Simplex Method: Maximization	Duality Principle: Minimization
1. Maximize objective function f subject to constraints involving $\le$.	**1.** Minimize objective function f subject to constraints involving $\ge$.
2. Form the initial simplex tableau.	**2.** Write down the matrix for minimization with constraints, A.
3. Apply the simplex method to obtain an optimal solution.	**3.** Write down the matrix for the dual maximization problem, A^T.
	4. Form the initial simplex tableau for the dual problem.
	5. Apply the simplex method to obtain an optimal solution. Use entries in the last row of columns corresponding to slack variables for the coordinates.

TABLE 2

7.5 EXERCISES

PRACTICE

For each given system of inequalities (constraints), rewrite as a system of equations by adding slack or subtracting surplus variables.

1. $\begin{cases} 5x_1 + x_2 \le 15 \\ 3x_1 + 2x_2 \ge 7 \end{cases}$

2. $\begin{cases} 2x_1 + 3x_2 \le 7 \\ 4x_1 - x_2 \ge 5 \\ x_1 + x_2 \le 4 \end{cases}$

3. $\begin{cases} 8x_1 + 3x_2 - x_3 \ge 10 \\ 6x_2 - x_3 \ge 7 \\ x_1 + x_2 + x_3 \le 8 \end{cases}$

4. $\begin{cases} x_1 + x_2 + 2x_3 \le 12 \\ 2x_1 + x_2 + 3x_3 \le 17 \\ x_1 - x_2 + x_3 \ge 7 \\ 2x_2 \le 5 \end{cases}$

5. $\begin{cases} 4x_1 - x_3 \ge 9 \\ x_2 + 2x_3 \le 13 \\ 5x_1 - x_2 + 4x_3 \ge 14 \\ 3x_1 + x_2 + x_3 \le 17 \end{cases}$

6. Suppose you want to minimize $f(x_1, x_2) = x_1 + 3x_2$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 \le 5 \\ 2x_1 - x_2 \ge 3 \end{cases}$$

Rewrite this as a maximization problem and rewrite the system of constraints using slack variables as in the simplex method.

7. Suppose you want to minimize $f(x_1, x_2, x_3) = 4x_1 + 7x_2 + 9x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \le 9 \\ -x_1 + 3x_3 \ge 6 \\ 5x_1 - x_2 - x_3 \ge 7 \end{cases}$$

Rewrite this as a maximization problem and rewrite the system of constraints using slack variables as in the simplex method.

8. Consider again the optimization problem in Exercise 6. Rewrite this as a maximization problem, but this time rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

9. Consider again the optimization problem in Exercise 7. Rewrite this as a maximization problem, but this time rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

10. Suppose you want to minimize $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 6x_3$ subject to the following constraints.

$$\begin{cases} 3x_1 + 2x_2 + x_3 \le 8 \\ 2x_1 + x_3 = 5 \\ -x_1 - x_2 + 4x_3 \ge 7 \end{cases}$$

Rewrite this as a maximization problem, and rewrite the system of constraints using slack, surplus, and artificial variables as in the big M method. Also, write the objective function that results after introducing the artificial variables and the constant *M*.

Use either the simplex method or the big M method to solve each given optimization problem. See Examples 1, 2, and 3.

11. Minimize $f(x_1, x_2) = 4x_1 + 6x_2$ subject to the following constraints.

$$\begin{cases} 2x_1 + 3x_2 \le 10 \\ 4x_1 - x_2 \ge 4 \end{cases}$$

12. Maximize $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 4x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 + x_3 \le 7 \\ 3x_1 - x_2 + 2x_3 \ge 4 \\ x_1 + x_3 \ge 5 \end{cases}$$

13. Maximize $f(x_1, x_2, x_3) = 2x_1 - 2x_2 + 6x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 \le 20 \\ x_1 + x_3 = 5 \\ x_2 + x_3 \ge 10 \end{cases}$$

