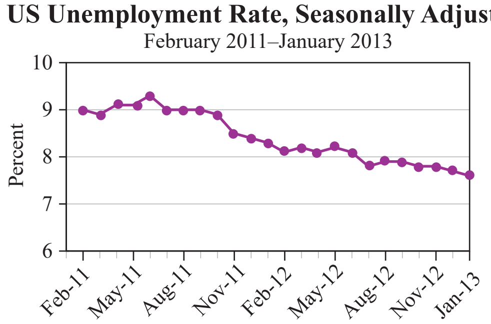
12. The following pie chart shows destinations of recent University of Kent mathematics graduates, including business, financial math, and statistics.[3]



a. Does it appear that any destination category accounts for more than 25% of the graduates?
b. Which pairs of destinations appear to have similar percentages of graduates?
c. How many mathematics graduates from the University of Kent were surveyed?
d. Is the graph misleading in any way?

2 InContext, http://www.incontext.indiana.edu
3 University of Kent, http://www.kent.ac.uk

13. The following graph shows the US unemployment rate from February 2011 to January 2013.[4]



a. Describe the trend of the unemployment percentage from February 2011 to January 2013.

b. Approximate the month and rate of the highest unemployment during this time period.

c. Approximate the month and rate of the lowest unemployment during this time period.

d. Is the graph misleading in any way?

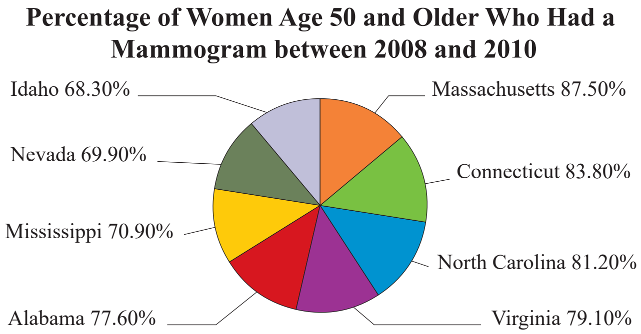
14. The following is a portion of a table about state health facts. It lists 8 of the 50 states along with the percentage of women age 50 and older who report having had a mammogram between 2008 and 2010.

Percentage of Women Age 50 and Older Who Had a Mammogram between 2008 and 2010

Massachusetts	87.50%
Connecticut	83.80%
North Carolina	81.20%
Virginia	79.10%
Alabama	77.60%
Mississippi	70.90%
Nevada	69.90%
Idaho	68.30%

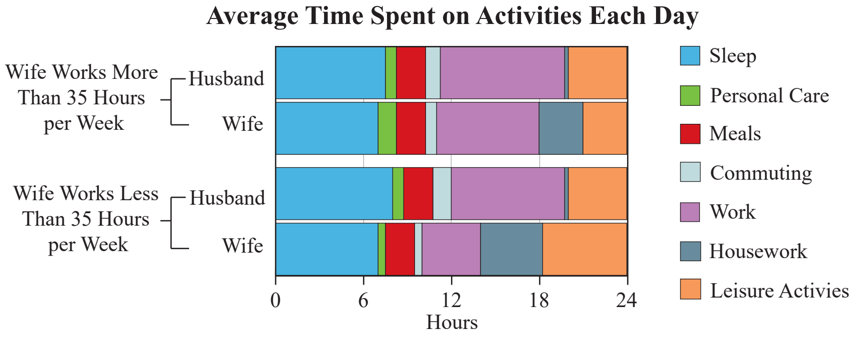
Source: Centers for Disease Control and Prevention (CDC). Behavioral Risk Factor Surveillance System Survey Data. Atlanta, Georgia: U.S. Department of Health and Human Services, Centers for Disease Control and Prevention, 2010, available at http://apps.nccd.cdc.gov/brfss/list.asp?cat=WH&yr=2010&qkey=4427&state=All

Is the following pie chart a good way to display this data? Explain why or why not.



4 Bureau of Labor Statistics, http://www.bls.gov

15. The stacked bar graph shows the average number of hours that married people in Japan spend each day doing various activities.[5]



a. For wives who spend less than 35 hours per week working, how many hours on average are spent each day for leisure activities?

b. For husbands whose wives work more than 35 hours per week, approximately how many hours on average are spent on sleep?

c. Compare the number of hours spent sleeping for wives in each category.

d. What type of graph could be used to represent these data in a clearer fashion?

5 Statistics Bureau (Japan), http://www.stat.go.jp

9.3 EXERCISES

PRACTICE

Find the mean, median, mode, range, and standard deviation for each data set. When applicable state whether the data set is unimodal, bimodal, or multimodal. Round answers to one more decimal place than the largest number of decimal places given in the data. All data sets are samples unless stated otherwise.

1. 19, 32, 15, 21, 25, 22, 22, 28, 27, 27, 26
2. \$11.40, \$32.00, \$22.50, \$12.01, \$10.08, \$18.30, \$18.40, \$32.00
3. 45, 21, 26, 26, 45, 37, 22, 33, 26, 21, 42, 37, 41, 43, 46, 35, 31, 29, 46
4. 310, 310, 310, 310, 310, 310
5. 9, 3, –5, –3, –7, 3, 0, 6, –9, –7, –3, –8
6. The following data represent sample ACT scores from students at a local high school.

ACT Scores

13	26
10	20
24	30
25	31
6	24
35	35
26	15

7. The following are lengths of each movie in the complete Harry Potter film series. Note that because these include all of the films in the series, this is a population.

Time Lengths for Harry Potter Film Series

Movie Title	Time (in Minutes)
Harry Potter and the Philosopher's Stone (2001)	152 minutes
Harry Potter and the Chamber of Secrets (2002)	161 minutes
Harry Potter and the Prisoner of Azkaban (2004)	141 minutes
Harry Potter and the Goblet of Fire (2005)	157 minutes
Harry Potter and the Order of the Phoenix (2007)	138 minutes
Harry Potter and the Half-Blood Prince (2009)	153 minutes
Harry Potter and the Deathly Hallows Part 1 (2010)	146 minutes
Harry Potter and the Deathly Hallows Part 2 (2011)	130 minutes

8. The following table shows the top 15 busiest airports from January to November 2011.

Top 15 Busiest Airports from January to November 2011

Airport	Total Passengers
Amsterdam Schiphol Airport	46,213,944
Beijing Capital International Airport	71,284,796
Dallas/Fort Worth International Airport	53,126,399
Denver International Airport	48,402,802
Dubai International Airport	46,287,234
Frankfurt Airport	52,191,355
Hartsfield-Jackson Atlanta International Airport	85,165,259
Hong Kong International Airport	48,587,000
John F. Kennedy International Airport	44,045,938
London Heathrow Airport	63,912,107
Los Angeles International Airport	56,819,805
Madrid Barajas Airport	46,019,110
O'Hare International Airport	61,370,268
Paris Charles de Gaulle Airport	56,254,938
Soekarno-Hatta International Airport	47,513,248
Tokyo International Airport	56,969,971

Source: Wikipedia, s.v. "World's busiest airports by passenger traffic," http://en.wikipedia.org/wiki/World%27s_busiest_airports_by_passenger_traffic

For each data set, determine the most appropriate measure of center.

9. Styles of houses in a suburb: ranch, colonial, bungalow, etc.

10. Grades on the final in Biology 210 at State University.

11. The ratings on a customer satisfaction survey: strongly disagree, disagree, neither agree nor disagree, agree, and strongly agree.

12. Salaries for janitorial staff at the state governmental buildings that include the Director of Sanitation's salary.

APPLICATIONS

Use the formula for the mean, $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$, to find the missing piece of data.

13. John knows that his first 4 tests grades were 84, 79, 82, and 88. Find John's grade on the fifth test if his average was 83.8.

14. A small boat that ferries visitors to a resort island has strict guidelines on the weight allowed for passenger luggage. Consequently the five vacationers are limited to a maximum average luggage weight of 40 pounds (lb). The following are the weights of three out of five pieces of luggage: 39 lb, 32 lb, and 43 lb. The two pieces of luggage that haven't been weighed will have to split the remaining weight allowance. Determine the maximum average possible weight allowance for each remaining bag.

Use the Empirical Rule to answer each question.

15. Although there is some controversy around the precise average body temperature of adults, new data suggest that the mean is 98.2° with a standard deviation of 0.6° and has a bell-shaped distribution.

a. According to this distribution, approximately what percentage of body temperatures are between 97° and 99.4°?

b. Approximately what percentage of temperatures are greater than 98.8°?

c. Approximately what percentage of temperatures are no more than 98.2°?

16. In 2011, high school seniors had the following mean and standard deviation on the mathematics portion of the SAT exam: $\mu = 514$ and $\sigma = 117$.

a. Approximately what percentage of scores were greater than 397 but less than 631?

b. What two scores have approximately 95% of the data between them?

c. Approximately what percentage of high school seniors had mathematics scores between the first and second quartiles?

d. Describe where the best and worst 0.3% of scores lie.

Solve each problem.

17. Marcel scored in the 91st percentile on the MCAT (Medical College Admissions Test). The medical school he is applying to only accepts students who score in the top 10% on the MCAT. Did Marcel score well enough to be considered for his school of choice?

18. The five-number summary is a numerical description of data that includes the minimum data point, the maximum data point, and the data points representing quartiles Q_1, Q_2, and Q_3. The following are house prices in one neighborhood.

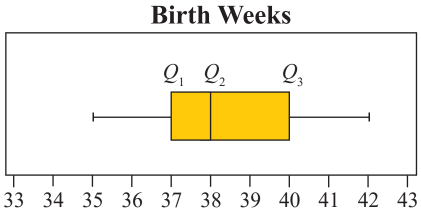
\$181,865	\$119,442	\$152,750	\$100,960	\$159,635
\$150,963	\$133,702	\$149,788	\$145,495	\$182,500
\$112,021	\$120,900	\$145,850	\$164,590	\$144,413

a. Find the five-number summary of the house prices.

b. What percentage of house prices is at or below \$159,635?

c. What is the range of house prices for this neighborhood?

19. The following graph contains a box plot. A box plot is a graphic display of a five-number summary, which was introduced in Exercise 18. The endpoints represent the minimum and maximum data values, while the lines sectioning off the box in the middle represent each of the quartiles as shown in the graph.



a. Based on the box plot, estimate each of the values in the five-number summary.

b. The midrange is the average of the minimum and maximum data values. Estimate the midrange from the box plot.

20. Using the information given in Exercise 19, calculate the values needed to construct a box plot for the following data as described in Exercise 19. Sketch a graph of the box plot.

310	320	450	460	470	500	520	540
580	600	650	700	710	840	870	900
1000	1200	1250	1300	1400	1720	2500	3700

21. Given the five-number summary for three data sets, sketch a box plot for each, side-by-side on the same graph. Then answer the following questions based on your box plots.

Committees and the Ages of Members

	Membership	Finance	Publicity
Min	23	26	25
Q_1	27	32	26
Q_2	29	38	27
Q_3	33	44	29
Max	35	46	33

a. Which committee has the largest range of ages?
b. Which committee has the least variation in the ages?
c. Which committee has the smallest median?

WRITING & THINKING

22. Given the following measures of center, decide the likely shape of the distribution: mean = 22.5, median = 17.0, mode = 17.0.

23. Accounting 101 has five class sections. All five classes took the same final. The mean scores on the final for each class were 72, 78, 76, 74, and 79. Can the mean final score for all students in Accounting 101 be found by averaging the mean scores in each class? Explain your answer.

24. Does the standard deviation of a data set equaling zero imply that all entries in the data set equal zero?

25. Is it possible for a data set to have a standard deviation of –2.5?

26. Explain the difference between Amelia making an 82 on her precalculus exam and scoring in the 82nd percentile in mathematics on the ACT test.

27. Describe two data sets, one that might have a large variation and one that might have a small variation.

28. Suppose that, in a list of data, 37% of the data are greater than 45. True or False: Q_1 must be greater than 45.

29. Lucas received an e-mail containing the five-number summary for the company sales data that he asked for. Unfortunately, the e-mail cut off the summary labels and scrambled their order. Can you still determine which number is the first quartile, Q_1? Explain your answer.

five-number summary: 11, 17.5, 9, 13.5, 19

30. Suppose that 110 male students are surveyed and that 52% have a height less than 1.776 m.

a. True or False: Of those surveyed, the mean height must be under 1.776 m.

b. True or False: Of those surveyed, the median height must be under 1.776 m.

31. If we know that a salary of \$65,300 was in the 67th percentile in a company survey, can we determine how many employees were in the sample? Why or why not?

9.4 EXERCISES

CONCEPT CHECK

1. Describe the characteristics of a binomial experiment.

2. What are the parameters of a binomial probability model?

3. Give an example of a binomial experiment, other than the one used in the section.

4. What is the formula for the binomial probability distribution function?

5. What influences the shape of the binomial probability distribution?

6. How do you calculate the expected value of a binomial random variable? The variance? The standard deviation?

PRACTICE

7. Calculate ${}_nC_x$ for each of the following combinations of x and n.
   a. $n = 5, x = 4$
   b. $n = 10, x = 8$
   c. $n = 15, x = 1$
   d. $n = 20, x = 0$

8. Calculate ${}_nC_x$ for each of the following combinations of x and n.
   a. $n = 4, x = 2$
   b. $n = 12, x = 8$
   c. $n = 18, x = 15$
   d. $n = 23, x = 20$

9. The random variable X is a binomial random variable with $n = 9$ and $p = 0.1$.
   a. Find the expected value of X.
   b. Find the standard deviation of X.
   c. Find the probability that X equals 2. (Use the formula for $P(X = x)$.)
   d. Find the probability that X is at most 3.
   e. Find the probability that X is at least 2.
   f. Find the probability that X is less than 5.

10. The random variable X is a binomial random variable with $n = 12$ and $p = 0.8$.
   a. Find the expected value of X.
   b. Find the standard deviation of X.
   c. Find the probability that X equals 7. (Use the formula for $P(X = x)$.)
   d. Find the probability that X is at most 4.
   e. Find the probability that X is at least 1.
   f. Find the probability that X is more than 10.

APPLICATIONS

11. A real estate agent has ten properties that she shows. She feels that there is a ten percent chance of selling any one property during a week. The chance of selling any one property is independent of selling another property.

a. What probability model would be appropriate for describing the number of properties sold each week?

b. Compute the expected number of properties to be sold in a week.

c. Compute the standard deviation of the number of properties sold each week.

d. Compute the probability of selling one property in one week.

e. Compute the probability of selling five properties in one week.

f. Compute the probability of selling at least three properties in one week.

12. A small commuter airline is concerned about reservation no-shows and, correspondingly, how much they should overbook flights to compensate. Assume their commuter planes will hold 15 people. Industry research indicates that 20% of the people making a reservation will not show up for a flight. Whether or not one person takes the flight is considered to be independent of other persons holding reservations.

a. What probability model would be appropriate for the number of passengers that actually take the flight?

b. If the airline decides to book 18 people for each flight, how often will there be at least one person who will not get a seat?

c. If they book 17 people, how often will there be at least one person who will not get a seat?

d. If they book 16 people, how often will there be at least one person who will not get a seat?

e. If they book 18 people for each flight, how often will there be one or more empty seats?

f. If they book 17 people, how often will there be one or more empty seats?

g. If they book 16 people, how often will there be one or more empty seats?

h. Based on the results from parts **b.** to **g.** above, which booking policy do you prefer? Explain your answer.

13. Seven plants are operated by a garment manufacturer. They feel there is a ten percent chance for a strike at any one plant and the risk of a strike at one plant is independent of the risk of a strike at another plant. Let X = the number of plants of the garment manufacturer that strike.

a. Determine the probability distribution for X.

b. Interpret the results for $P(X=0)$, $P(X=4)$, and $P(X=7)$.

c. Compute the expected value of X.

d. Compute the standard deviation for X. Is this value large in relation to the expected value? In what units is the standard deviation expressed?

14. A company that makes traffic signal lights buys switches from a supplier. Out of each shipment of 1000 switches, the company will take a random sample of 10 switches. Let X equal the number of defective switches in the sample.

a. The company has a policy of rejecting a lot if they find any defective switches in the sample. What is the probability that the shipment will be accepted if, in fact, 2% of the switches are actually defective?

b. What is the probability that the shipment will be accepted if the percentage of defective switches is actually 5%?

c. The company decides to change their policy and will accept the lot if they find no more than one defective switch. Repeat parts **a.** and **b.** for this new policy.

15. Parents have always wondered about the sex of a child before it is born. Suppose that the probability of having a male child was 0.5, and that the sex of one child is independent of the sex of other children.

a. Determine the probability of having exactly two girls out of four children.

b. What is the probability of having four boys out of four children?

16. A certain aspirin is advertised as being preferred by 4 out of 5 doctors. If the advertisement is assumed to be true, answer the following questions.

a. What is the probability that at least half of ten doctors chosen at random will prefer this brand of aspirin?

b. What is the probability that 9 out of 10 of the doctors will prefer this brand?

17. In manufacturing integrated circuits, the yield of the manufacturing process is the percentage of good chips produced by the process. The probability that an integrated circuit manufactured by the Ace Electronics Company will be defective is $p = 0.05$. If a random sample of 15 circuits is selected for testing, answer the following questions.

a. What is the probability that no more than one integrated circuit will be defective in the sample?

b. What is the expected number of defective integrated circuits in the sample?

18. The Alvin Secretarial Service procures temporary office personnel for major corporations. They have found that 90% of their invoices are paid within 10 working days. If a random sample of 12 invoices is checked, answer the following questions.

a. What is the probability that all of the invoices will be paid within 10 working days?

b. What is the probability that six or more of the invoices will be paid within 10 working days?

19. An experiment consists of rolling a pair of dice 10 times. On each roll the sum of the dots on the two dice is noted.

a. Find the probability that on any roll of the two dice the sum of the dots is either 7 or 11.

b. Find the probability that in the 10 rolls of the pair of dice, a 7 or 11 occurs 5 times.

c. Find the probability that in the 10 rolls of the pair of dice, a 7 or 11 does not occur at all.

d. Find the mean and variance of the number of times we see a 7 or 11 in the 10 rolls of the dice.

20. "Would you say you eat to live or live to eat?" was asked to each person in a sample of 1001 adults in a Gallup Poll taken in April 1996. Seventy-four percent of the respondents answered eat to live, 23% answered live to eat, and 3% had no opinion. Assuming these percents are accurate, find the probability, in 12 randomly chosen adults, that the number who would answer "eat to live" is

a. exactly 7,

b. no more than 10,

c. at most 11,

d. at least 3.

Solution

The first thing we need to do is find a z-score for the data point we are interested in of 2000 calories. Substituting into the formula, we have the following.

$$z = \frac{2000 - 2050}{175} \approx -0.29$$

Now, because we're interested in knowing the percentage that consumes more than 2000 calories, we let -0.29 be the lower bound. Our upper bound in this case is ∞. That gives us `normalcdf(-Ø.29,1E99)`. Calculating this using a TI-84 Plus gives a result of approximately `Ø.6141`. This means that 61.41% of females consume more than 2000 calories per day.

Skill Check Answers

1. $z \approx 0.68$

9.5 EXERCISES

PRACTICE

Calculate the z-score for each given value. Round your answer to the nearest hundredth.

1. $\mu = 57, \sigma = 11$
   a. $x_1 = 63$
   b. $x_2 = 38$
   c. $x_3 = 58$

2. $\bar{x} = 1123, s = 241$
   a. $x_1 = 1284$
   b. $x_2 = 900$
   c. $x_3 = 1364$
   d. $x_4 = 1123$

3. $\bar{x} = 3.19, s = 0.06$
   a. $x_1 = 3.13$
   b. $x_2 = 3.22$
   c. $x_3 = 3.00$

4. $\mu = 178.15, \sigma = 49.3$
   a. $x_1 = 73.9$
   b. $x_2 = 267.3$
   c. $x_3 = 199.5$

5. Scores on a test have a mean of 73 and a standard deviation of 11. Steve has a score of 68. Convert Steve's score to a z-score.

Answer each question thoughtfully.

6. The annual rainfall in a town has a mean of 47.22 inches and a standard deviation of 10 inches. Last year there was 51 inches of rain. How many standard deviations from the mean is that?

7. Mason's weekly poker winnings have a mean of \$144 and a standard deviation of \$51. Last week he won \$165. How many standard deviations from the mean is that?

Use the z-score formula to complete each table.

8. Find the missing value in each row of the table.

	z	x	μ	σ
a.		82.1	74.0	6.3
b.	1.05	162.3		8.9
c.	3.04		34.5	5.02
d.	–2.73	379	634	

9. Find the missing value in each row of the table.

	z	x	μ	σ
a.		4.33	6.10	2.04
b.	–2.39	–57		139.8
c.	0.58		118	21.2
d.	2.78	68	43	

Find the percentage of data points that lie below each z-score.

10. $z = -0.19$

11. $z = 1.46$

12. $z = 3.07$

13. $z = -2.22$

14. $z = 0$

Find the percentage of data points that lie above each z-score.

15. $z = 1.03$

16. $z = -1.87$

17. $z = -3.10$

18. $z = 2.84$

19. $z = 0$

Find the percentage of data points that lie between each pair of z-scores.

20. $z_1 = -1.00$, $z_2 = 1.00$

21. $z_1 = -2.40$, $z_2 = 1.73$

22. $z_1 = 2.00$, $z_2 = 3.00$

23. $z_1 = -3.01$, $z_2 = -0.56$

24. $z_1 = 0$, $z_2 = 2.61$

Find the percentage of data points that lie below z_1 and above z_2.

25. $z_1 = -1.10$, $z_2 = 1.10$

26. $z_1 = -2.84$, $z_2 = 2.84$

27. $z_1 = -1.75$, $z_2 = 0.53$

28. $z_1 = 1.09$, $z_2 = 2.88$

29. $z_1 = -0.01$, $z_2 = 0.02$

APPLICATIONS

30. Ava scored a 92 on a test with a mean of 71 and a standard deviation of 15. Charlotte had a score of 688 on a test with a mean of 493 and a standard deviation of 150. Which score was better with respect to their test? Assume the distributions of scores are approximately normal for both tests.

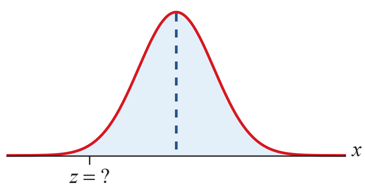
31. Avery started training to run a 5K. Her first race was a 5K for charity. She finished in 37.3 minutes. The average race time for the charity run was 36.42 with a standard deviation of 1.73 minutes. In her second race, Avery finished in 36.5 minutes. The race had a mean time of 33.02 minutes with a standard deviation of 2.45 minutes. In which race did Avery place higher in the list of finishers? Assume the distributions of finishing times are approximately normal for both races.

32. The average IQ score for adults is 100 with a standard deviation of 15. Assume that the distribution of IQ scores is approximately normal.
- **a.** Find the percentage of adults who have an IQ score less than 90.
- **b.** Find the percentage of adults who have an IQ score which exceeds the mean by at least 15 points.
- **c.** Find the percentage of adults who have an IQ score between 100 and 120.
- **d.** Find the percentage of adults who have an IQ score less than 55 or more than 145.

33. Assume the weights of offensive linemen in the NFL follow a normal distribution with a mean of 300 pounds and a standard deviation of 12.3 pounds.
- **a.** Find the percentage of linemen in the NFL who weigh more than 320 pounds.
- **b.** Find the percentage of linemen in the NFL who weigh between 275 and 325 pounds.
- **c.** Find the percentage of NFL linemen who weigh at least 260 pounds.
- **d.** Find the percentage of NFL linemen who weigh at most 315 pounds.

WRITING & THINKING

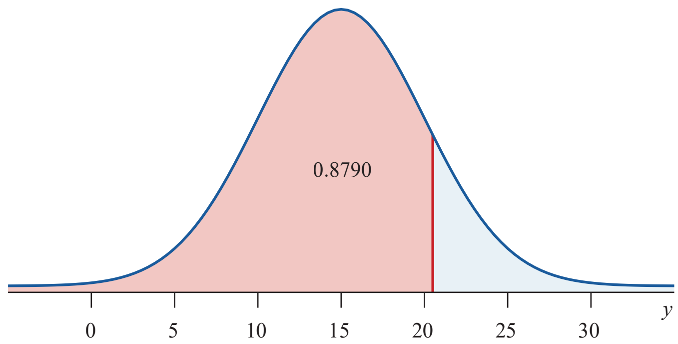
34. The mean score for a set of data is marked by the dotted line on the following graph. Which value is a likely z-score for the indicated value? Choose from **a.** -2.1, **b.** 0, or **c.** 2.7.



35. What is the minimum z-score that a piece of data would need to have in order to be in the top 10% of a normally distributed set of data?

36. What is an "average" z-score? Explain your answer.

37. What z-score represents the 1st quartile? 2nd quartile? 3rd quartile?



Thus, using the normal approximation and continuity correction, the probability that the restaurant will have no more than 20 no-shows is 0.8790. Notice that the continuity correction has a significant impact on the accuracy of the approximation. Using the binomial distribution, the exact probability is 0.8775.

9.6 EXERCISES

✔ CONCEPT CHECK

1. Why would you want to use the normal distribution to approximate a binomial distribution?

2. What are the parameters of a normal distribution used to approximate a binomial distribution?

3. What is continuity correction? How does it improve the normal approximation to the binomial?

PRACTICE

4. Consider the probability that fewer than 15 out of the 123 people watching a movie have already read the book. Assume that the probability of a given person having read the book is 40%. Verify that a normal distribution can be used to approximate the binomial probability, or show how the conditions have not been met.

5. Consider the probability that at most 2 out of 30 television sets on an assembly line are defective. Assume that the probability of a given television set being defective is 5%. Verify that a normal distribution can be used to approximate the binomial probability, or show how the conditions have not been met.

APPLICATIONS

Solve each problem. Use a normal distribution to approximate each probability.

6. Management at a small engineering company is considering the addition of a company cafeteria area. A random sample of 50 persons out of the total number of persons employed by the firm will be surveyed to see if they are in favor of the addition. Assume that the true percentage of persons that favor the addition is 90%.
   a. Find the expected number of employees in the sample who will favor the addition of the cafeteria area.
   b. Find the standard deviation of the number of employees in the sample who will favor the addition of the cafeteria area.
   c. What is the probability that between 35 and 37 employees (inclusive) in the sample will favor the cafeteria?
   d. What is the probability that more than 40 of the employees in the sample will favor the cafeteria?
   e. What is the probability that at most 38 of the employees in the sample will favor the cafeteria?

7. The accounting department of a large corporation checks the addition of expense reports submitted by executives before paying them. Historically, they have found that 15% of the reports contain addition errors. An auditor randomly selects 60 expense reports and audits them for addition errors.
   a. Find the expected number of reports in the sample that will have addition errors.
   b. Find the standard deviation of the number of reports sampled that will have addition errors.
   c. Find the probability that fewer than 10 of the sampled expense reports will have addition errors.
   d. Find the probability that at least 30 of the sampled expense reports will have addition errors.
   e. Find the probability that between 5 and 15 (inclusive) of the sampled expense reports will have addition errors.

8. A local electronics store purchased a market research study which suggests that 60 percent of all homes have gaming systems. A sample of 200 homes is selected to confirm the study's findings. If the marketing study is correct, answer the following questions.
   a. Find the expected number of homes sampled which will have gaming systems.
   b. Find the standard deviation of the number of homes in the sample which will have gaming systems.
   c. What is the probability that at most 80 of the sampled homes will have gaming systems?
   d. What is the probability that between 100 and 120 (inclusive) homes sampled will have gaming systems?
   e. What is the probability that at least 130 of the sampled homes will have gaming systems?

9. Suppose a virus is believed to infect two percent of the population. If a sample of 3000 randomly selected subjects are tested, answer the following questions.

a. Find the expected number of subjects sampled that will be infected.

b. Find the standard deviation of the number of subjects sampled that will be infected.

c. What is the probability that fewer than 30 of the subjects in the sample will be infected?

d. What is the probability that between 40 and 80 (inclusive) of the subjects in the sample will be infected?

e. Find the probability that at least 70 of the subjects in the sample will be infected.

Example 3: Finding Infinite One-Sided Limits

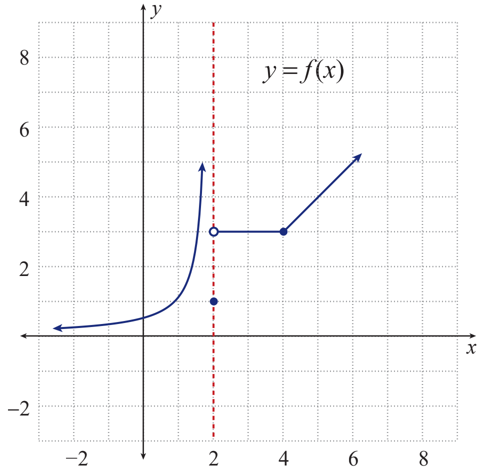
a. Find $\lim_{x\to-3^-}\left(\frac{x+5}{x+3}\right)$. **b.** Find $\lim_{x\to-3^+}\left(\frac{x+5}{x+3}\right)$.

Solution

a. As x approaches -3 from the left, the denominator will always be negative but will approach 0; the absolute value of the denominator will get smaller and smaller. For example, $-3.1+3=-0.1$, $-3.01+3=-0.01$, and so on. Meanwhile, the numerator will approach $-3+5=2$. Thus the fraction $\frac{x+5}{x+3}$ will become very large in the negative sense (or unbounded in the negative direction). So $\lim_{x\to-3^-}\left(\frac{x+5}{x+3}\right)=-\infty$.

b. Here, as x approaches -3 from the right, $x+3$ will always be positive. For example, $-2.9+3=+0.1$, $-2.99+3=+0.01$, and so on. Thus the denominator will approach 0 through positive values, and the numerator will approach $-3+5=2$. Therefore, the fraction $\frac{x+5}{x+3}$ will become unbounded in the positive direction, and we have $\lim_{x\to-3^+}\left(\frac{x+5}{x+3}\right)=+\infty$.

Example 4: Finding One-Sided Limits Using a Graph



Study the graph shown for $y=f(x)$ and find the following one-sided limits.

a. $\lim_{x\to2^-} f(x)$ **b.** $\lim_{x\to2^+} f(x)$ **c.** $\lim_{x\to4^-} f(x)$ **d.** $\lim_{x\to4^+} f(x)$

Solution

a. $\lim_{x\to2^-} f(x)=+\infty$ **b.** $\lim_{x\to2^+} f(x)=3$

Note: $f(2)=1$ according to the graph, but this fact does not affect either the left- or right-hand limits in parts **a.** and **b.**

c. $\lim_{x\to4^-} f(x)=3$ **d.** $\lim_{x\to4^+} f(x)=3$

10.1 EXERCISES

PRACTICE

In Exercises 1–6, determine the limits. In each case, make a suitable table, with four values, to support your answer. Choose the fourth value ±0.001 from the indicated *a*-value.

1. $\lim_{x\to7^-}\left(\frac{x^2-49}{x-7}\right)$ **2.** $\lim_{x\to7^+}\left(\frac{x^2+49}{x-7}\right)$

3. $\lim\limits_{x\to 3^+}\left(\dfrac{x^3-9x^2+27x-27}{x-3}\right)$

4. $\lim\limits_{h\to 0^+}\left(\dfrac{\sqrt{4+h}}{h}\right)$

5. $\lim\limits_{a\to 1^+}\left(\dfrac{a^{10}-1}{a-1}\right)$

6. $\lim\limits_{n\to \sqrt{2}^-}\left(\dfrac{n^2-2}{n-\sqrt{2}}\right)$

Given the table for $\lim\limits_{x\to a^+} f(x)$ in Exercise 7 and $\lim\limits_{x\to a^-} f(x)$ in Exercises 8–9, **a.** give the value for a and **b.** determine the limit, if there is one.

7.

x	y
2.500	0.2222
2.100	0.2439
2.010	0.2494
2.001	0.2499

8.

x	y
3.800	15.60
3.900	15.80
3.990	15.98
3.999	15.998

9.

x	y
3.000	3.43
3.100	11.99
3.140	313.90
3.141	843.60

10. a. Complete the table.

x	$f(x)=3x-1$
1	
1.4	
1.8	
1.9	
1.99	
1.999	

b. Find $\lim\limits_{x\to 2^-}(3x-1)$.

11. a. Complete the table.

x	$f(x)=x^2-2$
0	
–0.4	
–0.8	
–0.9	
–0.99	
–0.999	

b. Find $\lim\limits_{x\to -1^+}(x^2-2)$.

12. a. Complete the table.

x	$f(x)=\dfrac{x^2-1}{x+1}$
2	
1.6	
1.2	
1.1	
1.01	
1.001	

b. Find $\lim\limits_{x\to 1^+}\left(\dfrac{x^2-1}{x+1}\right)$.

13. a. Complete the table.

x	$f(x)=x^2+3$
2	
2.4	
2.8	
2.9	
2.99	
2.999	

b. Find $\lim\limits_{x\to 3^-}(x^2+3)$.

14. **a.** Complete the table.

x	$f(x)=\dfrac{1}{x-4}$
3	
3.4	
3.8	
3.9	
3.99	
3.999	

b. Find $\lim\limits_{x\to 4^-}\left(\dfrac{1}{x-4}\right)$.

15. **a.** Complete the table.

x	$f(x)=\dfrac{x}{x+2}$
−3	
−2.6	
−2.2	
−2.1	
−2.01	
−2.001	

b. Find $\lim\limits_{x\to -2^-}\left(\dfrac{x}{x+2}\right)$.

16. **a.** Complete the table.

x	$f(x)=\dfrac{x^2-4}{x+2}$
−1	
−1.4	
−1.8	
−1.9	
−1.99	
−1.999	

b. Find $\lim\limits_{x\to -2^+}\left(\dfrac{x^2-4}{x+2}\right)$.

17. **a.** Complete the table.

x	$f(x)=\dfrac{x-3}{x^2-2x-3}$
4	
3.6	
3.2	
3.1	
3.01	
3.001	

b. Find $\lim\limits_{x\to 3^+}\left(\dfrac{x-3}{x^2-2x-3}\right)$.

Find In Exercises 18–23, use the graph of $y=f(x)$ to find the limits.

18. $\lim\limits_{x\to -1^-} f(x)$

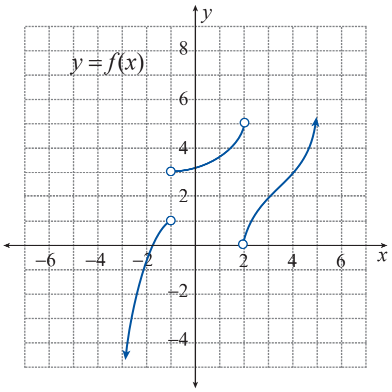
19. $\lim\limits_{x\to -1^+} f(x)$

20. $\lim\limits_{x\to 2^-} f(x)$

21. $\lim\limits_{x\to 2^+} f(x)$

22. $\lim\limits_{x\to 3^-} f(x)$

23. $\lim\limits_{x\to 3^+} f(x)$



In Exercises 24–29, use the graph of $y = f(x)$ to find the limits.

24. $\lim\limits_{x\to -1^-} f(x)$

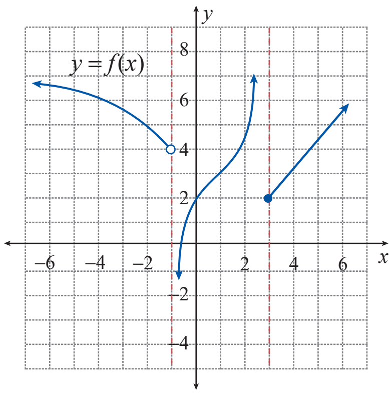
25. $\lim\limits_{x\to -1^+} f(x)$

26. $\lim\limits_{x\to 0^-} f(x)$

27. $\lim\limits_{x\to 0^+} f(x)$

28. $\lim\limits_{x\to 3^-} f(x)$

29. $\lim\limits_{x\to 3^+} f(x)$



In Exercises 30–35, use the graph of $y = f(x)$ to find the limits.

30. $\lim\limits_{x\to 0^-} f(x)$

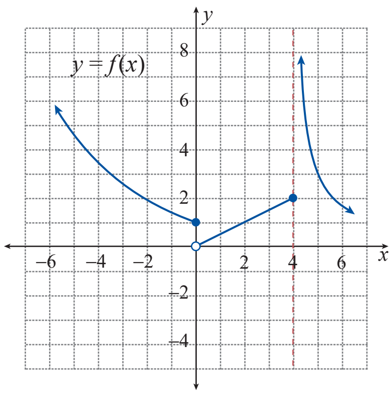
31. $\lim\limits_{x\to 0^+} f(x)$

32. $\lim\limits_{x\to 4^-} f(x)$

33. $\lim\limits_{x\to 4^+} f(x)$

34. $\lim\limits_{x\to 2^-} f(x)$

35. $\lim\limits_{x\to 2^+} f(x)$



Find the one-sided limits indicated in Exercises 36–59.

36. $\lim\limits_{x\to 2^+} (5x-3)$

37. $\lim\limits_{x\to -1^+} (2x+7)$

38. $\lim\limits_{x\to 0^-} (4-3x)$

39. $\lim\limits_{x\to 3^-} (1-6x)$

40. $\lim\limits_{x\to 2^-} (x^2-3x+1)$

41. $\lim\limits_{x\to -5^+} (x^2+4x-2)$

42. $\lim\limits_{x\to -4^+} (x^2-x+3)$

43. $\lim\limits_{x\to -3^-} (x^2+2x-3)$

44. $\lim\limits_{x\to 10^-} (0.01x^2+7x-30)$

45. $\lim\limits_{x\to 10^+} (0.2x^2-5x+6)$

46. $\lim\limits_{x\to 0^+} \left(\dfrac{x-3}{x}\right)$

47. $\lim\limits_{x\to 0^-} \left(\dfrac{2x+1}{x}\right)$

48. $\lim\limits_{x\to 1^+} \left(\dfrac{x-2}{x-1}\right)$

49. $\lim\limits_{x\to 1^-} \left(\dfrac{x-2}{x-1}\right)$

50. $\lim\limits_{x\to 2^-} \left(\dfrac{1}{x+2}\right)$

51. $\lim\limits_{x\to 2^+} \left(\dfrac{1}{x+2}\right)$

52. $\lim\limits_{x\to 3^+} \left(\dfrac{1}{x+1}\right)$

53. $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-5}\right)$

54. $f(x)=\begin{cases}2-3x & \text{if } x<2\\ x-1 & \text{if } x\geq 2\end{cases}$

a. $\lim\limits_{x\to 2^-} f(x)$

b. $\lim\limits_{x\to 2^+} f(x)$

55. $f(x)=\begin{cases}x^2+2 & \text{if } 0\leq x\leq 3\\ 2x+5 & \text{if } x>3\end{cases}$

a. $\lim\limits_{x\to 3^-} f(x)$

b. $\lim\limits_{x\to 3^+} f(x)$

56. $f(x)=\begin{cases}3x+1 & \text{if } 0\leq x\leq 4\\ x^2-3 & \text{if } x>4\end{cases}$

a. $\lim\limits_{x\to 4^-} f(x)$

b. $\lim\limits_{x\to 4^+} f(x)$

57. $f(x)=\begin{cases}x^3 & \text{if } x<2\\ x^2+5 & \text{if } x\geq 2\end{cases}$

a. $\lim\limits_{x\to 2^-} f(x)$

b. $\lim\limits_{x\to 2^+} f(x)$

58. $f(x)=\begin{cases}3-2x & \text{if } x<1\\ x & \text{if } 1\leq x\leq 4\\ \dfrac{1}{x-4} & \text{if } x>4\end{cases}$

a. $\lim\limits_{x\to 1^-} f(x)$ **b.** $\lim\limits_{x\to 1^+} f(x)$

c. $\lim\limits_{x\to 4^-} f(x)$ **d.** $\lim\limits_{x\to 4^+} f(x)$

59. $f(x)=\begin{cases}x^2-1 & \text{if } 0\leq x<2\\ 3 & \text{if } 2\leq x\leq 5\\ \dfrac{1}{x-5} & \text{if } x>5\end{cases}$

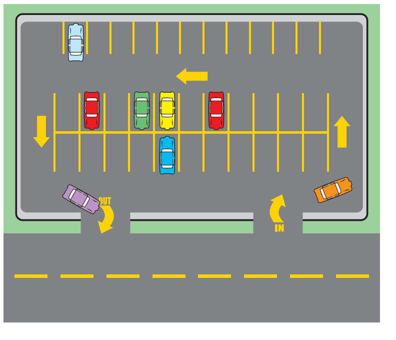
a. $\lim\limits_{x\to 2^-} f(x)$ **b.** $\lim\limits_{x\to 2^+} f(x)$

c. $\lim\limits_{x\to 5^-} f(x)$ **d.** $\lim\limits_{x\to 5^+} f(x)$

APPLICATIONS

60. Parking rates: The rate for parking in the short-term lot (maximum of 24 hours) at the airport is \$1.00 for the first hour plus \$0.75 for each additional hour or part thereof, with a maximum cost of \$7.00. The function for the cost of parking on this lot for t hours (up to 24 hours) is as follows.

$$C(t)=\begin{cases}1.00 & \text{for } 0<t\leq 1\\ 1.75 & \text{for } 1<t\leq 2\\ 2.50 & \text{for } 2<t\leq 3\\ 3.25 & \text{for } 3<t\leq 4\\ 4.00 & \text{for } 4<t\leq 5\\ 4.75 & \text{for } 5<t\leq 6\\ 5.50 & \text{for } 6<t\leq 7\\ 6.25 & \text{for } 7<t\leq 8\\ 7.00 & \text{for } 8<t\leq 24\end{cases}$$



a. Graph the function for $0<t\leq 24$ hr.

b. Find $\lim\limits_{t\to 3^-} C(t)$.

c. Find $\lim\limits_{t\to 3^+} C(t)$.

d. Find $\lim\limits_{t\to 8^-} C(t)$.

e. Find $\lim\limits_{t\to 8^+} C(t)$.

WRITING & THINKING

61. Suppose $f(x)$ and $g(x)$ are polynomials and $f(t)=0=g(t)=0$ for some t. If $\lim\limits_{x\to t^-}\dfrac{f(x)}{g(x)}=L$, must $\lim\limits_{x\to t^+}\dfrac{f(x)}{g(x)}$ also be L?

Indeterminate Form

A limit expression of the type $\lim\limits_{x\to a}\left(\dfrac{g(x)}{f(x)}\right)$ is called an **indeterminate form** of type $\dfrac{0}{0}$ if $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} f(x) = 0$.

A strategy of solving such a problem is given by the two-step method illustrated in the previous example.

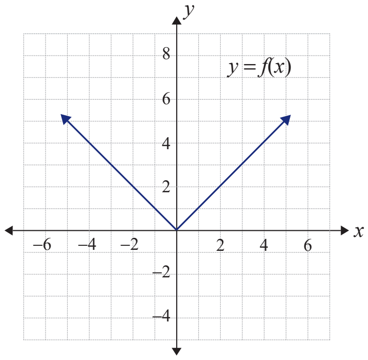
1. Replace the quotient with a simplified expression after factoring.
2. Evaluate the new limit problem by substitution if the denominator does not have a limit of 0 as $x \to a$.

10.2 EXERCISES

PRACTICE

In Exercises 1–10, use the graph to find the indicated limits, if they exist.

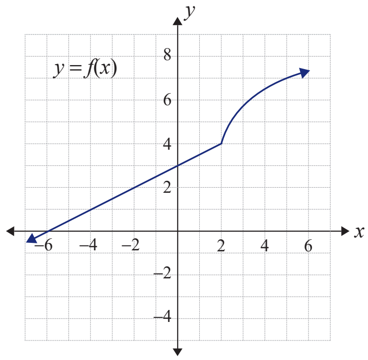
1.



a. $\lim\limits_{x\to 0^-} f(x)$ **b.** $\lim\limits_{x\to 0^+} f(x)$

c. $\lim\limits_{x\to 0} f(x)$ **d.** $\lim\limits_{x\to 2} f(x)$

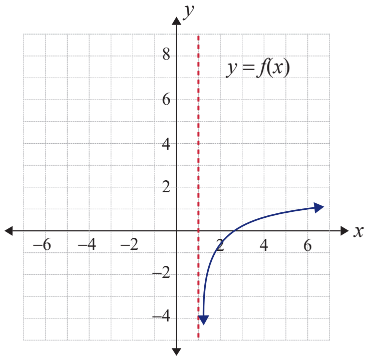
2.



a. $\lim\limits_{x\to 2^-} f(x)$ **b.** $\lim\limits_{x\to 2^+} f(x)$

c. $\lim\limits_{x\to 2} f(x)$ **d.** $\lim\limits_{x\to 0} f(x)$

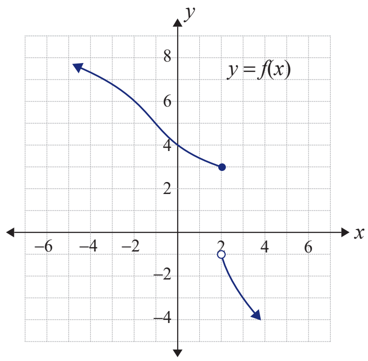
3.



a. $\lim\limits_{x\to 1^+} f(x)$ **b.** $\lim\limits_{x\to 6^-} f(x)$

c. $\lim\limits_{x\to 6^+} f(x)$ **d.** $\lim\limits_{x\to 6} f(x)$

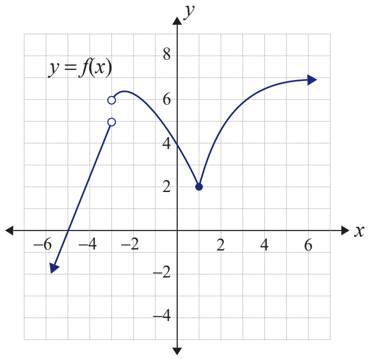
4.



a. $\lim\limits_{x\to 2^-} f(x)$ **b.** $\lim\limits_{x\to 2^+} f(x)$

c. $\lim\limits_{x\to 2} f(x)$ **d.** $\lim\limits_{x\to 0} f(x)$

5.

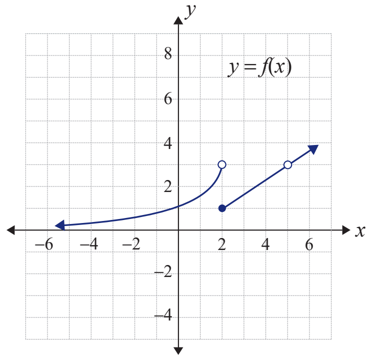


a. $\lim\limits_{x\to-3^-} f(x)$ **b.** $\lim\limits_{x\to-3^+} f(x)$

c. $\lim\limits_{x\to-3} f(x)$ **d.** $\lim\limits_{x\to1^-} f(x)$

e. $\lim\limits_{x\to1^+} f(x)$ **f.** $\lim\limits_{x\to1} f(x)$

6.

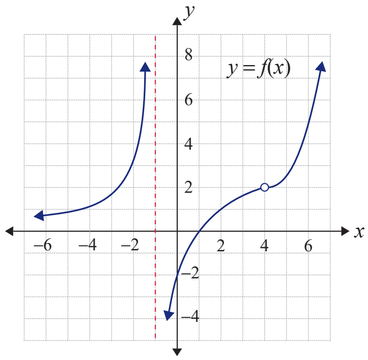


a. $\lim\limits_{x\to2^-} f(x)$ **b.** $\lim\limits_{x\to2^+} f(x)$

c. $\lim\limits_{x\to2} f(x)$ **d.** $\lim\limits_{x\to5^+} f(x)$

e. $\lim\limits_{x\to5^-} f(x)$ **f.** $\lim\limits_{x\to5} f(x)$

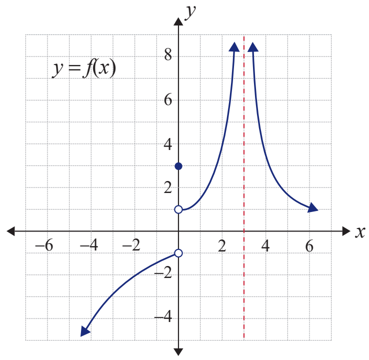
7.



a. $\lim\limits_{x\to-1^-} f(x)$ **b.** $\lim\limits_{x\to-1^+} f(x)$

c. $\lim\limits_{x\to4} f(x)$ **d.** $\lim\limits_{x\to1} f(x)$

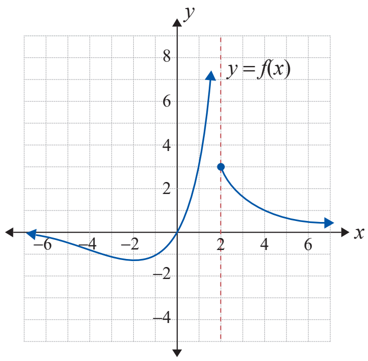
8.



a. $\lim\limits_{x\to0^-} f(x)$ **b.** $\lim\limits_{x\to0^+} f(x)$

c. $\lim\limits_{x\to3^+} f(x)$ **d.** $\lim\limits_{x\to3^-} f(x)$

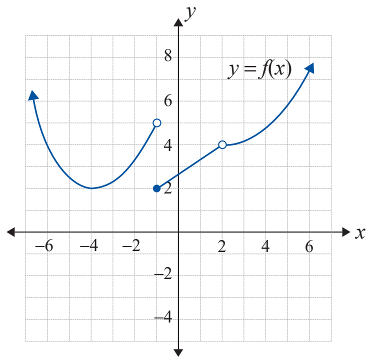
9.



a. $\lim\limits_{x\to2^-} f(x)$ **b.** $\lim\limits_{x\to2^+} f(x)$

c. $\lim\limits_{x\to2} f(x)$ **d.** $\lim\limits_{x\to0} f(x)$

10.



a. $\lim\limits_{x\to-1^-} f(x)$ **b.** $\lim\limits_{x\to-1^+} f(x)$

c. $\lim\limits_{x\to-1} f(x)$ **d.** $\lim\limits_{x\to2} f(x)$

In Exercises 11–16, determine the limit by first simplifying the expression algebraically.

11. $\lim\limits_{x \to 3} \left(\dfrac{3 - 13x + 4x^2}{x - 3} \right)$

12. $\lim\limits_{x \to 6} \left(\dfrac{x^2 - 36}{x - 6} \right)$

13. $\lim\limits_{x \to -7} \left(\dfrac{x - 7}{x^2 - 49} \right)$

14. $\lim\limits_{h \to 0} \left(\dfrac{f(3+h) - f(3)}{h} \right), f(x) = x^2 - 2$

15. $\lim\limits_{h \to 0} \left(\dfrac{f(2-h) - f(2)}{h} \right), f(x) = 1 - x + x^2$

16. $\lim\limits_{x \to 4} \left(\dfrac{x^4 - 256}{x^2 - 16} \right)$

APPLICATIONS

17. Salary: Erin is paid a weekly salary of \$12 per hour plus time-and-a-half for overtime (time in excess of 40 hours, but no more than 60 hours). Her salary is given by the function

$$S(t) = \begin{cases} 12t & \text{if } 0 < t \leq 40 \\ 480 + 18(t - 40) & \text{if } 40 < t \leq 60 \end{cases}$$

where t is the time in hours, $0 < t \leq 60$.

a. Find $\lim\limits_{t \to 40^-} S(t)$. **b.** Find $\lim\limits_{t \to 40^+} S(t)$. **c.** Find $\lim\limits_{t \to 40} S(t)$.

WRITING & THINKING

18. Suppose $f(x)$ and $g(x)$ are equal for all x-values except $x = t$.

a. Is $\lim\limits_{x \to t^-} f(x) = \lim\limits_{x \to t^-} g(x)$ true?

b. What about $\lim\limits_{x \to t^+} f(x) = \lim\limits_{x \to t^+} g(x)$?

c. Is $\lim\limits_{x \to t} f(x) = \lim\limits_{x \to t} g(x)$ necessarily true?

Example 7: Properties of Limits

Find $\lim\limits_{x\to-\infty}\left(\dfrac{x^3+x^2-x+1}{x^2-4}\right)$, if it exists.

Solution

$$\lim_{x\to-\infty}\left(\frac{x^3+x^2-x+1}{x^2-4}\right)=\lim_{x\to-\infty}\left(\frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}}{\frac{x^2}{x^3}-\frac{4}{x^3}}\right)$$

Divide each term by the highest power of x present, which in this function is x^3.

$$=\lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}{\frac{1}{x}-\frac{4}{x^3}}\right)$$

Simplify.

$$=\frac{1}{0}$$

Since $\frac{1}{0}$ is undefined, the limit is either $+\infty$ or $-\infty$. (See Example 3.)

Investigating the expression shows that the highest power is x^3. This term will dominate for very large values of x and will be negative for negative values of x. Thus

$$\lim_{x\to-\infty}\left(\frac{x^3+x^2-x+1}{x^2-4}\right)=-\infty.$$

Summary of Limits for Rational Functions as $x\to+\infty$ (or $x\to-\infty$)

Consider the function

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0},$$

where $a_n\neq 0$ and $b_m\neq 0$.

Case 1: For $m=n$, $\lim\limits_{x\to+\infty}f(x)=\dfrac{a_n}{b_m}$.

Case 2: For $m>n$, $\lim\limits_{x\to+\infty}f(x)=0$.

Case 3: For $m<n$, $\lim\limits_{x\to+\infty}f(x)=+\infty$. (Or $-\infty$ depending on the signs of a_n and b_m.)

10.3 EXERCISES

PRACTICE

In Exercises 1–28, find the indicated limit, if it exists.

1. $\lim\limits_{x\to-2}6$

2. $\lim\limits_{x\to4}2x$

3. $\lim\limits_{x\to3^-}\left(x^2+1\right)$

4. $\lim\limits_{x \to -3^+} \left(5 - 2x^2\right)$ **5.** $\lim\limits_{x \to 1^+} \left(\dfrac{x+2}{x-1}\right)$ **6.** $\lim\limits_{x \to 1^-} \left(\dfrac{x+2}{x-1}\right)$

7. $\lim\limits_{x \to \frac{1}{3}} \left(\dfrac{3x+1}{x+2}\right)$ **8.** $\lim\limits_{x \to 0^+} \left(\dfrac{x+4}{x-4}\right)$ **9.** $\lim\limits_{x \to 0^-} \left(\dfrac{x}{x^2+2x}\right)$

10. $\lim\limits_{x \to 0^-} \left(\dfrac{2x^2+x}{x}\right)$ **11.** $\lim\limits_{x \to +\infty} \left(\dfrac{x}{x^2+3}\right)$ **12.** $\lim\limits_{x \to +\infty} \left(\dfrac{2x^2+7}{3x^2-2}\right)$

13. $\lim\limits_{x \to -\infty} \left(\dfrac{x^3+64}{x^2-2x+1}\right)$ **14.** $\lim\limits_{x \to -\infty} \left(\dfrac{4x-x^3}{x^2+2x-7}\right)$ **15.** $\lim\limits_{x \to 2} \left(\dfrac{x^2-x-2}{x^2-4}\right)$

16. $\lim\limits_{x \to -3} \left(\dfrac{x^2-9}{x^2+2x-3}\right)$ **17.** $\lim\limits_{x \to 0^+} \left(4 - \dfrac{3}{x}\right)$ **18.** $\lim\limits_{x \to 1^-} \left(2x + \dfrac{5}{x-1}\right)$

19. $\lim\limits_{x \to +\infty} \left(8 + \dfrac{1}{x}\right)$ **20.** $\lim\limits_{x \to -\infty} \left(11 - \dfrac{2}{x^2}\right)$ **21.** $\lim\limits_{x \to 4} \sqrt{x+5}$

22. $\lim\limits_{x \to 2} \sqrt{3x+10}$ **23.** $\lim\limits_{x \to 1} \left(\sqrt{x}-3\right)$ **24.** $\lim\limits_{x \to 4} \left(\sqrt{x}+6\right)$

25. **a.** $\lim\limits_{x \to 4} \left(\dfrac{\sqrt{x}-2}{x-4}\right)$ [**Hint:** $x-4 = \left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)$]

b. $\lim\limits_{x \to 9} \left(\dfrac{x-9}{\sqrt{x}-3}\right)$ [**Hint:** $x-9 = \left(\sqrt{x}+3\right)\left(\sqrt{x}-3\right)$]

26. **a.** $\lim\limits_{h \to 0} \dfrac{\sqrt{2+h}-\sqrt{2}}{h}$ [**Hint:** Multiply by $\dfrac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$, simplify the numerator, and calculate the limit.]

b. $\lim\limits_{h \to 0} \dfrac{\sqrt{5x+h}-\sqrt{5x}}{h}$

27. $\lim\limits_{x \to +\infty} \left(\dfrac{x^2+3x-4}{x^2+5x-9}\right)$ **28.** $\lim\limits_{x \to -\infty} \left(\dfrac{x^2+x+1}{2x^3+3x^2+x-2}\right)$

29. For the function $f(x) = \begin{cases} -3 & \text{if } x \le 1 \\ x-4 & \text{if } x > 1 \end{cases}$, find the following:

a. $\lim\limits_{x \to 1^-} f(x)$ **b.** $\lim\limits_{x \to 1^+} f(x)$ **c.** $\lim\limits_{x \to 1} f(x)$ **d.** $\lim\limits_{x \to 2} f(x)$

30. For the function $f(x) = \begin{cases} 2x+1 & \text{if } x < 0 \\ x^2+1 & \text{if } x \ge 0 \end{cases}$, find the following:

a. $\lim\limits_{x \to 0^-} f(x)$ **b.** $\lim\limits_{x \to 0^+} f(x)$ **c.** $\lim\limits_{x \to 0} f(x)$ **d.** $\lim\limits_{x \to -2} f(x)$

31. For the function $f(x)=\begin{cases}5-x & \text{if } x\le 2\\ x^2-1 & \text{if } x>2\end{cases}$, find the following:

a. $\lim\limits_{x\to 2^-} f(x)$ **b.** $\lim\limits_{x\to 2^+} f(x)$ **c.** $\lim\limits_{x\to 2} f(x)$ **d.** $\lim\limits_{x\to 0} f(x)$

32. For the function $f(x)=\begin{cases}x^3+4 & \text{if } x\le -2\\ \sqrt{x^2+5} & \text{if } x>-2\end{cases}$, find the following:

a. $\lim\limits_{x\to -2^-} f(x)$ **b.** $\lim\limits_{x\to -2^+} f(x)$ **c.** $\lim\limits_{x\to -2} f(x)$ **d.** $\lim\limits_{x\to -1} f(x)$

APPLICATIONS

33. Utility costs: The Municipal Gas Company uses the following function for computing their customers' monthly gas bills:

$$C(x)=\begin{cases}0.37x+3.00 & \text{if } 0<x\le 24\\ 0.78x-6.84 & \text{if } x>24\end{cases},$$

where x is the number of therms (thermal units) used and $C(x)$ is the cost in dollars.

a. Find $\lim\limits_{x\to 24^-} C(x)$. **b.** Find $\lim\limits_{x\to 24^+} C(x)$. **c.** Find $\lim\limits_{x\to 24} C(x)$.

34. Income tax: A federal income tax schedule can be given by the function

$$T(x)=\begin{cases}0.15x & \text{if } 0<x\le 23{,}900\\ 0.28x-3107 & \text{if } 23{,}900<x\le 61{,}650\\ 0.33x-6189.50 & \text{if } 61{,}650<x\le 123{,}790\end{cases},$$

where x is the taxable income in dollars, $0<x\le 123{,}790$, and $T(x)$ is in dollars.

a. Find $\lim\limits_{x\to 23{,}900^-} T(x)$. **b.** Find $\lim\limits_{x\to 23{,}900^+} T(x)$.

c. Find $\lim\limits_{x\to 23{,}900} T(x)$. **d.** Find $\lim\limits_{x\to 61{,}650} T(x)$.

35. Average cost: A manufacturer of golf clubs estimates that if x sets of golf clubs are produced, then the average cost of producing each set is $A(x)=73+\dfrac{5780}{x}$ dollars. What will be the average cost of producing each set in the long run $\left(\lim\limits_{x\to+\infty} A(x)\right)$?

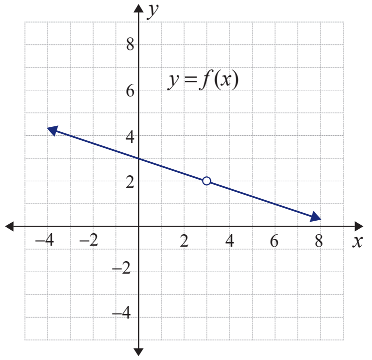
36. Dictation rate: It has been determined that after t weeks of class, a certain student in an intermediate shorthand class can take dictation at a rate of $W(t)=60+\dfrac{70t^2}{t^2+15}$ words per minute. What will be this student's rate of taking dictation in the long run $\left(\lim\limits_{t\to+\infty} W(t)\right)$?

10.4 EXERCISES

PRACTICE

In Exercises 1–8, use the graph of $y = f(x)$ to answer the questions regarding the function.

1.



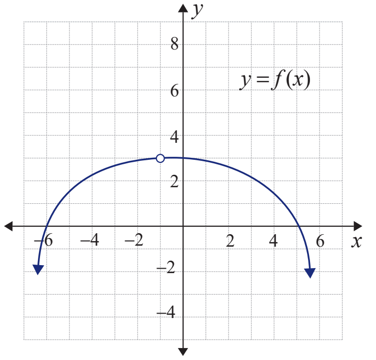
a. Find $\lim_{x \to 3^-} f(x)$.

b. Find $\lim_{x \to 3^+} f(x)$.

c. Find $f(3)$.

d. Is $f(x)$ continuous at $x = 3$? Explain.

2.



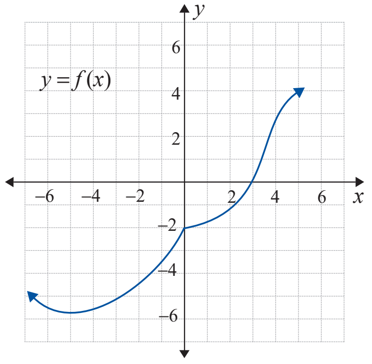
a. Find $\lim_{x \to -1^-} f(x)$.

b. Find $\lim_{x \to -1^+} f(x)$.

c. Find $f(-1)$.

d. Is $f(x)$ continuous at $x = -1$? Explain.

3.



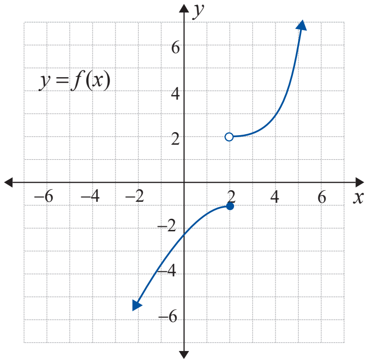
a. Find $\lim_{x \to 0^-} f(x)$.

b. Find $\lim_{x \to 0^+} f(x)$.

c. Find $f(0)$.

d. Is $f(x)$ continuous at $x = 0$? Explain.

4.



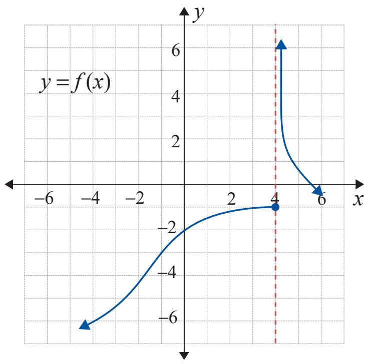
a. Find $\lim_{x \to 2^-} f(x)$.

b. Find $\lim_{x \to 2^+} f(x)$.

c. Find $f(2)$.

d. Is $f(x)$ continuous at $x = 2$? Explain.

5.



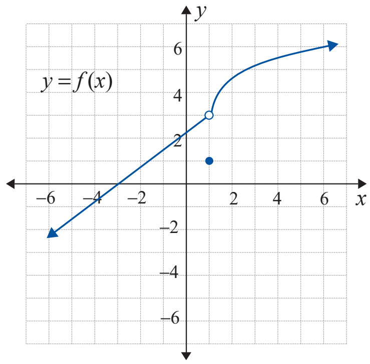
a. Find $\lim_{x \to 4^-} f(x)$.

b. Find $\lim_{x \to 4^+} f(x)$.

c. Find $f(4)$.

d. Is $f(x)$ continuous at $x = 4$? Explain.

6.



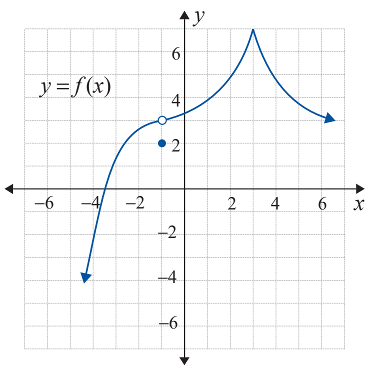
a. Find $\lim_{x \to 1^-} f(x)$.

b. Find $\lim_{x \to 1^+} f(x)$.

c. Find $f(1)$.

d. Is $f(x)$ continuous at $x = 1$? Explain.

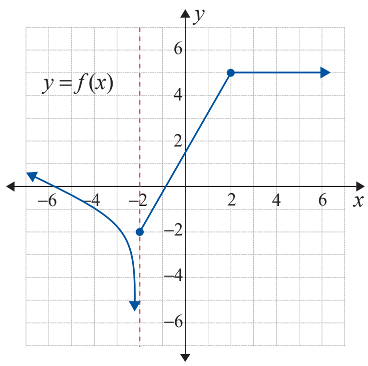
7.



a. Is $f(x)$ continuous at $x = -1$? Explain.

b. Is $f(x)$ continuous at $x = 3$? Explain.

8.

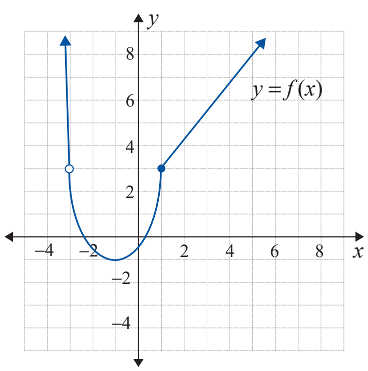


a. Is $f(x)$ continuous at $x = 2$? Explain.

b. Is $f(x)$ continuous at $x = -2$? Explain.

In Exercises 9–12, use the graph of $y = f(x)$ to find the points of discontinuity, if any exist. Determine what type of discontinuity each is.

9.



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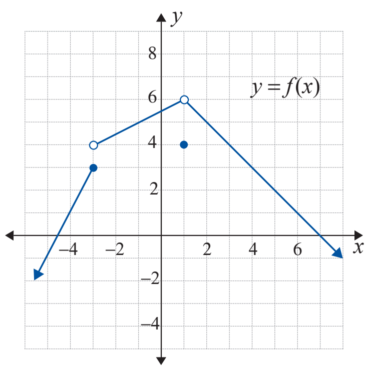

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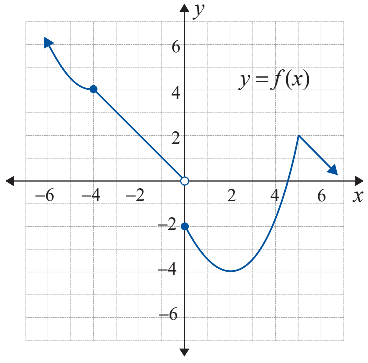

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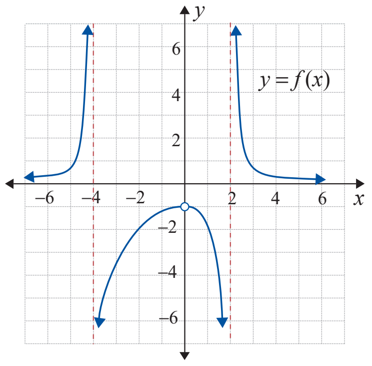


In Exercises 13–20, use the definition of continuity to determine whether or not the function is continuous at the given value of x.

13. $f(x) = 3 - 2x;\ x = 1$

14. $f(x) = 5x - x^2;\ x = 0$

15. $f(x) = \dfrac{x^2 - x - 2}{x - 2};\ x = 2$

16. $f(x) = \dfrac{x+1}{x^2 - 1};\ x = 0$

17. $f(x) = \begin{cases} 2x+1 & \text{if } x \le 1 \\ 3 & \text{if } x > 1 \end{cases};\ x = 1$

18. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases};\ x = 2$

19. $f(x) = \begin{cases} 1-3x & \text{if } x < 0 \\ 4x & \text{if } x \ge 0 \end{cases};\ x = 0$

20. $f(x) = \begin{cases} \dfrac{x}{x-3} & \text{if } x \ne 3 \\ 2 & \text{if } x = 3 \end{cases};\ x = 3$

In Exercises 21–28, find the points of discontinuity for each function, if any exist. Determine what type of discontinuity each is.

21. $f(x) = 2x^2 + 3x - 1$

22. $f(x) = 3x^2 - x + 7$

23. $f(x) = \dfrac{5}{x+3}$

24. $f(x) = \dfrac{x+8}{x}$

25. $f(x) = \dfrac{x}{x^2 - 9}$

26. $f(x) = \dfrac{2}{x^2 - 4x}$

27. $f(x) = \begin{cases} 2+3x & \text{if } x \le 1 \\ x^2+4 & \text{if } x > 1 \end{cases}$

28. $f(x) = \begin{cases} x^2+1 & \text{if } x \le 2 \\ 2x-1 & \text{if } x > 2 \end{cases}$

In Exercises 29–32, find a value for k so that the given function will be continuous at the indicated value for x.

29. $f(x) = \begin{cases} 3x & \text{if } x \le 2 \\ x^2+k & \text{if } x > 2 \end{cases};\ x = 2$

30. $f(x) = \begin{cases} 7 & \text{if } x < -3 \\ k-2x & \text{if } x \ge -3 \end{cases};\ x = -3$

31. $f(x) = \begin{cases} 3x-k & \text{if } x \le 1 \\ \dfrac{x^2-3x+2}{x-1} & \text{if } x > 1 \end{cases};\ x = 1$

32. $f(x) = \begin{cases} \dfrac{x^2+3x}{x} & \text{if } x < 0 \\ 2x^2-k & \text{if } x \ge 0 \end{cases};\ x = 0$

APPLICATIONS

33. Pricing: A leather craft store has the following pricing policy for a belt buckle:

$$C(x)=\begin{cases}0.79x & \text{if } 0<x<12\\ 0.71x & \text{if } 12\le x<50\\ 0.67x & \text{if } x\ge 50\end{cases},$$

where x is the number of buckles and $C(x)$ is in dollars.

a. Graph the function $C(x)$.

b. Is $C(x)$ a continuous function? Explain your answer.

34. Salary: A salesperson's weekly salary is determined by the function

$$S(x)=\begin{cases}550 & \text{if } 0<x<10{,}000\\ 0.06x & \text{if } x\ge 10{,}000\end{cases},$$

where x is the weekly sales.

a. Graph the function $S(x)$.

b. Is $S(x)$ a continuous function? Explain your answer.

35. Cost of telephone call: The cost of an overseas call is given by the function

$$C(x)=\begin{cases}9.00 & \text{if } 0<x\le 3\\ 0.95x+6.15 & \text{if } x>3\end{cases},$$

where $C(x)$ is in dollars and x is in minutes.

a. Graph the function $C(x)$.

b. Is $C(x)$ a continuous function? Explain your answer.

36. Revenue: The revenue from the sale of a particular model of wireless speaker is given by the function

$$R(p)=\begin{cases}84p & \text{if } 0<p\le 20\\ 144p-3p^2 & \text{if } p>20\end{cases},$$

where p is the price in dollars.

a. Graph the function $R(p)$.

b. Is $R(p)$ a continuous function? Explain your answer.

Solution

a. $\Delta d = d_3 - d_0 = 144 - 0 = 144$ ft Using data from Table 1.

b. $\dfrac{\Delta d}{\Delta t} = \dfrac{144 \text{ ft}}{3 \text{ sec}} = 48 \dfrac{\text{ft}}{\text{sec}}$ Using $\Delta t = t_3 - t_0 = 3 - 0 = 3$ sec.

c. We do not know how fast the ball is traveling when $t = 3$ seconds. This is a question of instantaneous velocity, which we will discuss later.

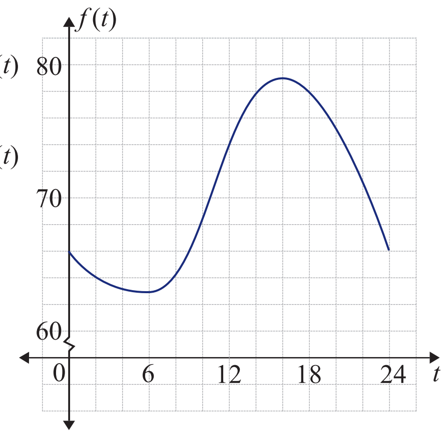
10.5 EXERCISES

PRACTICE

Use the graph to solve Exercises 1–2. The variable t is the number of hours since midnight and $f(t)$ is the temperature at time t.

1. What is the average rate of change of $f(t)$ from $t = 0$ to $t = 2$?

2. What is the average rate of change of $f(t)$ from $t = 6$ to $t = 12$?



In Exercises 3–10, find the average rate of change of the given functions between the given values of x_1 and x_2.

3. $f(x) = 5x + 3;\ x_1 = 1,\ x_2 = 3$

4. $f(x) = 3x + 8;\ x_1 = -2,\ x_2 = 1$

5. $f(x) = 2x^2 - x - 3;\ x_1 = 2,\ x_2 = 2.5$

6. $f(x) = 3x^2 - 2x - 1;\ x_1 = 1,\ x_2 = 1.5$

7. $f(x) = \dfrac{-2}{2x - 1};\ x_1 = 3,\ x_2 = 3.5$

8. $f(x) = \dfrac{2}{3x + 2};\ x_1 = 0.5,\ x_2 = 1$

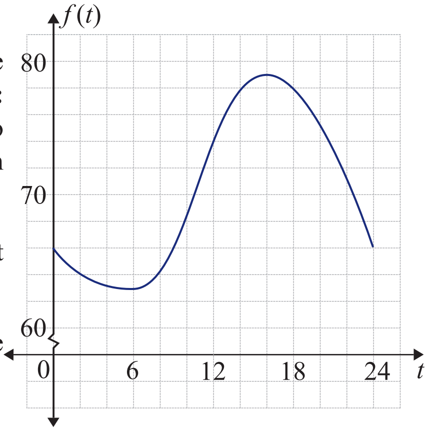
9. $f(x) = \sqrt{x};\ x_1 = 1,\ x_2 = 2.5$

10. $f(x) = \sqrt{x - 3};\ x_1 = 4,\ x_2 = 4.44$

10.6 EXERCISES

PRACTICE

Use the graph to solve Exercises 1–3. The variable t is the number of hours since midnight and $f(t)$ is the temperature at time t.



1. Roughly estimate the instantaneous rate of change of $f(t)$ at 3:00 p.m. (**Hint:** Extend an imaginary tangent line so as to come close to or to intersect points with integer coordinates.)

2. Estimate the time t for the lowest temperature $f(t)$.

3. Estimate the time t for the fastest increase in temperature $f(t)$.

4. Sketch one graph so that all of the following statements are true.

   (a) $f'(x)$ is positive for $-2 \le x \le 6$. **(b)** $f'(x) < 0$ for $x > 6$

   (c) $f'(6) = 0$ **(d)** $f(6) = 10$ and $f(0) = 1$

5. Interpret the meaning of $f(3) = 14$ and $f'(3) = 7$ for the function $f(x) = 2 + x + x^2$.

APPLICATIONS

In Exercises 6–15, determine what the slope f' represents in terms of the subject in the problem.

6. $f(t)$ is the distance in feet traveled by a car in t minutes.

7. $f(s)$ is the total money spent in a department store by s customers.

8. $f(n)$ is the number of birds nesting in woods with n trees per acre.

9. $f(x)$ is the total cost of manufacturing x toasters.

10. $f(u)$ is the total revenue from the sale of u car radios.

11. $f(t)$ is the speed of a race car after t seconds.

12. $f(v)$ is the total amount of information in bytes fed into a server at Castle Manufacturing Company in v seconds.

13. $f(x)$ is the vertical distance in meters traveled by a test rocket in x seconds.

14. $f(s)$ is the grade point average of freshmen at Sullivan Technical College where s is the average SAT score of the freshman class.

15. $f(t)$ is the cost of calculus books at college bookstores in the United States where t is the time in years since 1980.

16. Suppose $f(x)$ is the number of gallons of gas used by a car after it has traveled x miles.
- **a.** Suppose the car gets 20 miles/gallon. What is $f(100)$?
- **b.** Is $f'(100)$ positive or negative?
- **c.** Would $f'(100)$ be greater for a subcompact car or for an SUV?

17. Suppose $f(x)$ denotes the production units for input x in labor units (man-hours). Suppose $f(500) = 2000$ and $f'(500) = 3$.
- **a.** Interpret $f(500) = 2000$ and $f'(500) = 3$.
- **b.** Estimate the increased production if x is increased from 500 to 501.

18. Suppose $f(x)$ is the total number of students on a college campus that have the flu and x is the number of days after the first case is reported. Interpret $f(8) = 9$ and $f'(8) = 3$.

19. Suppose $f(x)$ is the cost of a Ford Taurus and x is the age of the car.
- **a.** Is $f'(x)$ positive or negative?
- **b.** Interpret the meaning of $f'(3) = -2500$.

20. Average prices for one-bedroom condominiums have steadily risen in Charleston, SC since 2000, according to local reports. Suppose $f(x) = 3000x + 72{,}000$ is the cost of a one-bedroom condo and x is the number of years since 2000.
- **a.** Interpret $f(0) = 72{,}000$.
- **b.** Interpret $f'(x) = 3000$.
- **c.** Interpret $f(3) = 81{,}000$.

21. Suppose $f(x)$ denotes the weight of a cancerous tumor x weeks after discovery. Interpret $f(3) = 4$ grams and $f'(3) = 0.4$ grams/week.

22. Water boils at 212° F and at 100° C. Water freezes at 32° F and 0° C. Let F denote temperature in degrees Fahrenheit and let x be temperature in degrees Celsius.
- **a.** Write a formula $F(x) = mx + b$, which can convert Celsius input x into Fahrenheit output F.
- **b.** Use the value of m to write a formula for $F'(x)$.

23. **Birth rate:** The fertility decline in many countries can be modeled by an appropriate equation. In Bangladesh, from 1970 to 2000, patterns of fertility changed according to the equation $y = -0.11x + 6.45$, where x is the time in years beginning in 1970 and y is the average number of children per woman. (**Source:** Lori Ashford, "World Population Highlights 2004," BRIDGE Population Reference Bureau, August 2004.)
- **a.** What number is $f(20)$ and what does it represent?
- **b.** What number is $f'(20)$ and what does it represent?

24. **Birth rate:** Patterns of fertility changed in India from 1970 to 2000 according to the equation $y = -0.068x + 5.22$, where x is the number of years after 1970 and y is the average number of children per woman. (**Source:** Lori Ashford, "World Population Highlights 2004," BRIDGE Population Reference Bureau, August 2004.)
- **a.** What number is $f(30)$ and what does it represent?
- **b.** What number is $f'(30)$ and what does it represent?

25. Birth rate: In China, from 1964 to the present, the death rate has remained nearly constant at about 8 deaths per 1000 persons. However, the yearly birth rate, in births per 1000 people, has declined in most years according to the formula $f(x) = -0.641x + 35.8$, where x is the number of years since 1964. (**Source:** Nancy E. Riley, "China's Population: New Trends and Challenges," *Population Bulletin*, Vol. 20, No. 2, June 2004.)

a. What number is $f(30)$ and what does it represent?

b. What number is $f'(30)$ and what does it represent?

(**Note:** See Exercise 36 in Section 10.8 for a similar problem using a more accurate model than the linear model given here. Both models are based on the same data.)

TECHNOLOGY

Use a graphing utility in Exercises 26–29 to find the slope of $f(x)$ at the given point. Sketch the graph of $f(x)$ and tangent line at the given point on your paper.

26. $f(x) = \dfrac{4+2x}{\sqrt{x}}$; $(16, 9)$

27. $f(x) = x^3$; $(3, 27)$

28. $f(x) = 2 - 3x + x^2$; $(1, 0)$

29. $f(x) = 10^x$; $(2, 100)$

30. For the function $f(x) = 4 - x - 2x^2$, find a window including the point (0, 4) so that the graph of the function and the tangent at (0, 4) are indistinguishable.

31. For the function $f(x) = x^3$, make a table with headings a, $f(a)$, and $f'(a)$. Then substitute numbers using $a = -1, 0, 1, 2, 3, 4$. Give a formula for $f'(x)$.

32. Locate (with a graphing utility) the x- and y-coordinates of the lowest point on the graph $f(x) = x^2 - 6x + 11$. What is the slope at the lowest point?

33. For the function $y = x^2$ add a column to Table 3 to include the y-intercepts of the tangent line. What curiosity do you observe in the table?

34. Find the LN button on your graphing calculator and sketch a graph of $y = \ln x$ on your calculator using the window [–2, 8] by [–0.5, 2]. What is the slope at the point (1, 0)? Sketch the graph and tangent on your paper.

35. On $f(x) = \sqrt{x}$, locate the x- and y-coordinates of the point at which the slope is exactly 1. (**Hint:** Find $(a, \sqrt{a})$ so that $f'(a) = 1$.)

36. A bacteria culture in a lab grows according to the formula $y = 1600(2^t)$ where t is time in hours and y is the quantity of bacteria.

a. Interpret the meaning of $f'(1)$.

b. Determine $f'(1)$ using a graphing utility.

37. Sketch $f(x) = (x+10)(x-5)(x-10)$ on a graphing utility. Give the window used.

a. Locate the points $(a, f(a))$ for which $f'(a) = 0$.

b. What is the value of $f'(2)$?

38. Sketch $y = \dfrac{1}{x}$ on a graphing utility. Find the x- and y-coordinates of any point with slope -25.

39. Suppose $f(x) = 2x^2$. Create a table of values for the slope of $f'(x)$ (see Table 4). Guess a formula for $f'(x)$.

10.7 EXERCISES

PRACTICE

Use the various rules of differentiation to find the derivative for each of the functions in Exercises 1–20.

1. $f(x) = 4$

2. $f(x) = 3x$

3. $f(x) = 7x - 2$

4. $y = 12$

5. $y = 4x^2$

6. $y = 8x^2$

7. $y = \dfrac{7}{x}$

8. $y = \dfrac{4}{x^5}$

9. $y = \dfrac{1}{2x^3}$

10. $g(x) = \dfrac{4}{3x^2}$

11. $g(x) = 3\sqrt{x}$

12. $h(x) = 2\sqrt[3]{x}$

13. $h(t) = t^{2.3}$

14. $h(t) = t^{-1.4}$

15. $f(x) = 3x^{0.8}$

16. $f(u) = 2u^{0.1}$

17. $f(u) = \dfrac{1}{\sqrt{u}}$

18. $f(x) = \dfrac{2}{\sqrt[4]{x}}$

19. $f(x) = -5x^{\frac{3}{4}}$

20. $f(x) = 6x^{-\frac{2}{3}}$

10.8 EXERCISES

PRACTICE

Use the various rules of differentiation to find the derivative for each of the functions in Exercises 1–10.

1. $y = x^3 - 7x$

2. $y = 4x^2 - 9x + 2$

3. $y = 0.3x^2 - 4x + 6$

4. $y = 120 + 8x - 0.2x^2$

5. $y = x^3 - 6x^2 + 5x + 2$

6. $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 3x + 4$

7. $y = 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - 5$

8. $y = 2x^{-\frac{2}{3}} + 3x^{\frac{1}{3}} + 7$

9. $f(t) = 2t^{-\frac{1}{2}} + t^{\frac{1}{2}} + t$

10. $f(x) = 3x^{-\frac{1}{3}} - 2x^{-\frac{1}{2}} + 1$

In Exercises 11–20, use algebraic techniques to rewrite each function as a sum or difference; then find the derivative.

11. $y = (x+1)(2x-3)$

12. $y = \sqrt{x}\left(2x^2 + x - 3\right)$

13. $f(v) = v^{\frac{3}{2}}\left(v^2 + 2v - 1\right)$

14. $f(v) = v^{\frac{1}{3}}\left(6 - 4v + v^2\right)$

15. $f(x) = \dfrac{x^4 + 5x^3}{x^2}$

16. $f(x) = \dfrac{6x^2 + 1}{x^3}$

17. $g(t) = \dfrac{t-2}{\sqrt{t}}$

18. $g(t) = \dfrac{t^2 + 3}{\sqrt[3]{t}}$

19. $g(x) = \dfrac{4x + 5x^{\frac{1}{2}} - 1}{\sqrt{x}}$

20. $g(x) = \dfrac{3\sqrt{x} + 4x - 2}{x^2}$

In Exercises 21–26, confirm your results with a graphing utility.

21. Let $f(x) = x^3 + 2x - 4$.
   a. Find the slope of the tangent line at $x = -1$.
   b. Find the equation of the tangent line at $x = -1$.

22. Let $f(x) = 2x^3 - x^2 - 3x$.
   a. Find the slope of the tangent line at $x = 2$.
   b. Find the equation of the tangent line at $x = 2$.

23. Let $g(x) = x^2 + 6x + 5$.
   a. Find the slopes of the tangent lines at $x = -1$, $x = -3$, and $x = -4$.
   b. Find the equations of the tangent lines at $x = -1$, $x = -3$, and $x = -4$.
   c. Sketch the graphs of the curve and the three tangent lines.

24. Let $g(x) = x^2 - 8x + 12$.
a. Find the slopes of the tangent lines at $x = 2$, $x = 4$, and $x = 5$.
b. Find the equations of the tangent lines at $x = 2$, $x = 4$, and $x = 5$.
c. Sketch the graphs of the curve and the three tangent lines.

25. Let $f(x) = 8 - x^2$.
a. Find the slopes of the tangent lines at $x = -2$, $x = 0$, and $x = 1$.
b. Find the equations of the tangent lines at $x = -2$, $x = 0$, and $x = 1$.
c. Sketch the graphs of the curve and the three tangent lines.

26. Let $f(x) = 10 - 3x - x^2$.
a. Find the slopes of the tangent lines at $x = -3$, $x = -2$, and $x = 0$.
b. Find the equations of the tangent lines at $x = -3$, $x = -2$, and $x = 0$.
c. Sketch the graphs of the curve and the three tangent lines.

APPLICATIONS

27. Velocity of a rocket: A model rocket is fired vertically upward. The height after t seconds is $s(t) = 192t - 16t^2$ feet.
a. Find the velocity at $t = 0$ seconds.
b. Find the velocity at $t = 4$ seconds.
c. When will the velocity be zero?

28. Velocity of a particle: A particle moving in a straight line is at a distance of $s(t) = 2.5t^2 + 18t$ feet from its starting point after t seconds, where $0 \le t \le 12$.
a. Find the velocity at $t = 6$.
b. Find the velocity at $t = 9$.

29. Population: A city's population t years from now can be estimated from the formula $P(t) = 9000 + 500t - 72\sqrt{t}$.
a. Find the rate at which the city is growing after 4 years.
b. Find the rate at which the city is growing after 9 years.

30. Cost: The total cost of producing x units of a product is given by $C(x) = 4000 + 25x - 0.2x^2$ dollars, where $0 \le x \le 50$.
a. Find the rate of change in the cost when $x = 10$.
b. Find the rate of change in the cost when $x = 30$.

31. Fuel consumption: When a factory operates from 6:00 a.m. to 6:00 p.m., its total fuel consumption varies according to the formula $f(t) = 0.9t^2 - 0.3t^{0.5} + 20$, where t is the time in hours after 6:00 a.m. and $f(t)$ is number of barrels of fuel oil.
a. How much fuel oil is consumed by noon?
b. What is the rate of consumption of fuel at 10:00 a.m.?
c. What is the average rate of consumption from 6:00 a.m. to 2:00 p.m.?

32. Population: The population of bacteria in a lab experiment for BIOL 403 at Nevada Tech is given by $f(x) = 2.2x^{1.5} - 0.7x + 2$, where x is the time in hours after 2:00 p.m. and $f(x)$ is population in suitable units.
a. What is the population at 2:00 p.m.?
b. The lab is over at 4:00 p.m. What is the new population of bacteria?
c. What is the average rate of change of bacteria from 2:00 p.m. to 4:00 p.m.?
d. What is the instantaneous rate of change of bacteria at 3:00 p.m.?

33. **Electrical charge:** The electrical charge on a new cell phone declines according to the formula $C(t)=15-0.1t^2-0.5t,$ where t is the time in hours following a full charge and $C(t)$ is a measure of the charge.
   - **a.** To the nearest hour, how long does one have until the charge is fully depleted?
   - **b.** What is the instantaneous rate of change, in charge units per hour, at $t=4$?
   - **c.** What is the average rate of change from $t=0$ to $t=4$?

34. **Spreading a rumor:** The number of college students at Salis Technical College who have not heard a new rumor is approximated by the formula $N(x)=300\left(1-0.004x^2\right),$ where x is the number of days following the start of a new rumor.
   - **a.** How many days does it take for 90 percent of the students to hear the rumor?
   - **b.** What is the instantaneous rate of change in students after one day? Interpret the meaning of this number.
   - **c.** Why is the slope negative in this exercise?

35. **Birth rate:** The fertility decline in many countries can be modeled by a quadratic equation. In China, from the late 1960s to the present, the number of births per woman has declined according to the formula $f(x)=0.00675x^2-0.3215x+5.585,$ where x is the number of years after 1969. (**Source:** Nancy E. Riley, "China's Population: New Trends and Challenges," *Population Bulletin*, Vol. 20, No. 2, June 2004.)
   - **a.** What was the number of births per woman in 1969?
   - **b.** What was the number of births per woman in 1999?
   - **c.** What was the rate of change of this fertility rate in 1979?

36. **Birth rate:** In China, from 1964 to the present, the death rate has remained nearly constant at approximately 8 deaths per 1000 people. The yearly birth rate, in births per 1000 people, has declined in most years according to the formula $f(x)=-0.00191x^3+0.134x^2-3.16x+44.5,$ where x is the number of years since 1964. (**Source:** Nancy E. Riley, "China's Population: New Trends and Challenges," *Population Bulletin*, Vol. 20, No. 2, June 2004.)
   - **a.** What number is $f(30)$ and what does it represent?
   - **b.** What number is $f'(30)$ and what does it represent?
   - **c.** In what year, according to the model, did the number of new births equal the number of deaths?

37. **Population growth:** The percentage of older persons in China (age 60 and over) has grown since the 1950s according to the formula $f(x)=0.003727x^2-0.105x+7.063,$ where x is the number of years since 1953. (**Source:** Nancy E. Riley, "China's Population: New Trends and Challenges," *Population Bulletin*, Vol. 20, No. 2, June 2004.)
   - **a.** What was the percentage of older persons in China in 1953?
   - **b.** What is the percentage of older persons projected to be in the year 2025?
   - **c.** At what rate was the percentage changing in 2000?

38. **Population growth:** The population in billions of people in the lesser developed countries has varied according to the formula $P(x)=\frac{4.953}{10^4}x^2-0.007352x+1.7748,$ where x is the number of years since 1900. (**Source:** Population Reference Bureau, "Transition's in World Population," *Population Bulletin*, Vol. 59, No. 1, 5, March 2004.)

a. What was the population in 1950?
b. At what rate was the population changing in 1950?
c. What population was projected for 2020 in these countries?

39. **Death rate:** The death rate in Mexico has varied since 1920 according to the formula $M(x)=-\frac{2.076}{10^4}x^3+0.033x^2-1.785x+43.07,$ where x is the number of years since 1920. (**Source:** Population Reference Bureau, "Transition's in World Population," *Population Bulletin*, Vol. 59, No. 1, 5, March 2004.)

a. What was the death rate in 1950?
b. At what rate was the death rate changing in 1950?
c. What was the death rate in 2000?
d. In what year did the formula predict that no one would die?

40. **Birth rate:** The function $f(x)=0.00375x^2-0.2355x+5.595$ gives the average number of births per woman in Thailand where x denotes the number of years since 1970. (**Source:** Lori Ashford, "World Population Highlights 2004," BRIDGE Population Reference Bureau, August 2004.)

a. What number is $f(10)$ and what does it represent?
b. What number is $f'(10)$ and what does it represent?

41. **Birth rate:** In Argentina, from 1970 to 2000, the average number of births per woman was given by the function $f(x)=-0.001x^2+0.014x+3.14,$ where x is the number of years after 1970. (**Source:** Lori Ashford, "World Population Highlights 2004," BRIDGE Population Reference Bureau, August 2004.)

a. What number is $f(30)$ and what does it represent?
b. What number is $f'(30)$ and what does it represent?

Example 5: Marginal Propensity

If the marginal propensity to save is $S'(x) = 0.02x$, what is the marginal propensity to consume?