14. Minimize $f(x_1, x_2) = 3x_1 + 4x_2$ subject to the following constraints.

$$\begin{cases} 4x_1 + 5x_2 \ge 18 \\ x_1 + x_2 = 4 \end{cases}$$

15. Minimize $f(x_1, x_2, x_3) = 7x_1 + 2x_2 + 3x_3$ subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \ge 6 \\ 3x_1 - x_2 + 2x_3 = 7 \\ -x_1 + x_3 \le 2 \end{cases}$$

16. Minimize $f(x_1, x_2) = 3x_1 + 5x_2$ subject to the following constraints.

$$\begin{cases} x_1 + 3x_2 \le 8 \\ 3x_1 - x_2 = 5 \end{cases}$$

17. Maximize $f(x_1, x_2, x_3) = 6x_1 + 4x_2 + 5x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 + 3x_3 \le 18 \\ x_1 + 2x_2 + 2x_3 \le 16 \\ x_1 + x_2 + x_3 = 10 \end{cases}$$

18. Minimize $f(x_1, x_2, x_3) = 10x_1 + 8x_2 + 15x_3$ subject to the following constraints.

$$\begin{cases} 2x_1 + x_2 - x_3 \ge 6 \\ x_1 + x_2 + 2x_3 = 6 \\ -x_1 + 4x_2 - x_3 \ge 24 \end{cases}$$

APPLICATIONS

Use either the simplex method or the big M method to solve each given application. See Example 4.

19. A truck driver is tasked with delivering stereos for an electronics company. Suppose this company has stereos at two warehouses; the warehouse in Charlotte has 500 stereos, and the warehouse in Charleston has 200 of them. A store in Raleigh needs 350 stereos, and a store in Spartanburg needs 250 stereos. Suppose the truck driver earns \$5 in profit per stereo delivered to Raleigh from Charlotte and \$4 for each stereo delivered to Spartanburg from Charlotte. Moreover, he earns \$7 for each stereo delivered to Raleigh from Charleston and \$5 for each stereo delivered to Spartanburg from Charleston. How many stereos should the truck driver deliver to each store from each warehouse to maximize his earnings?

20. A Mexican food truck produces and sells beef tacos and beef burritos. Suppose each taco uses 8 ounces of beef and 1 ounce of cheese, and suppose each burrito uses 12 ounces of beef and 5 ounces of cheese. The supply of beef is limited to 1180 ounces, but the supply of cheese is plentiful, and the workers on the truck wish to use up at least 200 ounces of cheese to minimize waste. If each taco sells for \$6 and each burrito sells for \$7, how many of each kind of food should the truck produce to maximize revenue?

21. A car company has two factories. Factory A has 600 cars (of a particular model) in stock, and factory B has 400 cars (of that same model) in stock. Two dealerships order this car model. Dealership I needs 300 cars and dealership II needs 400 cars. The cost of shipping per car from the two factories to the dealerships is given below.

	Dealership I	**Dealership II**
Factory A	\$40	\$30
Factory B	\$35	\$45

How should the company ship the cars from the two factories to the dealerships in order to minimize the shipping cost? (**Hint:** Use an approach similar to the one in Exercise 19.)

22. An ice cream company sells ice cream bars and ice cream sandwiches. One of its plants makes each treat and packs the bars and sandwiches into 12-count packages. There are enough lines running at this plant so that each package of bars takes 2 seconds to produce, and each package of sandwiches takes 5 seconds. Suppose the plant has at most 15 hours (54,000 seconds) available each day for producing and packaging the ice cream products. Also, limitations and product demand indicate that the number of packages bars plus the number of packages of sandwiches should be at least 18,000, and the number of bars should be at least 1.5 times the number of sandwiches. If the plant's manufacturing costs are \$2.50 for each package of bars and \$2 for each package of sandwiches, how many packages of each ice cream product should the plant produce to satisfy the constraints at a minimum cost?

23. Three water purification facilities can handle at most 8 million gallons during a certain time period. Plant I leaves 11% of certain impurities and costs $40,000 per million gallons. Plant II leaves 8% of certain impurities and costs $50,000 per million gallons. Plant III leaves 5% of certain impurities and costs $70,000 per million gallons. The desired level of impurities in the water from all plants combined is at most 8%. If plants I and III must handle at least 5 million gallons, find the number of gallons each plant should handle so the water is treated at a minimum cost.