Solution

$$\begin{aligned} C'(x) &= 1 - S'(x) \\ &= 1 - 0.02x \end{aligned}$$

We replace $S'(x)$ with $0.02x$.

10.9 EXERCISES

PRACTICE

In Exercises 1–10, create an appropriate function of the type indicated.

1. The cost of producing x leather belts is given by $C(x) = 220 + 0.4x$. Determine the average cost function.

2. The cost of manufacturing a certain class of screws for wall hangers is given by $C(x) = 800 + 0.0005x + 0.00002x^2$. Determine the average cost function.

3. A vendor charges \$3 for a hot dog. What is the demand function? What is the revenue function?

4. The Knoll Industrial Supply Company charges for 55-gallon drums using the demand function $p(x) = 22.5 - 0.5x$. What is the revenue function?

5. A supplier of souvenir T-shirts charges street vendors for each order based on a setup fee of \$50 and an item charge of \$1.15 per T-shirt. What is the cost function for the vendor?

6. A supplier of souvenir T-shirts has setup costs of \$25 and shirts cost \$0.40 each. What is the cost function?

7. The answer to Exercise 5 is also a revenue function for the supplier of T-shirts. Using the answers to Exercises 5 and 6, determine a profit function for the supplier of T-shirts.

8. For the supplier in Exercise 6, what is the average cost function?

9. A department store's cost estimate for a line of rocker-recliner chairs is given by $C(x) = 100 + 150x$. The store sells them for \$450 each.
   a. What is the average cost function?
   b. What is the demand function?
   c. What is the revenue function?
   d. What is the profit function?

10. A shoe store estimates a certain line of dress shoes has costs given by $C(x) = 1500 + 30x + 0.05x^2$. The store charges \$75 per pair.
 a. What is the average cost function?
 b. What is the demand function?
 c. What is the revenue function?
 d. What is the profit function?

11. Suppose that total national consumption is given by a function $C(x) = 200 - 0.6x - 0.05x^{0.6}$, where x is the total national income.
 a. Determine the marginal propensity to consume.
 b. Determine the marginal propensity to save.

12. If the marginal propensity to save of a certain country is given by $S'(x) = 0.4x + 0.3$, determine the marginal propensity to consume.

13. The average cost $\overline{C}(x)$ of a product is $\dfrac{C(x)}{x}$, where $C(x)$ is the total cost function.
 a. What is the average cost function if its total cost function is $C(x) = 30 + 2x + 0.003x^2$?
 b. What is the rate of change of average cost?
 c. What value of x results in a minimum average cost?

14. The average cost of a product is given by $A(x) = 20x^{-1} + 3$.
 a. Determine the cost function for the product.
 b. Determine the marginal cost function.

APPLICATIONS

15. The weekly cost of producing x electric drills is given by the function $C(x) = 2400 + 28x + 0.25x^2$.
 a. Find $C(10)$, $C(20)$, and $C(30)$.
 b. Find the marginal cost function.
 c. Find $C'(10)$, $C'(20)$, and $C'(30)$.
 d. Find the average cost function and the marginal average cost function.
 e. Find the marginal average cost when $x = 10$, $x = 20$, and $x = 30$.

16. The total cost function for producing x units of a product is given by $C(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 7x + 18$.
 a. Find $C(3)$, $C(4)$, and $C(6)$.
 b. Find the marginal cost function.
 c. Find $C'(3)$, $C'(4)$, and $C'(6)$.
 d. Find the average cost function and the marginal average cost function.
 e. Find the marginal average cost when $x = 3$, $x = 4$, and $x = 6$.

17. The total cost of producing x units of a commodity is given by $C(x) = 60 + 10x - 0.5x^2$.
 a. Find $C(4)$, $C(6)$, and $C(9)$.
 b. Find the marginal cost function.
 c. Find $C'(4)$, $C'(6)$, and $C'(9)$.
 d. Find the average cost function.
 e. Find the marginal average cost when $x = 4$, $x = 6$, and $x = 9$.

18. The total cost of producing x wireless speakers is given by the function $C(x) = 300 + 24x - 0.4x^2 + 0.1x^3$.

a. Find $C(2)$, $C(3)$, and $C(5)$.
b. Find the marginal cost function.
c. Find $C'(2)$, $C'(3)$, and $C'(5)$.
d. Find the average cost function and the marginal average cost function.
e. Find the marginal average cost if $x = 4$.

19. A manufacturer has determined that the revenue from the sale of x cell phones is given by $R(x) = 94x - 0.03x^2$ dollars. The cost of producing x cell phones is $C(x) = 10{,}800 + 34x$ dollars.

a. Find the profit function $P(x)$.
b. Find $P(200)$, $P(400)$, and $P(600)$.
c. Find the marginal profit function $P'(x)$.
d. Find $P'(200)$, $P'(400)$, and $P'(600)$.
e. Find any break-even points.

20. The revenue from the sale of x fire extinguishers is estimated to be $R(x) = 54x - 0.4x^2$ dollars. The total cost of producing x fire extinguishers is $C(x) = 400 + 30x - 0.2x^2$ dollars.

a. Find the profit function $P(x)$.
b. Find $P(20)$, $P(40)$, and $P(60)$.
c. Find the marginal profit function $P'(x)$.
d. Find $P'(20)$, $P'(40)$, and $P'(60)$.
e. Find any break-even points.

21. A company that produces and sells compact refrigerators has found that the revenue from the sale of x refrigerators is $R(x) = 100x - 0.1x^2$ dollars. The cost function is given by $C(x) = 2070 + 25x + 0.1x^2$ dollars.

a. Find the profit function $P(x)$.
b. Find $P(60)$, $P(80)$, and $P(100)$.
c. Find the marginal profit function $P'(x)$.
d. Find $P'(60)$, $P'(80)$, and $P'(100)$.
e. Find any break-even points.

22. A manufacturer has determined that the cost and the revenue of producing and selling x telescopes are $C(x) = x^2 + 20x + 1050$ dollars and $R(x) = 140x - 0.5x^2$ dollars, respectively.

a. Find the profit function $P(x)$.
b. Find $P(30)$, $P(35)$, and $P(40)$.
c. Find the marginal profit function $P'(x)$.
d. Find $P'(30)$, $P'(35)$, and $P'(40)$.
e. Find any break-even points.

23. The owner of a leather craft shop has determined that he can sell x attaché cases if the price is $p = D(x) = 46 + 0.25x$ dollars. The total cost for these cases is $C(x) = 0.15x^2 + 6x + 190$ dollars.

a. Find the revenue function $R(x)$.
b. Find the profit function $P(x)$.
c. Find $P(25)$, $P(30)$, and $P(40)$.
d. Find the marginal profit function $P'(x)$.
e. Find $P'(25)$, $P'(30)$, and $P'(40)$.

24. A firm can sell x items of a product when the price is $p = D(x) = 3.00 - 0.001x$ dollars. The total production costs are $C(x) = 0.002x^2 + 0.72x + 260$ dollars.
- **a.** Find the revenue function $R(x)$.
- **b.** Find the profit function $P(x)$.
- **c.** Find $P(300)$, $P(375)$, and $P(400)$.
- **d.** Find the marginal profit function $P'(x)$.
- **e.** Find $P'(300)$, $P'(375)$, and $P'(400)$.

25. A local publishing company prints a special magazine each month. It has been determined that x magazines can be sold monthly when the price is $p = D(x) = 5.50 - 0.0004x$. The total cost of producing the magazine is $C(x) = 0.0002x^2 + x + 4650$ dollars.
- **a.** Find the revenue function $R(x)$.
- **b.** Find the profit function $P(x)$.
- **c.** Find $P(3000)$, $P(3500)$, and $P(4000)$.
- **d.** Find the marginal profit function $P'(x)$.
- **e.** Find $P'(3000)$, $P'(3500)$, and $P'(4000)$.

26. A sales representative for a company that produces skateboards can sell x units of their deluxe model if the price is $p = D(x) = 79.9 - 0.03x$ dollars. The total cost for these skateboards is given by $C(x) = 0.08x^2 + 5.1x + 5800$ dollars.
- **a.** Find the revenue function $R(x)$.
- **b.** Find the profit function $P(x)$.
- **c.** Find $P(320)$, $P(340)$, and $P(350)$.
- **d.** Find the marginal profit function $P'(x)$.
- **e.** Find $P'(320)$, $P'(340)$, and $P'(350)$.

27. A certain model car has a valuation in dollars given by the formula $f(x) = 12{,}519.3 - 1391.1x$, for $0 \le x \le 7$, where x is the age of the car in years. $x = 0$ corresponds to this calendar year.
- **a.** What is $f(0)$? Interpret this number.
- **b.** What is the marginal valuation? Interpret this number.

28. Based on averaging results at a certain state college, a relationship between grades and SAT scores was found to be $f(s) = 1.36 + 0.00141s$, where s is a student's SAT score and $f(s)$ is the student's graduating GPA (GPA based on 4.0 maximum score).
- **a.** What is the expected GPA for a student with an SAT score of 1000?
- **b.** What is the marginal GPA? Interpret this number.

Exercises 29–31 deal with projections of world population based on estimates of fertility around the world. Use the following background information:

The Total Fertility Rate (TFR) is the average number of children a woman will have. The United Nations projects world population according to assumptions about TFR values. In general, lower values promote economic well-being and lower world population. (**Source:** Population Reference Bureau, "Transitions in World Population," *Population Bulletin*, Vol. 59, No. 1, 36, March 2004.)

29. Total fertility rate: Using a TFR of 1.5, the UN projects total world population, in billions, will be modeled by $F(t)$ where t is the number of years after 2000 and F is the function $F(t) = -\frac{2.1}{10^6}t^4 + \frac{1.6}{10^4}t^3 - 0.00457t^2 + 0.0994t + 6.03.$

a. Give the marginal population function $F'(t)$.
b. Determine the estimate of world population in 2030.
c. What is the marginal population in 2030? Interpret this number.

30. Total fertility rate: Using a TFR of 2.0, the UN projects total world population, in billions, will be modeled by $G(t)$ where t is the number of years after 2000 and G is the function $G(t) = -\frac{2.1}{10^6}t^4 + \frac{1.66}{10^4}t^3 - 0.0045t^2 + 0.113t + 6.03.$

a. Give the marginal population function $G'(t)$.
b. Determine the estimate of world population in 2030.
c. What is the marginal population in 2030? Interpret this number.

31. Total fertility rate: Using a TFR of 2.5, the UN projects total world population, in billions, will be modeled by $H(t)$ where t is the number of years after 2000 and H is the function $H(t) = -\frac{2.6}{10^6}t^4 + \frac{1.87}{10^4}t^3 - 0.00396t^2 + 0.116t + 6.05.$

a. Give the marginal population function $H'(t)$.
b. Determine the estimate of world population in 2030.
c. What is the marginal population in 2030? Interpret this number.

WRITING & THINKING

32. An economics professor claimed that the average cost was a minimum if the average cost equaled the marginal cost. Do you agree? Explain why or why not.

33. "The average revenue $\frac{R(x)}{x}$ is not usually studied in the context of business economics." Argue for or against this statement.

CAUTION

If the rules of differentiation have been followed, then an answer may be correct even though it does not "look like" the answer given. Be sure to check with your instructor to see how much algebraic simplification is expected and whether or not one form is preferred over another.

11.1 EXERCISES

PRACTICE

In Exercises 1–5, find $f'(x)$ two ways: (1) multiply the factors first, then find the derivative, and (2) use the Product Rule.

1. $f(x)=x^2\left(1+3x-2x^2\right)$

2. $f(x)=(x+3)(x-1)$

3. $f(x)=x^{\frac{1}{2}}\left(1+3x^2\right)$

4. $f(x)=x^{\frac{1}{2}}\left(1+x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)$

5. $f(x)=(2x+3)(2x-3)$

In Exercises 6–10, find $g'(x)$ two ways: (1) divide the factors first, then find the derivative, and (2) use the Quotient Rule and simplify the answer.

6. $g(x)=\dfrac{1+5x+x^2}{x}$

7. $g(x)=\dfrac{2+\sqrt{x}}{\sqrt{x}}$

8. $g(x)=\dfrac{x^2+1}{x^5}$

9. $g(x)=\dfrac{30x^2-10x^6}{5x}$

10. $g(x)=\dfrac{3x^{\frac{1}{2}}-5x^{\frac{3}{2}}+7x^{\frac{5}{2}}-9x^{\frac{7}{2}}}{x^{\frac{1}{2}}}$

In Exercises 11–34, use the Product Rule or Quotient Rule to find the derivative of each of the functions. Simplify your answers.

11. $f(x)=x^3\left(x^2+5\right)$

12. $f(x)=x^5\left(2x-x^3\right)$

13. $f(t)=t^{\frac{1}{2}}(4t+3)$

14. $f(t)=t^{\frac{2}{3}}\left(4t^2+1\right)$

15. $y=x^2\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)$

16. $y=x^{-2}\left(3x+x^{\frac{1}{3}}\right)$

17. $g(u)=\left(2u^2+3\right)(5-3u)$

18. $g(u)=\left(3u^2-8\right)\left(u^2+u\right)$

19. $g(t)=\left(5+\dfrac{1}{t}\right)\left(t^2+\dfrac{1}{5}\right)$

20. $f(t)=\left(1-\dfrac{3}{t^2}\right)\left(2t^2+t-1\right)$

21. $f(x)=\dfrac{3x}{x+6}$

22. $f(x)=\dfrac{7x^2}{2x-1}$

23. $f(x)=\dfrac{x+8}{x-7}$

24. $f(x)=\dfrac{x^2+2x-3}{x+2}$

25. $y=\dfrac{x^3-5}{x^2+1}$

26. $y=\dfrac{2x^2+3x}{x^3+6}$

27. $g(x)=\dfrac{\sqrt{x}}{x+9}$

28. $g(x)=\dfrac{6\sqrt{x}}{3x-4}$

29. $f(u)=\dfrac{u^2}{\sqrt{u}+1}$

30. $f(u)=\dfrac{7}{1-\sqrt[3]{u}}$

31. $f(t)=\dfrac{4-\sqrt{t}}{t^2+3}$

32. $f(t)=\dfrac{3-t}{4-5\sqrt{t}}$

33. $f(x)=\dfrac{x^2-5x}{1+2\sqrt[3]{x}}$

34. $f(x)=\dfrac{x\left(1+3\sqrt{x}\right)}{\sqrt{x}+6}$

In Exercises 35–44, you are given that $f(x)$ and $g(x)$ are differentiable functions and that $f(2)=3$, $f'(2)=-1$, $g(2)=-11$, and $g'(2)=6$. In each exercise, find the value of $h'(2)$.

35. $h(x)=x\cdot f(x)$

36. $h(x)=\dfrac{f(x)}{2x+1}$

37. $h(x)=\dfrac{f(x)+3x}{f(x)-3x}$

38. $h(x)=\dfrac{g(x)}{f(x)}$

39. $h(x)=\dfrac{g(x)}{3x+10}$

40. $h(x)=(3x+5)\cdot f(x)$

41. $h(x)=\dfrac{16x+1}{f(x)-11x+1}$

42. $h(x)=f(x)\cdot g(x)$

43. $h(x)=\dfrac{f(x)}{g(x)}$

44. $h(x)=g(x)\cdot(1+3x)$

In Exercises 45–50, find the equation of the line tangent to the graph of $f(x)$ at the given point.

45. $f(x)=\left(x+5x^{\frac{1}{2}}\right)\left(6x^2-12x+2\right);\ (4,700)$

46. $f(x)=\dfrac{\left(11x^2-3x+2\right)}{x^2+1};\ (1,5)$

47. $f(x)=\dfrac{2-3x}{5+2x};\ (0,0.4)$

48. $f(x)=\left(x^5-5\right)\left(x^3-x-1\right);\ (0,5)$

49. $f(x)=\dfrac{20}{17x+3};\ (1,1)$

50. $f(x)=\dfrac{\sqrt{x}+2}{x^2-1};\ \left(9,\dfrac{1}{16}\right)$

51. Given $f(x)=(1-x)\left(16-x^2\right)$, find the (x, y)-coordinates on the graph where the tangent line is horizontal.

52. Given $g(x)=(x-10)\left(x^2+2x+1\right)$, find any (x, y)-coordinates on $g(x)$ for which the tangent line is horizontal.

53. Find any point or points on the graph of $y=(x-5)(x+10)$ so that the slope equals 25. Sketch a graph of y and the tangent line or lines.

54. Find any point or points on the graph of $G(x)=(2x+1)(x-3)$ so that the slope is −20. Sketch a graph of G and the tangent line or lines.

55. Sketch a graph of $F(x)=\dfrac{30x}{2x^2+5}$ on the x-interval [−5, 10]. Determine the (x, y)-coordinates of any point with a horizontal tangent line, and sketch this (or these) horizontal tangent(s). Round to the nearest hundredth.

APPLICATIONS

56. Bacterial growth: It is estimated that the population of a bacterial culture after t hours is approximately $N(t)=\dfrac{t^2-2t}{3\sqrt{t}+2}$, where $N(t)$ is in thousands and $2\le t\le 10$. Find the rate of growth after 4 hours.

57. Marginal revenue: The demand function for a particular item is given by $D(x)=\dfrac{115}{3x+1}$. Find the marginal revenue when $x=3$.

58. Marginal profit: The profit from the sale of x items is given by $P(x)=(2-0.5x)(0.5x-5)$, where $P(x)$ is in hundreds of dollars and $2\le x\le 10$. Find the marginal profit when $x=5$.

59. Marginal cost: The cost of producing x items of a product is given by $C(x)=(0.1x+100)(0.1x+20)-600$. Find the marginal cost when $x=60$.

60. Velocity of a particle: A particle is moving slowly along a line. Its position after t seconds is $S(t)=\dfrac{t}{t^2+4}$ feet. Find the velocity when the particle has been moving for 3 seconds.

61. Population growth: It is estimated that t years from now the population of a city will be $P(t)=(0.6t-7)(0.5t+6)+85$ in thousands. How fast will the population be growing in 10 years?

To find how fast this measure was changing in 1988 (8 years since 1980), evaluate the derivative for $t = 8$.

$$P'(8) = \frac{8^3 + 72 \cdot 8}{\left(8^2 + 36\right)^{\frac{3}{2}}} = \frac{512 + 576}{(100)^{\frac{3}{2}}} = \frac{1088}{1000} = 1.088$$

In 1988, the air pollution measure was increasing at a rate of 1.088 units per year.

11.2 EXERCISES

PRACTICE

Find the derivative for each function given in Exercises 1–40 and simplify your answer.

1. $f(x) = (2x - 5)^4$

2. $f(x) = (7x + 2)^3$

3. $f(x) = (1 - 4x)^3$

4. $f(x) = (3 - 5x)^5$

5. $g(x) = \left(x^2 + 4\right)^{-2}$

6. $g(x) = \left(x^2 - 8\right)^{-1}$

7. $h(t) = \left(2t^2 + 3t\right)^{-3}$

8. $h(t) = \left(4t^2 - t\right)^{-2}$

9. $y = \left(2x^2 + 5x - 7\right)^2$

10. $y = \left(4x^2 + 9x - 3\right)^3$

11. $y = \left(x^3 + 1\right)^{\frac{1}{2}}$

12. $y = \left(2x^3 - 5\right)^{\frac{1}{3}}$

13. $y = \sqrt[3]{4x^2 + 1}$

14. $y = \sqrt{7 + 4x^2}$

15. $y = \sqrt[4]{1 - 2x^3}$

16. $y = \sqrt[3]{5x^3 - 4}$

17. $f(t) = 5t\left(t^3 + 3\right)^4$

18. $f(x) = -7x\left(x^4 - 2\right)^3$

19. $f(x) = 2x^3\left(x^2 - 8\right)^3$

20. $f(t) = t\left(4 - 3t^2\right)^2$

21. $g(x) = \dfrac{1}{\sqrt{x^2 - 6}}$

22. $g(x) = \dfrac{5}{\sqrt{x^3 + 4}}$

23. $g(t) = \dfrac{t}{\sqrt{t^2 + 8}}$

24. $g(x) = \dfrac{x^2}{\sqrt[3]{x^2 + 6}}$

25. $h(x) = \dfrac{\sqrt[3]{2x + 3}}{x^2}$

26. $h(x) = \dfrac{\sqrt{5x - 2}}{x^3}$

27. $y = (2x + 1)\sqrt{3x - 4}$

28. $y = (4x + 3)\sqrt{x^2 + 3}$

29. $y = (3x - 2)^2(5x + 1)^{-2}$

30. $y = (2x + 7)^3(3x + 1)^{-4}$

31. $f(t) = \dfrac{5t + 1}{(t - 1)^{\frac{2}{3}}}$

32. $f(t) = \dfrac{t^2 + t + 1}{\sqrt{t^4 - 1}}$

33. $g(t) = \left(\dfrac{2t + 5}{t + 1}\right)^3$

34. $g(t) = \left(\dfrac{5t + 4}{t^2 - 3}\right)^4$

35. $y = \left(\dfrac{x + 3}{4 - 2x}\right)^{\frac{1}{2}}$

36. $y = \left(\dfrac{x^2}{4x + 1}\right)^{\frac{1}{3}}$

37. $y = \sqrt{\dfrac{x + 2}{3x - 1}}$

38. $y = \sqrt{\dfrac{x^2 + 6}{x^3}}$

39. $y=\dfrac{x^2+x}{\sqrt{7-2x}}$

40. $y=\dfrac{\left(x^2+2\right)^2}{\sqrt{5x-3}}$

In Exercises 41–50, find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$. Then evaluate $\frac{dy}{dx}$ for the given value of x.

41. $y=u^2+2,\ u=3x^2+1;\ x=-1$

42. $y=\sqrt{u+4},\ u=x^2+x-1;\ x=2$

43. $y=\dfrac{1}{u^2},\ u=2x^3-3x+3;\ x=1$

44. $y=\sqrt[3]{u},\ u=2x^3-4x;\ x=2$

45. $y=u^{\frac{3}{2}},\ u=x^3-2x^2;\ x=3$

46. $y=\dfrac{1}{u^3},\ u=3x+1;\ x=1$

47. $y=\sqrt[3]{u},\ u=7x^2+1;\ x=-3$

48. $y=u^2+3u+4,\ u=x^3-5x-2;\ x=-2$

49. $y=2u^2-5u+3,\ u=5x+6;\ x=2$

50. $y=2u^3-3u+1,\ u=x^3+8;\ x=1$

In Exercises 51–56, use the given information to find $h'(2)$: $f(2)=3$, $f'(2)=-1$, $g(2)=4$, $g'(2)=10$, $g(3)=8$, and $g'(3)=7$.

51. $h(x)=\left(f(x)+1\right)^3$

52. $h(x)=\left(\dfrac{f(x)}{g(x)}\right)^2$

53. $h(x)=g\left(f(x)\right)$

54. $h(x)=\sqrt{f(x)+g(x)}$

55. $h(x)=\left(g(x)\right)^3$

56. $h(x)=\left(2+3\cdot f(x)\right)\left(g(x)\right)^3$

Determine the equation of the tangent line for $f(x)$ at the x-value indicated in Exercises 57–60.

57. $f(x)=(3x-1)^3;\ x=1$

58. $f(x)=\left(\dfrac{x^2+1}{x+1}\right)^3;\ x=2$

59. $f(x)=\left(3x^2+2x+8\right)^{\frac{1}{2}};\ x=4$

60. $f(x)=\sqrt{10x+1};\ x=8$

APPLICATIONS

61. Marginal revenue: A dealer of microwave ovens estimates that he can sell x ovens per month when the demand function (price) is $p=D(x)=20\sqrt{280-4x}$ dollars.

a. Find the revenue function $R(x)$.

b. Find $R(21)$.

c. Find the marginal revenue function $R'(x)$.

d. Find $R'(21)$.

62. **Marginal revenue:** The demand function (price) for a particular product is given by $D(x)=8\sqrt{25-5x+0.25x^2}$ dollars, where x is the number of units (in hundreds) sold.
   a. Find the revenue function $R(x)$.
   b. Find $R(2)$.
   c. Find the marginal revenue function $R'(x)$.
   d. Find $R'(2)$.

63. **Population growth:** It is estimated that t years from now the population of Castle City will be $P(t)=10(40+2t)^2-1600t$.
   a. What will the population be in 8 years?
   b. Find the rate of change in population in 8 years.

64. **Air pollution:** It is estimated that t years from now the level of air pollution in Bohrberg will be $P(t)=\dfrac{0.6\sqrt{8t^2+11t+60}}{(t+1)^2}$ parts per million. Find the rate of change in the pollution level in 7 years.

65. **Pollution:** After a sewage spill, the level of pollution in San Remo Bay is estimated by $P(t)=\dfrac{200t^2}{\sqrt{t^2+11}}$, where t is the time in days since the spill occurred. How fast is the level changing after 5 days? Round to the nearest whole number.

66. **Bacterial growth:** It is estimated that in t hours the population of bacteria in a culture will be $P(t)=\dfrac{8000}{\sqrt{8-0.5t}}$.
   a. What will be the population in 8 hours?
   b. Find the rate of change in the population in 8 hours.

67. **Rate of change of cost:** A manufacturer of vacuum cleaners estimates that the total cost of producing x vacuum cleaners is given by $C(x)=-0.5x^2+56x+800$ dollars. Records show that after t hours on a typical day, the number of units produced is given by $x=5\sqrt{t^2+5t}$. Find the rate of change of total cost with respect to time at the end of
   a. 4 hours.
   b. 5 hours.

68. **Rate of change of profit:** A manufacturer has determined that the weekly profit from the sale of x items is given by $P(x)=-x^2+280x-4000$ dollars. It is estimated that after t days in any week, $x=0.5t^2+5t$ items will have been produced. Find the rate of change of profit with respect to time at the end of
   a. 4 days.
   b. 5 days.

69. **Pollution:** Studies show that the average level of certain pollutants in the air is given by $L = 1 + 0.2x + 0.001x^2$ parts per million when the population is x thousand people. It is estimated that t years from now the population will be $x = \dfrac{200}{\sqrt{7 - 0.5t}}$ in thousands. Find the rate of change of the level of pollutants after
   a. 6 years.
   b. 12 years.

70. **Security costs:** The annual cost for campus security is given by $C(x) = 3x^2 - 32x + 16$ in thousands of dollars. It is estimated that the enrollment in t years will be $x = 16 + 0.5t + 0.02t^2$ in thousands. Find the rate of change in security costs after
   a. 3 years.
   b. 4 years.

71. **Baseball attendance:** The average home attendance per week at a Class AA baseball park varied according to the formula $N(t) = (3 + 0.2t)^{\frac{1}{2}}$, where t is the number of weeks into the season ($0 \le t \le 12$) and N is in thousands of persons.
   a. What was the attendance during the first week into the season?
   b. Determine the number $N'(5)$.
   c. Interpret the meaning of $N'(5)$.

72. **Weekly attendance:** The semester after its student team won an intercollegiate Duplicate Bridge championship, the average weekly attendance at the University Union Building varied according to the formula $B(t) = 100 - 50\left(1 - \dfrac{t}{16}\right)^{\frac{3}{2}}$, where t is the number of weeks after the championship ($0 \le t \le 16$) and B is the number of persons.
   a. What is $B(0)$ and what does it represent?
   b. Determine $B'(t)$.
   c. What is $B'(7)$? Interpret the meaning of this number.

73. **Class registration:** The annual registration in university calculus classes varies according to the formula $P(t) = 1 + \left(1 - \dfrac{t}{30}\right)^{2.5}$, where t is the number of years since 1995 and P is in millions of students.
   a. Determine $P(0)$ and explain its meaning.
   b. Determine $P'(t)$.
   c. Calculate $P'(10)$ and explain its meaning.

74. **Travel:** The number of passengers traveling from California to Central America and back on cruise ships is given by $C(t) = (10t + 50)^{1.5}$, where t is the number of years since 1990 and C is passenger count in thousands.
   a. When did the number of passengers hit 1,000,000 people ($C = 1000$)?
   b. Compute $C'(t)$.
   c. Calculate $C'(12)$ and intrepret its meaning.

$$0=(16)^{\frac{1}{4}}\cdot\frac{3}{4}(81)^{-\frac{1}{4}}\frac{dy}{dt}+(81)^{\frac{3}{4}}\cdot\frac{1}{4}(16)^{-\frac{3}{4}}(4)$$

$$0=\cancel{2}\cdot\frac{\cancel{3}}{{}_2\cancel{4}}\cdot\frac{1}{\cancel{3}}\cdot\frac{dy}{dt}+27\cdot\frac{1}{\cancel{4}}\cdot\frac{1}{8}\cdot\cancel{4}$$

$$0=\frac{1}{2}\cdot\frac{dy}{dt}+\frac{27}{8}$$

$$-\left(\frac{1}{2}\cdot\frac{dy}{dt}\right)=\frac{27}{8}$$

$$\frac{dy}{dt}=\left(-\cancel{2}\right)\frac{27}{\cancel{8}_4}$$

$$\frac{dy}{dt}=-\frac{27}{4}=-6.75$$

Because the derivative is negative, the capital must **decrease** at a rate of $\frac{27}{4}=6.75$ units per month to keep the current level of production.

11.3 EXERCISES

PRACTICE

Use implicit differentiation to find $\frac{dy}{dx}$ for each of the equations in Exercises 1–20.

1. $2x^2+y^2=4$

2. $x^3+y^3=5$

3. $2x^3+y^3=8$

4. $x^2-y^2=16$

5. $\sqrt{x}+\sqrt{y}=1$

6. $x-\sqrt{y}=2$

7. $x^2y=2$

8. $xy^2=-1$

9. $x^2+xy+y^2=-1$

10. $x^3+2xy-y^2=3$

11. $4x^2+3xy+y^2=2x$

12. $x^3+y^3=3xy$

13. $\frac{1}{x}+\frac{x}{y}=2x$

14. $x^2+\frac{2x}{y}=\frac{1}{x^2}$

15. $x^2+\sqrt{xy}=2y^2$

16. $2y+\sqrt{xy}=5x^2$

17. $x^2y^2+xy^3=x^4$

18. $x^3y+xy^3=3x^3$

19. $x^2+(y-2)^2=16$

20. $x^2+4(y+3)^2=9$

In Exercises 21–30, use implicit differentiation to find $\frac{dy}{dx}$ for the given equations; then find the slope of the tangent line at the given point.

21. $4x^2 - 8y^3 = 24;\ (2, -1)$

22. $3x^3 + 5y^2 + x = 1;\ (-1, 1)$

23. $x^2y - y^2 + 4x + 8 = 0;\ (1, 4)$

24. $5x^2 + xy^2 + 2x = 8;\ (-2, 2)$

25. $x^3 + 2xy - y^2 = 0;\ (3, -3)$

26. $4x^2 - 3xy + y^2 = 7;\ (2, 3)$

27. $\frac{1}{x} + \frac{x}{y^2} = 2x;\ (1, -1)$

28. $3x - \frac{2x}{y^2} + x^2 = 12;\ (3, 1)$

29. $2y + \sqrt{xy} = 5x^2;\ (2, 8)$

30. $x^2 - \sqrt{xy} = 2y^2 + 3x;\ (4, 1)$

In Exercises 31–40, *x* and *y* are functions of a third variable, *t*. Use implicit differentiation to find an expression for $\frac{dy}{dt}$.

31. $x^2 - 4y^2 = 16$

32. $3x^2 + y^4 = 4$

33. $x^3 + 5y^2 = 2x$

34. $6x^2 + 5x = 2y^2$

35. $\sqrt{xy} = 4$

36. $\sqrt{x} + \sqrt{y} = 3$

37. $x^2 + xy + y^2 = 3$

38. $x + 2xy - y^2 = 6$

39. $2xy + x^2 = y^3$

40. $x^2y^2 - y^3 + 4x = 0$

APPLICATIONS

41. Retail sales: The manager of an audio electronics store has determined that the number of stereo receivers and the number of speaker systems sold weekly are related by the equation $0.9y^2 = 10x + xy$, where *x* is the number of receivers and *y* is the number of speaker systems. Find $\frac{dy}{dx}$ if *x* = 12 and *y* = 20, and interpret your answer.

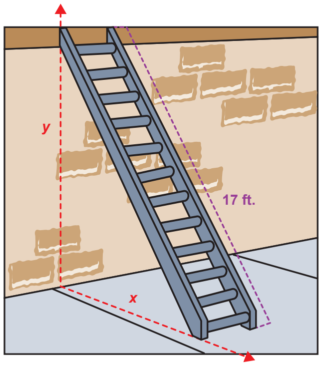
42. Retail sales: The number of pairs of trousers *x* and the number of shirts *y* sold at a department store are related by the equation $36x = 11y + 0.01x^2y$. Find $\frac{dy}{dx}$ when *x* = 10 and *y* = 30, and interpret your answer.

43. Cobb-Douglas production: The level of production of a company is given by $P = 30x^{\frac{1}{3}}y^{\frac{2}{3}}$ units monthly, where *x* is the units of labor and *y* is the units of capital. The company is currently utilizing 64 units of labor and 27 units of capital. If labor is increased by 2 units per week, what will be the change in units of capital per week to maintain the current level of production?

44. Cobb-Douglas production: The level of production of a company is given by $P = 18x^{0.3}y^{0.7}$ units monthly, where *x* is the units of labor and *y* is the units of capital. The company is currently utilizing 35 units of labor and 24 units of capital. If capital is increased by 3 units per week, what will be the change in units of labor per week to maintain the current level of production?

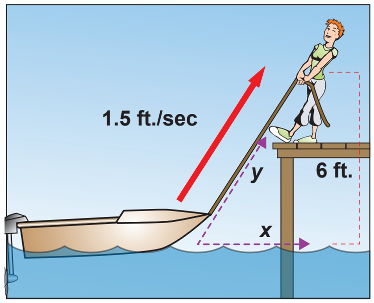
45. **Rate of increase in cost:** The cost of producing x units of a product is given by the function $C(x) = 0.02x^3 - x^2 + 8x + 200$. The factory is currently producing 60 units per week but plans to increase production at a rate of 3 units per week. What will be the rate of increase in the total cost?

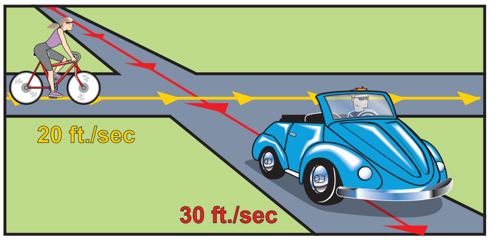
46. **Rate of decrease in cost:** The cost of producing x units of a commodity is given by the function $C(x) = x^2 - 2x^{\frac{3}{2}} + 7x + 180$. Currently, the production level is 36 units per day. The company plans to decrease production at a rate of 2 units per day. What will be the rate of decrease in the total cost?



47. **Sliding ladder:** A 17 ft ladder is leaning against a wall. The bottom of the ladder is pulled away from the wall at a rate of 3 ft/sec. How fast is the top of the ladder moving down the wall when the top is 8 feet above the ground?

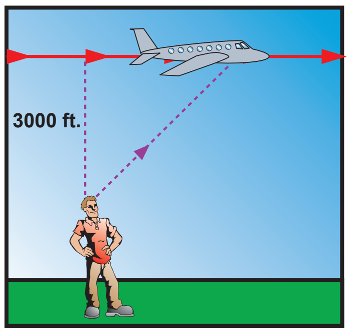
48. **Velocity:** Marijean is standing on a boat dock pulling in her boat by means of a rope attached to a boat at water level. Her hands are 6 feet above the water and she is pulling in the rope at a rate of 1.5 ft/sec. How fast is the boat approaching the dock if there are 10 feet of rope still out?

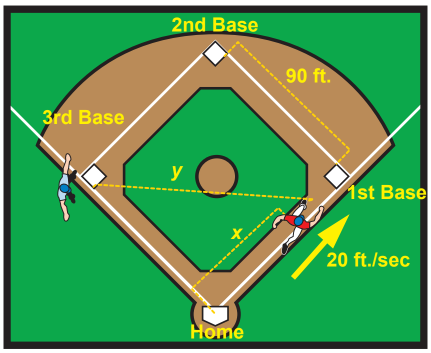




49. **Driving:** A car traveling south at 30 ft/sec crosses an intersection. When the car is 90 feet past the intersection, a bicyclist crosses the intersection traveling east at a rate of 20 ft/sec. How fast is the distance between the car and the bicycle increasing 5 seconds after the bicycle crosses the intersection?

50. **Distance to an airplane:** An airplane traveling at a height of 3000 feet crosses directly over an observer. The speed of the plane is 400 ft/sec. How fast is the distance between the observer and the plane changing after 10 seconds?





51. **Baseball:** A base runner heads towards first base with a speed of 20 ft/sec. A baseball diamond is a square, 90 feet on each side.
   a. How long will it take for the runner to reach first base?
   b. What is the runner's rate of change of distance from the umpire standing on third base (we will call this $\frac{dy}{dt}$)? (**Hint:** Use the diagram to get a relation between x and y.)
   c. What is the runner's speed when he arrives at first base?

52. Sailing: A Coast Guard radar monitoring station on shore observed a sailboat on its radar grid. It was determined that the boat's east-west distance along the shoreline changed by the formula $x = 12 + 0.1t$ and its seaward distance (north-south) changed by the formula $y = 20 - 0.3t$. Here, x and y are in miles and t is in minutes.

a. What was the sailboat's position at $t = 0$ minutes? Sketch this situation.

b. Determine $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

c. What is $\frac{dy}{dx}$ at $t = 10$? Interpret this number.

d. Where and when will the sailboat hit the shore (assuming it keeps its present heading)?

53. Bags of oranges: Suppose the wholesale price in Miami of bags of oranges satisfies a demand equation of $xp + 20p = 1040$, where x is the number of bags supplied and p is the demand (unit price) in dollars.

a. If 500 bags are available today, what is the unit price?

b. If the supply is increasing at the rate of 100 bags per day, at what rate is the price changing?

54. Dripping water: The radius of a pan of water is 7 cm and water from a tap drips in at the rate of 10 cubic centimeters per minute.

a. Determine $\frac{dh}{dt}$, where h is the height of water in the pan.

b. How long will it take to fill the pan to a height of 7 cm?

55. Demand: The Arrow Marketing Group sells teddy bears with college logos to college and university gift shops. Their demand equation is $px - 2800 = x$, where x is the quantity of bears and p is the price in dollars that each sells for.

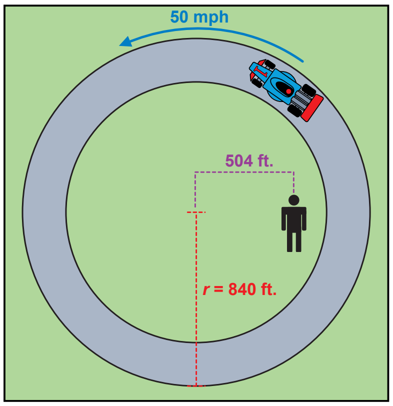
a. How many bears can be sold (or "demanded") at \$4.50 apiece?

b. Determine a formula $\frac{dx}{dp}$ and evaluate for p and x as in part a.

56. Electricity costs: Electricity costs per semester at Mount State University are calculated by the formula $C = 0.05x + 0.03y - 0.08xy$, where x is a measure of student size and activity, y is dependent on the usage of various buildings and their efficiencies, and C is in millions of dollars.

a. Determine C if $x = 7$ and $y = 9$.

b. Determine a general formula for $\frac{dy}{dx}$ and evaluate for $x = 7$ and $y = 9$.



57. Race track: A circular race track has a radius of 840 feet. At a certain point in time, t_0, an observer in a maintenance pit 504 feet from the center of the track clocks a car (when it is directly north of him) traveling counterclockwise along the track at a speed of 50 miles per hour from his right to left $\left(\frac{dx}{dt}\right)$. (Use 1 mile = 5280 feet.)

a. Determine the location of the car on the track. Assume the center of the track is at the origin.

b. Determine a general equation for $\frac{dy}{dx}$.

c. What is $\left.\frac{dy}{dx}\right|_{t=t_0}$? Interpret its meaning.

d. What is $\frac{dy}{dt}$ at $t = t_0$?

58. **Chlorination costs:** Chlorination costs for the swimming pool at a spa are given by $C = 0.27x + 2y - 0.001xy^2$, where x is the number of weekly swimmers, y is the number of special functions, and C is in dollars.

  a. Determine C if $x = 500$ and $y = 4$.

  b. Determine a formula for $\frac{dy}{dx}$ and evaluate at $x = 500$ and $y = 4$.

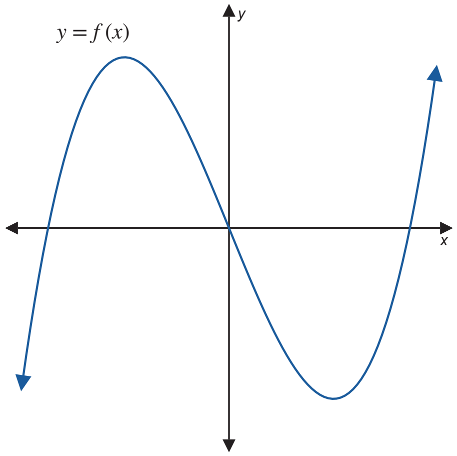
59. **Probability measurement:** The standard deviation, S, of a binomial random variable is given by $S = \sqrt{np(1-p)}$, where n is the number of trials or repetitions of an experiment and p is the probability of success on one outcome. S is a measure of how the data tends to vary from the center (or mean).

  a. For $n = 768$ and $p = 0.25$, determine S.

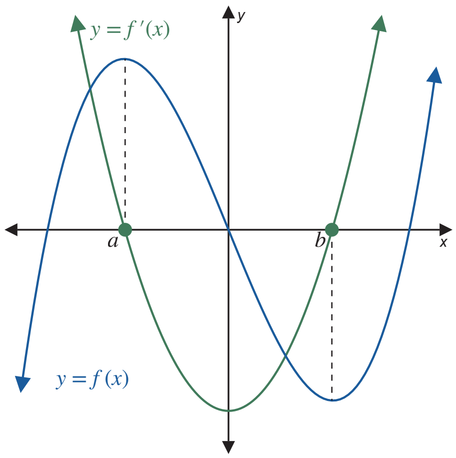
  b. Suppose we consider S as a fixed quantity. Determine $\frac{dp}{dn}$ for n and p as in part a.

Example 5: Graphing $f'(x)$

The graph for $y = f(x)$ is given below. Identify the portions which show where $f'(x)$ is negative, positive, or zero. Draw a possible graph of $f'(x)$ by estimating the absolute values of the slopes on $f(x)$.



Solution



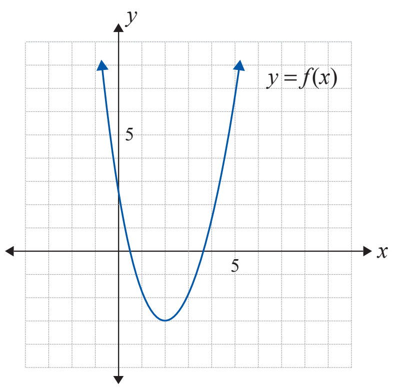
Let $x = a$ and $x = b$ (say $a < b$) denote the two x-values corresponding to the two local extremes. Then $f'(a) = f'(b) = 0$. In between a and b, the y-values decrease on the interval (a, b); therefore y' is negative. At $x = b$, the y-values start to increase (and y' becomes positive). Thus the graph of y' includes points $(a, 0)$ and $(b, 0)$ and lies below the x-axis (as y' is negative) from $x = a$ to $x = b$. For $x > b$ and $x < a$, the graph of y' lies above the x-axis. We draw a smooth curve from $(a, 0)$ to $(b, 0)$ and extend it in both directions above the x-axis. The low point for y' seems to occur near $x = 0$ because the graph of y is steepest near $(0, 0)$.

11.4 EXERCISES

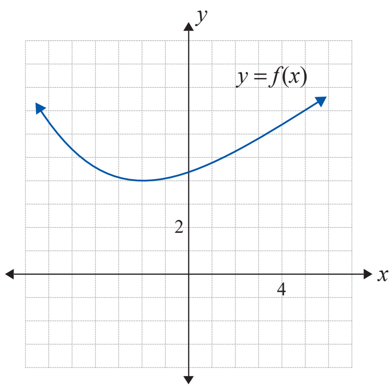
PRACTICE

In Exercises 1–10, find the open intervals on which **a.** f is increasing, and **b.** f is decreasing.

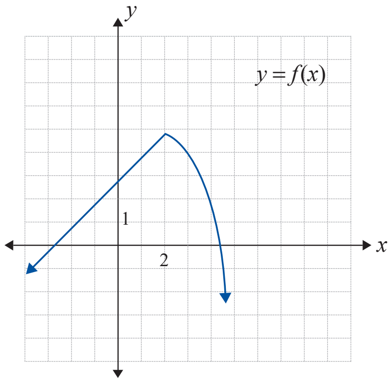
1.



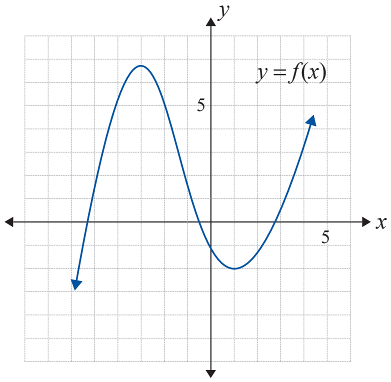
2.



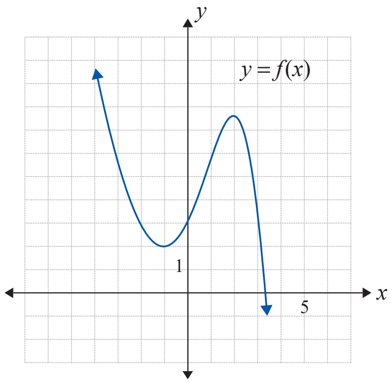
3.


y
y = f(x)
1
2
x

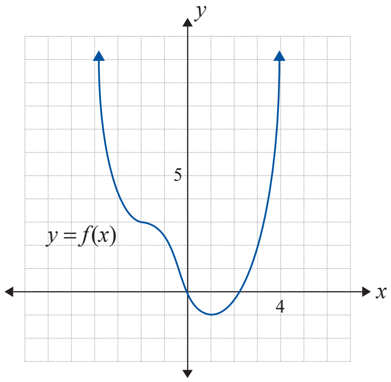
4.


y
y = f(x)
5
5
x

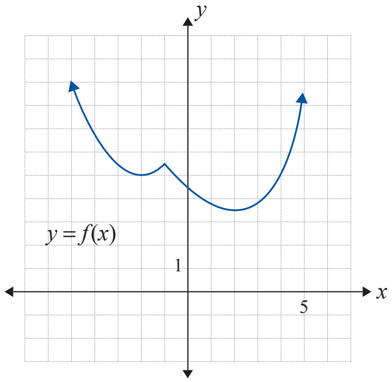
5.


y
y = f(x)
1
5
x

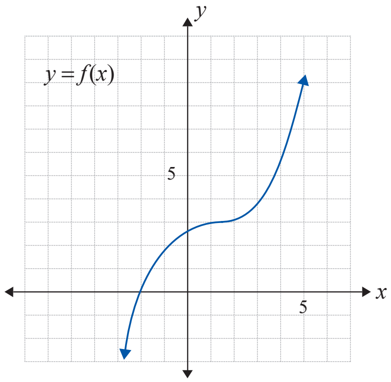
6.


y
5
y = f(x)
4
x

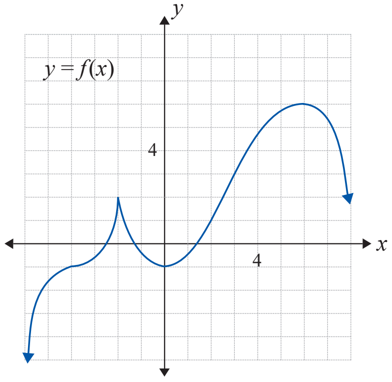
7.


y
y = f(x)
1
5
x

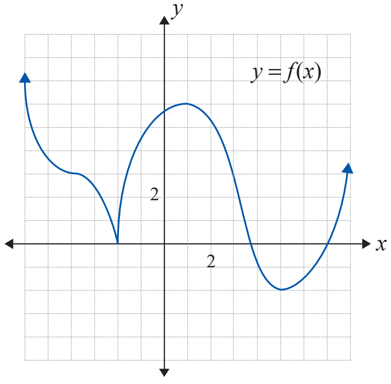
8.


y
y = f(x)
5
5
x

9.

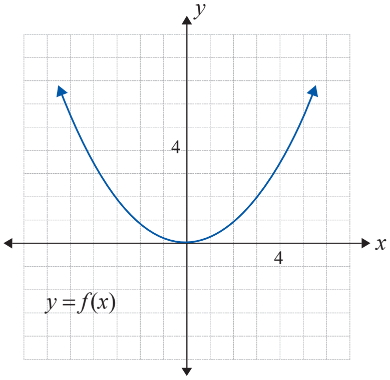

y
y = f(x)
4
4
x

10.

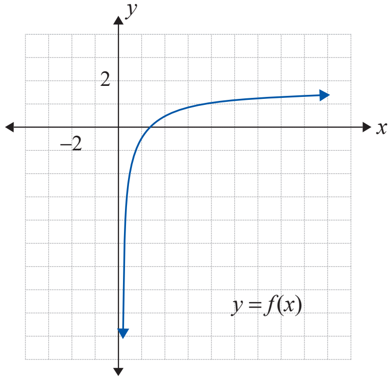

y
y = f(x)
2
2
x

For each of the graphs of $f(x)$ in Exercises 11–13, **a.** determine the open intervals on which the function is increasing and the open intervals on which it is decreasing, and **b.** sketch a possible graph of $f'(x)$.

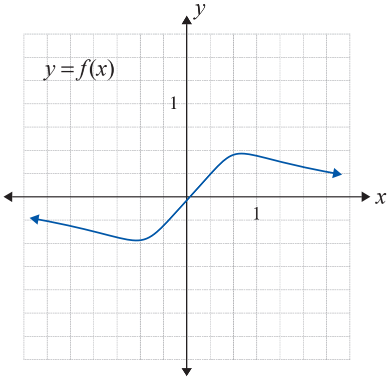
11.



12.



13.



For each of the functions in Exercises 14–33, **a.** find all values of x that correspond to horizontal tangent lines, **b.** find the open intervals on which the function is increasing and the open intervals on which it is decreasing, and **c.** graph the function.

14. $f(x)=x^2-8x+3$

15. $f(x)=2x^2+12x-1$

16. $f(x)=5-3x-x^2$

17. $f(x)=7x-2x^2$

18. $f(x)=2-4x-2x^2$

19. $f(x)=3x^2-4x+2$

20. $f(x)=(2x+3)^2$

21. $f(x)=(3x-2)^2$

22. $f(x)=2x^3-5$

23. $f(x)=3x^3+4$

24. $f(x)=x^3-3x^2+7$

25. $f(x)=x^3-6x^2-4$

26. $f(x)=x^3-3x^2-9x+12$

27. $f(x)=x^3-x^2-x$

28. $f(x)=x^3+\frac{1}{2}x^2-2x+3$

29. $f(x)=x^3-x^2-5x+2$

30. $f(x)=\frac{1}{3}x^3-x^2-8x+10$

31. $f(x)=\frac{1}{3}x^3-2x^2+3x-6$

32. $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x-6$

33. $f(x)=\frac{1}{3}x^3+\frac{5}{2}x^2+4x+11$

For each of the functions in Exercises 34–43, **a.** find all values of x that correspond to horizontal tangent lines, and **b.** find the open intervals on which the function is increasing and the open intervals on which it is decreasing.

34. $f(x)=\dfrac{x-1}{x}$

35. $f(x)=\dfrac{x}{x-2}$

36. $f(x)=\dfrac{x^2-4}{x}$

37. $f(x)=\dfrac{x^2-9}{x}$

38. $f(x)=\dfrac{x^3+16}{x}$

39. $f(x)=\dfrac{2x^3-27}{x^2}$

40. $f(x)=\dfrac{x+2}{x-1}$

41. $f(x)=\dfrac{x-5}{x+3}$

42. $f(x)=2x-\dfrac{125}{x^2}$

43. $f(x)=x^2+\dfrac{128}{x}$

APPLICATIONS

44. Revenue: A store manager has determined that the revenue from the sale of x units of a product is given by $R(x)=32x-0.4x^2$ dollars, where $0\le x\le 80$. On what interval of sales is the revenue increasing, and on what interval of sales is it decreasing?

45. Revenue: A producer of computer software has determined that the revenue from the production and sale of x units is given by $R(x)=48x-0.003x^2$ dollars, where $0\le x\le 10{,}000$. For what interval of production is the revenue increasing, and for what interval is it decreasing?

46. Profit: The revenue from the sale of x coffee makers is given by $R(x)=40x-0.4x^2$ dollars. The total cost is given by $C(x)=370+16x-0.2x^2$ dollars, where $0\le x\le 100$. Determine the interval(s) where the profit is increasing and where it is decreasing.

47. Profit: The revenue from the sale of x 50-gallon aquariums is given by $R(x)=54x-0.3x^2$ dollars. The total cost function is given by $C(x)=0.1x^2+4x+200$ dollars, where $0\le x\le 100$. Determine the interval of sales for which the profit is increasing and the interval for which it is decreasing.

48. Population: The population of the inner-city district of a city is given in thousands by $P(t)=24-0.3t+0.01t^2$, where t is the number of months after the implementation of an urban renewal project. How long will it be before the population starts to increase?

49. Wildlife management: In an attempt to naturally control the elk population in a national park, the U.S. Fish and Game Department has reintroduced the wolf into the area. It is estimated that the population of the elk herd will be $P(t)=600+12t-4t^{\frac{3}{2}}$, where t is the number of years after the reintroduction of the wolf. How long will it be before the elk population begins to decrease?

50. **Average cost:** The cost of producing x wireless speakers is given in dollars by $C(x) = 320 + 30x + 0.2x^2$, where $x \geq 0$. Determine the interval of production for which the average cost function is increasing.

51. **Average cost:** The cost of producing x units of a product is given in dollars by $C(x) = 250 + 45x - 0.2x^2$, where $x \geq 0$. Show that the average cost function is always decreasing. (This case corresponds to situations in which increased production distributes the cost so that the cost per unit decreases.)

11.5 EXERCISES

PRACTICE

For each of the functions in Exercises 1–20, **a.** find the critical values, and **b.** use the First Derivative Test to find any local extrema.

1. $f(x)=4x-x^2$

2. $f(x)=9x-x^2$

3. $f(x)=x^2+6x-2$

4. $f(x)=x^2-10x+12$

5. $f(x)=\frac{1}{2}x^2-4x+3$

6. $f(x)=-\frac{1}{2}x^2+3x-2$

7. $f(x)=x^3+x^2-x+3$

8. $f(x)=x^3+2x^2+x-2$

9. $f(x)=-x^3-\frac{3}{2}x^2+18x+6$

10. $f(x)=2x^3+x^2-4x$

11. $f(x)=x^3-3x+6$

12. $f(x)=x^3+3x^2-4$

13. $f(x)=x+\frac{9}{x}$

14. $f(x)=x-\frac{4}{x}$

15. $f(x)=\frac{x^2-16}{x}$

16. $f(x)=\frac{4x^2-9}{x}$

17. $f(x)=9x+x^{-1}$

18. $f(x)=25x+x^{-1}$

19. $f(x)=16x+x^{-2}$

20. $f(x)=54x-x^{-2}$

APPLICATIONS

21. Rate of dictation: It has been determined that after t weeks of class, the average students in an intermediate shorthand class can take dictation at a rate of $W(t)=60+\frac{70t^2}{t^2+15}$ words per minute. Show that the rate of dictation is an increasing function which approches an upper bound.

22. Court reporting: A typical student in an intermediate court reporting class can reach a level of recording $W(t)=40+\frac{35t^2}{t^2+20}$ words per minute after t hours of instruction and practice. Show that the number of words recorded per minute increases up to a certain level.

23. Marathon running speed: The speed at which a marathon runner travels varies over time. The function $F(t)=5+\frac{5t^2}{t^2+200}$ describes the velocity of a particular runner at time t ($F(t)$ is in miles/hour). Determine the intervals over which the function is increasing or decreasing. Is this particular runner an experienced runner? (Experienced runners run faster near the end of the race.)

24. Heating: A frozen pizza is placed in the oven at $t=0$. The function $F(t)=30+\frac{320t^2}{t^2+100}$ approximates the temperature of the pizza at time t. Show that the temperature approaches an upper bound. (The pizza will approach oven temperature over time.)

25. **Cooling:** A cup of hot coffee is placed in a room. The temperature in degrees Fahrenheit of the coffee is approximated by the function $F(t)=180-\dfrac{100t^2}{t^2+40}$ where t is the number of minutes the coffee has been in the room. Show that the temperature function is a decreasing function, and find the temperature of the room. (The coffee approaches room temperature over time.)

26. **Velocity and acceleration:** The velocity of a car varies according to the function $F(t)=\dfrac{t^3}{1875}-\dfrac{119t^2}{1500}+\dfrac{53t}{15}+5,$ where t is time ($0<t<100$). Determine any local maximum and minimum velocities, and the times at which they occur. (Be sure to specify whether each is a maximum or mininum.)

27. **Skydiving:** The velocity of a skydiver after parachute deployment is given by the formula $F(t)=180-\dfrac{165t^2}{t^2+10},$ where t is the time after deployment. Show that the function is decreasing, and determine the terminal velocity for the diver-parachute system. (The velocity approaches terminal velocity as t goes to infinity.)

28. **Pressure:** The pressure in a pressure cooker is given by the function $F(T)=1+\dfrac{3T^2}{T^2+180},$ where T is the temperature inside the kettle. Show that the function is increasing, and determine the upper bound for pressure.

29. **Space probe closing speed:** In order to minimize travel time, a fictional probe on its way to a distant star accelerates for part of the voyage, and then decelerates to enter orbit safely. The closing speed of the probe with the star is given by the function $F(t)=\dfrac{t^2}{25}-4t+0.5,$ where t is the number of years after launch and $F(t)$ is measured in percentage of light speed. At what time does the probe begin decelerating? What is the probe's maximum closing speed? (**Hint:** Closing speed is negative.)

30. **Computation speed:** An experimental supercomputer is undergoing testing to determine whether it will meet the necessary specifications. The number of computations it can perform per second is found to be modeled by the function $F(t)=\dfrac{89t^3}{441{,}000}-\dfrac{757t^2}{22{,}050}+\dfrac{2227t}{1470},$ where $0<t<100$ is the number of minutes after startup and $F(t)$ is in quadrillions of computations/sec. Find all relative maximums and minimums for the function over the interval. Round to the nearest tenth. If the number of computations is zero at any time after startup, the computer crashes. Does the computer crash during testing?

31. **Fuel economy:** The fuel consumption of an automobile is not constant. Fuel economy depends largely on the speed of the vehicle. The function $F(v)=-\dfrac{8v^3}{19{,}125}+\dfrac{28v^2}{85}-\dfrac{288v}{85}+80$ ($F(v)$ is in miles per gallon) describes the fuel consumption of a new hybrid vehicle, where $0<v<85$ is the velocity of the vehicle. On what intervals is the consumption increasing? Which velocities yield maximum efficiencies? (Remember, high efficiency means low consumption. Round to the nearest tenth.)

Example 5: Maximizing Profits

A company finds that its profit in dollars for producing x units of a product in one week is given by $P(x) = -2x^2 + 1600x$. If the company is set up so that no more than 500 units can be manufactured in any one week, how many units should the company produce to maximize profit?

Solution

The production restrictions indicate that x is in the closed interval [0, 500]. Find the critical values by setting $P'(x) = 0$ and solving for x (Step 1).

$$\begin{aligned} P'(x) &= -4x + 1600 \\ 0 &= -4x + 1600 \\ 4x &= 1600 \\ x &= 400 \end{aligned}$$

Note that P' is defined for all x in [0, 500].

Now evaluate $P(x)$ for $x = 0$, $x = 400$, and $x = 500$ (Step 2).

$x = 0$	$P(0) = -2(0)^2 + 1600 \cdot 0 = 0$	
$x = 400$	$P(400) = -2(400)^2 + 1600 \cdot 400 = 320{,}000$	Absolute max
$x = 500$	$P(500) = -2(500)^2 + 1600 \cdot 500 = 300{,}000$	

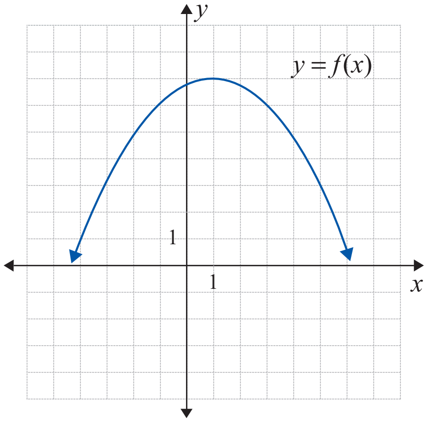
The profit is maximized at \$320,000 when 400 units are produced (Step 3).

11.6 EXERCISES

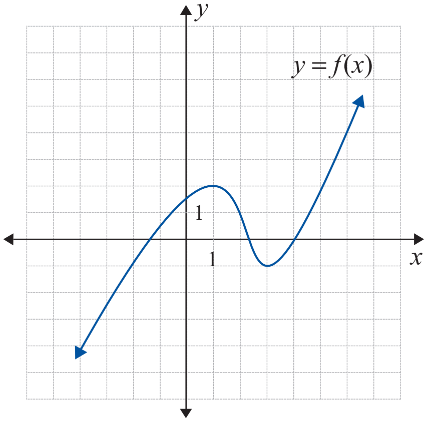
PRACTICE

In Exercises 1–8, find the absolute extrema for each graph of $f(x)$ on the given interval.

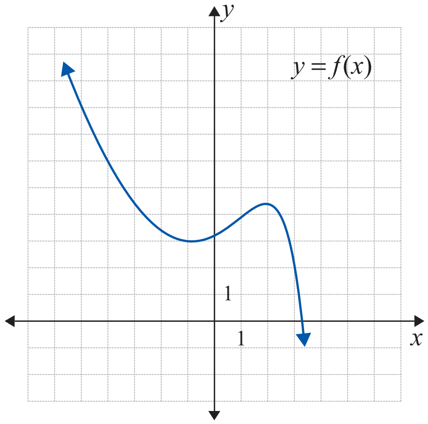
1. [3, 5]



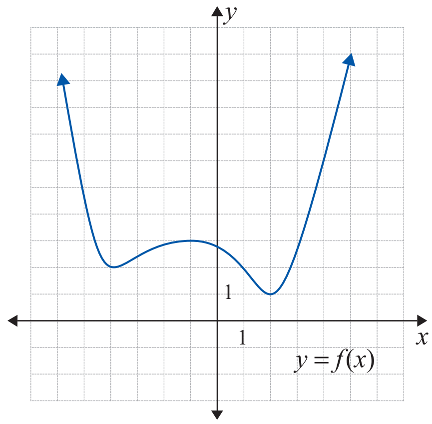
2. [−1, 4]



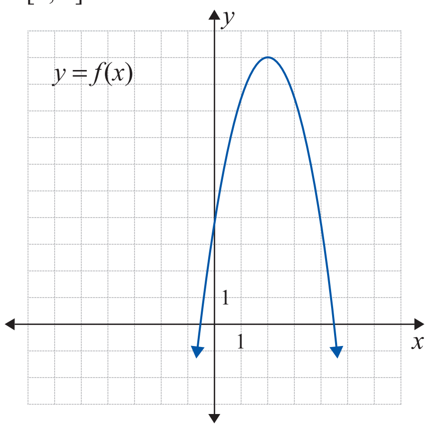
3. $[-5, 3]$



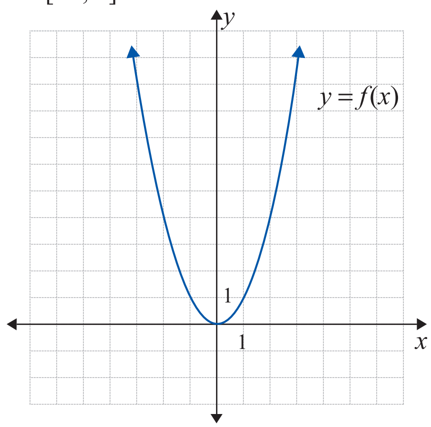
4. $[-4, 4]$



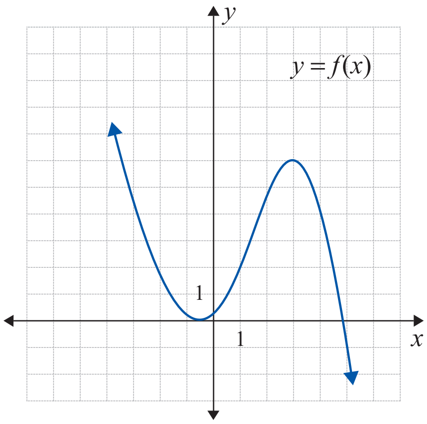
5. $[1, 4]$



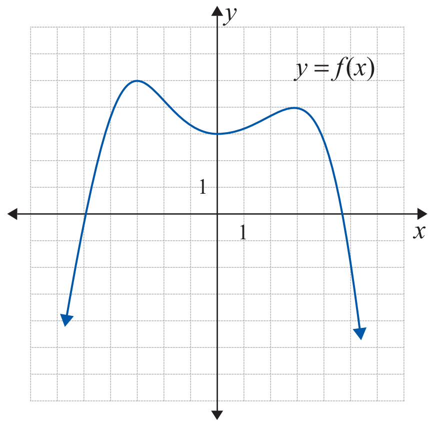
6. $[-2, 2]$



7. $[-3, 5]$



8. $[-4, 5]$



In Exercises 9–38, find the absolute extrema for each function on the given interval.

9. $f(x) = x^2 - 8x;\ [0, 5]$

10. $f(x) = 3x^2 - 12x;\ [0, 4]$

11. $f(x) = 6 + 10x - x^2;\ [3, 6]$

12. $f(x) = 11 - 4x - x^2;\ [-3, 0]$

13. $f(x) = 14 - 3x;\ [0, 4]$

14. $f(x) = 7 + \frac{1}{2}x;\ [-2, 4]$

15. $f(x) = 8 - x^3;\ [-1, 3]$

16. $f(x) = x^3 + 4;\ [-2, 2]$

17. $f(x)=x^3-12x;\ [-3,4]$

18. $f(x)=x^3-3x;\ [-2,3]$

19. $f(x)=2x^3-x^2;\ [0,2]$

20. $f(x)=x^3+2x^2;\ [0,3]$

21. $f(x)=9x-3x^2-x^3;\ [-4,2]$

22. $f(x)=x^3-3x^2-24x;\ [-3,3]$

23. $f(x)=2x^3-3x^2-12x-10;\ [0,4]$

24. $f(x)=x^3+3x^2-24x;\ [0,3]$

25. $f(x)=x^{\frac{2}{3}}-4;\ [-1,8]$

26. $f(x)=3x^{\frac{2}{3}}+2;\ [-1,4]$

27. $f(x)=3x^{\frac{1}{3}}-4x;\ [0,1]$

28. $f(x)=3x^{\frac{2}{3}}+x;\ [-9,1]$

29. $f(x)=\sqrt{x^2+4};\ [-1,2]$

30. $f(x)=\sqrt{9-x^2};\ [-1,2]$

31. $f(x)=\sqrt[3]{x^2-1};\ [-2,2]$

32. $f(x)=\left(x^2-1\right)^{\frac{2}{3}};\ [-2,2]$

33. $f(x)=x+\frac{4}{x};\ \left[\frac{1}{2},3\right]$

34. $f(x)=2x+\frac{18}{x};\ [1,4]$

35. $f(x)=4x+\frac{9}{x};\ [1,3]$

36. $f(x)=9x+\frac{16}{x};\ [1,2]$

37. $f(x)=x^2+\frac{16}{x};\ \left[\frac{1}{2},4\right]$

38. $f(x)=x^2+\frac{2}{x};\ [1,4]$

APPLICATIONS

39. Revenue: The weekly revenue from the sale of x units of a product is given by $R(x)=24x-0.5x^2$ dollars. If the company is set up so that it can produce no more than 40 units per week, how many units should the company produce to maximize revenue?

40. Revenue: The revenue from the sale of x units of a product is given by $R(x)=12x-0.04x^2$ thousand dollars, where $0\le x\le 250$. How many units should be sold to maximize the revenue?

41. Profit: A manufacturer of telescopes has determined that the revenue from the production and sale of x telescopes is $R(x)=140x-0.5x^2$ dollars. The cost function is given by $C(x)=x^2+20x+1050$ dollars. Find the level of production and sales that will maximize the profit if $0\le x\le 70$.

42. Profit: A marketing analyst for a company that produces skateboards has determined that if the company sells x units of the deluxe model, the revenue function is $R(x)=79.9x-0.03x^2$ dollars and the cost function is $C(x)=0.08x^2+5.1x+5800$ dollars. Find the sales level that will yield maximum profit if $0\le x\le 380$.

43. Average cost: The cost of producing x compact refrigerators is given by $C(x)=2880+35x+0.2x^2$ dollars. Find the value of x that minimizes the average cost function if $0\le x\le 150$.

44. Average cost: The cost of producing x electronic games is given by $C(x)=1080+42x+0.3x^2$ dollars. Find the value of x that minimizes the average cost function if $0\le x\le 90$.

45. **Air quality:** The Air Quality Management District monitors the level of pollution in the air. On a good day, the level is approximately $P(t) = 35 + \frac{126t}{0.5t^2 + 18}$ PSI (Pollution Standard Index), where t is the number of hours after 7:00 a.m. and $0 \le t \le 11$. At what time will the pollution level be a maximum?

46. **Air quality:** On a moderately smoggy day, the level of nitrogen dioxide in the air is approximately $N(t) = 0.126 + \frac{0.36t}{2t^2 + 40.5}$ ppm (parts per million), where t is the number of hours after 8:00 a.m. and $0 \le t \le 10$. At what time will the level of nitrogen dioxide reach its maximum?

47. **Bacteria:** It is estimated that t hours after a particular bacterium is introduced into a culture, the population of bacteria in the culture will be $P(t) = \frac{4800}{\sqrt{12 - 0.5t}}$, where $0 \le t \le 6$. What will be the absolute maximum population? At what time t will this occur?

48. **Population:** It is estimated that t years from now the population of a small community will be $P(t) = \frac{5000}{\sqrt{25 + 0.4t}}$ people, where $0 \le t \le 10$. What will be the maximum population?

49. **Altitude:** The altitude of an airplane following a certain flight path is given by the function $F(t) = \frac{13t}{20} - \frac{3t^2}{200}$, where t is in minutes, and $F(t)$ is in thousands of feet. Find the absolute maximum of the function over the interval [0, 43].

50. **Velocity of a car:** The velocity of a car in miles per hour varies according to the function $F(x) = \frac{x^2}{100} - \frac{19x}{25} + \frac{5349}{100}$, where x denotes time in seconds. Find the absolute maximum and minimum velocities over the interval [0, 100]. Find the car's velocity at each endpoint.

51. **Tire distortion:** When a car accelerates, its tires are distorted by the force exerted on them by the motor. For a certain model car, the distortion after the driver floors the accelerator is modeled by the function $F(t) = 0.00026t^3 - 0.0533t^2 + 2.74014t$, where $0 < t < 100$ is the number of milliseconds after acceleration begins, and $F(t)$ is a percentage of the tire's maximum flexibility (at $F(t) = 100$, the tire tears into pieces). Find the time when the maximum distortion occurs, and find the percentage that the tires are distorted. Do the tires survive the acceleration?

52. **Pollution:** The amount of pollution (measured in parts per million) in a small river is given by the function $F(t) = \frac{450t^3}{169} - \frac{18{,}225t^2}{169} + \frac{23{,}625t}{23} + \frac{260{,}675}{169}$, where t is the number of years after 1980. Sometime after 1980, an environmental protection law was enacted to reduce the amount of pollution in the river, and in the same year the pollution levels began to decrease. Find the maximum pollution for $0 \le t \le 20$, and determine the year the law was enacted. (The largest integer less than the t-value specifies the year of enactment.)

53. **Wind resistance:** An aeronautics company is testing a new airframe. The company wishes to determine an ideal cruising speed, and one step in the process is to find the speed at which the airframe experiences the least wind resistance (other than when it is not moving). After performing a number of tests, the engineers determine that the wind resistance can be modeled by a function $G(v)=\frac{v^3}{3{,}645{,}000}-\frac{8v^2}{6075}+\frac{350v}{243}+\frac{614{,}500}{729}$, where $300 \le v \le 4000$ is the wind velocity in feet per second and $G(v)$ is in newtons. Determine the ideal cruising speed for the airframe based on this model.

54. **Bacteria:** A bacteriologist doing research on antibiotics has discovered that a certain type of disease-causing bacteria can be effectively treated using a cocktail of two different antibiotics administered at specific intervals. The population changes in response to the antibiotics according to the function $F(t)=-0.0224t^3+4.5676t^2-252.4610t+5000$, where t is in the interval [0, 200]. The maximum values of the function correspond to the times when the antibiotics were administered. At what times, t, are the antibiotics administered and what is the population of bacteria at those times?

55. **Gravitational pull:** A rocket traveling to the moon is affected by the gravity of Earth and of the moon. The total gravitational force exerted on the rocket is approximated by the function $F(h)=\frac{43{,}750h^2}{3}-\frac{70{,}000h}{3}+10{,}000$, where $0 < h < 1$ is the height of the object, given in percentage of the distance between Earth and the moon, and $F(h)$ is measured in newtons. Find the height at which the minimum occurs and the gravitational force at that altitude. Additionally, there is an onboard experiment which can only be performed if the gravitational force falls below 1000 newtons. Can this experiment be performed?

56. **Thrown object:** A cell phone is thrown into the air. The position of the phone is given by the function $F(t)=-\frac{49t^2}{10}+\frac{297t}{10}$, where $0 < t < 10$ is the number of seconds after the phone is thrown, and $F(t)$ is measured in feet. Find the maximum height the phone attains. Also find the time when the phone hits the ground. The phone will shatter if it hits the ground with a speed greater than 32 fps (feet per second). Does the phone shatter?

57. **Acceleration:** Due to many variable factors in car engines, acceleration is never constant. The acceleration of a particular car in a particular test is approximated by the function $F(t)=-\frac{260t^3}{137}+\frac{2418t^2}{137}-\frac{7371t}{137}+75$, where $0 \le t \le 5$ is the number of seconds after acceleration begins. What is the maximum acceleration of the car during this test? When does it occur? What is the minimum acceleration, and when does it occur?

58. **Power consumption:** A certain piece of equipment in a chemistry lab draws power according to the function

$$G(m)=\left(-6.087\times10^{-6}\right)m^3+\left(8.641\times10^{-3}\right)m^2-1.751m+1000,$$

where $0 < m < 1000$ is the mass of the sample to be analyzed in grams and G is power consumption in watts. Find the sample mass which causes the equipment to draw the most power. How much power does the equipment draw for a sample of this mass?

59. **Chemistry:** A certain chemical procedure requires the addition of reactants and catalysts at precise times to maintain reaction rates. The rate of reaction is given by the function

$$H(t) = \left(9.423\times10^{-7}\right)t^3 - \left(1.230\times10^{-4}\right)t^2 + \left(4.384\times10^{-3}\right)t + 0.001,$$

where $0 < t < 100$ is the number of seconds after beginning the reaction, and $H(t)$ is measured in number of moles formed per second. Find the time at which a relative maximum reaction rate is reached and the number of moles per second being formed at that time. At what time was the second reactant/catalyst mixture added? (The reactant/catalyst mixture is added at a relative minimum.) Round to two decimal places.

60. **Phone traffic:** Phone traffic varies greatly over the course of a day. A phone provider estimates that the number of international calls active per minute on a certain holiday is given by the function $P(t) = -0.001t^3 + t^2 + 50t + 150{,}000$, where $0 < t < 1000$ is the number of minutes after 6.00 a.m. and $P(t)$ is the number of active international calls. For what t does maximum phone traffic occur? How many calls are active at this time?

61. **Stored energy:** While a rubber ball is moving, it has kinetic energy. When the ball impacts a hard surface, the kinetic energy is converted to potential energy, and then back to kinetic energy. This happens quite rapidly. The amount of potential energy after impact is approximated by the function $E(t) = -\frac{3t^2}{80} + \frac{3t}{2}$, where t is the number of nanoseconds after impact and t is in [0, 40]. When does the ball have maximum potential energy? How much potential energy does it have at its maximum?

62. **Photosynthesis:** Plants absorb different amounts of light, depending on the wavelength of the light. A study of Rhododendron bushes show that they absorb light according to the formula $L(w) = \frac{-2.22w^3}{10^{10}} + \frac{4.44w^2}{10^6} - \frac{2w}{10^3} + 0.444$, where $100 \le w \le 650$ is the wavelength of light in nanometers, and $L(w)$ is the proportion of the light shone on the bush. What wavelength does Rhododendron absorb the best? What proportion of light of this wavelength does Rhododendron absorb?

63. **Package delivery:** A study says that the package flow in the Southeast USA during the month of August follows the function $D(t) = \frac{7t^3}{9300} - \frac{7t^2}{248} + \frac{7t}{31} + 1$, where $1 \le t \le 31$ is the day of the month, and $D(t)$ is given in millions of packages. On which day are the most packages delivered? How many packages are delivered on this day?

Thus $x = 0$ and $x = \frac{4}{3}$ will make $f''(x)$ equal to zero. Both values give inflection points for $f(x)$.

We can use $f'(x) = 0$ to locate the maximum and minimum points for $f(x)$.

$$\begin{aligned} f'(x) &= 0 \\ 8x^2 - 4x^3 &= 0 \\ 4x^2(2 - x) &= 0 \\ x &= 0, 2 \end{aligned}$$

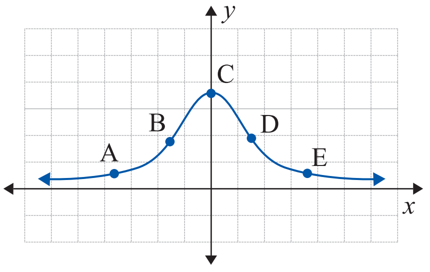
In Figure 8, we observe $x = 0$ corresponds to an inflection point (neither a maximum nor a minimum) and $x = 2$ corresponds to the maximum point $\left(2, \frac{31}{3}\right)$.

In this problem, the algebraic solutions of $f'(x) = 0$ and $f''(x) = 0$ are of a familiar type. When these equations are too difficult for an algebraic solution, a graphing utility is invaluable.

12.1 EXERCISES

PRACTICE

1. At each point marked on the graph of f, determine if f' is positive, negative, or zero. Determine if f'' is positive, negative or zero.



Draw a graph that satisfies the given conditions in Exercises 2–5.

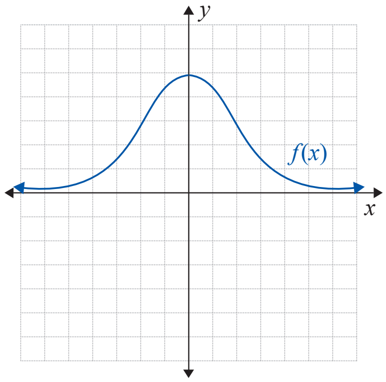
2. $f(5) = 9, \quad f'(5) = 2, \quad f''(5) = -2$
3. $f(-5) = -9, \quad f'(-5) = 2, \quad f''(-5) = 2$
4. $f(5) = -9, \quad f'(5) = 0, \quad f''(5) = 3$
5. $f(0) = 12, \quad f'(0) = 0, \quad f''(0) = -3$

For Exercises 6–13, find $f''(x)$. Then evaluate $f''(0)$, $f''(1)$, and $f''(4)$, if they exist.

6. $f(x) = x^3 + x^2 + 3$
7. $f(x) = x^3 - x^2 + 7$
8. $f(x) = x^2 - 5\sqrt{x} + 1$
9. $f(x) = x^2 + 2\sqrt{x} - 3$
10. $f(x) = \sqrt{x - 4}$
11. $f(x) = \sqrt{2x + 1}$
12. $f(x) = \frac{x}{x + 5}$
13. $f(x) = \frac{x - 2}{x + 4}$

In Exercises 14–17, sketch a possible graph for $f'(x)$ on the same coordinate axes as $f(x)$. Then locate all inflection points on the graph of $f(x)$.

14.



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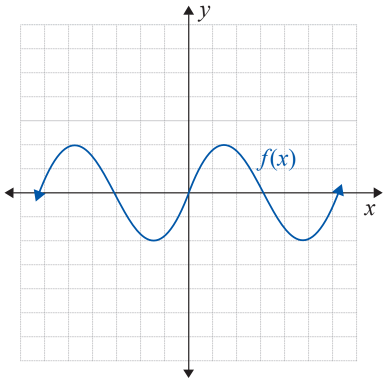

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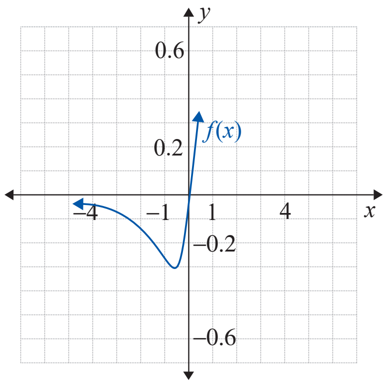

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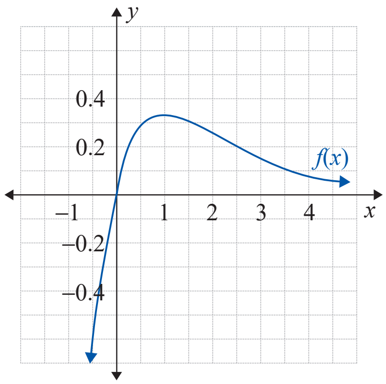
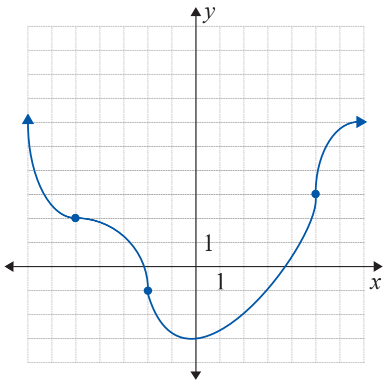


For each of the graphs in Exercises 18–21, list the interval(s) **a.** on which *f* is concave upward and **b.** on which *f* is concave downward; then **c.** locate all points of inflection.

18.



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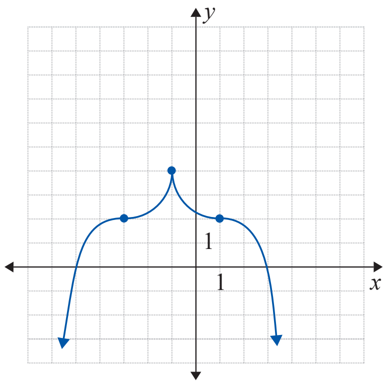
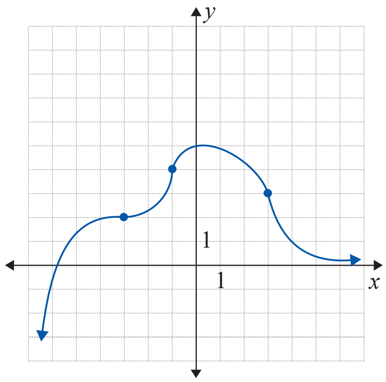
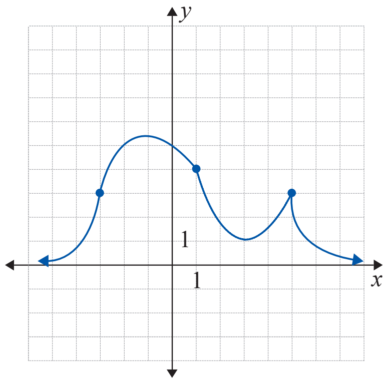


20.



21.



In Exercises 22–33, determine the intervals on which each function is **a.** concave upward and **b.** concave downward; then **c.** locate all points of inflection. Use the information gathered to sketch the function. Confirm the details with a graphing calculator.

22. $f(x) = 2x^2 + 5x - 9$

23. $f(x) = 5x^2 + 8x - 1$

24. $f(x) = x^3 - 3x^2 + 7$

25. $f(x) = x^3 + 6x^2 - 10$

26. $f(x) = x^3 + 11x - 4$

27. $f(x) = 5x^3 + 7x + 2$

28. $f(x) = \frac{1}{3}x^3 - 2x^2 + x - 3$

29. $f(x) = \frac{1}{3}x^3 + 3x^2 + 2x - 5$

30. $f(x) = \sqrt[3]{2x + 3}$

31. $f(x) = \sqrt[3]{5x - 3}$

32. $f(x) = \dfrac{x}{x^2 - 4}$

33. $f(x) = \dfrac{4x}{x^2 - 5}$

WRITING & THINKING

In Exercises 34–37, give an example of a polynomial function that satisfies the conditions.

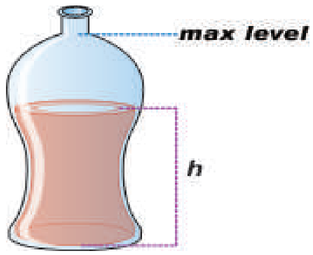
34. $F(5) = 15$; $F'(x)$ is nonzero, but $F''(x) = 0$ for all x.

35. $G(0) = 0$, $G'(0) = 0$, and $G''(0) = 0$; $G(x)$ is concave upward everywhere and has no inflection points.

36. $H(4) = 0$; $H'(x)$ is positive for $x > 4$ and negative for $x < 4$. $H(x)$ has no inflection points.

37. $J(4) = 0$; $J'(4)$ is zero but $J'(x)$ is positive if $x \neq 4$; $J''(4) = 0$.

APPLICATIONS



38. Filtrate: In a chemistry lab a filtrate drips slowly but continuously at a constant rate into a glass container shaped like the one shown. The container eventually fills to the base of the neck. Let t denote the passage of time and h be the height of the liquid.

a. Describe at what points on the bottle $\dfrac{dh}{dt}$ will be a maximum and a minimum.

b. Sketch a graph of $\dfrac{dh}{dt}$. Are there any inflection points on a graph of $y = h(t)$?

c. Add a sketch of $y = h(t)$ on the same coordinate axes as in part **b.**

Testing shows that

$$S''(x) > 0 \quad \text{for} \quad 0 < x < 50$$

and

$$S''(x) < 0 \quad \text{for} \quad 50 < x < 80.$$

Concavity changes from upward on the left side of $S(50)$ to downward on the other, signifying it is a point of diminishing returns.

Thus the point of diminishing returns is at $(50, S(50)) = (50, 5100)$. At the point of diminishing returns, \$50,000 are spent on advertising, and sales in tires are \$5,100,000.

12.2 EXERCISES

PRACTICE

Find both the first and second derivatives for each of the functions in Exercises 1–12. Locate any relative maximum or minimum points and any points of inflection. Determine the intervals on which the function is concave upward or concave downward.

1. $f(x) = 7x^2 - 28x + 8$

2. $f(x) = 5x^2 - 9x + 2$

3. $f(x) = 2x^3 + 5x - 1$

4. $f(x) = 3x^3 + 6x - 8$

5. $f(x) = x^3 + 2\sqrt{x} + 5$

6. $f(x) = x^4 - 3\sqrt{x} + 2$

7. $f(x) = (x^2 + 7)^2$

8. $f(x) = (2x^2 - 5)^2$

9. $f(x) = \sqrt{x^2 + 3}$

10. $f(x) = \sqrt[3]{x^2 + 9}$

11. $f(x) = \dfrac{3x}{x^2 + 1}$

12. $f(x) = \dfrac{2x + 1}{x^2 - 4}$

In Exercises 13–16, find all inflection points. Apply the Second Derivative Test at possible maximum/minimum points. Make a sketch of the graph and confirm your results with a graphing calculator.

13. $f(x) = (x + 5)\sqrt[3]{x}$

14. $f(x) = (x^2 + 1)\sqrt[3]{x}$

15. $f(x) = 2x\sqrt[3]{x + 1}$

16. $f(x) = (x + 10)\sqrt[3]{x^2 + 10}$

In Exercises 17–30, use the Second Derivative Test to find all local extrema, if the test applies. Otherwise, use the First Derivative Test.

17. $f(x) = x^2 - 3x + 5$

18. $f(x) = 8 + 7x - 2x^2$

19. $f(x) = x^3 - 3x^2 + 8$

20. $f(x) = x^3 + 6x^2 - 10$

21. $f(x) = x^3 - 12x + 3$

22. $f(x) = x^3 - 3x + 4$

23. $f(x) = \dfrac{2}{3}x^3 - x^2 - 4x - 2$

24. $f(x) = \dfrac{1}{3}x^3 + x^2 - 3x - 1$

25. $f(x)=x^4-8x^2+7$

26. $f(x)=x^4-2x^2+3$

27. $f(x)=x^4+2x^3-4$

28. $f(x)=x^4-6x^3+8$

29. $f(x)=2x+\frac{8}{x}$

30. $f(x)=\frac{x^2+9}{x}$

APPLICATIONS

31. Point of diminishing returns: Find the point of diminishing returns for the sales function $S(x)=112+1.8x^2-0.1x^3$, where x represents thousands of dollars spent on advertising, $0 \le x \le 10$, and S is sales in thousands of dollars.

32. Point of diminishing returns: The sales function for a product is given by $S(x)=204+6.3x^2-0.25x^3$, where x represents thousands of dollars spent on advertising, $0 \le x \le 12$, and S is sales in thousands of dollars. Find the point of diminishing returns.

33. Marginal cost: The cost function for a particular product is given by $C(x)=0.1x^3-2.4x^2+24x+190$ dollars, where $0 \le x \le 12$. Find the minimum marginal cost.

34. Marginal cost: Find the minimum marginal cost of a product if the cost function is given by $C(x)=0.0001x^3-0.036x^2+16.8x+1900$ dollars, where $0 \le x \le 150$.

35. Law enforcement: Due to the rapid increase in major crimes, the mayor of a large city plans to organize a major crime task force. It is estimated that for every 1000 persons in the city, the numbers of major crimes will be $N(t)=56+3t^2-0.8t^{\frac{5}{2}}$, where t is the number of months after the task force is organized and $0 \le t \le 12$.

a. Find the maximum $N(t)$.

b. Find the maximum rate of increase in $N(t)$.

36. Meteorology: Meteorology records for a certain city suggest that for the month of June, the temperature between midnight and 6:00 p.m. can be approximated by $T(t)=-0.04t^3+1.14t^2-7.2t+66$ degrees, where t is the number of hours after midnight and $0 \le t \le 18$.

a. Find the maximum and minimum temperatures.

b. Find the maximum rate of increase in the temperature.

WRITING & THINKING

37. Given $f(x)=px^3+bx+10$, answer the following questions.

a. Suppose p and b are positive numbers. What can be said about maximum/minimum points and points of inflection?

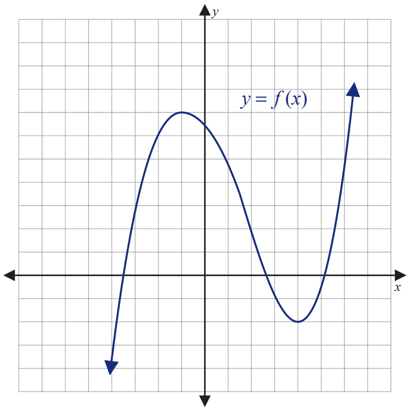
b. Suppose p and b have opposite signs (one is positive and the other negative). What can be said about maximum/minimum points and points of inflection?

c. If the constant term 10 is changed to some other value, do your responses to parts **a.** and **b.** change?

d. Put your answers to parts **a.**, **b.**, and **c.** together in a "Lab Report" which discusses the coefficients in the given polynomial $y=px^3+bx+c$.

Example 5: Graphing the Derivative

Consider the given graph of a function.

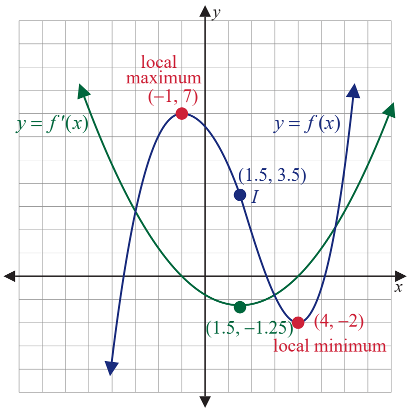


a. Identify the local extrema and locate the point(s) of inflection.

b. Determine the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing, and identify the intervals on which $f(x)$ is concave upward and concave downward.

c. Sketch on the same coordinate plane a possible graph of $f'(x)$.

Solution



a. A local maximum is located at (–1, 7), and a local minimum is located at (4, –2). There is a point of inflection at about (1.5, 3.5).

b. The function is increasing on the intervals $(-\infty, -1)$ and $(4, +\infty)$. It is decreasing on the interval $(-1, 4)$. The function is concave downward on the interval $(-\infty, 1.5)$ and concave upward on $(1.5, +\infty)$.

c. The slope of $f(x)$ is positive but decreasing from $-\infty < x < -1$, and is 0 at $x = -1$. The slope then becomes negative and continues decreasing until $x = 1.5$ at which point slope is a minimum. Then the slope starts to increase from some negative value (which we estimate to be -1.25) to 0 (at $x = 4$). It then becomes positive and continues increasing.

12.3 EXERCISES

PRACTICE

In Exercises 1–16, sketch the graph of a continuous function that satisfies all the given conditions.

1.
- **a.** $f(-1) = 2$
- **b.** $f'(-1) = 0$
- **c.** $f'(x) < 0$ if $x < -1$
- **d.** $f'(x) > 0$ if $x > -1$
- **e.** $f''(x) > 0$ for all x

2.
- **a.** $f(3) = 4$
- **b.** $f'(3) = 0$
- **c.** $f'(x) < 0$ if $x > 3$
- **d.** $f'(x) > 0$ if $x < 3$
- **e.** $f''(x) < 0$ for all x

3.
- **a.** $f(-2) = 4, f(-1) = 1, f(1) = -1$
- **b.** $f'(-2) = 0, f'(1) = 0$
- **c.** $f'(x) < 0$ if $-2 < x < 1$
- **d.** $f'(x) > 0$ if $x < -2$ or $x > 1$
- **e.** $f''(-1) = 0$
- **f.** $f''(x) < 0$ if $x < -1$
- **g.** $f''(x) > 0$ if $x > -1$

4.
- **a.** $f(0) = -2, f(2) = 0, f(3) = 3$
- **b.** $f'(0) = 0, f'(3) = 0$
- **c.** $f'(x) < 0$ if $x < 0$ or $x > 3$
- **d.** $f'(x) > 0$ if $0 < x < 3$
- **e.** $f''(2) = 0$
- **f.** $f''(x) < 0$ if $x > 2$
- **g.** $f''(x) > 0$ if $x < 2$

5. **a.** $f(-3)=5, f(-1)=2, f(0)=-1$
b. $f'(-3)=0, f'(0)=0$
c. $f'(x)<0$ if $x<0$ and $x \neq -3$
d. $f'(x)>0$ if $x>0$
e. $f''(-3)=0, f''(-1)=0$
f. $f''(x)<0$ if $-3<x<-1$
g. $f''(x)>0$ if $x<-3$ or $x>-1$

6. **a.** $f(1)=2, f(2)=3, f(4)=4, f(6)=2$
b. $f'(1)=0, f'(4)=0$
c. $f'(x)<0$ if $x>4, x<1$
d. $f'(x)>0$ if $1<x<4$

7. **a.** $f(x)=ax^3+bx+c$
b. $f(0)=0$
c. $f(1)=15$
d. $f'(-1)=0$ and $x=-1$ is a local max
e. $f''(x)>0$ if $x<10$

8. **a.** $f(10)=5$
b. $f'(5)=0$
c. $f''(5)=10$
d. $f''(x)<0$ if $x>10$

9. **a.** $f''(x)>0$ if $x<5$
b. $f''(5)=0$
c. $f''(x)<0$ if $x>5$
d. $f'(x)>0$ for all x

10. **a.** $f(4)=8$
b. $f'(4)=0$
c. $f''(4)=8$

11. **a.** $f(-5)=4$
b. $f'(-5)=0$
c. $f''(-5)=-2$

12. **a.** $f(x)=ax^2+bx+c$
b. $f'(-3)=0$
c. $f''(-3)=2$

13. **a.** $f(x)=ax^3+bx^2+cx+d$
b. $f(0)=25$
c. $f'(4)=0, f'(-4)=0$
d. $f''(4)=48, f''(-4)=-48$

14. **a.** $f(x)=ax^2+bx+c$
b. $f(0)=79$
c. $f'(5)=0$
d. $f''(x)=6$

15. **a.** $f(x)=ax^3+bx^2+cx+d$
b. $f(0)=2$
c. $f'(0)=5$
d. $f''(0)=4$
e. $f''(1)=12$

16. **a.** $f(x)=ax^3+bx$
b. $f'(0)=-12$
c. $f'(2)=0$

For each of the functions in Exercises 17–36, determine $f'(x)$ and $f''(x)$. Then complete a summary table like those in Examples 3 and 4. Use this table to sketch the graph of the function. (If available, use a graphing utility or calculator to obtain a suitable window and confirm the accuracy of your calculations.)

17. $f(x)=x^2-4x+7$

18. $f(x)=x^2+6x-8$

19. $f(x)=6+5x-x^2$

20. $f(x)=2+3x-2x^2$

21. $f(x)=x^3+3x^2-6$

22. $f(x)=\frac{1}{3}x^3-4x+3$

23. $f(x)=\frac{1}{3}x^3+x^2-3x+5$

24. $f(x)=2x^3-3x^2-12x+5$

25. $f(x)=x^4-2x^2+4$

26. $f(x)=x^4-8x^2-3$

27. $f(x)=\frac{1}{4}x^4-x^3+5$

28. $f(x)=x^4+4x^3+12$

29. $f(x)=x^4-4x+7$

30. $f(x)=3x^4-4x^3+3$

31. $f(x)=(x+5)(x-3)^2$

32. $f(x)=(x+1)^2(x-10)^2$

33. $f(x)=(2x+1)(x-8)^3$

34. $f(x)=(x-5)(x-10)(x+3)$

35. $f(x)=2x(5x+8)^3$

36. $f(x)=16x(21+x)^3$

37. Suppose that $f(x)=mx^2+6x+4$. Determine a value of m so that $f(x)$ has a minimum at $x=-1$.

38. Given $y=4x^2+nx+8$, determine a value for n so that y has a minimum at $x=2$.

39. Determine a value for m such that at $x=1$ the tangent to the function $f(x)=mx^2+6x+1$ has an equation of $y=12x-2$.

40. Determine a value for m so that $y=4x^3+mx^2$ has an inflection point at $x=-10$.

APPLICATIONS

41. In an action movie, the hero is seen fighting the villain inside a plane, which has a large hole in its side. The hero (actually a movie stunt man) is then thrown from the plane. He falls quickly and soon reaches a constant velocity. The hero opens his parachute but it deploys slowly, as if he is having trouble, but finally, in triumph, all is well and he drifts steadily and slowly to the ground.

a. Draw a graph of the hero's vertical distance to the ground, represented by y, versus time t (in seconds).

b. Describe any interesting points on the graph with points in the movie narrative.

42. The effectiveness of a certain medical injection is modeled by $E(t)=0.01t(100-t)$, where t is time in minutes and E is a measure of concentration in the bloodstream. Effectiveness readings above 9.0 are satisfactory and readings above 30 are dangerous.

a. If an injection is given at midnight, when are the readings satisfactory? When do they become too low?

b. How high do the effectiveness readings get?

c. The supervising nurse and the resident pharmacologist must assign a schedule for injections. For the next week, assuming injected dosages are additive, give a reasonable schedule for injections so that the patient's E-reading stays at or above 9 but never exceeds 30.

43. The productivity rating of an individual worker at the Cruz Corporation assembly line is based on the number of tasks accomplished, mistakes made, and responsiveness to difficulties encountered. The average of all scores allows the company to use a simple model based on time on the floor given by $\text{PR}=-0.4x^3+2x^2+10x+5$, where x is in hours at work. A PR score of 20 is acceptable and a score of 40 is highly unusual.

a. When are workers' scores the highest?

b. Design an 8-hour day where workers do the most demanding jobs for about 6 hours and have 2 hours for less stressful work. Explain your reasoning.

WRITING & THINKING

44. Is it possible for a polynomial $y = ax^2 + bx + c$ to have an inflection point?

12.4 EXERCISES

PRACTICE

For each of the rational functions in Exercises 1–12, find **a.** any vertical asymptotes, **b.** any horizontal asymptotes, and **c.** any oblique asymptotes.

1. $f(x)=\dfrac{1}{x-4}$

2. $f(x)=-\dfrac{3}{x+6}$

3. $f(x)=\dfrac{2x}{x+8}$

4. $f(x)=\dfrac{5x}{2x+1}$

5. $f(x)=\dfrac{x+2}{x^2+1}$

6. $f(x)=\dfrac{x-7}{x^2+3}$

7. $f(x)=\dfrac{5x^2}{3x^2-2x-1}$

8. $f(x)=\dfrac{2x^2}{x^2+3x}$

9. $f(x)=\dfrac{x^2-4}{x}$

10. $f(x)=\dfrac{3x^2+2}{x}$

11. $f(x)=\dfrac{x^2+1}{x+1}$

12. $f(x)=\dfrac{x^2-5}{x-2}$

In Exercises 13–22, sketch the graph of each rational function. Show any asymptotes on each graph.

13. $f(x)=-\dfrac{2}{x+5}$

14. $f(x)=\dfrac{4}{x-3}$

15. $f(x)=\dfrac{2x}{x+1}$

16. $f(x)=\dfrac{3x}{x-2}$

17. $f(x)=\dfrac{x-2}{x-1}$

18. $f(x)=\dfrac{x+4}{2x+1}$

19. $f(x)=2x+\dfrac{2}{x}$

20. $f(x)=3x+\dfrac{12}{x}$

21. $f(x)=\dfrac{3x^2+6}{x}$

22. $f(x)=\dfrac{2x^2+1}{3x}$

APPLICATIONS

23. Junker Renovation completely overhauls junked or abandoned cars. Data shows their 1970s models hold their value quite well. The value $F(x)$ of one of these cars is given by $F(x)=70-\dfrac{15x}{x+1}$ where x is the number of years since repurchase and F is in hundreds of dollars.

- **a.** What is the initial resale price of a car?
- **b.** Find all asymptotes.
- **c.** Sketch the function.
- **d.** What is the long term value of one of these cars?

24. The average cost $A(x)$ is the total cost $C(x)$ divided by the quantity x. Thus $A(x)=\dfrac{C(x)}{x}$. If the total cost function for a product is $C(x)=3x+12$, graph the average cost $A(x)$. If there are any asymptotes, locate them and interpret their meaning.

25. A product's total costs are given by $C(x) = 0.03x^2 + 24x + 10$.
  a. Graph the average cost function, locating any asymptotes.
  b. What is the meaning of the asymptotes for average cost?

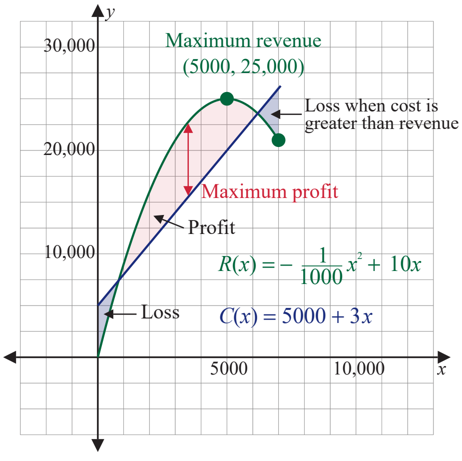
26. The Polar Pollution Control Company removes debris from old motors. Suppose the cost $F(x)$ of removing x percent of the pollutants is given by $F(x) = \dfrac{100{,}000}{100 - x}$, where x is a percentage, $0 \le x < 100$, and F is in dollars.
  a. Determine $\lim\limits_{x \to 0^+} F(x)$ and $\lim\limits_{x \to 100^-} F(x)$ and interpret their meanings.
  b. What percentage can be removed at a cost of \$3000?
  c. Show that $F(x)$ is always increasing. Does this make sense in the context of the problem?

27. The cost of camel rides in Tunisia is modeled by the function $F(x) = 40 - \dfrac{20x}{x+3}$, where x is the number of years since 2000 and F is a national average cost in dinars.
  a. What was the cost of a camel ride in 2002?
  b. What are the asymptotes for F and which is significant in the problem?
  c. When was the average cost 26 dinars?

28. The sugar level concentration in the bloodstream of a certain diabetes patient is modeled by $S(t) = 1 + \dfrac{0.2t}{t^2 + 2}$, where S is in suitable units and t is the time in hours following a meal of allowed carbohydrate content.
  a. Which asymptotes play a role here?
  b. For $0 \le t \le 6$, what is the highest S-value and when does it occur? (If this level exceeds 4, the patient will become ill.)
  c. Are there any inflection points? What is the meaning in the context of the problem of an inflection point?

29. Data suggests a professional football team will win $F(x)$ games (out of 16) if the salary of the superstar players increases. For one team, the function F is given by $F(x) = 8 + \dfrac{6x}{0.125x^2 + 2}$, where x is the average salary (in millions) of the superstars (players earning at least one million dollars).
  a. Are there any asymptotes of consequence in the problem?
  b. What average salary gives the biggest return on games won, according to this model? (Here return is total games won.)

30. If administrative assistants at Bookworm Publications make phone follow-ups after textbook reviews, more colleges and universities will adopt a new statistics book. The publisher noticed that new book sales varied in the second year according to $S(x) = A_0 - \dfrac{(x+200)}{x^2}$, where S is total sales and x is the number of phone calls made to colleges which have adopted the book. A_0 denotes the sales from the previous year.
  a. Assuming $A_0 = 2500$, what sales can Bookworm Publications expect if they make 100 follow-up phone calls?
  b. What is the horizontal asymptote and what is its significance?



$$P'(x) = -\frac{1}{500}x + 7$$

Set $P'(x)$ equal to 0, and solve for x.

$$-\frac{1}{500}x + 7 = 0$$
$$-x = -3500$$
$$x = 3500$$

Thus, a maximum profit occurs if the company produces and sells 3500 calculators, which is only half its production capabilities. The price for each calculator would be

$$p = -\frac{1}{1000}(3500) + 10 = \$6.50.$$

The graph shown illustrates the relationship between profit, revenue, and cost. Note that maximum revenue and maximum profit do not necessarily occur at the same level of production and sales.

12.5 EXERCISES

APPLICATIONS

1. **Minimizing inventory costs:** An appliance store owner estimates that he will sell 125 vacuum cleaners of a particular model. It costs \$12 to store one vacuum cleaner for one year. There is a fixed cost of \$30 for each order. Find the lot size and the number of orders per year that will minimize inventory costs.

2. **Minimizing inventory costs:** A hardware store sells 96 chainsaws per year. It costs \$5 to store one chainsaw for one year. There is a fixed reordering cost of \$15. Find the lot size and the number of orders per year that will minimize inventory costs.

3. **Minimizing inventory costs:** An art gallery owner expects to sell 90 copies of a limited-edition print during the next year. It costs \$1.50 to store one copy for one year. For each order she places, there is a fixed cost of \$7.50, plus \$0.50 for each copy. Find the lot size and the number of times the gallery owner should order per year to minimize her inventory costs.

4. **Minimizing inventory costs:** The owner of Lamps-4-U expects to sell 180 brass lamps during the year. For each order he places, there is a fixed cost of \$18, plus \$2 for each lamp ordered. It costs \$5 to store one lamp for one year. In what lot size and how many times per year should he reorder to minimize the inventory costs?

5. **Minimizing inventory costs:** A T-shirt company sells 4000 sweatshirts per year. To reorder, there is a fixed cost of \$6 plus \$0.80 for each sweatshirt. It costs \$1.20 to store one sweatshirt for one year. In what lot size and how many times per year should an order be placed to minimize inventory costs?

6. **Minimizing inventory costs:** An office supply store sells 7500 pink highlighters per year. It costs \$0.15 to store one pink highlighter for one year. To reorder these highlighters, there is a fixed cost of \$22.50, plus \$0.10 for each highlighter. In what lot size and how many times per year should an order be placed to minimize inventory costs?

7. **Minimizing inventory costs:** A snowmobile dealer in Minnesota expects to sell 960 snowmobiles during the next year. It costs \$9 to store one snowmobile for one year. To reorder, there is a fixed cost of \$67.50, plus \$7.50 for each snowmobile. In what lot size and how many times per year should an order be placed to minimize inventory costs?

8. **Minimizing inventory costs:** A car dealer expects to sell 1320 cars during the next year. It costs \$660 to store one car for one year. To reorder, there is a fixed cost of \$225, plus \$304 for each car. Find the lot size and the number of orders that should be placed so inventory costs will be minimized.

9. **Maximizing revenue:** A chain of discount stores sells 84 weather radios per month at \$20 each. The owners estimate that for each \$1 increase in price, they will sell 3 fewer radios per month. How much should they charge for their weather radios to maximize their revenue?

10. **Maximizing revenue:** A farmer estimates that if he plants 30 grapefruit trees per acre, the average yield per tree will be 480 pounds. For each additional tree planted per acre, the yield per tree will be reduced by 12 pounds. How many trees should be planted per acre to maximize the yield?

11. **Maximizing revenue:** Sam operates a chain of convenience stores. He estimates that he can sell 600 small packs of gum per day if he charges 75 cents each. Sam determines that for each 10-cent reduction in price, he will sell an additional 80 packs per day. How much should he charge for the small packs of gum to maximize his revenue?

12. **Maximizing revenue:** A sporting goods store sells 200 baseball gloves per month at \$36 each. The owner estimates that for each \$2 increase in price, he will sell 5 fewer gloves. Find the price that will maximize revenue.

13. **Maximizing revenue:** A sports arena has 40 roaming soda salespeople, each of whom sells 200 sodas per event. Management estimates that for each additional salesperson, the yield per salesperson decreases by 4. How many additional salespeople should management hire to maximize the number of sodas sold?

14. **Maximizing revenue:** Ms. Wills owns a 16-rack dry stack boat storage facility. The unit rent per rack is currently \$400 per month, and all racks are rented. Each time rent is increased by \$20, one boat owner will move out. Find the rental price that will maximize Ms. Wills' revenue.

15. **Linear demand function:** A local amusement park found that if the admission was \$7, about 1000 customers per day were admitted. When the admission was dropped to \$6, the park had about 1200 customers per day. Assuming a linear demand function, determine the admission price that will yield maximum revenue.

16. **Linear demand function:** A department store manager has determined that when the price of a tank top was \$12, she sold 100 tank tops per month. However, only 80 tank tops were sold per month when the price was raised to \$14. Assuming a linear demand function, determine the price that would maximize the revenue.

17. **Linear demand function:** The cost of producing x units of an item is $C(x) = 10x + 20$. When the selling price is \$20, twenty-one items are sold. However, when the price is \$16, twenty-three items are sold. Assuming the demand function is linear, determine the price per unit and the number of units sold that will maximize the profit.

18. **Linear demand function:** The manager of a bakery knows he can sell 60 small bags of donut holes when the price is \$1.20 each. If the price is \$1.50, only 48 bags are sold. The total cost function for x bags is $C(x) = 0.70x + 15$ dollars. Assuming a linear demand function, determine the price per bag and the number of bags sold that will maximize the profit.

19. **Linear demand function:** The manufacturer of microwave ovens can sell 800 to his dealers at \$392 each. If the price is \$380, he can sell 1000. The total cost of producing x microwaves is $C(x) = 3600 + 250x - 0.01x^2$ dollars. Assuming the demand function is linear, find the price per microwave and the number of microwaves sold that will maximize profit.

20. **Linear demand function:** A candy store can sell 180 lollipops at 62 cents each. The store can sell 220 lollipops if the price is 54 cents each. The total cost of producing x lollipops is $C(x) = 3050 - 10x + 0.04x^2$ cents. Find the number of lollipops that should be produced to maximize profit.

21. **Profit:** Suppose $P(x)$ represents profit on the sales of x cell phones. Suppose $P(25{,}000) = 12{,}000$, $P'(25{,}000) = 2$, and $P''(25{,}000) = -3$.
    a. Is the company making money or losing it? How much?
    b. If sales are increased, will the profits rise or fall? By how much?
    c. What is the meaning of $P''(25{,}000) = -3$?

22. **Profit:** Suppose the monthly marginal profit from John's online tutoring service is $P'(x) = 3$, where x is the number of subscribers.
    a. The profit function is (choose one): linear, quadratic, or a polynomial of degree 3 or higher.
    b. Suppose monthly costs are $C(x) = 9x + 20$. Assuming initial revenue is 0, what is the revenue function?

23. **Profit:** Suppose the demand for a product is \$12 and the total costs are $C(x) = 0.3x^2 + 2x + 5$.
    a. What is the revenue function?
    b. What is the profit function?
    c. What is the maximum value of the profit?

24. Average cost: $A(x)=\dfrac{C(x)}{x}$ gives average cost.

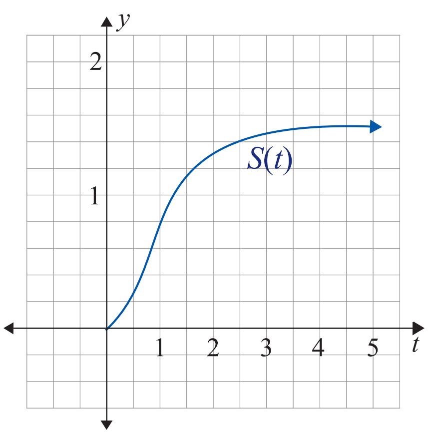
a. Calculate a formula for $A'(x)$.
b. Set $A'(x)=0$ and solve for $C'(x)$.
c. If average costs are minimal, describe a relationship between average cost and marginal cost. That is, interpret the result of part **b.**

25. Average cost: Suppose that a company's average cost is $A(x)=0.2x+3$.
a. Determine the cost function.
b. Determine the marginal average cost.
c. Determine the marginal cost.

26. Earnings: Suppose a company's earnings are given by $E(x)=P(x)+I(x)$, where x is the number of years since 2018, $P(x)$ is the annual profit function, and $I(x)$ is the intangible growth (the growth in value of the company's intangible assets such as its good name). If $P(x)=1.3x+2$ and $I(x)=0.25x+1$ for a certain company, determine the following.
a. The marginal earnings for year x
b. The actual earnings for 2020
c. The average earnings formula (earnings per year since 2018)

27. Value: The value of a new business franchise grows according to the formula $V(x)=10+\dfrac{10x}{1+0.5x}$. Here V is in thousands of dollars and x is the number of years after 2012.
a. What is the expected value in 2022?
b. Is the value V increasing or decreasing in 2022?
c. Is the rate of increase in value changing? What has this to do with V' or V''?

28. Value: The timber value of a small stand of pine trees is given by $V(x)=50\left(1-\dfrac{1}{x+2}\right)$, where x is the number of years after 2000 and V is in dollars.
a. What was the value in 2000?
b. At what rate was the value changing in 2010?
c. What asymptotes are present and what is their significance in the problem?



WRITING & THINKING

29. Sales function: A sales function $S(t)$ for a new product is shown in the figure. $S(t)$ is total sales (quantity of items) and t is time in months since the product's release. Copy this graph onto your paper and add a graph of a possible $S'(t)$. Locate approximately the point of inflection on your curve for S.

30. Cost function: A certain cost function $C(x)$ satisfies $C(10)=20$, $C(20)=40$, and $C(30)=60$. Suppose $C''(10)=-2$, $C''(20)=0$, and $C''(30)=2$. Draw a suitable function.

b. The projectile hits the ground when $s(t) = 0$.

$$
\begin{aligned}
-16t^2 + 160t + 176 &= 0 && \text{Solve } s(t) = 0 \text{ for } t. \\
-16\left(t^2 - 10t - 11\right) &= 0 \\
-16(t - 11)(t + 1) &= 0 \\
t = 11 \text{ or } \cancel{t = -1} & && \text{Time cannot be negative, so we discard the } t = -1.
\end{aligned}
$$

Thus the projectile hits the ground 11 seconds after it is fired.

c. The projectile's velocity when it hits the ground is $v(11)$. From part **a.** we know

$$v(t) = -32t + 160.$$

So

$$
\begin{aligned}
v(11) &= -32(11) + 160 \\
&= -352 + 160 \\
&= -192 \text{ ft/sec}. && \text{The negative sign indicates the projectile is falling.}
\end{aligned}
$$

The projectile will be falling at a rate of 192 ft/sec when it hits the ground.

12.6 EXERCISES

APPLICATIONS

1. **Volume of a box:** A rectangular box with no top is to be made from a piece of cardboard that is 24 in. by 24 in. Equal squares are to be cut from each corner and the sides folded to form the box. What size should the squares be to maximize the volume of the box?

2. **Volume of a box:** A rectangular box with no top is to be made from a piece of cardboard that is 20 in. by 20 in. Equal squares are to be cut from each corner and the sides folded to form the box. What size should the squares be to maximize the volume of the box?

3. **Volume of a box:** Equal squares are to be cut from each corner of a rectangular piece of thin sheet metal, and the sides are to be folded to form a box. If the piece of metal is 8 in. by 15 in., find the dimensions of the box having maximum volume.

4. **Volume of a box:** Equal squares are to be cut from each corner of a rectangular piece of thin sheet metal, and the sides are to be folded to form a box. If the piece of metal is 10 in. by 16 in., find the dimensions of the box having maximum volume.

5. **Surface area:** A rectangular box is designed to have a square base and an open top. The volume is to be 864 in.^{3}
   a. What dimensions will give a minimum surface area?
   b. What is the minimum surface area?

6. **Surface area:** A rectangular box is designed to have a square base and an open top. The volume is to be 256 in.^{3}
   a. What dimensions will give a minimum surface area?
   b. What is the minimum surface area?

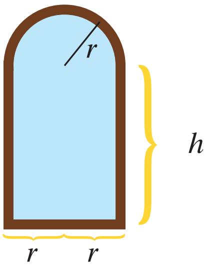
7. **Container design:** A container manufacturer is asked to design a closed rectangular shipping crate with a square base. The volume is 10 ft^3. The material for the top and sides costs \$2 per square foot and the material for the bottom costs \$3 per square foot. Find the dimensions of the box that will minimize the total cost.

8. **Container design:** A container manufacturer is asked to design a closed rectangular shipping crate with a square base. The volume is 36 ft^3. The material for the top costs \$1 per square foot, the material for the sides costs \$0.90 per square foot, and the material for the bottom costs \$1.40 per square foot. Find the dimensions of the box that will minimize the total cost of material.

9. **Container design:** An investor plans to manufacture rectangular box containers whose bottom and top measure x by $3x$. The box must contain 18 cubic feet. The top and bottom will cost \$2 per square foot, and the four sides will cost \$3 per square foot. What should the height h be so as to minimize costs?

10. **Area:** A rectangular plot is to be enclosed with an existing block wall as one side. If there are 680 ft of fencing available for the other three sides, find the dimensions that will maximize the area.

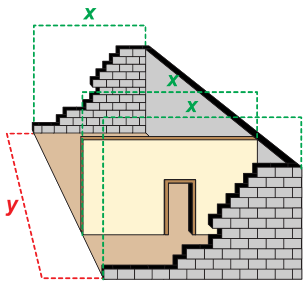
11. **Area:** A rectangular play area is to be enclosed with the side of a house as one of the sides. If there are 74 ft of fencing available for the other three sides, find the dimensions that will maximize the area.

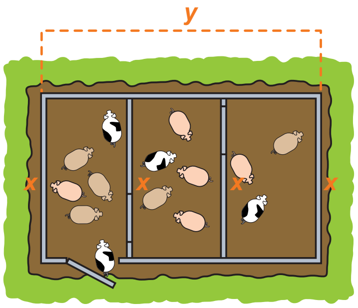


12. **Window area:** A front window on a new home is designed as a rectangle with a semicircle on the top. If the window is designed to let in a maximum amount of light, and the architect fixes the perimeter of the entire window at 600 inches, determine the radius r and rectangular height h so as to maximize the area.

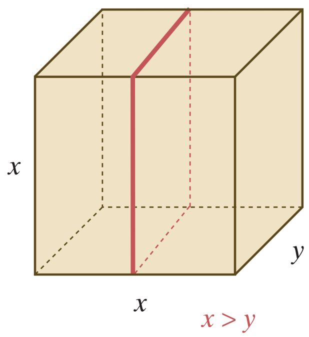
13. **Wall construction:** An old stone wall makes two legs of a right angle, one 40 feet long and the other 20 feet long. A constructor is told to add 220 feet of new stone fence to complete a rectangular fence. How should he complete the fence so as to maximize the enclosed area? Determine the maximum enclosed area he may obtain.

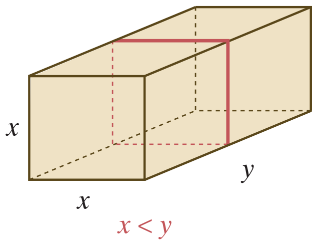
14. **Construction:** A warehouse is being constructed with a total floor area of 2200 ft^2. A single partition is built to divide the building into storage area and office space. The exterior walls cost \$160 per foot, and the interior wall costs \$120 per foot. Find the dimensions of the warehouse that will minimize the cost.





15. **Construction:** A farmer wants to build a rectangular pen and then divide it with two interior fences. The total area is to be 2484 ft^2. The exterior fence costs \$18 per foot, and the interior fence costs \$16.50 per foot. Find the dimensions of the pen that will minimize the cost.

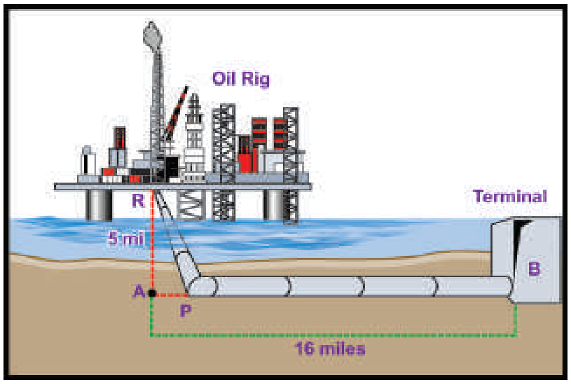




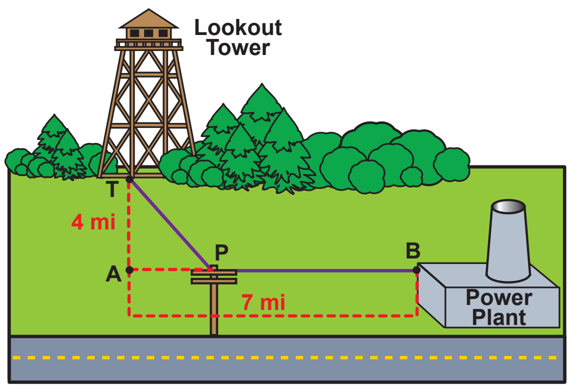
16. Shipping: The Postal Service has a limit of 108 in. on the combined length and girth of a rectangular package to be sent by parcel post. Find the dimensions of the package of maximum volume that has a square cross section. (**Hint:** There are two different answers, depending on the shape of the box. The two shapes are shown here. The girth is defined to be the smallest perimeter of a rectangular cross section of the box.)

17. Shipping: An independent parcel service has a limit of 130 in. on the combined length and girth of a rectangular package it will ship. Find the dimensions of the package of maximum volume that has a square cross section. (**Hint:** There are two different answers. The girth is defined to be the smallest perimeter of a rectangular cross section of the box.)

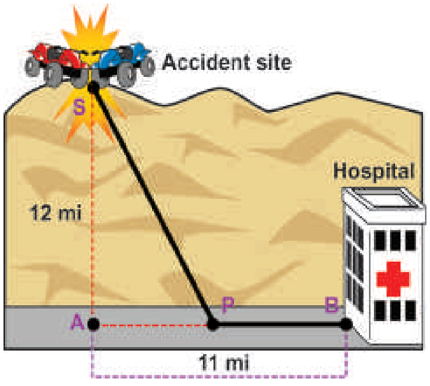
18. Pipeline construction: An oil company wishes to run a pipeline from a drilling platform located 5 miles offshore to a shipping terminal 16 miles down the coast. The costs are \$130,000 per mile to lay the pipeline underwater and \$120,000 per mile to lay the pipeline over land. Find the location of point P (as illustrated in the diagram) so that the total cost of laying pipe will be minimized.



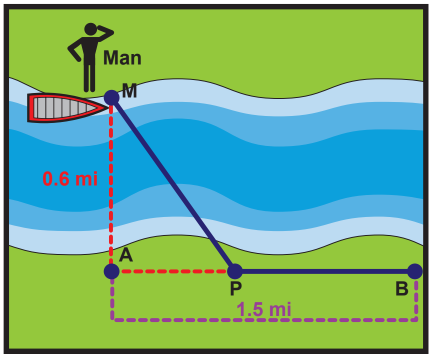
19. Power line construction: The U.S. Forest Service wishes to run a power line to a fire lookout tower located in a wooded area. The tower is 4 miles from the nearest road and the power source is 7 miles down that road. It costs \$5000 per mile to run the line through the forest and \$3000 per mile to run the line along the road. Find the location of point P (as illustrated in the diagram) so that the total cost of running the power line will be minimized.



20. **Minimum time:** The Off-Roaders, an all-terrain vehicle club, were driving their four-wheelers in the desert when one member had a serious accident. At the time, they were 12 miles from the nearest paved road. The nearest hospital was located 11 miles down the paved road. If they can average 20 mph on the desert and 52 mph on the road, locate point P on the road toward which they should drive in order to minimize the time needed to get to the hospital.



21. **Minimum time:** A man is on the bank of a river that is 0.6 miles wide. He wants to reach a point on the opposite shore that is 1.5 miles downstream. If he can row a boat across the river at 4 mph and walk at 5 mph, find the location P, on the opposite shore, to which he should row in order to minimize the total time he would need to reach the point downstream.



22. **Distance and velocity:** A particle is moving along a straight line such that the distance traveled at the end of t seconds is given by $s(t) = 7t^2 + 30t$ feet.
   a. Find the velocity if $t = 2$ seconds.
   b. How far has the particle traveled?

23. **Distance and velocity:** A ball is rolled down an incline. The distance (in feet) of the ball from the starting point after t seconds is given by $s(t) = 19t + 8t^2$.
   a. Find the velocity after 3 seconds.
   b. How far has the ball traveled in 3 seconds?

24. **Distance and velocity:** A projectile is fired vertically, and its height (in feet) after t seconds is given by $s(t)=104t-16t^2$.

a. Find the maximum height of the projectile.
b. When does the projectile hit the ground?
c. How fast is the projectile moving when it hits the ground?

25. **Distance and velocity:** A stone is projected vertically. The height (in feet) of the stone at time t (in seconds) is given by $s(t)=-16t^2+112t+128$.

a. What is the maximum height of the stone?
b. When will the stone hit the ground?
c. What is the speed of the stone when it hits the ground?

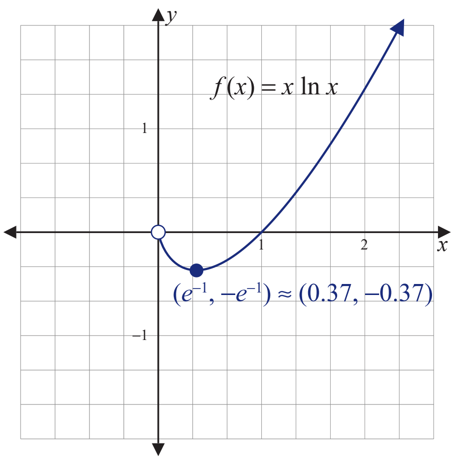
26. **Distance and velocity:** A child rolls a hoop down a hilly street. The distance traveled (in feet) after t seconds is given by $s(t)=4t+t^2$.

a. How far has the hoop traveled in 3 seconds?
b. What was the speed at 3 seconds?
c. At what rate was the speed changing at $t=3$?

A local (and absolute) minimum occurs at the critical value $x = e^{-1}$, and

$$f\left(e^{-1}\right) = e^{-1}\ln e^{-1}$$
$$= e^{-1}(-1) = -e^{-1}.$$

Note: One easy point to get without a calculator is (1, 0) (recall that $\ln 1 = 0$). Therefore, it is possible to notice that $1 \cdot \ln 1 = 0$, and (1, 0) is an x-intercept.



13.1 EXERCISES

PRACTICE

Find the derivative of each of the functions in Exercises 1–28.

1. $f(x) = x^2 - \ln x$

2. $f(x) = 4x^2 + \ln x^2$

3. $f(x) = 25 + x\ln x$

4. $f(x) = \left(x^2+1\right)\ln x$

5. $y = \dfrac{\ln x}{x}$

6. $y = \dfrac{x^2}{\ln x}$

7. $y = (\ln x)^3$

8. $y = \sqrt{\ln x}$

9. $f(x) = \ln(5x+3)$

10. $f(x) = \ln(7x-2)$

11. $f(x) = \ln\left(x^2+2\right)$

12. $f(x) = \ln\left(x^2+3x\right)$

13. $f(x) = \ln\sqrt{2x^2-1}$

14. $f(x) = \ln\sqrt{x^3+4}$

15. $y = \ln(4x+3)^2$

16. $f(x) = \ln\left(x^2-4\right)^3$

17. $y = \sqrt{x}\ln\left(x^2+2\right)$

18. $y = \dfrac{\ln(5x+2)}{x^3}$

19. $y = \dfrac{\ln\left(x^2+2x-1\right)}{x}$

20. $y = x^{-2}\ln\left(x^2-3x+4\right)$

21. $f(x) = \ln\left((3x+1)\left(x^2+3\right)\right)$

22. $f(x) = \ln\left(x^2(4x-1)\right)$

23. $f(x) = \ln\left((2x-1)^2\left(x^2-2\right)\right)$

24. $f(x) = \ln\left(\sqrt{4x-7}\,(6x+7)\right)$

25. $y = \ln\left(\dfrac{x+1}{x-2}\right)$

26. $y = \ln\left(\dfrac{3x-1}{x+5}\right)$

27. $f(x) = \ln\left(\dfrac{x^2-5}{2x+9}\right)$

28. $f(x) = \ln\sqrt{\dfrac{4x+3}{x^2-6}}$

For Exercises 29–34, determine a formula for $f''(x)$. Use your calculator (if necessary) to solve the equation $f''(x) = 0$ and locate any possible inflection points.

29. $f(x) = \dfrac{x}{\ln x}$

30. $f(x) = (2x + x^2)\ln x$

31. $f(x) = 3x^2 \ln x$

32. $f(x) = \ln(x^3)$

33. $f(x) = (\ln x)^3$

34. $f(x) = \ln x + x^2 + 3x + 2$

Use logarithmic differentiation to find the derivatives of the functions in Exercises 35–42.

35. $y = (2x-5)^3\sqrt{x^2-2x}$

36. $y = (4-5x)^4(7x+2)^{-\frac{2}{3}}$

37. $y = (x^2+2)^4(x^2-1)^{-\frac{1}{3}}$

38. $y = (3x-2)^5\sqrt{x^2+x}$

39. $y = \dfrac{(2x+x^2)^3}{(4x-9)^2}$

40. $y = \dfrac{\sqrt[3]{x^2+4}}{(2-5x)^3}$

41. $y = \dfrac{(x+2)^2(3x+4)^2}{\sqrt{x-6}}$

42. $y = \dfrac{(x^2-3)^2\sqrt{x^2+3x}}{x+7}$

In Exercises 43–48, find the absolute extrema for each of the functions on the indicated interval.

43. $f(x) = x - \ln x;\ [0.5, 2]$

44. $f(x) = x^2 - 8\ln x;\ [0.3, 4]$

45. $f(x) = \dfrac{x}{\ln x};\ [1.2, 3]$

46. $f(x) = \dfrac{\ln x}{x^2};\ [1, 2]$

47. $f(x) = x^2 - \ln x^3;\ [1, e]$

48. $f(x) = \ln(3 + 2x - x^2);\ [0, 2.5]$

Using the graph-sketching techniques discussed in Chapter 12, sketch the graph of each function in Exercises 49–54.

49. $f(x) = 4x\ln x$

50. $f(x) = x^2 \ln x$

51. $f(x) = x - 3\ln x$

52. $f(x) = 4x - \ln x^2$

53. $f(x) = \dfrac{\ln x}{x}$

54. $f(x) = \dfrac{\ln x}{x^2}$

APPLICATIONS

55. Marginal revenue: A retailer has determined that the revenue from the sale of x end tables is given by the function $R(x) = 96x + \ln(4x^2 + 15)$ dollars. Find the marginal revenue.

56. Advertising: An automobile dealer has estimated that he can sell $N(x) = 420 + 72\ln(1 + 0.5x)$ cars annually, where x (in thousands of dollars) is the amount spent on advertising.

a. Find the number of cars sold if \$6000 is spent on advertising.

b. Find the rate of change in number of cars sold if \$6000 is spent on advertising.

57. Revenue: The daily demand for a product is given by $p = -8\ln(0.01x)$, where p is the price in dollars when x units are sold and $0 < x \le 100$. Find the maximum daily revenue.

58. Revenue: The demand for a product is given by $p = 14 - 6.5\ln x$, where x is the number of units (in thousands) that can be sold at a price p dollars and $2 \le x \le 8$. Find the maximum revenue.

59. Insect population: Mediterranean fruit flies are discovered in a citrus orchard. The Department of Agriculture has determined that the population of flies t hours after the orchard has been sprayed with pesticide is approximated by $N(t) = 25 - 5t\ln(0.04t) - t$, where $0 < t \le 25$.

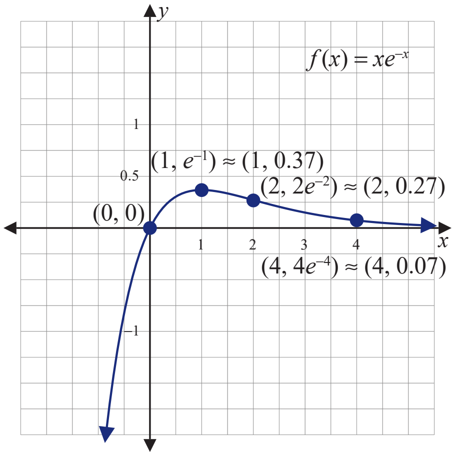
a. Find $N(3)$, $N(10)$, $N(25)$.

b. What will be the maximum number of flies in the orchard?

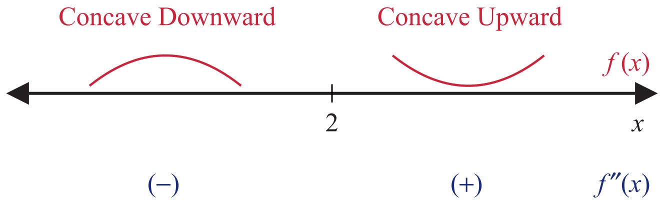
60. Air quality: The Air Quality Management Board estimates that t hours after midnight in a major city the level of ozone in the air is about $N(t) = 0.013 - 0.007t\ln(0.026t)$ parts per million, where $0 < t \le 18$.

a. Find $N(6)$, $N(10)$, $N(18)$.

b. Find the maximum level of pollution.



So there is a local max at $(1, e^{-1})$.



There is a point of inflection at $(2, 2e^{-2})$.

13.2 EXERCISES

PRACTICE

Find the first and second derivative of each of the functions in Exercises 1–6.

1. $f(x) = 3e^x$

2. $f(x) = -6e^x$

3. $f(x) = x^2 + 5e^x$

4. $f(x) = 4x^2 - 2e^x$

5. $f(x) = xe^x$

6. $f(x) = -7x^2e^x$

For Exercises 7–14, find a formula for $f'(x)$ and determine the slope $f'(a)$ at the point where $x = a$ is given.

7. $f(x) = e^x \ln x; \quad x = 1$

8. $f(x) = e^x \ln(x+4); \quad x = 1$

9. $f(x) = \dfrac{e^x}{e^x - 1}; \quad x = 2$

10. $f(x) = \dfrac{e^x}{\ln x}; \quad x = e$

11. $f(x) = 2e^{4x}; \quad x = -1$

12. $f(x) = 8e^{-3x}; \quad x = 4$

13. $f(x) = 3e^{2x+1}; \quad x = 0$

14. $f(x) = 5e^{\frac{x}{2}}; \quad x = 0$

For Exercises 15–18, find a formula for $f'(x)$ and use it to determine the intervals on which $f(x)$ is increasing or decreasing.

15. $f(x) = e^{1-x^2}$

16. $f(x) = e^{-0.04x^2}$

17. $f(x) = \left(e^{2x} - 4\right)^2$

18. $f(x) = \left(e^{4x} + 2\right)^3$

For Exercises 19 and 20, determine $f'(x)$ and use it to determine the intervals on which $f(x)$ is increasing or decreasing. Determine for each function if there is a horizontal asymptote. Confirm your results with a graphing calculator.

19. $f(x) = \sqrt{e^{-0.2x} + 11}$

20. $f(x) = \dfrac{1}{\sqrt{3e^x + 1}}$

Find $f'(x)$ and use it to argue whether or not there is an oblique asymptote for each of the functions in Exercises 21 and 22.

21. $f(x)=\ln\left(e^x+1\right)$

22. $f(x)=\ln\sqrt{5+e^{2x}}$

For each of the following functions in Exercises 23–30, find the absolute extrema on the indicated interval.

23. $f(x)=xe^{2x};\ [-2, 1]$

24. $f(x)=xe^{\frac{x}{3}};\ [-4, 0]$

25. $f(x)=x^2e^{-x};\ [-1, 2]$

26. $f(x)=2x^2e^{-x};\ [-2, 3]$

27. $f(x)=5e^{1-x^2};\ [-2, 1]$

28. $f(x)=3xe^{-x^2};\ [-2, 2]$

29. $f(x)=(2x+3)e^{-0.2x};\ [1, 4]$

30. $f(x)=(4x-1)e^{-0.5x};\ [0, 3]$

For each of the functions in Exercises 31–36, **a.** find any critical values, **b.** find any hypercritical values, **c.** find all intervals of concavity, and **d.** sketch the graph of the function. If available, confirm your results with a graphing utility.

31. $f(x)=xe^{-0.4x}$

32. $f(x)=2xe^{-0.5x}$

33. $f(x)=4x^2e^{-x}$

34. $f(x)=3x^2e^{-x}$

35. $f(x)=e^x+e^{-x}$

36. $f(x)=\dfrac{e^x}{x}$

APPLICATIONS

37. Revenue: The demand for a product is given by $D(x)=140e^{-0.05x}$, where x is the number of units sold each week and $0\le x\le 30$.

a. Find the number of units sold that will yield maximum revenue.

b. Find the price per unit that will yield maximum revenue.

38. Revenue: The demand equation for a certain product is given by $D(x)=210e^{-0.025x}$, where x is the number of units sold each week and $0\le x\le 60$.

a. Find the number of units sold that will yield the maximum revenue.

b. Find the price per unit that will yield maximum revenue.

39. Advertising: An automobile manufacturer is planning a television advertisement campaign to introduce a new model for their truck. It is estimated that $N(t)=600\left(1-e^{-0.02t}\right)$ people (in thousands) will have seen the advertisement after t days of advertising. How fast is N increasing at the end of 7 days?

40. Insect population: The mosquito population of a pool of water is estimated to be $P(t)=400+1400e^{-0.3t}$, where t is the number of hours after the pool has been treated. Find the rate of change in the population at the end of 5 hours.

41. Bacterial population: The population of bacteria in an experimental culture is estimated by $N(t)=\dfrac{10{,}000}{1+9e^{-0.14t}}$, where t is the number of hours after the experiment begins. How fast is the population changing at the end of 5 hours?

42. **Disease control:** The elk herd in a national park has been infected by a contagious disease. The number of infected animals is estimated by $N(t)=\dfrac{600}{1+49e^{-0.36t}}$, where t is the number of days after the disease was discovered. How fast is the disease spreading after 4 days?

43. Suppose the value of the inventory of original Winchester rifles at Bill's Antique Firearms Company has increased according to the formula $r(t)=\dfrac{8500}{1+10e^{-0.6t}}$, where r is the average value (in dollars) of one of their rifles and t is the number of years since 2000.

a. What was the average value of a rifle in 2000? In 2005?
b. At what rate was r changing in 2005? In 2006?
c. If there is an inflection point for $r(t)$, locate it and explain its significance in the application.
d. When is the rate of increase of r at a maximum?

44. Suppose an advertising campaign for the sale of a new magazine, *Dungeons and Creeps*, causes sales to vary according to the formula $S(t)=8\left(1-0.3e^{-0.2t+1}\right)$, where S is monthly sales in thousands of magazines and t is time in months since the ad campaign started.

a. What were the monthly sales when the ad campaign started?
b. What was the rate at which sales were changing after 4 months into the campaign?
c. What are the long-term monthly sales expectations?

45. A research scientist determines that a mass of algae in a pond grows according to $A(t)=1+2te^{-0.5t}$, where A represents the mass-density of algae in the pond in suitable units and t is the time in months ($t=0$ corresponds to April 1st).

a. What day of the year corresponds to a maximum A-value?
b. When does the rate of decline in algae reach its maximum?

46. A certain calculus student recalls information according to the formula $p=70e^{-0.6x}+30$, where p is the percentage of information retained after x weeks.

a. After 4 weeks, what percentage of a lesson is retained?
b. After 4 weeks, at what rate is the percentage changing?
c. What does the model predict a few months after the calculus course is over?

47. Inexpensive videos detailing the championship basketball season of Castle High School are sold locally by a civic club to raise money for next year's team. The total sales are given by $S=\dfrac{12{,}500}{1+15e^{-0.5x}}$, where S is the total number of videos sold after x weeks.

a. What are the total sales after 3 weeks?
b. What is the rate of change of sales after 3 weeks?
c. After about how many weeks will the total sales begin to level off?
d. When is the sales rate increasing fastest? Illustrate this point graphically.

WRITING & THINKING

48. **a.** Find the equation of the tangent line to $f(x) = 2e^{-x^2+1} + 4$ at the point where $x = 2$.

b. Discuss the advantages and disadvantages of using the tangent line to get values of $f(x)$ for $x \geq 2$ rather than the function itself.

49. Determine k in the equations that follow by finding $f'(0)$. Use logarithmic differentiation.

a. Let $f(x) = 10^x$. Determine the value of k in the formula $f'(x) = k \cdot 10^x$.

b. Let $y = f(x) = \pi^x$. Determine k in the formula $f'(x) = ky$.

TECHNOLOGY

Use a graphing calculator to graph $f(x)$ and $f'(x)$. Then locate all extrema and all inflection points, if any.

50. $f(x) = e^{-x^2} \ln(x^2 + 2)$

51. $f(x) = \ln \dfrac{2 + 3e^{-x}}{x + 2}$

13.3 EXERCISES

APPLICATIONS

1. **Population:** The population of a city is growing exponentially at a rate of 3.5 percent per year. The population was 8400 in 2000.
   a. Find an exponential function that represents the population t years after 2000.
   b. What was the population in the year 2010?
   c. When was the population 12,800?

2. **Bee population:** A swarm of bees grows exponentially at a rate of 4 percent hourly. Initially, there were 900 bees in the swarm.
   a. Find an exponential function for the number of bees in the swarm after t hours.
   b. How many bees are in the swarm after 6 hours?
   c. How many hours will it take for the swarm to double in size? Round your answer to the nearest tenth.

3. **Cost:** In 2018, the cost of a medium pizza was about \$9.00. In 2021, the cost was \$12.00. If the cost is growing exponentially, predict the cost of a medium pizza in 2027?

4. **Ant colony:** A colony of ants is growing exponentially. When first observed, the colony contained about 400 ants. If at the end of 9 days there are about 700 ants, approximately how many ants will be present at the end of 15 days?

5. **Bacterial population:** A bacteria culture grows at a rate proportional to its size. If the population doubles every 6 hours, how long will it take for the population to be three times its initial size?

6. **Demand for oil:** The demand for oil in the United States doubles every 8 years. How long will it take for the demand to triple?

7. **Inflation:** The amount of goods and services that costs \$100 on January 1, 2015 costs \$139.10 on January 1, 2018. Estimate the cost of the same goods and services on January 1, 2025. Assume the cost is growing exponentially.

8. **Interest compounded continuously:** One thousand dollars is deposited in a savings account where the interest is compounded continuously. After 4 years, the balance will be \$1366.15. When will the balance be \$1870.00?

9. **Half-life:** The decay rate for a radioactive isotope is 2.6 percent per year. Find its half-life.

10. **Half-life:** The decay rate of a radioactive isotope is 6.5 percent per year. Find its half-life.

11. **Archaeological dating:** A wooden carving found at an archaeological dig contains about 34 percent of its carbon-14. Approximately how old is the carving?

12. **Archaeological dating:** Bones from the skeleton of an animal have lost 62 percent of their carbon-14. Estimate the age of the bones.

13. **Atmospheric pressure:** As the elevation above sea level is increased, the atmospheric pressure declines exponentially. The pressure at sea level is approximately 15 lb/in.^{2} and the pressure at 3000 feet of elevation is about 13 lb/in.^{2} Find the pressure at 5000 ft.

14. **Drug concentration:** The concentration of a drug in the body fluids is known to decline exponentially. If 20 mg of a drug is administered and 8 mg remains after 3 hours, how much will remain after 5 hours?

15. **Depreciation:** It is determined that the value of a piece of machinery declines exponentially. A machine that was purchased 5 years ago for $65,000 is worth $35,000 today. What will be the value of the machine 5 years from now?

16. **Population:** The population of a certain economically depressed union is declining exponentially at a rate of 1.5 percent. If the population in 2010 was 30,000, estimate the population in 2030.

17. **Reliability:** Studies show that the fractional part P of light bulbs that has burned out after t hours of use is given by $P = 1 - e^{-0.03t}$. What fractional part of the bulbs has burned out after 50 hours? How long will it be before half of the bulbs have burned out?

18. **Advertising:** A radio station estimates that during an intense advertising campaign, the number of people N who will hear a commercial is given by $N = A\left(1 - e^{-0.02x}\right)$, where A is the number of people in the broadcasting area and x is the number of times the commercial is run. If there are 60,000 people in the area,
   a. How many people will hear the commercial if it is run 20 times?
   b. How many times should the station plan to run the commercial to be certain that at least 30,000 people hear it?

19. **Ecology:** The Department of Fisheries has begun a reclamation project at a lake where the fish population was nearly destroyed by agricultural chemicals. They estimate that the population of fish in t years will be $P = 6000 - 5200e^{-0.28t}$.
   a. What was the initial population?
   b. What will be the population after 4 years?
   c. How long will it take for the population to be 5000 fish?

20. **Advertising:** The manager of The Sound Lab has determined that after an intense advertising campaign, the monthly sales of a particular wireless speaker can be approximated by $N = 300 + 180e^{-0.04t}$ units, where t is the number of months after the campaign.
   a. Find the monthly sales initially.
   b. Find the monthly sales when $t = 6$.
   c. When will the monthly sales be 400 units?

21. **Skills development:** Beverly is making a small souvenir to give to each person attending her family reunion. The length of time, in minutes, she takes to make the n^{th} one is given by the function $T(n) = 12 + 30e^{-0.1n}$. How long will it take her to make the 30^{th} souvenir?

22. **Dairy farming:** The number of dairy farmers in a particular state who are feeding a new supplement to their milking cows is given by the function $W(t) = 340\left(1 - e^{-0.09t}\right)$, where t is the number of months the supplement has been available. How long will it be before 200 farmers are feeding the supplement to their cows?

23. **Cost:** The total cost function for a local company is given by $C(t) = 12 - ce^{-kt}$ in thousands of dollars, where t is the time in months. The fixed costs are \$5000 and the total cost after 2 months is \$10,200. Find the total cost at the end of 6 months.

24. **Skills development:** The time that it takes a service attendant to change a tire is given by the function $T(x) = 4.4 + Ce^{-kx}$ minutes, where x is the number of tires the attendant has changed before. It takes Patrick 15 minutes to change the first tire ($x = 0$) and 9.3 minutes to change the seventh tire. How long will it take him to change the eleventh tire?

Solution

a. The quantity demanded is x and $x = F(p)$.

$$\begin{aligned} x &= F(1.70) \\ &= 180 - 30(1.70) \\ &= 129 \end{aligned}$$

b. $E = -\dfrac{pF'(p)}{F(p)} = -\dfrac{p(-30)}{180-30p} = \dfrac{p}{6-p}$

c. $E(1.70) = \dfrac{30(1.70)}{180-30(1.70)} = \dfrac{51}{129} = \dfrac{17}{43}$

d. Since $E < 1$, the demand is inelastic and so an increase in price will bring a decrease in demand and an increase in revenue.

e. $$\begin{aligned} R = xp &= (180-30p)p \\ &= 180p - 30p^2 \end{aligned}$$

Thus $R' = 180 - 60p$. Setting $R' = 0$, we obtain $60p = 180$, so $p = \$3.00$ per loaf.

Since $R'' = -60$, the function R is concave down at $p = 3$ and this price gives a maximum for the revenue.

13.4 EXERCISES

PRACTICE

For each of the demand functions in Exercises 1–16, find **a.** the function describing the elasticity of demand and **b.** the value of x that maximizes the revenue.

1. $p = D(x) = 84 - 3x$

2. $p = D(x) = 144 - 1.5x$

3. $p = D(x) = 520 - 2.6x$

4. $p = D(x) = 480 - 3.2x$

5. $p = D(x) = 200e^{-0.2x}$

6. $p = D(x) = 67e^{-0.1x}$

7. $p = D(x) = 88e^{-0.025x}$

8. $p = D(x) = 130e^{-0.04x}$

9. $p = D(x) = \sqrt{150 - x}$

10. $p = D(x) = \sqrt{180 - 2x}$

11. $p = D(x) = \sqrt{162 - 3x}$

12. $p = D(x) = \sqrt{255 - 2.5x}$

13. $p = D(x) = 18 - \sqrt{x}$

14. $p = D(x) = 21 - 2\sqrt{x}$

15. $p = D(x) = 363 - x^2, \quad x \le 18$

16. $p = D(x) = 600 - 0.5x^2, \quad x \le 34$

APPLICATIONS

17. **Maximum revenue:** The demand function for an electric pencil sharpener is given by $p = D(x) = 19.2 - 0.4x$ dollars. Find the level of production for which the revenue is maximized.

18. **Maximum revenue:** The demand function for a popular stereo receiver is given by $p = D(x) = 540 - 0.05x^2$ dollars. Find the level of production for which the revenue is maximized.

19. **Elastic demand:** The demand function for an exclusive wool blanket is given by $p = D(x) = 33 - 2\sqrt{x}$ dollars, where x is in thousands of blankets. Find the level of production for which the demand is elastic.

20. **Elastic demand:** Find the levels of production for which the demand is elastic if the demand is given by $p = D(x) = \sqrt{207 - 3x}$ dollars.

21. **Elastic demand:** An arcade sells video games and determines that $x = 30\left(1 - e^{-\frac{p}{10}}\right)$, where x is the number of video games demanded for a unit price p.
    - **a.** Determine the quantity demanded when $p = \$10$ per game.
    - **b.** Determine E and interpret the result at $p = \$10$.
    - **c.** What revenue is generated at $p = \$10$?

22. **Elastic demand:** Lucky Blooms sells a new rose variety which has established a demand of $x = f(p) = \dfrac{e^{\frac{p}{3}} + 350}{e^{\frac{p}{2}}}$.
    - **a.** Determine the quantity demanded when $p = \$3$.
    - **b.** Determine E and interpret the result at $p = \$3$.

23. **Elastic demand:** The demand for a product is given by $x = F(p) = \dfrac{1800}{10 + \ln(1+p)}$.
    - **a.** If $p = 20$, determine E and interpret the results.
    - **b.** What is the revenue function R?
    - **c.** Use R' to determine if R is increasing at $p = 20$. Is your answer consistent with part **a.**?

24. **Elastic demand:** Suppose a product has a demand function $x = F(p) = 300e^{-\frac{p}{10}}$.
    - **a.** Find the elasticity function.
    - **b.** Is the demand elastic or inelastic at $p = \$20$?
    - **c.** Determine the unit price which maximizes revenue.
    - **d.** Discuss whether or not your answers to **b.** and **c.** are consistent.

25. **Elastic demand:** Suppose the demand for a product is $p = D(x) = 300e^{-\frac{x^2}{200}}$.
    - **a.** Determine the unit price if the quantity $x = 5$.
    - **b.** What is formula for $D'(x)$?
    - **c.** What is the elasticity at $x = 5$?
    - **d.** Determine the value of x which maximizes revenue.

Example 9: L'Hôpital's Rule and the Indeterminte Form ∞^0

Determine $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Solution

The base has a limit of ∞ and the exponent has a limit of 0. We proceed as in the last two examples.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x} \qquad \text{Indeterminate form } \frac{\infty}{\infty}$$

Applying l'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

and therefore $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^0 = 1$.

13.5 EXERCISES

PRACTICE

Evaluate the limit using previous techniques. Then decide whether l'Hôpital's Rule is applicable and, if so, use it to check your answer.

1. $\lim_{x\to 3} \frac{2x^2-18}{x-3}$

2. $\lim_{x\to -2} \frac{x^3+8}{x+2}$

3. $\lim_{x\to\infty} \frac{6x^2-x+7}{x-3x^2}$

4. $\lim_{x\to -\infty} \frac{5x^2-2x+1}{2.5x^3-3x^2+6}$

5. $\lim_{x\to 0} \frac{2x}{\sqrt{x+3}-\sqrt{3}}$

6. $\lim_{x\to 0^+} \left(\sqrt{x}\right)^{1/x}$

7. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x\sqrt{x+1}}\right)$

Two functions are in competition to determine the indicated limit. Identify the type of the indeterminate form, and fill out the table to decide which function dominates.

8. $\lim_{x\to\infty} f(x)$, where $f(x) = \frac{\sqrt{5x^3+7}}{0.2x^2+1}$

x	1	10	100	1000	10,000	100,000
$f(x)$						

9. $\lim\limits_{x\to\infty} g(x)$, where $g(x)=\dfrac{0.5\sqrt{x}}{\ln(x+1)}$

x	1	10	100	1000	10,000	100,000
$g(x)$						

10. $\lim\limits_{x\to\infty} h(x)$, where $h(x)=x^{100}e^{-x}$

x	1	10	100	1000	10,000	100,000
$h(x)$						

Check whether l'Hôpital's Rule applies to the given limit. If it does, use it to determine the value of the limit. If it does not, find the limit some other way. (When necessary, apply l'Hôpital's Rule several times.)

11. $\lim\limits_{x\to\infty}\dfrac{2x+5}{x^2-7}$

12. $\lim\limits_{x\to\infty}\dfrac{4-2.5x}{x+3}$

13. $\lim\limits_{x\to-\infty}\dfrac{1.5x^3-2x^2+x+9}{x^2+2.1x-4}$

14. $\lim\limits_{x\to-\infty}\dfrac{4.5x^4+x^3-2}{3-1.5x^4}$

15. $\lim\limits_{x\to0^+}\dfrac{\sqrt{x}}{\ln x}$

16. $\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+2}{2x+1}$

17. $\lim\limits_{t\to0}\dfrac{t}{\sqrt{2t+9}-3}$

18. $\lim\limits_{x\to\infty}\dfrac{\ln x}{\ln(x^2+3x)}$

19. $\lim\limits_{x\to0}\dfrac{\log_{10}(x^2+2x+1)}{\log_{10}(x+1)}$

20. $\lim\limits_{x\to0}\dfrac{x}{3^{x/2}-1}$

21. $\lim\limits_{x\to\infty}\dfrac{2^x}{x^2-3x+4}$

22. $\lim\limits_{x\to\infty}\dfrac{x+2^x}{5^x-x}$

23. $\lim\limits_{x\to\infty}\dfrac{4^x+x^2}{3^x-x}$

24. $\lim\limits_{x\to\infty}\dfrac{\ln(\ln x)}{x\ln x}$

25. $\lim\limits_{x\to\infty}\dfrac{\log_4(2x+1)}{\log_5(x-4)}$

26. $\lim\limits_{x\to0^+}\dfrac{\log_4(x+1)}{\log_3 x}$

27. $\lim\limits_{x\to0}\dfrac{3^x-1}{x3^x}$

Identify the indeterminate product, quotient, difference, or power, and use l'Hôpital's Rule to find the limit. If the limit is not of indeterminate form, say so and find it by other means.

28. $\lim\limits_{x\to0^+} x\ln x$

29. $\lim\limits_{x\to\infty}\dfrac{\sqrt{2x^2+1}}{x+3}$

30. $\lim\limits_{x\to\infty}(\ln x)^{-1/x}$

31. $\lim\limits_{x\to0^+}\left(\dfrac{1}{x}\right)^x$

32. $\lim\limits_{x\to0^+}(-\ln x)^x$

33. $\lim\limits_{x\to1^+}\left(\dfrac{1}{\ln x}-\dfrac{2}{x-1}\right)$

34. $\lim\limits_{x\to0^+}\left(\dfrac{1}{x}+\ln x\right)$

35. $\lim\limits_{x\to4^+}\left(\dfrac{32}{x^2-16}-\dfrac{x}{x-4}\right)$

36. $\lim\limits_{x\to0^+} x^{(x^2)}$

37. $\lim\limits_{x\to0^+}(2^x-x)^{1/x}$

38. $\lim\limits_{x\to0^+}(1-x)^{1/x}$

39. $\lim\limits_{x\to\infty}\left(\sqrt{x^2-3x}-\dfrac{3}{x^2+1}\right)$

40. $\lim_{x\to\infty}(x-1)^{1/x}$ **41.** $\lim_{x\to\infty}\frac{\ln x}{x^{7/5}}$ **42.** $\lim_{x\to\infty}\frac{x^{100}}{3^x}$

43. $\lim_{x\to\infty}\frac{\ln(100x^2+e^x)}{100x}$ **44.** $\lim_{x\to 0}(1+2x)^{1/x}$ **45.** $\lim_{x\to 1}x^{1/(1-x)}$

Find the limit. If applicable, use l'Hôpital's Rule (as many times as appropriate).

46. $\lim_{x\to\infty}\frac{2x^5+x^3-4}{e^x}$ **47.** $\lim_{x\to\infty}x^{1/x^3}$ **48.** $\lim_{x\to 0^+}x^{x^x}$

49. $\lim_{x\to 0^+}(x^x)^x$ **50.** $\lim_{x\to\infty}x^{1/x^n},\quad n\in\mathbb{Z}^+$ **51.** $\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}$

52. $\lim_{x\to\infty}\frac{x^2+1}{2^x}$ **53.** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ **54.** $\lim_{x\to\infty}\sqrt[x]{x}$

55. $\lim_{x\to\infty}\frac{2^x+5^x}{6^x}$

Convince yourself that the initial use of l'Hôpital's Rule is not helpful in finding the limit. If possible, try to find a way to make use of the theorem, or evaluate the limit in some other way.

56. $\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x}}$ **57.** $\lim_{x\to\infty}\frac{\sqrt[3]{x+1}-2}{\sqrt{x^2+2}}$ **58.** $\lim_{x\to\infty}\frac{2^x+3^x}{5^x}$

59. $\lim_{x\to\infty}\frac{5^x-6^x}{7^x+8^x}$ **60.** $\lim_{x\to\infty}\frac{2^{-x}}{x^{-1}}$ **61.** $\lim_{x\to\infty}\left(\frac{1}{x+1}\right)^{-x^3}$

62. $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{e^{-x}}$ **63.** $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

64. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$ **65.** $\lim_{x\to\infty}2^{-x}x\ln x$

WRITING & THINKING

Find the error(s) in the limit calculation.

66. $\lim_{x\to 2}\frac{x^2-2}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4\quad$ (Incorrect!)

67. $\lim_{x\to -\infty}\frac{5^x+1}{5^x}=\lim_{x\to -\infty}\frac{(\ln 5)5^x}{(\ln 5)5^x}=1\quad$ (Incorrect!)

Use l'Hôpital's Rule to prove the assertion.

68. $\lim_{x\to\infty}\frac{p(x)}{e^{kx}}=0\quad$ ($p(x)$ is a polynomial, $k>0$)

69. $\lim_{x\to\infty}\frac{(\ln x)^n}{x^k}=0\quad(n\in\mathbb{N},\ k>0)$ **70.** $\lim_{x\to\infty}\frac{a^x}{x^n}=\infty\quad(a>1,\ n\in\mathbb{N})$

Find the value(s) of *c* satisfying the conclusion of Cauchy's Mean Value Theorem. If the theorem doesn't apply, explain why.

71. $f(x)=x, \quad g(x)=x^2+1; \quad [0,1]$

72. $f(x)=x^3-1, \quad g(x)=x^2+2x; \quad [-1,1]$

73. $f(x)=x^3-x, \quad g(x)=-x^2+2x+3; \quad [-1,3]$

74. $f(x)=x^3, \quad g(x)=-x^2; \quad [-2,3]$

75. $f(x)=x^2+3x, \quad g(x)=3x^2-5x+3; \quad [-1,3]$

76. $f(x)=\frac{1}{x}, \quad g(x)=\ln x; \quad [1,2]$

77. $f(x)=x^2-5x-9, \quad g(x)=x^3+x+10; \quad [-3,2]$

78. Recall the compound interest formula for the value of an investment of P dollars after t years, compounded n times a year at an annual interest rate of r:

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

Use l'Hôpital's Rule to prove that if we let $n\to\infty$, we obtain the continuous compounding formula $A=Pe^{rt}$.

79. The strength of an electric field due to a disk charge is obtained from the formula

$$E(x)=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

where σ is the electric charge per unit area (in $\mathrm{C/m^2}$), $\varepsilon_0=8.85\cdot10^{-12}\ \mathrm{C^2/Nm^2}$, R is the radius of the ring, and x is the distance to the charge in meters. Use l'Hôpital's Rule to confirm that $E(x)\to 0$ as $x\to\infty$. How is E affected by σ and R at a given distance? What happens to the rate of change of E as x increases? (**Hint:** Apply l'Hôpital's Rule to dE/dx as $x\to\infty$.)

80. Marquis de l'Hôpital first illustrated the rule named after him in his 1696 textbook, *Analyse des Infiniment Petits*. He used an example where the objective was to find

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

for $a>0$. Determine the above limit.

TECHNOLOGY

Use a graphing utility to graph the function for different values of the parameter *c*. Examine how the values of the parameter affect the indicated limit.

81. $\lim_{x\to\infty}\left(1+\frac{1}{cx}\right)^x$; What happens to the limit when $|c|\to\infty$?

82. $\lim_{x\to 0^+}\frac{1-c^x}{cx}$; What happens to the limit when $c\to\infty$?

13.6 EXERCISES

PRACTICE

Find the differential for each of the functions in Exercises 1–14.

1. $y = x^3 + 5$

2. $y = 4x^3 + x - 7$

3. $u = \left(2t^2 + 1\right)^2$

4. $u = (5t + 9)^2$

5. $A = \pi r^2$

6. $A = x(44 - 2x)$

7. $V = \dfrac{4}{3}\pi r^3$

8. $V = x^3$

9. $S = 4x^2 + \dfrac{1350}{x}$

10. $C = 75 + 10x - 0.6\sqrt{2x}$

11. $C = 40 + 3x + 0.4\sqrt{x}$

12. $S = 2\pi r^2 + \dfrac{90\pi}{r}$

13. $P = -0.2x^2 + 75x - 2400$

14. $P = -0.3x^2 + 84x - 870$

For each of the functions given in Exercises 15–22, use the given values for x and Δx to find **a.** Δy, **b.** dy, and **c.** $\Delta y - dy$.

15. $y = x^2 - 3x + 4,\ x = 3,\ \Delta x = 0.2$

16. $y = x^2 + 5x - 9,\ x = 2,\ \Delta x = 0.15$

17. $y = \left(2x^2 - 4\right)^3,\ x = -2,\ \Delta x = 0.04$

18. $y = \left(x^2 + x - 1\right)^3,\ x = -3,\ \Delta x = 0.05$

19. $y = 20\left(x - \dfrac{24}{x}\right),\ x = 2,\ \Delta x = -0.12$

20. $y = 2x^2 + \dfrac{125}{x^2},\ x = 5,\ \Delta x = -0.15$

21. $y = \sqrt{3x + 4},\ x = 7,\ \Delta x = 0.5$

22. $y = \sqrt{12 - 5x},\ x = 2,\ \Delta x = 0.07$

Use differentials to approximate the indicated roots in Exercise 23–30. Express your answer as $a \pm {}^{b}\!/_{c}$, where a is the nearest integer.

23. $\sqrt{37}$

24. $\sqrt{65}$

25. $\sqrt[3]{26}$

26. $\sqrt[3]{126}$

27. $\sqrt{50.4}$

28. $\sqrt{79.5}$

29. $\sqrt[3]{62.3}$

30. $\sqrt[3]{218.3}$

APPLICATIONS

31. Cost: A total cost function (in dollars) is given by $C(x) = 375 + 9x + 0.01x^2$. Use differentials to estimate the change in cost when the level of production is increased from 60 to 62 units.

32. Cost: A total cost function (in dollars) is given by $C(x) = 930 + 15x + 0.2x^2$. Using differentials, estimate the change in cost from $x = 100$ to $x = 101$.

33. Profit: The weekly revenue (in dollars) from the sale of x coffee makers is given by $R(x) = 40x$. The total cost function is given by $C(x) = 370 + 16x + 0.2x^2$. Use differentials to approximate the change in profit if the weekly sales are increased from 25 to 28 coffee makers.

34. **Profit:** The monthly revenue from the sale of x 50-gallon aquariums is given by $R(x) = 54x - 0.3x^2$ dollars. The total cost function is given by $C(x) = 0.1x^2 + 4x + 200$ dollars. Using differentials, find the approximate change in profit if the monthly sales are increased from 40 to 44 aquariums.

35. **Population:** It is estimated that t years from now the population of a city will be $P(t) = 10(4000 + 2t^2) - 1600t$. Use differentials to estimate the change in population as t changes from 6 to 6.25 years.

36. **Bacterial population:** It is estimated that t hours from now the population of bacteria in a culture will be $P(t) = \dfrac{8000}{\sqrt{8 - 0.5t}}$. Use differentials to estimate the change in population as t changes from 8 to 8.3 hours.

37. **Volume:** The edge of a cube measures 18 in. with a possible error in measurement of 0.02 in. Use differentials to estimate the possible error in computing the volume.

38. **Fiberglass coating:** A cube is 12 in. on a side. It is to be covered with a fiberglass coating 0.25 in. thick. Use differentials to estimate the volume of the fiberglass coating.

39. **Measurement error:** A manufacturer of cargo containers receives an order for a cube-shaped container. The specifications state that the volume should be 125 ft^3. with a maximum error of no more than 1 ft^3. Using differentials, find the possible error in the length of the edges.

40. **Melting ice:** A block of ice is in the form of a 10 in. cube. If it melts uniformly until the volume changes to 972 in.^{3}, approximate the change in the length of each edge by using differentials.

41. **Volume of a weather balloon:** A spherical weather balloon is being inflated. Use differentials to find the approximate change in the volume if the radius changes from 20 to 21.5 inches. (**Hint:** $V = \frac{4}{3}\pi r^3$.)

42. **Volume of a tumor:** A spherical cancer tumor is being treated with an experimental drug. The radius of the tumor has been reduced from 1.6 to 1.4 cm. Use differentials to estimate the change in the volume of the tumor. (**Hint:** $V = \frac{4}{3}\pi r^3$.)

speed at $t = 0$. Thus $C_1 = -200$ m/s. We use -200 rather than $+200$ since the direction of the meteor is toward the origin (so s is decreasing due to the speed). Therefore,

$$v(t) = -30t^2 + 64t - 200.$$

Since $v = \frac{ds}{dt}$, we can say $ds = v \cdot dt$. Integrating again, we have

$$\int ds = \int v \cdot dt = \int \left(-30t^2 + 64t - 200\right) \cdot dt.$$

Integrating a second time gives

$$s = -30\left(\frac{1}{3}t^3\right) + 64\left(\frac{1}{2}t^2\right) - 200t + C_2$$
$$s = -10t^3 + 32t^2 - 200t + C_2.$$

Once again there is a constant of integration to evaluate. If at the time of initial measurement the distance of the meteor from Earth is 50,000 kilometers, then $s(0) = 50{,}000{,}000$ meters.

$$50{,}000{,}000 = s(0) = -10(0)^3 + 32(0)^2 - 200(0) + C_2 = C_2$$

Here we see $C_2 = 50{,}000{,}000$ meters, the initial distance of the meteor from Earth.

It is typical of "acceleration" problems that there are two integrations, two constants of integration to evaluate, and two additional pieces of data necessary for this evaluation. It is common to use the notation v_0 and s_0 to denote initial ($t = 0$) values of velocity and distance.

One case of special interest is a body falling due to Earth's gravity. In this case the acceleration a is a constant. As before, $a = \frac{dv}{dt}$ so $dv = a \cdot dt$. Integrating both sides gives

$$v = at + C.$$

The constant C is v_0, the initial velocity. That is, $v = at + v_0$. Since $\frac{ds}{dt} = v$, $ds = v \cdot dt$. One more integration gives

$$s = \int ds = \int \left(at + v_0\right) dt = a\left(\frac{1}{2}t^2\right) + v_0 t + s_0$$
$$s = \frac{1}{2}at^2 + v_0 t + s_0.$$

14.1 EXERCISES

PRACTICE

In Exercises 1–12, show that the function $F(x)$ is an antiderivative of the function $f(x)$ by differentiating F.

1. $F(x) = 4x - 1, \quad f(x) = 4$

2. $F(x) = 6x, \quad f(x) = 6$

3. $F(x) = 3x^2 + 5x + 2, \quad f(x) = 6x + 5$

4. $F(x)=\frac{1}{2}x^2-4x+e^{2x}-1,\quad f(x)=x-4+2e^{2x}$

5. $F(x)=\ln x-\frac{1}{x}-4e^{x^2},\quad f(x)=\frac{1}{x}+\frac{1}{x^2}-8xe^{x^2}$

6. $F(x)=\ln x^3+\frac{1}{x^2}+6,\quad f(x)=\frac{3}{x}-\frac{2}{x^3}$

7. $F(x)=\left(x^2+3\right)^4-1,\quad f(x)=8x\left(x^2+3\right)^3$

8. $F(x)=3(5x-1)^{\frac{2}{3}}+8,\quad f(x)=\frac{10}{\sqrt[3]{5x-1}}$

9. $F(x)=\frac{5}{3}\left(e^x-4\right)^3+e,\quad f(x)=5e^x\left(e^x-4\right)^2$

10. $F(x)=3e^{x^2-1}-7,\quad f(x)=6xe^{x^2-1}$

11. $F(x)=\ln\left(x^2+5x-3\right)-\sqrt{5},\quad f(x)=\frac{2x+5}{x^2+5x-3}$

12. $F(x)=\ln\left(e^{3x}-x\right)+\sqrt{11},\quad f(x)=\frac{3e^{3x}-1}{e^{3x}-x}$

Find the indefinite integrals in Exercises 13–32.

13. $\int 7\,dx$

14. $\int \frac{2}{3}\,dx$

15. $\int 5x^4\,dx$

16. $\int -2x^{-3}\,dx$

17. $\int \left(x^2-3\right)dx$

18. $\int \left(x^4+5\right)dx$

19. $\int \left(\frac{1}{3}-e^t\right)dt$

20. $\int \left(e^t+t\right)dt$

21. $\int \left(\frac{1}{y}+y^3\right)dy$

22. $\int \left(\frac{1}{\sqrt{y}}-\frac{1}{y}\right)dy$

23. $\int \left(4x^2+\frac{2}{x}+\frac{1}{x^2}\right)dx$

24. $\int \left(9x-\frac{3}{x}-\frac{1}{\sqrt{x}}\right)dx$

25. $\int \left(2\sqrt[3]{x}+5\sqrt{x}\right)dx$

26. $\int \left(\sqrt{x}+6e^x-\frac{5}{x}\right)dx$

27. $\int \left(4e^y-2y^5-\frac{1}{5}\right)dy$

28. $\int \left(y^{\frac{3}{2}}+5y^{-\frac{2}{3}}-y^{-1}\right)dy$

29. $\int \left(\frac{2}{\sqrt[3]{t}}+\frac{7}{t^3}\right)dt$

30. $\int \left(2t^{\frac{5}{2}}+\frac{4}{t}-\sqrt{3}\right)dt$

31. $\int \left(\frac{2}{3}e^x+x^{-\frac{3}{2}}-7x^{-1}\right)dx$

32. $\int \left(0.25e^x+4x^{-\frac{1}{4}}\right)dx$

In Exercises 33–38, perform the indicated multiplication and then integrate.

33. $\int x^3(2x-1)\,dx$

34. $\int x^2(3x-5)\,dx$

35. $\int (3t+2)^2\,dt$

36. $\int (5x+6)^2\,dx$

37. $\int \sqrt{y}\left(y^2+2y-1\right)dy$

38. $\int \sqrt{y}\left(4-3y-2y^2\right)dy$

In Exercises 39–44, simplify the indicated quotient and then integrate.

39. $\int \frac{3x^2+5x-4}{x^2}\,dx$

40. $\int \frac{x^3-6x^2+x}{x^2}\,dx$

41. $\int \frac{4+\sqrt{x}-3x}{x}\,dx$

42. $\int \frac{5x^2-2x+3}{\sqrt{x}}\,dx$

43. $\int \frac{x^{\frac{3}{2}}+6-2xe^x}{x}\,dx$

44. $\int \frac{4x^2+4\sqrt{x}-7x}{x^2}\,dx$

45. Find the antiderivative $F(x)$ that satisfies the given condition.

a. $F'(x)=x^2-e^x, \quad F(0)=1$

b. $F'(x)=6x^2+x-10, \quad F(0)=0$

c. $\frac{dF}{dx}=\frac{10}{\sqrt{x}}, \quad F(1)=20$

d. $\frac{dF}{dx}=6e^x-2, \quad F(0)=-10$

46. Given that $f'(x)=x^2-2$, determine the function $f(x)$ with the given constant of integration C. Draw all three functions on the same coordinate system.
a. $C=-1$
b. $C=1$
c. $C=3$

47. Given $f'(x)=6x^2-24x$.
a. Determine the x-values at which $f(x)$ has a local maximum or minimum.
b. Determine whether there is an inflection point.
c. Given $f(1)=-9$, sketch f and determine if the answers to part a. are correct.

APPLICATIONS

48. Cost: The weekly marginal cost of producing x ice cream makers is $28 + 0.05x$ dollars per ice cream maker. Find the cost function if the fixed costs are $2400.

49. Cost: The marginal cost of producing x clock radios is $0.3x^2-0.8x+24$ dollars per clock radio. The fixed costs are $1500. Find the cost function.

50. Revenue: The marginal revenue from selling x irons is $94 - 0.06x$ dollars per iron. Find the revenue function. (**Hint:** $R(0) = 0$.)

51. Revenue: The marginal revenue from selling x floor lamps is $100 - 0.2x$ dollars per lamp. Find the revenue function. (**Hint:** $R(0) = 0$.)

52. **Profit:** The marginal profit from the production and sale of x cameras is estimated to be $24 - 0.4x$ dollars per unit.
    - **a.** Find the firm's profit function if the profit from the production and sale of 80 units is $240.
    - **b.** What is the profit from the sale of 90 units?

53. **Profit:** A manufacturer has determined that the marginal profit from the production and sale of x wireless speakers is approximately $120 - 3x$ dollars per speaker.
    - **a.** Find the profit function if the profit from the production and sale of 30 speakers is $1200.
    - **b.** What is the profit from the sale of 40 speakers?

54. **Population:** The population of a community is growing at a rate given by $\frac{dP}{dt} = 120 - 15t^{\frac{1}{2}}$ people per year. Find a function to describe the population t years from now if the present population is 8600 people.

55. **Rodent control:** Animal control officers have implemented a program to eliminate rats in a community. They estimate that the population of rats is changing at a rate of $\frac{dP}{dt} = 24t^{\frac{1}{2}} - 40t$ rats per month. Find a function for the rat population t months from now if the current population is estimated to be 6300 rats.

56. **Air quality:** The air quality control office estimates that for a population of x thousand people, the level of pollution in the air is increasing at a rate of $\frac{dL}{dx} = 0.2 + 0.002x$ parts per million per thousand people. Find a function to estimate the level of the pollutants if the level is 5.4 parts per million when the population is 20,000 people.

57. **Ecology:** Biologists are treating a stream contaminated with bacteria. The level of contamination is changing at a rate of $\frac{dN}{dt} = -\frac{960}{t^2} - 240$ bacteria per cubic centimeter per day, where t is the number of days since the treatment began. Find a function $N(t)$ to estimate the level of contamination if the level after 1 day was about 5720 bacteria per cubic centimeter.

58. **Height:** An object is projected vertically so that the velocity after t seconds is given by $v(t) = 96 - 32t$ feet per second.
    - **a.** Find the height function $s(t)$ if $s(0) = 18$ feet.
    - **b.** What will be the height after 3 seconds?

59. **Distance:** A vehicle travels in a straight line for t minutes with a velocity $v(t) = 72t - 6t^2$ feet per minute, for $0 \le t \le 10$.
    - **a.** Find the distance function $s(t)$ if $s(0) = 0$.
    - **b.** How far will the vehicle travel in 5 minutes?
    - **c.** How far will the vehicle travel in 10 minutes?

60. Distance: A particle moves along an axis with velocity given by $v(t) = 3t - 1$, where t is in seconds and v is in ft/s.

a. Determine the acceleration, $a(t)$.

b. What is the distance function, $s(t)$, if $s(0) = 5$?

61. Distance: A meteor falls partly under the influence of Earth's gravity at a velocity given by $v(t) = 200 + 30t + 24t^{\frac{1}{2}}$ for $0 \le t \le 24$, where t is in hours and v is in miles per hour.

a. Determine the acceleration.

b. Determine the distance function if $s(0) = 5000$ miles.

WRITING & THINKING

62. a. Compute the derivative of $y = e^{mx+b}$ where m and b are constants. Use your answer to determine an integration formula for $\int e^{mx+b}\,dx$.

b. Compute the derivative of $y = \dfrac{1}{e^{mx+b}}$ where m and b are constants. Use your answer to determine an integration formula for $\int \dfrac{1}{e^{mx+b}}\,dx$.

14.2 EXERCISES

PRACTICE

In Exercises 1–36, use the technique of substitution to perform each integration.

1. $\int (x+4)^7\,dx$

2. $\int (y-6)^{-3}\,dy$

3. $\int 2x\left(x^2-1\right)^{\frac{1}{2}}dx$

4. $\int 3x^2\left(x^3-5\right)^{\frac{1}{3}}dx$

5. $\int \frac{1}{t+2}\,dt$

6. $\int \frac{1}{x-11}\,dx$

7. $\int \frac{3t^2}{t^3+4}\,dt$

8. $\int \frac{1}{y^2-8}\cdot 2y\,dy$

9. $\int e^{y+5}\,dy$

10. $\int e^{x-9}\,dx$

11. $\int e^{-0.2x}(-0.2)\,dx$

12. $\int e^{0.5t}(0.5)\,dt$

13. $\int \frac{1}{5x+3}\,dx$

14. $\int (3y-2)^{-2}\,dy$

15. $\int \frac{1}{\sqrt{4x-1}}\,dx$

16. $\int e^{-4x}\,dx$

17. $\int xe^{2x^2}\,dx$

18. $\int \frac{x}{2x^2+5}\,dx$

19. $\int \frac{x}{\left(3x^2-1\right)^2}\,dx$

20. $\int y^2e^{-2y^3}\,dy$

21. $\int \frac{2t+1}{t^2+t-4}\,dt$

22. $\int \left(x^2+3x-1\right)^4(2x+3)\,dx$

23. $\int 5ye^{-y^2}\,dy$

24. $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

25. $\int 6x\sqrt{5+2x^2}\,dx$

26. $\int \frac{e^t}{e^t-1}\,dt$

27. $\int \frac{4}{x^2}e^{\frac{1}{x}}\,dx$

28. $\int 4y\sqrt[3]{3y^2+7}\,dy$

29. $\int e^{2x}\left(1-3e^{2x}\right)^2dx$

30. $\int \frac{4x+10}{x^2+5x+2}\,dx$

31. $\int \frac{\ln x}{x}\,dx$

32. $\int \frac{\ln 4x}{x}\,dx$

33. $\int \frac{1}{x \ln x} dx$

34. $\int \frac{(\ln x)^2}{x} dx$

35. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

36. $\int \frac{7x}{e^{x^2}} dx$

In Exercises 37–40, divide first and then integrate.

37. **a.** $\int \frac{x-1}{x-2} dx$ $\left(\textbf{Hint: } \frac{x-1}{x-2} = 1 + \frac{1}{x-2}\right)$

b. $\int \frac{x+3}{x+1} dx$ $\left(\textbf{Hint: } \frac{x+3}{x+1} = 1 + \frac{2}{x+1}\right)$

38. **a.** Rework Exercise 37a and substitute $u = x - 2$.

b. Rework Exercise 37b and substitute $u = x + 1$.

39. $\int \frac{x+2}{x+4} dx$

40. $\int \frac{x+5}{x-3} dx$

In Exercises 41–44, find a function $f(x)$ given $f'(x)$ and one (x, y)-value.

41. $f'(x) = 10xe^{5x^2}$; $(0, 11)$

42. $f'(x) = \frac{3}{x-5}$; $(6, 5)$

43. $f'(x) = (2x+2)^5$; $(-1, 10)$

44. $f'(x) = \frac{12x^2}{(x^3+6)^2}$; $\left(0, -\frac{2}{3}\right)$

APPLICATIONS

45. Appreciation: The value V of a painting is increasing at a rate of $4500(25-1.8t)^{-\frac{3}{2}}$ dollars per year. The painting originally sold for \$1000.

a. Write a function for its value t years after the original sale.

b. What will the painting be worth 5 years after the original sale?

46. Skills development: It is estimated that after t weeks of practice, students in a typing class can increase their speed $24e^{-0.2t}$ words per minute for each additional week of practice.

a. If their initial speed was 0 words per minute ($S(0) = 0$), write a function to represent their speed after t weeks of practice.

b. How fast can they type after 10 weeks (to the nearest word)?

47. Price: After the NBA championship basketball game, the price of a souvenir cap changes at the rate of $-\frac{1}{2x}$ dollars per cap, where x is the number of caps sold (in thousands). Write a function for the price p if $p(10) = 12.85$.

48. Profit: The marginal profit from the production and sale of x units of a product is estimated to be $\frac{100}{1+0.5x} - 4.5$ dollars per unit. If $P(0) = -60$, find the profit function $P(x)$.

49. Daily production: Records show that t hours after starting work on a typical day, an employee can assemble bikes at a rate of $\frac{6t+15}{\sqrt{t^2+5t}}$ per hour. Find the daily production function $N(t)$ if $N(0)=0$.

50. Position of a particle: A particle is moving in a straight line with the velocity $v(t)=\sqrt{2t+7}$ feet per second, where t represents time in seconds. Find the position function $s(t)$ if $s(0)=0$.

51. Position of a projectile: The velocity of a projectile moving in a straight line t seconds after it is fired is given by $v(t)=36+\frac{60}{(t+1)^2}$ feet per second. Find the position function $s(t)$ if $s(1)=10$.

52. The marginal value for a tract of land is $V'=\frac{20}{2t+1}$, where t is time in years since 2000 and V is in thousands of dollars.
- **a.** Determine the function V in terms of t if $V(0)=20$.
- **b.** What was the value of the land in 2015?

53. A point mass moving on a horizontal axis has a deceleration given by $a(t)=-48(2t+1)^2$, where t is time in seconds and a is in feet per second per second.
- **a.** If $v(0)=6$ ft/s, determine a velocity function $v(t)$.
- **b.** If $s(0)=-\frac{3}{4}$ feet, determine a distance function $s(t)$.

54. The number of viewers of a new TV show grew at a rate $V'(t)=900e^{0.3t-4}$ all summer long, where V is the total number of viewers in thousands and t is the number of weeks since June 1st.
- **a.** Determine $V(t)$ if $V(0)=100$.
- **b.** At what rate is V changing when $t=5$?
- **c.** When will the show hit 1,000,000 viewers ($V=1000$)?

55. Suppose in a medieval country, from 1200 to 1300, the life expectancy of a female serf changed at the rate $f'(t)=\frac{0.3}{1+0.01t}$, where t is time in years after 1200 and $f(t)$ is the average age of death.
- **a.** What are the units of $f'(t)$?
- **b.** Determine $f(t)$ if $f(0)=30$.
- **c.** What was the life expectancy of a female serf in 1300?

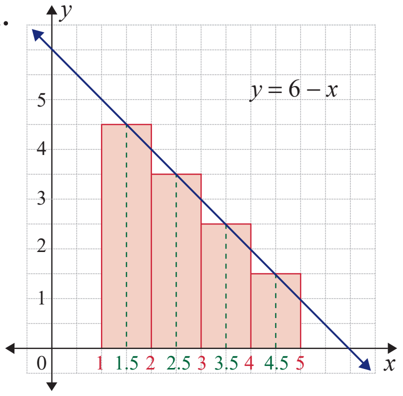
56. A new evening school is growing at the rate of $p'(t)=\frac{150}{\sqrt{1+0.2t}}$, where $p(t)$ is the total evening school student population and t is the time in years after 2000. The initial enrollment was 1500 students.
- **a.** How fast was enrollment changing in 2002?
- **b.** What was the expected enrollment in 2005?

14.3 EXERCISES

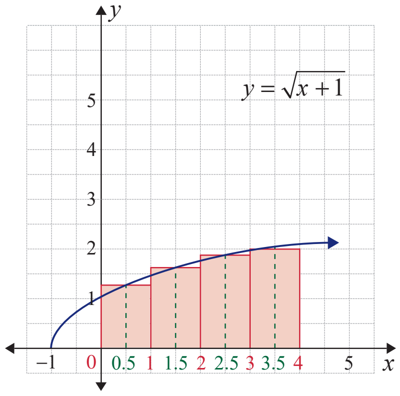
PRACTICE

In Exercises 1–2, find the area of the shaded region.

1.



2.



In Exercises 3–6, find the Riemann sum S_n for the given function, interval, and value of n.

3. $f(x)=x^2$; $[a, b]=[0, 4]$; $n=4$; $c_1=0.5, c_2=1.5, c_3=2.5, c_4=3.5$

4. $f(x)=9-x^2$; $[a, b]=[-3, 2]$; $n=5$; $c_1=-2.5, c_2=-1.5, c_3=-0.5, c_4=0.5, c_5=1.5$

5. $f(x)=\sqrt{2-x}$; $[a, b]=[-2, 2]$; $n=4$; Use the midpoint of each subinterval for the value of each c_k.

6. $f(x)=\dfrac{1}{x+2}$; $[a, b]=[-1, 3]$; $n=4$; Use the midpoint of each subinterval for the value of each c_k.

WRITING & THINKING

Use four rectangles to estimate the area between the graph of the given function and the x-axis on the given interval. Construct three estimates for the function: the first using the left endpoints of the subintervals as the sample points, the second using the right endpoints of the subintervals, and the third using the midpoints of the subintervals. Can you tell which are guaranteed to be underestimates or overestimates? (**Hint:** Consider the increasing/decreasing and concavity features of the graph. It is helpful to make a sketch.)

7. $f(x)=\sqrt{x}$ on $[0,4]$

8. $f(x)=\dfrac{x^3}{16}$ on $[0,4]$

9. $f(x)=\dfrac{1}{x}$ on $[1,5]$

10. $f(x)=\sqrt{4-x^2}$ on $[-2,2]$

11. $f(x)=e^{2-x}$ on $[0,2]$

14.4 EXERCISES

PRACTICE

In Exercises 1–32, evaluate each definite integral.

1. $\int_{-1}^{0} 5x^2\,dx$

2. $\int_{1}^{4} \frac{1}{x}\,dx$

3. $\int_{0}^{3} (1+e^x)\,dx$

4. $\int_{0}^{2} 6e^x\,dx$

5. $\int_{3}^{5} \frac{1}{x-2}\,dx$

6. $\int_{1}^{9} \sqrt{x}\,dx$

7. $\int_{-4}^{-2} \frac{1}{x^2}\,dx$

8. $\int_{2}^{4} (7x+2)\,dx$

9. $\int_{-1}^{3} (4x+1)\,dx$

10. $\int_{-2}^{1} \frac{4}{x+3}\,dx$

11. $\int_{-1}^{3} e^{x+1}\,dx$

12. $\int_{2}^{3} (x^2+2x-4)\,dx$

13. $\int_{2}^{4} x(x^2-3)\,dx$

14. $\int_{1}^{8} \left(1+\frac{1}{\sqrt[3]{x}}\right)dx$

15. $\int_{0}^{3} \frac{1}{3x+1}\,dx$

16. $\int_{1}^{3} 2e^{-1.5x}\,dx$

17. $\int_{3}^{5} \frac{1}{(3x+1)^2}\,dx$

18. $\int_{-1}^{0} \sqrt{3x+4}\,dx$

19. $\int_{2}^{3} (3-2x)^4\,dx$

20. $\int_{0}^{2} \frac{x}{\sqrt[3]{x^2+4}}\,dx$

21. $\int_{2}^{6} \frac{3x}{x^2-3}\,dx$

22. $\int_{3}^{5} \frac{x+2}{x^2+4x+3}\,dx$

23. $\int_{0}^{1} e^x(e^x+1)\,dx$

24. $\int_{0}^{5} xe^{-0.24x^2}\,dx$

25. $\int_{1}^{3} xe^{x^2-1}\,dx$

26. $\int_{1}^{2} (x-1)(2x^2-4x+1)^2\,dx$

27. $\int_{6}^{7} \frac{x-3}{\sqrt{x^2-6x+4}}\,dx$

28. $\int_{1}^{8} \left(2x^{\frac{2}{3}}-x^{-2}\right)dx$

29. $\int_{0}^{1} \frac{e^x}{e^x+1}\,dx$

30. $\int_{1}^{4} \frac{\ln x}{x}\,dx$

31. $\int_{1}^{3} \frac{1+\ln x}{x}\,dx$

32. $\int_{2}^{4} \frac{1}{x^2}e^{\frac{1}{x}}\,dx$

For Exercises 33–38, find the average value of the function on the given interval.

33. $f(x)=x^2+6;\ [1, 4]$

34. $f(x)=4x^2-3x+1;\ [-1, 3]$

35. $f(x)=\sqrt{x+1};\ [3, 8]$

36. $f(x)=\sqrt[3]{2x+1};\ [0, 13]$

37. $f(x)=2e^{-0.25x};\ [0, 4]$

38. $f(x)=1+e^{-0.4x};\ [0, 5]$

APPLICATIONS

39. Pollution: The level of pollution in San Felipe Bay, due to an oil spill, is estimated to be $f(t)=\dfrac{1800t}{\sqrt{t^2+11}}$ parts per million, where t is the time in days since the spill occurred. Find the average level of pollution during the first 5 days after the spill occurred.

40. Bacterial population: It is estimated that the number of bacteria present in a culture t hours after bacteria are introduced to the culture is given by $N(t)=\dfrac{8000}{\sqrt{8-0.5t}}$. Find the average number of bacteria present during the first 8 hours.

41. Average production: The daily production level for a product is given by $N(t)=240-240e^{-0.2t}$ units, where t is the time in hours after production begins. Find the average production during the first 4 hours.

42. Average marginal profit: The marginal profit from the production and the sale of x barbecue grills is given by $P'(x)=52-0.8x$ dollars per grill. Find the average marginal profit for the first 40 grills produced and sold.

b. Integrate from $x = 200$ to $x = 400$.

$$\begin{aligned}\int_{200}^{400}(20-0.05x)\,dx &= 20x-0.025x^2\Big]_{200}^{400}\\ &= \left[20(400)-0.025(400)^2\right]-\left[20(200)-0.025(200)^2\right]\\ &= (8000-4000)-(4000-1000)\\ &= 1000\end{aligned}$$

His profit changes by \$1000 when sales increase from 200 to 400 frames.

Note that from parts a. and b. we see that the profit on sales for the first 200 frames ($x = 0$ to $x = 200$) is greater than the profit for the second 200 frames ($x = 200$ to $x = 400$). This result is quite reasonable because the marginal profit, $P'(x) = 20 - 0.05x$, is decreasing by 5 cents per frame. In this problem, the fixed costs are relevant. Suppose the fixed costs are \$1000. Then $P(x) = 20x - 0.025x^2 - 1000$ dollars, and $P(200) = \$2000$, which is the **net profit**. The integral from $x = 0$ to $x = 200$ gives the increase in profits, $P(200) - P(0) = 2000 - (-1000) = 3000$.

14.5 EXERCISES

PRACTICE

For Exercises 1–18, find the total area bounded by the *x*-axis and the curve $y = f(x)$ on the indicated interval.

1. $f(x) = 3x + 1,\ [0, 5]$

2. $f(x) = 7 - 2x,\ [-1, 3]$

3. $f(x) = x^2 + 1,\ [-2, 2]$

4. $f(x) = 0.5x^2 + 2,\ [1, 4]$

5. $f(x) = x^3 + 2,\ [-1, 1]$

6. $f(x) = 2x^3 - 1,\ [1, 2]$

7. $f(x) = x^2 + x + 1,\ [-1, 3]$

8. $f(x) = x^2 + 2x - 3,\ [1, 3]$

9. $f(x) = \dfrac{4}{x+1},\ [0, 3]$

10. $f(x) = \dfrac{3}{2x+1},\ [0, 2]$

11. $f(x) = 3e^{0.6x},\ [0, 5]$

12. $f(x) = 1 + e^{-0.3x},\ [0, 4]$

13. $f(x) = x^2 - 2x - 8,\ [2, 5]$

14. $f(x) = x^2 + 3x - 4,\ [0, 4]$

15. $f(x) = \begin{cases} 2 - x & \text{if } -1 \le x \le 2 \\ x^2 - 4 & \text{if } 2 \le x \le 3 \end{cases}$

16. $f(x) = \begin{cases} x^2 & \text{if } -2 \le x \le 1 \\ 2x - 1 & \text{if } 1 \le x \le 2 \end{cases}$

17. $f(x) = \begin{cases} x + 2 & \text{if } -2 \le x \le 0 \\ \sqrt{x+4} & \text{if } 0 \le x \le 5 \end{cases}$

18. $f(x) = \begin{cases} 1 - 2x & \text{if } -2 \le x \le 0 \\ e^{2x} & \text{if } 0 \le x \le 1.5 \end{cases}$

APPLICATIONS

19. **Profit:** The marginal profit for a certain style of sports jacket is given by $P'(x) = 56 - 0.8x$ dollars per jacket, where x is the number of jackets produced and sold weekly. Find the profit for the first 50 jackets that are produced and sold. (Ignore any fixed costs.)

20. **Profit:** The marginal profit of an important product is given by $P'(x) = 10 - 0.015e^{0.6x}$ dollars per item, where x is the number of items produced and sold. Find the profit for the first 8 items. (Ignore any fixed costs.)

21. **Cost:** The marginal cost of a product is given by $15 + \frac{4}{\sqrt{x}}$ dollars per unit, where x is the number of units produced. The current level of production is 100 units weekly. If the level of production is increased to 169 units weekly, find the increase in the total costs.

22. **Revenue:** The marginal revenue from the sale of x bottles of a wine is given by $8.4 - 0.3\sqrt{x}$ dollars per bottle. Find the increase in total revenue if the number of bottles sold is increased from 225 to 350.

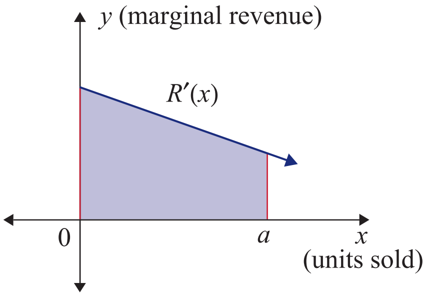
23. **Wildlife management:** The manager of a wildlife preserve has started a management program to control the population of the preserve's bison herd. It is estimated that the population will continue to grow according to the function $N'(t) = 15 - 6t^{\frac{1}{2}}$ bison per year, where t is the number of years after implementation of the plan and $0 \le t \le 5$. Find the increase in the population during the first 4 years of the program.

24. **Bacterial population:** It is estimated that t hours after some particular bacteria are introduced into a culture, the population will be increasing at a rate of $P'(t) = \frac{1200}{(12 - 0.5t)^{\frac{1}{2}}}$ bacteria per hour. Find the increase in the population during the first 6 hours.

WRITING & THINKING

In Exercises 25 and 26, explain the meaning of the shaded region in each graph.

25.



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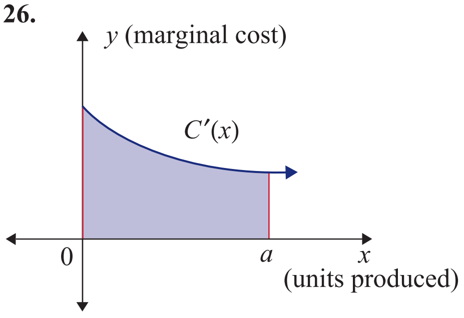


1. The domain is [0, 1].

2. The range is [0, 1].

3. $f(0) = 0$ and $f(1) = 1$.

4. $f(x) \le x$ for all x in the interval [0, 1].

Example 4: Using a Lorenz Curve

Suppose that $f(x) = 0.6x^2 + 0.4x$ represents a Lorenz curve for some country.

a. What percent of the country's total income is earned by the lower 50 percent of the families in this country?

b. Find the coefficient of inequality.

Solution

a. $f(0.5) = 0.6(0.5)^2 + 0.4(0.5) = 0.35$

Since we want the lower 50% of families, set $x = 0.5$.

The lower 50 percent of the families earn 35 percent of the country's total income.

b.

$$\begin{aligned} &2\int_0^1 \left[x - \left(0.6x^2 + 0.4x\right)\right]dx \\ &= 2\int_0^1 \left[\left(0.6x - 0.6x^2\right)\right]dx \\ &= 2\left(\frac{0.6x^2}{2} - \frac{0.6x^3}{3}\right)\Bigg]_0^1 \\ &= 2\left[\left(\frac{0.6(1)^2}{2} - \frac{0.6(1)^3}{3}\right) - \left(\frac{0.6(0)^2}{2} - \frac{0.6(0)^3}{3}\right)\right] \\ &= 2\left[(0.3 - 0.2) - 0\right] \\ &= 0.2 \end{aligned}$$

Combine terms in the integrand before finding the antiderivative.

The coefficient of inequality is 0.2.

14.6 EXERCISES

PRACTICE

In Exercises 1–16, find the area of the region bounded by the graphs of the given equations.

1. $y = x^2,\ y = x - 1,\ x = -1,\ x = 4$

2. $y = x^2 + 2,\ y = x,\ x = 2,\ x = 5$

3. $y = x^3,\ y = x^2,\ x = 0,\ x = 1$

4. $y = x^2 + 1,\ y = 1 - 2x,\ x = 0,\ x = 3$

5. $y=\sqrt{2x+1},\ y=3x+2,\ x=0,\ x=2$

6. $y=e^{x-1},\ y=x,\ x=1,\ x=4$

7. $y=e^{-x},\ y=x+1,\ x=0,\ x=3$

8. $y=x^2+1,\ y=e^{-0.2x},\ x=0,\ x=4$

9. $y=\frac{1}{x},\ y=\frac{5}{2}-x,\ x=\frac{1}{2},\ x=2$

10. $y=\frac{1}{x+1},\ y=e^{0.7x},\ x=0,\ x=2$

11. $y=x+1,\ y=x^2+x$

12. $y=x^2+1,\ y=6-x$

13. $y=\sqrt{x},\ y=x^2$

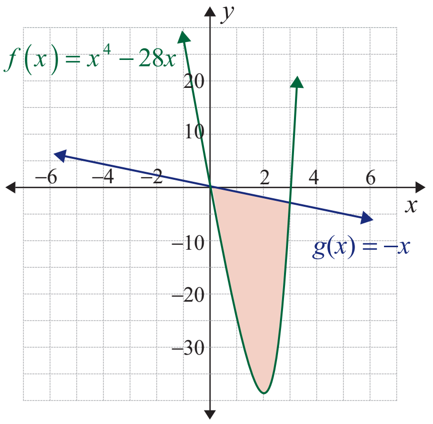
14. $y=x^2-6x,\ y=-x^2$

15. $y=x^2-2x-3,\ y=2x+2$

16. $y=x^2+5x-1,\ y=2-x^2$

For Exercises 17–21, determine the area pictured (check each answer using a graphing utility if possibe). In each case you must determine the limits of integration if necessary.

17.



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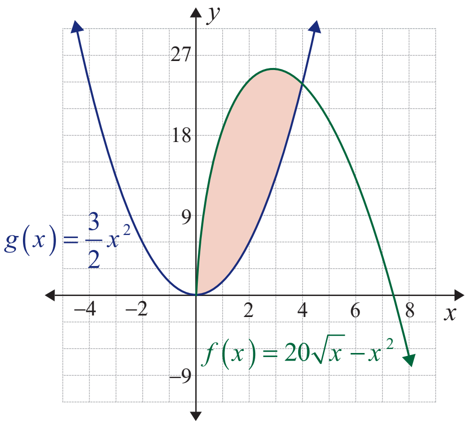

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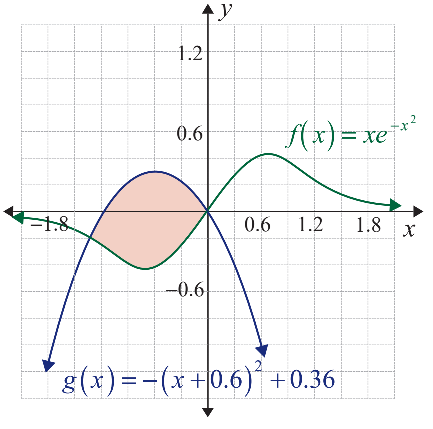

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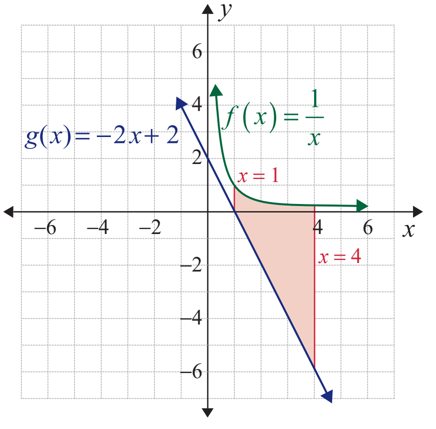

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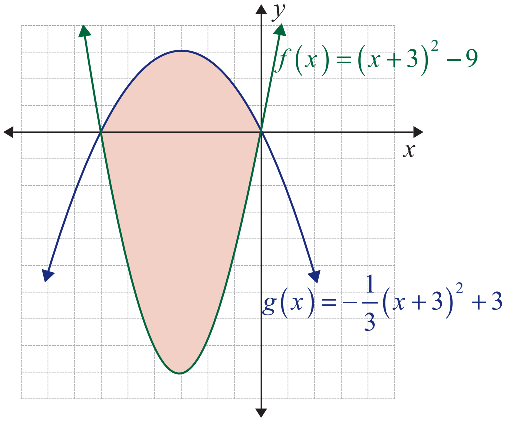


For each of the demand and supply functions in Exercises 22–25, find **a.** the equilibrium point, **b.** the consumers' surplus, and **c.** the producers' surplus.

22. $D(x)=18-0.4x,\ S(x)=3+0.1x,\ 0\le x\le 40$

23. $D(x)=24-0.2x,\ S(x)=10+0.5x,\ 0\le x\le 100$

24. $D(x)=1000-30x,\ S(x)=200+0.5x^2,\ 0\le x\le 30$

25. $D(x)=66-5\sqrt{x},\ S(x)=16+x,\ 0\le x\le 120$

APPLICATIONS

26. Consumers' surplus: The demand function for a particular product is given by the function $D(x)=24-0.6x-0.03x^2$. If x_E = 10 units, find the consumers' surplus.

27. Consumers' surplus: Find the consumers' surplus for a product if the demand function is given by $D(x)=\dfrac{800}{x+4}$ and $x_E=4$ units.

28. Producers' surplus: Find the producers' surplus for a product if the supply function is given by $S(x)=9e^{0.4x}$ and $x_E=5$ units.

29. Producers' surplus: The supply function for a product is given by the function $S(x)=\sqrt{16+1.5x}$. If $x_E=6$ units, find the producers' surplus.

30. Consumers' and producers' surplus: The demand curve for a product is given by $D(x) = 18 - 3x$ and the corresponding supply curve is $S(x) = 3x + 6$. Find the consumers' surplus and the producers' surplus.

31. Consumers' and producers' surplus: The demand curve for a product is given by $D(x) = 125 - 15x$ and the corresponding supply curve is $S(x) = 50 + 10x$. Find the equilibrium point, the CS, and the PS.

32. Lorenz curve: The income distribution of a small country is estimated by the Lorenz curve $f(x)=\frac{13}{18}x^2+\frac{5}{18}x$.

a. What percentage of the country's total income is earned by the lower 80 percent of its families? Round to the nearest percentage point.

b. Find the coefficient of inequality.

33. Lorenz curve: The Lorenz curve for estimating the income distribution of a country is given by $f(x)=\frac{7}{16}x^2+\frac{9}{16}x$.

a. What percentage of the country's total income is earned by the lower 70 percent of its families? Round to the nearest percentage point.

b. Find the coefficient of inequality.

34. Lorenz curve: A study shows that the income distribution of farmers in a certain state is estimated by $f(x)=0.47x^3+0.24x^2+0.29x$.

a. What percentage of the state's farming income is earned by the lower 60 percent of the state's farmers? Round to the nearest percentage point.

b. Find the coefficient of inequality.

35. Lorenz curve: In a certain state the income distribution for the lumber and the logging industry is estimated by $f(x)=1.16x^3-0.82x^2+0.66x$.

a. What percentage of the state's lumber and logging income is earned by the lower 50 percent of the companies? Round to the nearest percentage point.

b. Find the coefficient of inequality.

36. Lorenz curve: The income distribution for a certain country in 1996 was estimated by the function $f(x)=0.34x+0.66x^2$. In 2000 the income distribution was estimated by the function $f(x)=0.3x+0.72x^2-0.02x^3$.

a. Find the coefficient of inequality for each of the years.

b. Which year had a more equitable income distribution?

37. Lorenz curve: The income distribution for country A is estimated by the function $f(x)=0.24x+0.72x^2+0.04x^3$. The income distribution for country B is estimated by the function $f(x)=0.28x+0.69x^2+0.03x^3$.

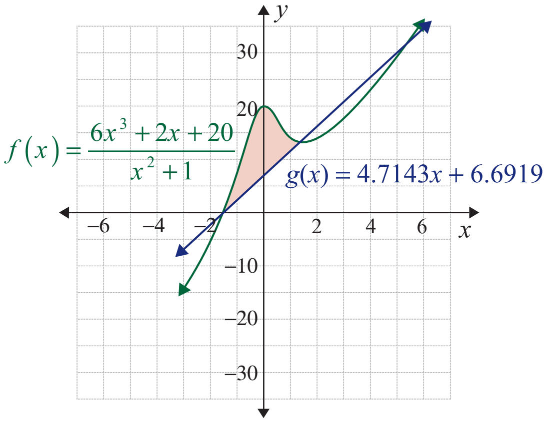
a. Find the coefficient of inequality for each of the two countries.

b. Which country has a more equitable income distribution?

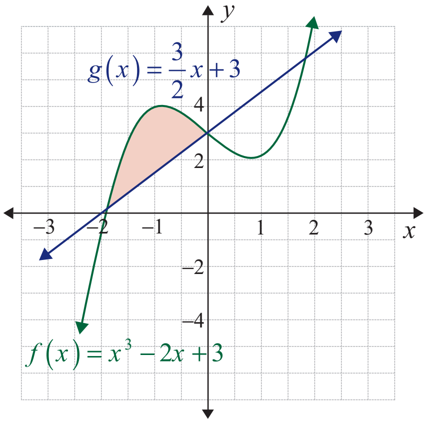
TECHNOLOGY

For Exercises 38–41, use a graphing utility to determine the area.

38.



39.



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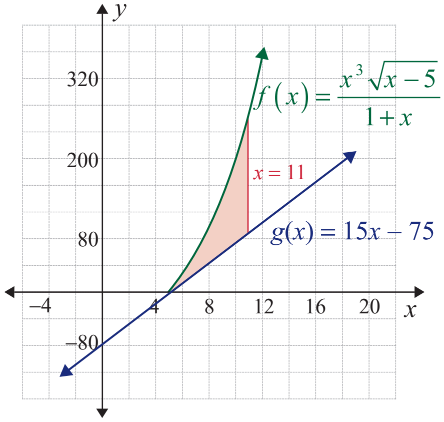
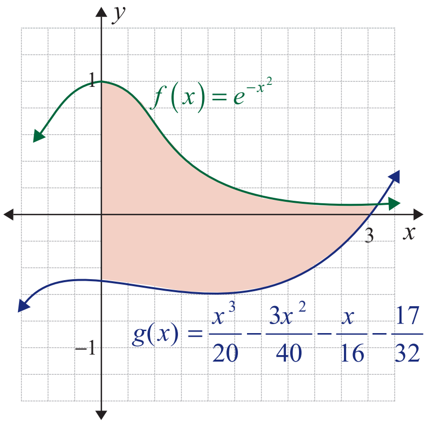


41.



Thus the final form for $P(t)$ is $P(t)=\dfrac{10{,}000}{1+99e^{-0.14t}}$.

Table 1 contains the general solutions of differential equations that arise in three basic growth applications.

Application	Differential Equation	General Solution
Unbounded growth	$\dfrac{dy}{dt}=ky$	$y=Ce^{kt}$
Bounded growth	$\dfrac{dy}{dt}=k(M-y)$	$y=M+Ce^{-kt}$
Logistic curve	$\dfrac{dy}{dt}=ky(M-y)$	$y=\dfrac{M}{1+Ce^{-Mkt}}$

TABLE 1: General Solutions for Growth Applications

14.7 EXERCISES

PRACTICE

In Exercises 1–12, verify that the differential equation has the given function as a particular solution.

1. $\dfrac{dy}{dx}=6,\ \ y=6x-1$

2. $\dfrac{dy}{dx}=3x-2,\ \ y=\dfrac{3}{2}x^2-2x-4$

3. $\dfrac{dy}{dx}=3+y,\ \ y=e^x-3$

4. $x\dfrac{dy}{dx}-x+2=0,\ \ y=x-2\ln x+7$

5. $\dfrac{dy}{dx}=y^{\frac{3}{2}},\ \ y=\dfrac{4}{(5-x)^2}$

6. $\dfrac{dy}{dx}=-0.5y,\ \ y=3e^{-0.5x}$

7. $2\dfrac{dy}{dx}+3y=1,\ \ y=\dfrac{1}{3}-2e^{-1.5x}$

8. $x\dfrac{dy}{dx}=xy+y,\ \ y=4xe^x$

9. $2x^2y''-xy'-2y=4-15x,\ \ y=3x^2+5x-2$

10. $x^2y''+xy'-y+\ln x=0,\ \ y=x+\ln x$

11. $x^2y''-xy'+y=0,\ \ y=x\ln x$

12. $y''-2y'+y=0,\ \ y=e^x(x+2)$

In Exercises 13–20, find the solution of each separable differential equation.

13. $\dfrac{dy}{dx}=3x+\dfrac{1}{x}$

14. $\dfrac{dy}{dx}=-3xy$

15. $\dfrac{dy}{dx}=\dfrac{2y-1}{x+1}$

16. $x\dfrac{dy}{dx}=\dfrac{x^2+1}{y^2}$

17. $\dfrac{dy}{dx}=-0.4y$

18. $\dfrac{dy}{dx}=-2(26-y)$

19. $\dfrac{dy}{dx}=(1-5x)y^2$

20. $\dfrac{dy}{dx}=(x^2+1)e^{-y}$

In Exercises 21–32, solve each initial-value problem or obtain a general solution as indicated. (Refer to Table 1 in the text if necessary.)

21. $\dfrac{dy}{dx}=0.6y(20-y)$

22. $\dfrac{dy}{dx}=0.3y(50-y)$

23. $\dfrac{dy}{dx}=0.25y,\ \ y=5 \text{ when } x=0$

24. $\dfrac{dy}{dx}=-3x,\ \ y=10 \text{ when } x=2$

25. $x\dfrac{dy}{dx}=y+1,\ \ y=14 \text{ when } x=3$

26. $\dfrac{dy}{dx}=-2xy,\ \ y=18 \text{ when } x=0$

27. $\dfrac{dy}{dx}=-6x^2y^2,\ \ y=2 \text{ when } x=1$

28. $\dfrac{dy}{dx}=\dfrac{xy}{x^2+1},\ \ y=7 \text{ when } x=0$

29. $\dfrac{dy}{dx}=8x+2xy,\ \ y=10 \text{ when } x=0$

30. $\dfrac{dy}{dx}=0.3(80-y),\ \ y=60 \text{ when } x=0$

31. $\dfrac{dy}{dx}=0.8y(40-y),\ \ y=30 \text{ when } x=0$

32. $\dfrac{dy}{dx}=0.04y(60-y),\ \ y=10 \text{ when } x=0$

APPLICATIONS

33. Elasticity of demand: The elasticity of demand for a product is given by $E=1.5$. Find the demand function $p=D(x)$ if $D(8)=24$.

34. Elasticity of demand: The elasticity of demand for a product is given by $E=2$. Find the demand function $p=D(x)$ if $D(25)=30$.

35. Elasticity of demand: The elasticity of demand for a product is given by $E=\dfrac{2(120-x)}{x}$. Find the demand function $p=D(x)$ if $D(20)=180$.

36. Elasticity of demand: The elasticity of demand for a product is given by $E=\dfrac{60-0.4x}{0.2x}$. Find the demand function $p=D(x)$ if $D(70)=28$.

37. **Resale value:** The resale or salvage value V of a machine decreases at a rate proportional to its value. Thus $\frac{dV}{dt} = -kV$, where t is the machine's age in years and k is its rate of decrease in value.
    a. Find the expression for the value when the machine is t years old if the original value was \$24,000 and the rate of decrease is 6 percent.
    b. Find the value of the machine when it is 7 years old.

38. **Drug concentration:** The amount A of a drug remaining in a body t hours after an injection decreases at a rate proportional to the amount present. This suggests the differential equation $\frac{dA}{dt} = -kA$. The amount of a certain drug decreases at a rate of 3 percent per hour. Find the amount of the drug remaining in the body 4 hours after an injection of 20 cc of the drug.

39. **Newton's Law of Cooling:** Newton's Law of Cooling states that the rate at which the temperature T of an object changes is proportional to the difference between the temperature of the object and the temperature of the surrounding medium. That is $\frac{dT}{dt} = -k(T - M)$, where k is the constant of proportionality, t is time, and M is the constant temperature of the medium.
    a. Solve the differential equation for $T(t)$.
    b. Find $T(5)$, if $T(0) = 78°$, $M = 26°$, and $k = 0.3$.

40. **Newton's Law of Cooling:** The temperature of a roast was 160° when it was removed from an oven and placed in a room with constant temperature of 76°. After 10 minutes, the temperature of the roast was 152°. Find the temperature 20 minutes after the roast was removed from the oven. (See Exercise 39.)

41. **Spread of a rumor:** In a small community with a population of 2800, a rumor about the mayor was started. The rate at which the rumor spread was approximated by $\frac{dN}{dt} = 0.0003N(2800 - N)$ people per day, where N is the number of people who have heard the rumor t days after the rumor was started.
    a. Write an equation for $N(t)$, assuming that 20 people have heard the rumor at $t = 0$.
    b. How many days will it take for 1500 people to hear the rumor? Round to the nearest day.

42. **Spread of a disease:** The population of seals on an island is about 600. Biologists estimate that there are 12 seals with a very infectious disease. The disease will spread at a rate $\frac{dN}{dt} = 0.006N(600 - N)$ seals per day, where N is the number of infected seals t days after the discovery of the disease.
    a. Write a function $N(t)$ for the number of seals infected t days after the discovery of the disease.
    b. At what time t will 300 seals will be infected? Round to the nearest day.

$$\int u\,dv = uv - \int v\,du$$

$$\int_1^5 x\sqrt{x-1}\,dx = x \cdot \frac{2}{3}(x-1)^{\frac{3}{2}}\Big]_1^5 - \int_1^5 \frac{2}{3}(x-1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}x(x-1)^{\frac{3}{2}}\Big]_1^5 - \frac{2}{3}\cdot\frac{2}{5}(x-1)^{\frac{5}{2}}\Big]_1^5$$

$$= \frac{2}{3}x(x-1)^{\frac{3}{2}} - \frac{4}{15}(x-1)^{\frac{5}{2}}\Big]_1^5$$

$$= \left[\frac{10}{3}(4)^{\frac{3}{2}} - \frac{4}{15}(4)^{\frac{5}{2}}\right] - (0)$$

$$= \frac{80}{3} - \frac{128}{15}$$

$$= \frac{400}{15} - \frac{128}{15}$$

$$= \frac{272}{15}$$

15.1 EXERCISES

PRACTICE

In Exercises 1–16, use the technique of integration by parts to evaluate the integrals.

1. $\int xe^{2x}dx$

2. $\int 3xe^{-x}dx$

3. $\int 2ye^{0.5y}dy$

4. $\int 5te^{0.4t}dt$

5. $\int \ln t\,dt$

6. $\int y^2 \ln y\,dy$

7. $\int x^3 \ln 5x\,dx$

8. $\int 8x \ln 3x\,dx$

9. $\int x\sqrt{x+2}\,dx$

10. $\int x\sqrt{x-3}\,dx$

11. $\int x(x+4)^{-2}dx$

12. $\int x(x-1)^{-3}dx$

13. $\int \frac{t}{2e^{0.6t}}dt$

14. $\int y^2e^{3y}dy$

15. $\int \sqrt{x}\ln 7x\,dx$

16. $\int 3x(x-6)^{-\frac{2}{3}}dx$

In Exercises 17–22, use the technique of integration by parts to evaluate each definite integral. Round your answer to the nearest hundredth.

17. $\int_0^2 xe^{-2x}dx$

18. $\int_0^3 (x+1)e^{-0.5x}dx$

19. $\int_0^1 (x+2)e^{-4x}dx$

20. $\int_0^4 (1-2x)e^{1.2x}dx$

21. $\int_{-2}^3 \frac{x}{\sqrt{6+x}}dx$

22. $\int_0^4 x\sqrt{1+2x}\,dx$

In each of Exercises 23–30, identify the *u* and *dv* which would solve the integral using integration by parts. Then evaluate the integral and round your answer to the nearest hundredth.

23. $\int_0^1 4x(3x+1)^5\,dx$

24. $\int_1^2 \frac{x}{\sqrt{2x+5}}\,dx$

25. $\int_{-1}^2 (x+1)(x+2)^{\frac{3}{2}}\,dx$

26. $\int_1^4 \sqrt{x}\ln x\,dx$

27. $\int_1^5 x^2\ln x\,dx$

28. $\int_1^3 \frac{\ln t}{t^2}\,dt$

29. $\int_0^6 \ln(x+1)\,dx$

30. $\int_1^2 (2x+1)\ln x\,dx$

In Exercises 31–40, use the technique of substitution or integration by parts to evaluate the integrals.

31. $\int 5te^{-2t}\,dt$

32. $\int 5te^{-2t^2}\,dt$

33. $\int \sqrt{3x}\ln x\,dx$

34. $\int \frac{\ln x}{x}\,dx$

35. $\int 3x(2x^2-1)^{\frac{3}{2}}\,dx$

36. $\int 3x(2x-1)^{\frac{3}{2}}\,dx$

37. $\int \frac{(\ln x)^2}{x}\,dx$

38. $\int x\ln x^2\,dx$

39. $\int \frac{e^x}{1-e^x}\,dx$

40. $\int \frac{x}{\sqrt{5x^2-3}}\,dx$

APPLICATIONS

41. Demand for a natural resource: The demand for a natural resource t years from now will be increasing at a rate of $te^{0.01t}$ million units per year. If the current demand is 80 million units, write a function for the demand t years from now.

42. Revenue: The marginal revenue for x units of a product is given by $R'(x)=(200-30x)e^{-0.15x}$ dollars per unit. Find the revenue function $R(x)$ if $R(0)=0$.

43. Revenue: The marginal revenue for x units of a product is given by $R'(x)=18-0.4\ln x$ dollars per unit, where $x \geq 1$. Find the revenue function if $R(1)=\$18.40$.

44. Resale value: The value of a machine depreciates at a rate of $-200t(t+1)^{-2}$ dollars per year, where t is the age (in years) of the machine. If the original cost of the machine is \$540, find a function for the value of the machine when it is t years old.

$$
\begin{aligned}
PV &= (50{,}000 + 2000t)\left(-10e^{-0.10t}\right)\Big]_0^5 - \int_0^5 \left(-10e^{-0.10t}\right)(2000\,dt) \quad \text{Using the parts formula, } uv - \int v\,du \\
&= (50{,}000 + 2000t)\left(-10e^{-0.10t}\right)\Big]_0^5 + 20{,}000\int_0^5 e^{-0.10t}\,dt \\
&= (50{,}000 + 2000t)\left(-10e^{-0.10t}\right)\Big]_0^5 + \left(-200{,}000e^{-0.10t}\right)\Big]_0^5 \\
&= 60{,}000\left(-10e^{-0.5}\right) + 500{,}000 - 200{,}000\left(e^{-0.5}\right) + 200{,}000 \\
&= 700{,}000 - 800{,}000e^{-0.5} \\
&\approx 700{,}000 - 800{,}000(0.6065307) \\
&\approx 700{,}000 - 485{,}225 = 214{,}775
\end{aligned}
$$

The present value of the new store's income stream is approximately $214,775.

15.2 EXERCISES

APPLICATIONS

1. **Annuity:** Estimate the amount of an annuity if $1000 is deposited annually for 10 years at a rate of 8 percent compounded continuously.

2. **Annuity:** An amount of $6000 is invested in an account each year for 8 years. Find the approximate balance at the end of the 8 years if the account pays interest at a rate of 7 percent compounded continuously.

3. **Annuity:** Christine has decided to invest $2000 each year into an IRA account that pays interest at the rate of 9 percent compounded continuously. Find the amount in the account at the end of 15 years.

4. **Annuity:** Bob and Ann plan to deposit $4000 per year into their retirement account. If the account pays interest at a rate of 8.4 percent compounded continuously, approximately how much will be in their account after 12 years?

5. **Annuity:** Bryan plans to deposit $1200 each year into an annuity. If the account pays interest at a rate of 7.5 percent compounded continuously, find the approximate balance of his account after 10 years.

6. **Income stream:** Find the value of an income stream after 7 years if the rate of flow is estimated to be $200,000 annually and the income is invested at a rate of 8 percent compounded continuously.

7. **Income stream:** The owner of a local convenience store estimates that the store will generate an annual income of $340,000 for the next 4 years. If the rate of interest is 9 percent compounded continuously, find the value of the income stream.

8. **Income stream:** A real estate investment is expected to generate an income flow of $12,000 annually for the next 6 years. Find the amount of the income stream if the interest rate is 7.8 percent compounded continuously.

9. **Income stream:** Find the value of an income stream after 5 years if $R(t) = 3600e^{0.02t}$ is the rate of flow of revenue and the income is deposited at a rate of 7 percent compounded continuously.

10. **Income stream:** A certain investment has a continuous flow of money at a rate of $R(t) = 7200e^{0.01t}$. Find the value of this flow after 4 years if the interest rate is 8.2 percent compounded continuously.

11. **Income stream:** Find the value of an income stream if $R(t) = 50 + 0.2t$ is the rate of flow of revenue reinvested at 6 percent compounded continuously for 8 years.

12. **Income stream:** Find the value of an income stream if $R(t) = 80 + 1.2t$ is the rate of flow of revenue reinvested at 6.4 percent compounded continuously over the next 6 years.

13. **Income stream:** The profit from a number of soft drink machines is estimated to be at the rate of $R(t) = 15 + 0.8t$ thousand dollars per year. If the profits are deposited into an account paying 6.5 percent compounded continuously, find the amount of the income stream after 7 years.

14. **Income stream:** It is estimated that a computer will save accounting fees at a small company at a rate of $R(t) = 4 + 0.6t$ thousand dollars per year. If the savings are reinvested at 5 percent compounded continuously, find the amount of the income stream after 4 years.

15. **Income stream:** Find the present value of an income stream with $R(t) = 60 - 0.4t$, $r = 8$ percent, and $T = 20$.

16. **Income stream:** Find the present value of an income stream with $R(t) = 150 - t$, $r = 12$ percent, and $T = 10$.

17. **Income stream:** The rate of flow of an income stream is estimated by $R(t) = 6000e^{0.015t}$ for the next 4 years. Find the present value of this flow if the interest rate is 6 percent compounded continuously.

18. **Income stream:** The rate of flow of an income stream for the next 6 years is estimated by $R(t) = 10{,}000e^{-0.01t}$. Find the present value of this flow if the interest rate is 8.5 percent compounded continuously.

19. **Income stream:** Sandy estimates that the profits from his ice cream store will be $R(t) = 24 + 3.6t$ thousand dollars per year for the next 5 years. Find the present value of the store if the current interest rate is 10 percent compounded continuously.

20. **Income stream:** Elco Grain Company expects their profits to be $R(t) = 30 + 12e^{0.02t}$ thousand dollars per year for the next 4 years. If the current interest rate is 8 percent compounded continuously, find the present value of the company.

WRITING & THINKING

21. In Figure 1, replace the column information with $100e^{0.01(5)}$, $100e^{0.01(4)}$, $100e^{0.01(3)}$, $100e^{0.01(2)}$, $100e^{0.01(1)}$, $100e^{0.01(0)}$. This suggests that the future value of an annuity could be given by $\int_0^N Pe^{rt}\,dt$. Does this give the same result as the formula for the future value of an annuity shown in the lesson? Explain why or why not.

$$\int \frac{1}{(x+2)(x+3)}dx = \frac{1}{3-2}\ln\left|\frac{x+2}{x+3}\right| + C_2$$
$$= \ln\left|\frac{x+2}{x+3}\right| + C_2$$
$$= \ln|x+2| - \ln|x+3| + C_2$$

Now combining the parts gives the result.

$$\int \frac{x+1}{x^2+5x+6}dx = 3\ln|x+3| - 2\ln|x+2| + C_1 + \ln|x+2| - \ln|x+3| + C_2$$
$$= 2\ln|x+3| - \ln|x+2| + C \qquad \text{Where } C = C_1 + C_2$$

15.3 EXERCISES

PRACTICE

Use Table 1 to find the following integrals.

1. $\int \frac{1}{4x+3}dx$

2. $\int \sqrt{9x+2}\,dx$

3. $\int e^{-0.15x}dx$

4. $\int \ln x\,dx$

5. $\int \frac{1}{(2x-5)^2}dx$

6. $\int \frac{x}{x+6}dx$

7. $\int x\sqrt{3x-4}dx$

8. $\int \frac{x}{(2x+1)(x-2)}dx$

9. $\int \sqrt{x^2+36}\,dx$

10. $\int x^4 \ln x\,dx$

11. $\int \frac{1}{x^2-16}dx$

12. $\int \frac{1}{x(4x-3)}dx$

13. $\int \frac{1}{(x+8)(5x-1)}dx$

14. $\int \frac{1}{2+e^{3x}}dx$

15. $\int 7x^5 \ln x\,dx$

16. $\int x^4 e^{-2x}\,dx$

17. $\int \frac{1}{8-5e^{-0.7x}}dx$

18. $\int \frac{1}{(0.3x+2)^2}dx$

19. $\int \frac{2}{x(3x-1)}dx$

20. $\int \frac{4}{x^2-8}dx$

21. $\int 14\sqrt{6x-5}\,dx$

22. $\int \frac{13}{(4x-1)(2x+3)}dx$

23. $\int \frac{8}{x(0.4x+1)}dx$

24. $\int 2x\sqrt{3x-4}\,dx$

25. $\int x^3 e^{1.5x}\,dx$

26. $\int \sqrt{x^2+9}\,dx$

27. $\int \frac{x}{(x-7)(5x+2)}dx$

28. $\int \sqrt{x^2-15}\,dx$

29. $\int \frac{1}{\sqrt{x^2-12}}dx$

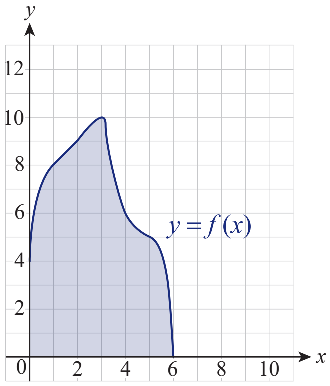
30. $\int \frac{6}{24-9e^{3.1x}}dx$

15.4 EXERCISES

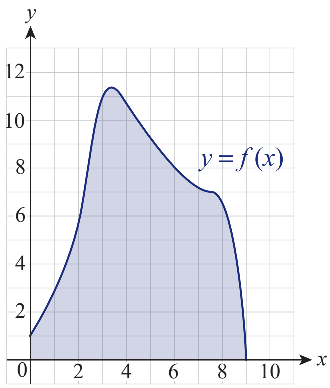
PRACTICE

The function $f(x)$ is given by its graph. Use the Trapezoidal Rule and Simpson's Rule, respectively, to approximate the shaded area $\int_a^b f(x)\,dx$ by **a.** T_6 and **b.** S_6.

1.



2.



Use the Trapezoidal Rule and Simpson's Rule with $n = 8$ to approximate the integral. Then find the exact value and compare your answers.

3. $\int_0^8 x^4\,dx$

4. $\int_1^5 \frac{1}{x}\,dx$

5. $\int_1^5 \frac{1}{x^2}\,dx$

6. $\int_0^4 \sqrt{x}\,dx$

7. $\int_0^4 x^3\,dx$

8. $\int_{-2}^6 \sqrt[3]{x+2}\,dx$

9. $\int_0^2 e^x\,dx$

10. $\int_1^5 \ln x\,dx$

11. $\int_{-2}^6 \left(4 - \frac{1}{2}x\right)dx$

12. $\int_{-4}^4 \left(16 - x^2\right)dx$

13. $\int_0^4 x\sqrt{x^2+2}\,dx$

14. $\int_0^{16} \frac{1}{\sqrt{x+1}}\,dx$

15. $\int_0^8 \frac{x}{\sqrt{x^2+1}}\,dx$

Use **a.** the Trapezoidal Rule and **b.** Simpson's Rule to approximate the definite integral for the indicated value of *n*.

16. $\int_0^4 \sqrt[4]{x}\,dx; \quad n = 4$

17. $\int_0^5 \sqrt{x^4+4}\,dx; \quad n = 10$

Some texts discuss the "Midpoint Rule" as a numerical integration method. The idea is simply forming a Riemann sum by choosing the midpoint of each subinterval as the sample point.

Use the "Midpoint Rule" with $n = 8$ to approximate the integral and compare your answers to those in Exercises 3–5.

18. $\int_0^8 x^4\,dx$

19. $\int_1^5 \frac{1}{x}\,dx$

20. $\int_1^5 \frac{1}{x^2}\,dx$

21. Use Simpson's Rule with $n = 6$ to approximate $\ln 2 = \int_1^2 \frac{1}{x}\,dx$.

22. Use Simpson's Rule with $n = 6$ to approximate $\pi = \int_0^1 \frac{4}{x^2+1}\,dx$.

23. Use Simpson's Rule with $n = 24$ to approximate the area of the region bounded by the graphs of $y = \sqrt{1+x^4}$, $x = -6$, $x = 6$, and the x-axis. (Notice that this problem leads to a nonelementary integral.)

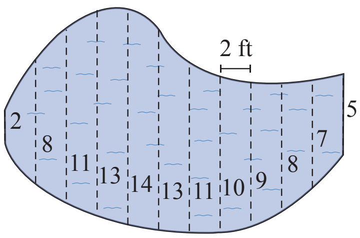
APPLICATIONS

24. The following table summarizes acceleration data for the 2012 Ford Mustang Boss 302 Laguna Seca. Use Simpson's Rule to estimate the total distance traveled by "the Boss" during its timed 0–120 mph run. (**Hint:** Sketching a graph similar to the one in Example 2 is useful. Be sure to identify which area you can approximate and how it yields the answer to the problem.)

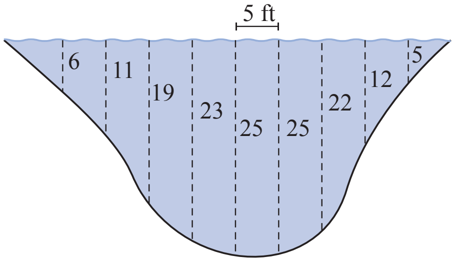
Time to Speed

Miles per Hour	Seconds
0–120	13.0
0–110	10.9
0–100	9.1
0–90	7.6
0–80	6.3
0–70	5.2
0–60	4.1
0–50	3.3
0–40	2.4
0–30	1.7
0–20	1.1
0–10	0.4

Source: *Road & Track*



25. Use the Trapezoidal Rule to estimate the amount of water needed to raise the water level by two inches in a pool with the shape shown in the figure. At 2-foot intervals, the distances across the pool (in feet) are as indicated in the diagram.



26. The figure shows the vertical cross-section of the Lazee river where the Dinkatown ferry docks. The depth of the river is indicated at 5-foot intervals in the diagram. If the river flows at 5 ft/s, use Simpson's Rule to estimate the amount of water passing by the dock every second.

WRITING & THINKING

27. Prove that Simpson's Rule actually gives the exact answer for definite integrals of all polynomials of degree 3 or less.

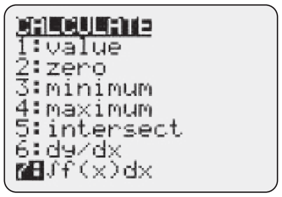


FIGURE 4

Step 2: Press 2nd trace and select item 7:∫f(x)dx (this is the integration symbol). (See Figure 4.)

At the prompt, type the lower limit 1 and enter. At the next prompt, type the upper limit 3. The area is shaded and the decimal answer .Ø4398Ø93 appears at the bottom of the screen (see Figure 5). This is an approximation to the actual value, but when using this method, the upper and lower limits must be in the range of x-values plotted on the screen.

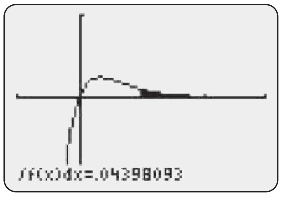


FIGURE 5

Method 2

Step 1: Press mode and select CLASSIC.

Step 2: From the home screen, press math and select item 9:fnInt((function integral). (See Figure 6.)

Type the function, the variable of integration x, the lower limit, the upper limit, a right parenthesis, and enter. The four items within the parentheses must be separated by commas. (**Note:** You may use any number for the upper limit; for this function 100 works well.)

The calculator will return .Ø442551719, a more accurate answer than the result from Method 1 (see Figure 7).

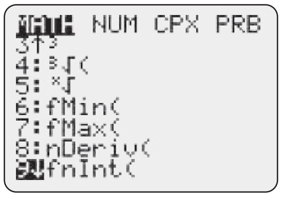


FIGURE 6

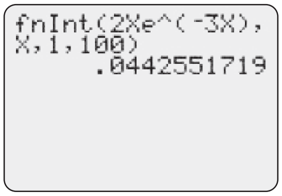


FIGURE 7

15.5 EXERCISES

PRACTICE

In Exercises 1–10, find the limit if it exists.

1. $\lim\limits_{b\to+\infty}\dfrac{1}{b}$

2. $\lim\limits_{b\to+\infty}\dfrac{1}{\sqrt[3]{b}}$

3. $\lim\limits_{b\to+\infty}\dfrac{\sqrt{b}}{20}$

4. $\lim\limits_{b\to+\infty}e^{0.1b}$

5. $\lim\limits_{b\to+\infty}e^{-4b}$

6. $\lim\limits_{b\to+\infty}(-12\ln b)$

7. $\lim\limits_{b\to+\infty}\left(2+\dfrac{9}{\sqrt{3b+1}}\right)$

8. $\lim\limits_{b\to+\infty}\left(5+e^{-2b}\right)$

9. $\lim\limits_{b\to+\infty}7b^4e^{-b}$

10. $\lim\limits_{b\to+\infty}(7b+2)^{-\frac{2}{3}}$

In Exercises 11–34, determine whether the improper integrals are convergent or divergent, and evaluate those which are convergent.

11. $\int_{2}^{+\infty} \frac{4}{x^3}\,dx$

12. $\int_{1}^{+\infty} \frac{1}{\sqrt[3]{x}}\,dx$

13. $\int_{8}^{+\infty} x^{-\frac{2}{3}}\,dx$

14. $\int_{4}^{+\infty} 5x^{-\frac{3}{2}}\,dx$

15. $\int_{20}^{+\infty} 3e^{-x}\,dx$

16. $\int_{4}^{+\infty} e^{-2x}\,dx$

17. $\int_{2}^{+\infty} e^{-\frac{x}{3}}\,dx$

18. $\int_{2}^{+\infty} 4e^{-0.5x}\,dx$

19. $\int_{2}^{+\infty} e^{1.5x}\,dx$

20. $\int_{-1}^{+\infty} \frac{1}{80}e^{0.16x}\,dx$

21. $\int_{0}^{+\infty} \frac{1}{(x+3)^2}\,dx$

22. $\int_{0}^{+\infty} \frac{4}{\sqrt{3x+1}}\,dx$

23. $\int_{-1}^{+\infty} \frac{2}{\sqrt[3]{2x+3}}\,dx$

24. $\int_{2}^{+\infty} (3x+2)^{-\frac{4}{3}}\,dx$

25. $\int_{0}^{+\infty} \frac{5}{x+1}\,dx$

26. $\int_{0}^{+\infty} (5x+4)^{-\frac{3}{2}}\,dx$

27. $\int_{0}^{+\infty} x^2e^{-x^3}\,dx$

28. $\int_{0}^{+\infty} -4xe^{x^2}\,dx$

29. $\int_{1}^{+\infty} xe^{1-x^2}\,dx$

30. $\int_{0}^{+\infty} 7xe^{-x^2}\,dx$

31. $\int_{2}^{+\infty} \frac{1}{x(\ln x)^3}\,dx$

32. $\int_{e}^{+\infty} \frac{1}{x\ln x}\,dx$

33. $\int_{0}^{+\infty} xe^{-x}\,dx$

34. $\int_{0}^{+\infty} xe^{-0.2x}\,dx$

In Exercises 35–38, find the area, if it exists, of the region under the curve $y = f(x)$ on the given interval of the x-axis.

35. $f(x) = \frac{4}{x^2},\ x \geq 2$

36. $f(x) = 3e^{-x},\ x \geq 0$

37. $f(x) = \frac{3}{x},\ x \geq 6$

38. $f(x) = 2e^{0.8x},\ x \geq 0$

WRITING & THINKING

39. The integral $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converges if and only if (choose all that apply):

a. $0 < p < 1$
b. $p \neq 1$
c. p is an integer greater than or equal to 2
d. $p > 1$
e. p is positive
f. none of the above

TECHNOLOGY

40. Integrate $\int_{1}^{\infty} 2xe^{-3x}\,dx$ by evaluating the limit and compare your answer to the calculator values obtained at the end of the section.

We have considered only solids of revolution in which a region is rotated about the x-axis. The formula we used would need some adjustment if the region were rotated about some other horizontal line, since the radius of revolution would be represented by some expression other than $f(x)$. Such solids of revolution will not be considered in this course.

15.6 EXERCISES

PRACTICE

Find the volume of the solid generated when the regions bounded by the graphs of the given equations and the x-axis are rotated about the x-axis.

1. $y = x,\ x = 0,\ x = 2$

2. $y = 3x,\ x = 1,\ x = 3$

3. $y = 2\sqrt{x},\ x = 0,\ x = 4$

4. $y = \sqrt[3]{x},\ x = 0,\ x = 8$

5. $y = e^x,\ x = -1,\ x = 2$

6. $y = e^{-x},\ x = -1,\ x = 2$

7. $y = 1 - x^2,\ x = -1,\ x = 1$

8. $y = 4 - x^2,\ x = -2,\ x = 2$

9. $y = \left(16 - x^2\right)^{\frac{1}{2}},\ x = 0,\ x = 4$

10. $y = \sqrt{3 - x^2},\ x = 0,\ x = 1$

11. $y = \frac{4}{x},\ x = 1,\ x = 3$

12. $y = \frac{2}{x},\ x = 1,\ x = 2$

13. $y = \frac{1}{\sqrt{x}},\ x = 1,\ x = 6$

14. $y = \frac{2}{\sqrt{x}},\ x = 1,\ x = 5$

15. $y = x + \sqrt{x},\ x = 1,\ x = 4$

16. $y = \sqrt{x} - x,\ x = 0,\ x = 1$

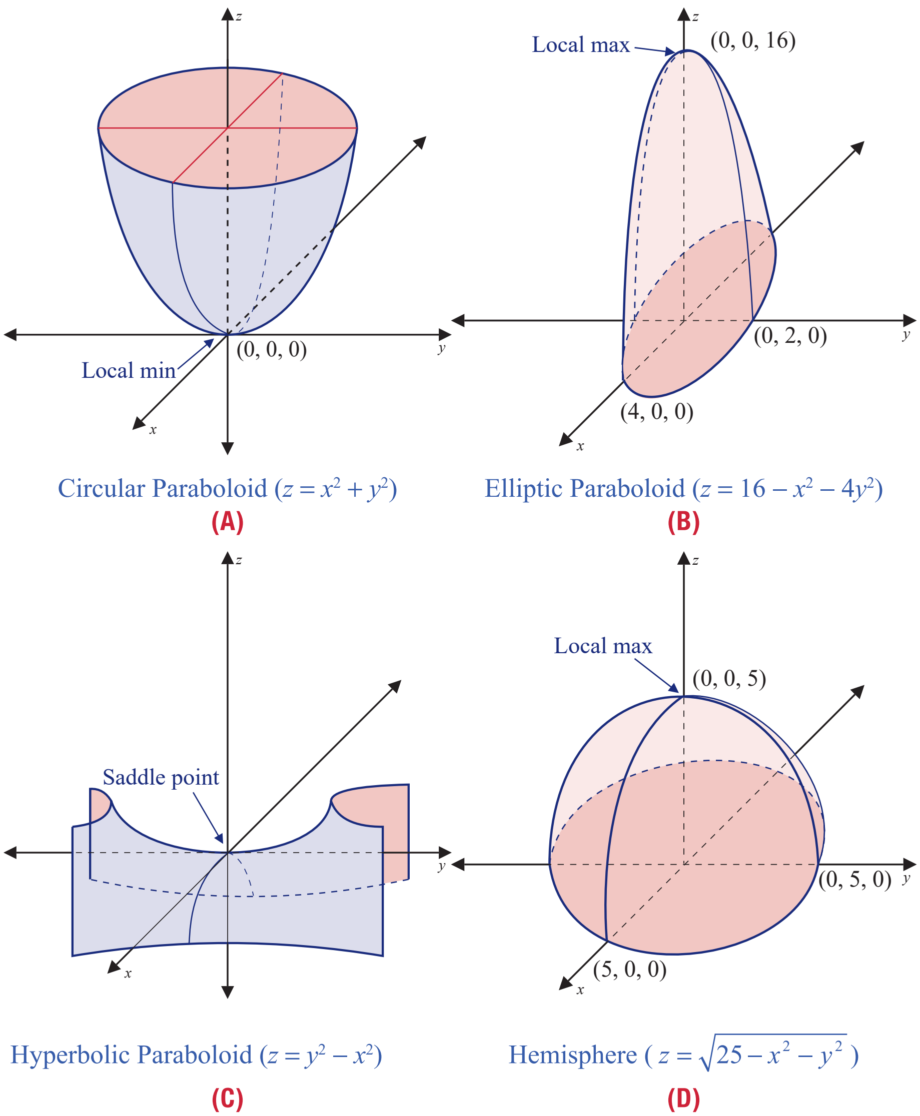


FIGURE 4

16.1 EXERCISES

PRACTICE

In Exercises 1–15, find the indicated function values, if possible.

1. $f(x, y) = 12x - 3y + xy$
- **a.** $f(1, 3)$
- **b.** $f(0, 4)$

2. $f(x, y) = 7xy - 11x + 9y$
- **a.** $f(-2, 1)$
- **b.** $f(3, -2)$

3. $f(x, y) = 4x^2 - 3xy + y^2$
- **a.** $f(2, 5)$
- **b.** $f(0, 3)$

4. $g(x, y) = 2xy + 5x^2y + y^3$
- **a.** $g(-2, 2)$
- **b.** $g(-1, -1)$

5. $g(x, y) = \dfrac{4x + y}{x - y}$
  a. $g(3, -1)$
  b. $g(2, 2)$

6. $f(x, y) = \dfrac{2x - y}{x^2 + y}$
  a. $f(6, 2)$
  b. $f(3, -9)$

7. $g(x, y) = \dfrac{8x^2 y}{\sqrt{2x + y}}$
  a. $g(1, -4)$
  b. $g(-2, 6)$

8. $f(x, y) = \dfrac{5x^2 + y^3}{\sqrt{4x + y^2}}$
  a. $f(-1, 3)$
  b. $f(3, 2)$

9. $g(x, y) = 3xe^{x+y}$
  a. $g(2, 1)$
  b. $g(-1, 0)$

10. $g(x, y) = ye^{4x} + 2xy$
  a. $g(1, 5)$
  b. $g(-1, 3)$

11. $f(x, y) = x\ln xy + y\ln x$
  a. $f(1, e)$
  b. $f(e^2, 1)$

12. $f(x, y) = 2x^3 y + \ln y$
  a. $f(-2, e)$
  b. $f(3, 1)$

13. $A(P, r, t) = Pe^{rt}$
  a. $A(1000, 0.08, 4)$
  b. $A(500, 0.06, 5)$

14. $A(P, r, t) = P\left(1 + \dfrac{r}{4}\right)^{4t}$
  a. $A(1500, 0.08, 6)$
  b. $A(800, 0.06, 10)$

15. $S(l, w, h) = 2lw + 2lh + 2wh$
  a. $S(18, 15, 9)$
  b. $S(14, 9, 11)$

In Exercises 16–19, draw a three-dimensional coordinate system, and graph and label the given points.

16. $A(0, 0, 3);\ B(2, -1, 0);\ C(4, 2, 1);\ D(3, -2, -2)$

17. $A(0, 1, 0);\ B(3, 0, -4);\ C(1, 1, -3);\ D(-2, 3, 0)$

18. $A(-3, 0, 2);\ B(0, 0, 2);\ C(2, -2, 2);\ D(0, -2, 2)$

19. $A(-3, -2, 1);\ B(4, 0, -1);\ C(0, 0, 4);\ D(-1, 4, -1)$

In Exercises 20–25, draw a three-dimensional coordinate system, and graph the given planes.

20. $x = 0$ **21.** $y = 0$ **22.** $z = 0$

23. $x = 3$ **24.** $y = 2$ **25.** $z = 4$

APPLICATIONS

26. **Stock yield:** The yield of a stock is given by the function $Y(d, p) = \dfrac{d}{p}$, where d is the dividend per share of stock and p is the price per share. Find the yield of a stock that sells for \$5.88 if the dividend is \$1.00.

27. **Intelligence quotient:** The intelligence quotient (IQ) of a person is determined by $f(M, C) = 100 \cdot \frac{M}{C}$, where M is the mental age (determined by tests) and C is the actual or chronological age. Find the IQ of a child who is 13 years old and has a mental age of 15.4 years. (Round to the nearest integer.)

28. **Cobb-Douglas production:** The number of units of a product that are manufactured by a company is given by $f(L, K) = 300L^{0.4}K^{0.6}$, where L is the units of labor and K is the units of capital.
    a. How many units of a product will be manufactured by utilizing 30 units of labor and 24 units of capital? (Round to the nearest unit.)
    b. How many units will be produced if the number of units of labor and capital are doubled? (Round to the nearest unit.)

29. **Cost:** A company manufactures two lawn mower models, standard and self-propelled. The cost of producing each standard mower is \$80, and the cost of producing each self-propelled mower is \$140. If the fixed costs are \$5200, the total cost function is given by $C(x, y) = 5200 + 80x + 140y$, where x is the number of standard and y is the number of self-propelled mowers.
    a. Find $C(30, 20)$. **b.** Find $C(36, 25)$.

30. **Cost:** The cost function for producing two models of a product is found to be $C(x, y) = 850 + 32x + 20y$, where x is the number of model A and y is the number of model B. The cost for model A is \$32, the cost for model B is \$20, and the fixed costs are \$850 per week.
    a. Find $C(40, 24)$. **b.** Find $C(60, 38)$.

31. **Cost:** The cost of producing the standard model of a video camera is \$160. The cost of producing the deluxe model is \$220.
    a. If a company has weekly fixed costs of \$1360, find the cost function $C(x, y)$, where x is the number of standard models and y is the number of deluxe models.
    b. Find $C(15, 12)$.

32. **Cost:** A company makes two grades of paint, grade I, guaranteed for 5 years, and grade II, guaranteed for 10 years. A gallon of grade I costs \$3.20 to make, while a gallon of grade II costs \$3.90 to make. The weekly fixed costs are \$4500.
    a. Find the cost function $C(x, y)$ for making x gallons of grade I and y gallons of grade II.
    b. What is the cost of making 200 gallons of grade I and 140 gallons of grade II?

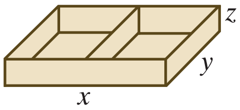
33. **Revenue:** A grocery store sells two brands of a product, the store brand and a name brand. The manager estimates that if she sells the store brand for x dollars and the name brand for y dollars, she will be able to sell $64 - 20x + 18y$ units of the store brand and $52 + 16x - 22y$ units of the name brand.
    a. Find the revenue function $R(x, y)$.
    b. What is the revenue if she sells the store brand for \$4.00 and the name brand for \$4.50?

34. Revenue: A pharmacy sells two cold remedies, one a generic remedy and the other a name brand. The store manager has determined that he can sell $26 - 6x + 8y$ bottles of the generic remedy and $22 + 5x - 9y$ bottles of the name brand if the prices are x dollars per bottle and y dollars per bottle, respectively.

a. Find the revenue function $R(x, y)$.

b. What is the revenue if the generic remedy is priced at \$6.20 per bottle and the name brand is priced at \$7.00 per bottle?

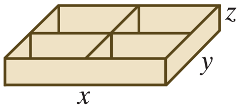
35. Volume and surface area: A rectangular box has no top and one partition (see diagram).



a. Write a function of three variables for the number of cubic units in the volume of the box.

b. Write a function of three variables for the number of square units of material needed to construct the box.

36. Volume and surface area: A rectangular box has no top and two intersecting partitions (see diagram).



a. Write a function of three variables for the number of cubic units in the volume of the box.

b. Write a function of three variables for the number of square units of material needed to construct the box.

37. Compound interest: A deposit of \$1000 is made into a savings account earning interest compounded quarterly. The amount $A(r, t)$ after t years is given by $A(r,t) = 1000\left(1+\frac{r}{4}\right)^{4t}$, where r is the interest rate in decimal form. Use this function of two variables to complete the following table.

		Number of Years (t)		
		3	**5**	**10**
Rate (r)	**0.06**			
	0.08			
	0.10			

38. Interest compounded continuously: A deposit of \$1000 is made into a savings account earning interest compounded continuously. The amount $A(r, t)$ after t years is given by $A(r, t) = 1000e^{rt}$, where r is the interest rate in decimal form. Use this function of two variables to complete the following table.

		Number of Years (t)		
		5	**8**	**12**
Rate (r)	**0.080**			
	0.085			
	0.100			

c. Treat x and y as constants. In this case, the entire expression $2ye^{xy}$ is treated as a constant.

$$\frac{\partial w}{\partial z} = 0 + 2z = 2z$$

16.2 EXERCISES

PRACTICE

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the functions in Exercises 1–28.

1. $f(x,y) = 4x + 7y - 10$

2. $f(x,y) = 11x - 19y + 2$

3. $f(x,y) = 2x^2 + 5y^2$

4. $f(x,y) = 5x^3 - 6y^4$

5. $f(x,y) = x^2y + 4xy^3 + 6$

6. $f(x,y) = x^3 - 90x^2y^3 - 9$

7. $f(x,y) = y\sqrt{25 + x^2}$

8. $f(x,y) = x^3\sqrt{y^2 + 5}$

9. $f(x,y) = \sqrt{49 - x^2 - y^2}$

10. $f(x,y) = \sqrt{16 + 2x^2 + y^2}$

11. $f(x,y) = 4e^{x-y}$

12. $f(x,y) = 7e^{xy}$

13. $f(x,y) = x \ln y$

14. $f(x,y) = \ln(x^2 + y^2)$

15. $f(x,y) = \ln(x^2 + 3xy)$

16. $f(x,y) = \frac{y}{\ln x}$

17. $f(x,y) = \frac{2x^2}{y^2 + 1}$

18. $f(x,y) = \frac{x^2 + 3}{5y - 9}$

19. $f(x,y) = \frac{x^2}{xy + 3}$

20. $f(x,y) = \frac{x + y}{4x - y}$

21. $f(x,y) = x^3e^y + ye^{x^2}$

22. $f(x,y) = xe^{y^3} - y^2e^x$

23. $f(x,y) = x\sqrt{xy + 2}$

24. $f(x,y) = y\sqrt{x^2 + y^3}$

25. $f(x,y) = x^4e^{xy}$

26. $f(x,y) = y^3e^{-xy}$

27. $f(x,y) = y^5 \ln(x^2 - 5y^2)$

28. $f(x,y) = 3x \ln xy^4$

In Exercises 29–32, find $\frac{\partial S}{\partial m}$ and $\frac{\partial S}{\partial b}$.

29. $S(m,b) = (2m + b - 9)^2 + (4m + b - 13)^2 + (5m + b - 18)^2$

30. $S(m,b) = (8m + b - 17)^2 + (9m + b - 23)^2 + (10m + b - 28)^2$

31. $S(m,b) = (6m + b - 40)^2 + (8m + b - 49)^2 + (9m + b - 55)^2 + (10m + b - 62)^2$

32. $S(m,b) = (12m + b - 81)^2 + (13m + b - 88)^2 + (14m + b - 96)^2 + (15m + b - 101)^2$

In Exercises 33–44, find all second-order partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

33. $f(x,y)=3xy+x^2y^3-19$

34. $f(x,y)=5x^3y-3x^3y^2-13$

35. $f(x,y)=x^4y^{\frac{2}{3}}$

36. $f(x,y)=(xy)^{\frac{3}{4}}$

37. $f(x,y)=xe^{2y}$

38. $f(x,y)=\dfrac{e^{x^3}}{y^4}$

39. $f(x,y)=(4x-3y)^{\frac{5}{3}}$

40. $f(x,y)=\sqrt{7x^3+y^2}$

41. $f(x,y)=\dfrac{3x+1}{5y+3}$

42. $f(x,y)=\dfrac{6y^2-5}{2x+7}$

43. $f(x,y)=\dfrac{2xy}{x-y}$

44. $f(x,y)=\dfrac{x-y}{xy}$

In Exercises 45–48, find f_x, f_y, and f_z.

45. $f(x,y,z)=xy+2xz+9yz$

46. $f(x,y,z)=3x^2y+2xyz+7xz^2$

47. $f(x,y,z)=(8x^2+5y^2-2z^2)^2$

48. $f(x,y,z)=\sqrt{x^2+2y^2+4z^2}$

In Exercises 49–52, find $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}$, and $\dfrac{\partial F}{\partial \lambda}$. (Note that λ is the Greek letter lambda.)

49. $F(x,y,\lambda)=8x+15xy-2y^2+\lambda(x+y-60)$

50. $F(x,y,\lambda)=3x^2+12y^2+\lambda(x+2y-84)$

51. $F(x,y,\lambda)=5x^2+3xy-10y^2+\lambda(14x+17y-49)$

52. $F(x,y,\lambda)=7x^2-2xy+9y^2+\lambda(8x+15y-120)$

APPLICATIONS

53. Marginal productivity: The number of units of a product that are manufactured by a company is given by $f(L,K)=80L^{\frac{2}{3}}K^{\frac{1}{3}}$, where L is the units of labor and K is the units of capital. Find the marginal productivity of labor and the marginal productivity of capital if the company is currently utilizing 27 units of labor and 64 units of capital.

54. Marginal productivity: The productivity of a company is approximated by $f(L,K)=20L^{\frac{2}{5}}K^{\frac{3}{5}}$, where L is the units of labor and K is the units of capital. Find the marginal productivity of labor and the marginal productivity of capital if the company is currently utilizing 32 units of labor and 32 units of capital.

55. Marginal cost: A company manufactures two products, product A and product B. The cost of producing x units of A and y units of B is $C(x,y)=3000+7x+5.8y+0.03x^2-xy+0.02y^2$.

a. Find the marginal cost with respect to x.

b. Find the marginal cost with respect to y.

56. **Marginal profit:** The profit from the sale of two products is given by the function $P(x, y) = 88x + 54y - 0.02x^2 - 0.015y^2 - 68$, where x is the number of units of product A sold, and y is the number of units of product B sold.
   a. Find the marginal profit with respect to x.
   b. Find the marginal profit with respect to y.

57. **Marginal profit:** A company produces two models of a product. The cost function is given by $C(x, y) = x^2 - 2xy + 2y^2 + 4x + 3y + 8$ and the revenue function is given by $R(x, y) = 20x + 15y$, where x is the number of units of model A and y is the number of units of model B produced and sold.
   a. Find the profit function.
   b. Find $P_x(20,14)$ and $P_y(20,14)$ and interpret the results.

58. **Marginal profit:** A firm produces and sells x units of product A and y units of product B. Its revenue function is given by $R(x, y) = 80x + 100y$ and its cost function is given by $C(x, y) = x^2 + 1.5y^2 - xy + 1500$. Find $P_x(50, 25)$ and $P_y(50, 25)$ and interpret the results.

59. **Marginal profit:** A marketing manager of a department store has determined that revenue is related to the number of units of television advertising x and the number of units of newspaper advertising y by the function $R(x, y) = 500(20x + 5y + 20xy - x^2)$. Each unit of television advertising costs \$5000 and each unit of newspaper advertising costs \$2500.
   a. Find the marginal profit with respect to x.
   b. Find the marginal profit with respect to y.

60. **Marginal profit:** A firm manufactures and sells two models of electric mowers. The standard model of the mower sells for \$300, and the self-propelled model of the mower sells for \$400. The total cost function is $C(x, y) = 90{,}000 + 0.05x^2 + 0.1y^2 + 0.125xy$, where x is the number of standard models and y is the number of self-propelled models.
   a. Find the marginal profit with respect to x.
   b. Find the marginal profit with respect to y.

16.3 EXERCISES

PRACTICE

For each of the Exercises 1–24, find all local maxima, local minima, and saddle points.

1. $f(x,y)=x^2+y^2-6x+2y-4$

2. $f(x,y)=x^2+2y^2+8x-4y+2$

3. $f(x,y)=x^2-y^2-10x+2y+9$

4. $f(x,y)=12x+8y-x^2-y^2-7$

5. $f(x,y)=5x+8y-x^2-y^2+11$

6. $f(x,y)=y^2-x^2+6x-y-10$

7. $f(x,y)=x^2+xy-y^2-5y-8$

8. $f(x,y)=x^2-2xy+4y^2-6y+3$

9. $f(x,y)=2x^2-3xy+3y^2+5x-13$

10. $f(x,y)=10x-2x^2+2xy-3y^2+5$

11. $f(x,y)=x^2-xy+y^2-2x-2y+1$

12. $f(x,y)=3x^2-2xy+y^2-16x+4y+14$

13. $f(x,y)=-x^2+xy-y^2+4x-5y+6$

14. $f(x,y)=x^2+3y^2+5xy+4x-3y+15$

15. $f(x,y)=x^3+y^2-3x-6y+11$

16. $f(x,y)=x^3-3x^2-2y^2-9x+8y+7$

17. $f(x,y)=x^2-y^3+9y^2-4x-15y-14$

18. $f(x,y)=2x^2+y^3-3y^2-12x-24y+21$

19. $f(x,y)=x^3-3x^2y+12y$

20. $f(x,y)=9x-xy^2+2y^3$

21. $f(x,y)=2x^2-x^2y+y^2$

22. $f(x,y)=x^2-2xy^2+4y^2$

23. $f(x,y)=xy+\dfrac{2}{x}+\dfrac{4}{y}$

24. $f(x,y)=xy+\dfrac{9}{x}+\dfrac{3}{y}$

APPLICATIONS

25. Profit: An automobile agency sells two models of a car. The annual profit is estimated by $P(x,y)=-0.1x^2-0.2y^2+6x+10y-160$ in thousands of dollars. Find the number of each model that should be sold to maximize profit.

26. **Sales:** The owner of a small business advertises in the newspaper and on radio. He has found that the number of units that he sells is approximated by $N(x, y) = -0.5x^2 - y^2 + 8x + 12y + 240,$ where x (in thousands of dollars) is the amount spent on newspaper advertising and y (in thousands of dollars) is the amount spent on radio advertising. How much should he spend on each to maximize the number of units sold?

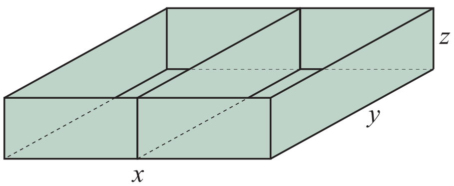
27. **Profit:** A firm produces two kinds of magazine racks, one selling for \$50 and the other for \$45. The total cost of producing x of the \$50 racks and y of the \$45 racks is given by $C(x, y) = 0.15x^2 + 0.1y^2 - 10x - 3y + 4760$ dollars. Find the number of each kind that should be produced and sold to maximize the profit.

28. **Profit:** A company makes two types of work gloves, leather and cloth. The leather gloves sell for \$5.80 and the cloth gloves for \$1.60. The total cost function is $C(x, y) = 0.25x^2 + 0.03y^2 + 1.3x + 0.4y + 14$ in thousands of dollars, where x (in thousand pairs) is the number of leather gloves and y (in thousand pairs) is the number of cloth gloves. How many gloves of each type should be produced and sold to maximize profits?

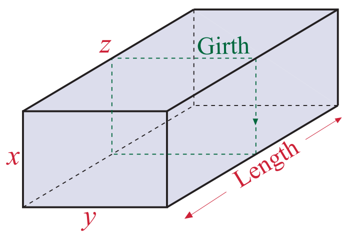
29. **Revenue:** A department store sells two types of T-shirts, adult and youth. The store manager has determined that he can sell $23 - 6x + 8y$ adult T-shirts and $26 + 5x - 9y$ youth T-shirts if the price is x dollars for the adult and y dollars for the youth. Find the price of each that will yield maximum revenue.

30. **Revenue:** A grocery store sells two brands of a product, a name brand and a store brand. The manager estimates that if she sells the name brand for x dollars per unit and the store brand for y dollars per unit, she will be able to sell $62 - 20x + 18y$ units of the name brand and $53 + 16x - 22y$ units of the store brand. Find the price of each that will yield maximum revenue.

31. **Construction:** A rectangular box is to be constructed without a top and with one partition. The volume of the box must be 162 in.^3 Find the dimensions that will minimize the material required to construct the box.



32. **Packaging:** The Postal Service has a limit of 108 inches on the combined length and girth of a rectangular package to be sent by parcel post. Length is the measurement for the longest side and girth is the distance around the package perpendicular to the length. Find the dimensions of the package of maximum volume that can be sent.



16.4 EXERCISES

PRACTICE

In Exercises 1–8, use the method of Lagrange multipliers to find the minimum value of f subject to the given constraint.

1. $f(x, y) = x^2 + y^2$, subject to $x + y - 4 = 0$
2. $f(x, y) = 4x^2 + 3y^2$, subject to $x + y - 7 = 0$
3. $f(x, y) = 5x^2 + 4y^2 - 2x$, subject to $x - y - 2 = 0$
4. $f(x, y) = 2x^2 + y^2 - 18x$, subject to $3x - y - 8 = 0$
5. $f(x, y) = 6x^2 + 5y^2 - xy$, subject to $2x + y = 24$
6. $f(x, y) = 2x^2 + 3y^2 - 3xy$, subject to $x + y = 16$
7. $f(x, y) = x^3 + y^3$, subject to $x + y = 8$
8. $f(x, y) = x^3 - y^3$, subject to $x - y = 10$

In Exercises 9–16, use the method of Lagrange multipliers to find the maximum value of f subject to the given constraint.

9. $f(x, y) = 2x^2 - 5y^2$, subject to $x - y = 3$
10. $f(x, y) = 5y^2 - 8x^2$, subject to $x + y = 6$
11. $f(x, y) = 8xy - 3x^2$, subject to $x + 2y = 14$
12. $f(x, y) = 6x^2 - 5xy$, subject to $2x - y = 8$
13. $f(x, y) = x^2 + y^2 + 4xy$, subject to $3x + 4y = 23$
14. $f(x, y) = x^2 - 4y^2 + 84xy$, subject to $5x + 2y = 18$
15. $f(x, y) = 15x^{0.4}y^{0.6}$, subject to $10x + 8y = 200$
16. $f(x, y) = 8x^{\frac{1}{2}}y^{\frac{1}{2}}$, subject to $6x + 15y = 450$

In Exercises 17–20, use the method of Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint.

17. $f(x, y) = 4xy$, subject to $x^2 + 4y^2 = 72$
18. $f(x, y) = 5xy$, subject to $9x^2 + y^2 = 162$
19. $f(x, y) = x^3 + 4y^3$, subject to $x + y = 6$
20. $f(x, y) = 3x^3 + y^3$, subject to $3x + y = 8$

APPLICATIONS

21. **Cost:** A company has a plant in Los Angeles and a plant in Oklahoma City. The firm is committed to produce a total of 40 units of a product each week. The cost function is given by $C(x, y) = 0.3x^2 + 0.2y^2 + 20x + 7y + 200,$ where x is the number of units produced in Los Angeles and y is the number of units produced in Oklahoma City. How many units should be produced in each plant to minimize the total weekly costs?

22. **Profit:** A department store sells two styles of a jacket, lined and unlined. During the month of January, the management expects to sell exactly 250 jackets. The profit function is given by $P(x, y) = -0.3x^2 - 0.4y^2 - 0.3xy + 80x + 65y - 1000,$ where x is the number of lined jackets sold and y is the number of unlined jackets sold. How many of each type should be sold to maximize the profit?

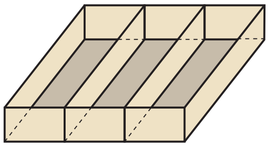
23. **Production:** The production function for a certain product is given by $f(x, y) = 75x^{0.3}y^{0.7},$ where x is the number of units of labor and y is the number of units of capital. Each unit of labor costs \$300, and each unit of capital costs \$200. If the company's budget allows a total of \$20,000 for labor and capital, find the maximum level of production.

24. **Production:** The management of a company has determined that x units of labor and y units of capital are required to produce $f(x, y) = 130x^{0.4}y^{0.6}$ units of a product. Each unit of labor costs \$450, and each unit of capital costs \$360. Find the maximum number of units that can be produced if a total of \$90,000 is available for labor and capital.

25. **Sales:** A sales representative for a textbook publishing company estimates her monthly sales for March to be $S(x, y) = 30x + 18y - 1.2x^2 - 0.6y^2$ in thousands of dollars, where x and y represent the number of days spent in each of the two metropolitan areas that comprise her sales territory. If she plans to work 20 days during the month, how many days should she spend in each area to maximize her sales?

26. **Revenue:** The marketing manager of a department store has determined that revenue, in dollars, is related to the number of units of television advertising x and the number of units of newspaper advertising y by the function $R(x, y) = 500(20x + 5y + 6xy - x^2).$ Each unit of television advertising costs \$3000, and each unit of newspaper advertising costs \$1500. If the advertising budget is \$30,000, find the maximum revenue.

27. **Construction:** A farmer wants to build a rectangular pen and then divide it with two interior fences. The total area enclosed is to be 2484 ft^2. The exterior fence costs \$18 per foot, and the interior fence costs \$16.50 per foot. Find the dimensions of the pen that will minimize the cost of fencing.



28. **Shipping:** A container manufacturer is asked to design a closed rectangular shipping crate with a square base. The volume of the crate is 36 ft^3. The material for the top costs \$1 per square foot, the material for the sides costs \$0.90 per square foot, and the material for the bottom costs \$1.40 per square foot. Find the dimensions that will minimize the total cost of building the crate.

16.5 EXERCISES

PRACTICE

In Exercises 1–4, **a.** find the equation of the regression line for the given points, and **b.** draw the scatter diagram and graph the regression line.

1. (0, 3), (1, 5), (2, 7), (3, 8), (5, 9), (6, 9)

2. (1, 10), (2, 8), (3, 7), (4, 6), (5, 5), (6, 5), (7, 4)

3. (1, 9.6), (2, 8.7), (3, 7.7), (4, 6.1), (5, 5.0)

4. (1, 5.2), (2, 6.4), (3, 8.1), (4, 9.2), (5, 10.6)

In Exercises 5–10, find the equation of the regression line for the given points.

5. (10, 6.5), (20, 5.8), (30, 5.6), (40, 3.1), (50, 1.8)

6. (1, 0.2), (2, 0.4), (3, 0.3), (4, 0.6), (5, 0.6)

7. (1, 236), (2, 248), (3, 270), (4, 285), (5, 291)

8. (1, 0.45), (3, 0.71), (4, 0.82), (5, 0.94)

9. (0.6, 4.8), (0.8, 5.0), (1.0, 4.8), (1.2, 5.2), (1.4, 5.8)

10. (3.2, 0.10), (4.1, 0.15), (4.8, 0.20), (5.1, 0.23), (6.0, 0.29)

APPLICATIONS

11. Advertising budget: During the last 5 years, the advertising manager for a corporation has gathered the following data that shows the relationship between the advertising budget (in millions of dollars) and the total sales (in thousands of units).

Advertising Budget (x) (in millions)	\$4.5	\$6.5	\$3.5	\$3.2	\$2.6
Sales (y) (in thousands)	37	46	42	32	29

a. Find the regression line for the data.
b. Estimate the sales if \$4 million is budgeted for advertising.

12. Price: Records at a company for the last 5 years show the following relationship between the units sold (in thousands) and the price of a product.

Price (p)	\$8.80	\$8.00	\$7.50	\$6.90	\$6.20
Quantity Sold (x) (in thousands)	3.8	5.2	7.3	8.0	9.6

a. Find the regression line for the price in terms of units.
b. Estimate the price that should be charged in order to sell 10,000 units.

13. Construction: The following data shows the amount spent on office building construction (in thousands) for a particular county during a six-month period.

Month	Apr	May	June	July	Aug	Sept
Amount (in thousands)	\$24	\$19	\$30	\$49	\$68	\$69

a. Find the regression line for the data.
b. Estimate the amount spent in October.

14. Revenue: The annual revenue (in millions of dollars) for a corporation is given in the following table.

Year	1999	2000	2001	2002	2003	2004
Revenue (in millions)	\$66	\$82	\$127	\$201	\$310	\$392

a. Find the line of regression for the data.
b. Estimate the revenue for 2005.

15. Livestock futures: The price of livestock futures is the estimated market price of livestock on the delivery date (end of the indicated month). The cattle futures (in cents per pound) for the months February through July are as follows.

Month	Feb	Mar	Apr	May	June	July
Price (¢ per pound)	79.10	76.02	71.80	71.45	71.45	72.50

a. Find the line of regression for the data.
b. Estimate the price for August.

16. Tourism: The total number of foreign tourists visiting the United States, as reported by the U.S. Travel and Tourism Administration, is shown in the following table.

Year (x)	2000	2001	2002	2003	2004
Tourists (y) (in millions)	25.7	26.3	29.7	34.2	38.3

a. Find the regression line for the data.
b. Estimate the number of foreign tourists that visited the United States in 2005.

16.6 EXERCISES

PRACTICE

In Exercises 1–10, evaluate the given double integral.

1. $\int_0^1\int_1^2 (x+1)\,dydx$

2. $\int_0^2\int_1^3 (4-x)\,dydx$

3. $\int_{-1}^2\int_1^4 (3x+2y)\,dydx$

4. $\int_{-2}^2\int_3^4 (2x-y)\,dydx$

5. $\int_1^2\int_2^3 \frac{2y}{x}\,dydx$

6. $\int_1^3\int_{-1}^2 \frac{3y}{2x}\,dydx$

7. $\int_1^3\int_{-2}^2 (x^2+3y^2-1)\,dxdy$

8. $\int_2^3\int_{-1}^1 (2x^2+y^2-x)\,dxdy$

9. $\int_0^2\int_0^1 e^{x+y}\,dxdy$

10. $\int_0^1\int_{-1}^2 ye^{xy}\,dxdy$

In Exercises 11–16, evaluate the double integral on the given rectangular region.

11. $\iint_R (x-y^2)\,dA$ $\quad R: 0\le x\le 2$ and $0\le y\le 1$

12. $\iint_R (xy+x)\,dA$ $\quad R: 0\le x\le 3$ and $0\le y\le 3$

13. $\iint_R y\sqrt{x+1}\,dA$ $\quad R: 0\le x\le 3$ and $1\le y\le 5$

14. $\iint_R x^2\sqrt{3+y}\,dA$ $\quad R: -1\le x\le 4$ and $1\le y\le 6$

15. $\iint_R e^{x+2y}\,dA$ $\quad R: 0\le x\le 3$ and $0\le y\le 4$

16. $\iint_R e^{2x+y}\,dA$ $\quad R: 0\le x\le 2$ and $0\le y\le 1$

In Exercises 17–24, evaluate the double integral.

17. $\int_0^2\int_0^{3x} xy^2\,dydx$

18. $\int_0^1\int_{2x}^{x^2} x^2y^2\,dydx$

19. $\int_1^4\int_0^{x^2} \sqrt{\frac{y}{x}}\,dydx$

20. $\int_1^4\int_x^{x^2} \sqrt{\frac{x}{y}}\,dydx$

21. $\int_1^3\int_1^{e^y} \frac{y}{x}\,dxdy$

22. $\int_0^2\int_0^y e^{y^2}\,dxdy$

23. $\int_0^4\int_0^y \sqrt{9+y^2}\,dxdy$

24. $\int_0^2\int_0^{4-y^2} y\sqrt{x}\,dxdy$

In Exercises 25–34, evaluate the double integral on the given region.

25. $\iint_R 2xy\,dA$ $\quad R: 0\le x\le 1$ and $x^2\le y\le \sqrt{x}$

26. $\iint_R 3xy^2\,dA$ $\quad R: 0\le x\le 1$ and $x^3\le y\le \sqrt[3]{x}$

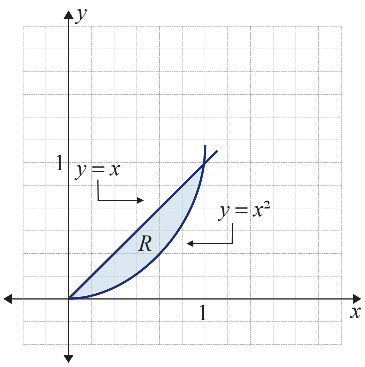
27. $\iint_R (x^2-y)\,dA$ $\quad R: 1\le x\le 2$ and $x\le y\le x^2$

28. $\iint_R (3-2x-2y)\,dA \qquad R: 0 \le x \le 1 \text{ and } 0 \le y \le (2-x)$

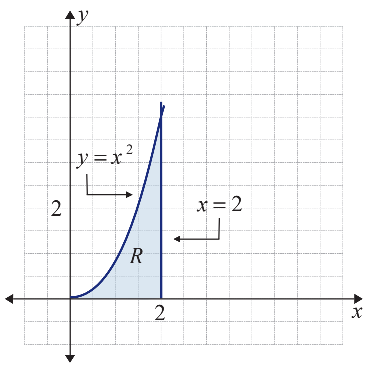
29. $\iint_R e^y\,dA \qquad R: 0 \le x \le 2 \text{ and } x \le y \le 3x$

30. $\iint_R e^y\,dA \qquad R: 0 \le x \le 1 \text{ and } 0 \le y \le 2x$

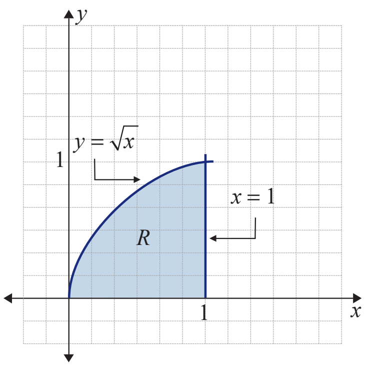
31. $\iint_R (x+y)\,dA$



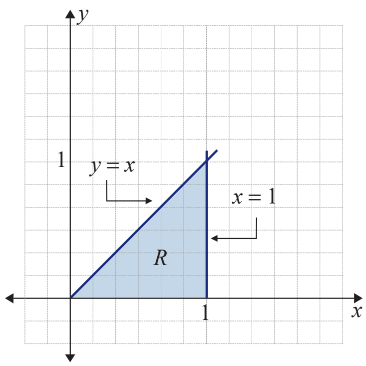
32. $\iint_R (2xy+x)\,dA$



33. $\iint_R (3-2xy)\,dA$



34. $\iint_R \sqrt{4-x^2}\,dA$



35. Find the volume of the solid bounded above by the graph of $f(x,y) = 8 - x^2 - y^2$ and below by the rectangle $R: -1 \le x \le 2$ and $0 \le y \le 2$.

36. Find the volume of the solid bounded above by the graph of $f(x,y) = 2 + x^2 + y^2$ and below by the rectangle R: $0 \le x \le 1$ and $0 \le y \le 3$.

37. Find the volume of the solid bounded above by the graph of $f(x,y) = 8 - 4x - 2y$ and below by the triangle with vertices (0, 0, 0), (2, 0, 0), and (0, 4, 0).

38. Find the volume of the solid bounded above by the graph of $f(x,y) = 3 + x + 2y$ and below by the triangle with vertices (0, 0, 0), (0, 2, 0), and (2, 0, 0).

39. Find the volume of the solid bounded above by the graph of $f(x,y) = 2xy$ and below by the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$.

40. Find the volume of the solid bounded above by the graph of $f(x,y) = 4x^2y$ and below by the region bounded by $y = x^2$, $y = 0$, and $x = 1$.