24. A silverware manufacturer produces forks, knives, and spoons. The manager feels that restricting the types of silverware produced could increase revenue. The following table gives the price of each piece of silverware, the raw materials cost for each piece, and the profit made.

	Price	Materials Cost	Profit
Knives	$4	$0.10	$0.60
Forks	$6	$0.10	$0.25
Spoons	$5	$0.15	$1.10

The monthly revenue must be at least $12,000 and the raw material costs should be no more than $280. Also, suppose there should be at least as many forks produced as spoons. How many of each type of silverware should be produced to maximize profit?

25. A candidate for president has budgeted a maximum of $900,000 for political advertisements during her presidential campaign. Suppose each minute of television time costs $30,000 and each one-page newspaper ad costs $12,000. Each television ad is expected to be viewed by 2 million voters, and each newspaper ad is expected to be seen by 200 thousand voters. The candidate is advised to use at least 8 television ads and at least 5 newspaper ads. How should the candidate allocate the advertising budget to reach as many voters as possible?

26. An aquarium contains several feeding areas for dolphins, sharks, and stingrays. There are 3 foods (A, B, and C) available at each of these feeding areas. The following table describes how many pounds of each type of food is required to feed one animal.

	Food A	Food B	Food C
Dolphin	3	1	2
Shark	2	6	6
Stingray	1	2	2

Suppose the aquarium has 300 pounds of food A available and 400 pounds of food B available. Also suppose the aquarium has a surplus of food C and would like to use at least 450 pounds of it. How many of each type of animal can the aquarium support so that the number of dolphins, sharks, and stingrays is a maximum?

27. A family has a side business of selling three kinds of snow cones from their trailer: Tropical Paradise, Arctic Chill, and Paradise Blast. Each Tropical Paradise snow cone uses 3 ounces of mango syrup, 1 ounce of blueberry syrup, and 8 ounces of shaved ice. Each Arctic Chill snow cone uses 3 ounces of blueberry syrup and 8 ounces of shaved ice. Each Paradise Blast snow cone uses 2 ounces of mango syrup, 2 ounces of blueberry syrup, and 8 ounces of shaved ice. Suppose 225 ounces of blueberry syrup and 52.5 pounds (840 ounces) of shaved ice are available, and suppose that all of the ice has to get used up. In addition, suppose there is plenty of mango syrup in stock, and the family would like to use at least 150 ounces of it. If the family makes a profit of $0.40 for each Tropical Paradise snow cone, $0.60 profit for each Arctic Chill snow cone, and $0.50 profit for each Paradise Blast snow cone, how many of each kind of snow cone should the family sell in order to maximize their profit?

28. A company produces three kinds of 3-ring binders depending on their thickness: 1.5-inch, 2-inch, and 3-inch. The number of units of steel, cardboard, and plastic needed to make each kind of 3-ring binder are in the table below.

	1.5-inch	**2-inch**	**3-inch**
Steel	2	3	6
Cardboard	10	18	30
Plastic	8	12	16

Suppose the company wants to use no more than 280 units of steel, no less than 1520 units of cardboard, and exactly 1000 units of plastic per day. If it costs the company $2 to produce each 1.5-inch binder, $4 to produce each 2-inch binder, and $6 to produce each 3-inch binder, then how many of each kind of binder should the company produce in order to minimize cost?

29. A plant that makes popsicles has a mixing tank. Currently, it contains 60 gallons of a mixture that is 15% orange syrup, 18% high fructose corn syrup, and 41% water. The foreman at the plant wants to add more mixture to the tank before the lines run out of fluid to make the popsicles. He has 3 different mixes available. Mix I contains 20% orange syrup, 40% high fructose corn syrup, and 35% water. Mix II contains 10% orange syrup, 30% high fructose corn syrup, and 40% water. Mix III contains 25% orange syrup, 40% high fructose corn syrup, and 15% water. The desired final mixture should have at most 18% orange syrup, exactly 30% high fructose corn syrup, and at least 35% water. If the costs of each mix per gallon are $60 (I), $45 (II), and $30 (III), how many gallons of each mix should the foreman use to minimize the cost? **Hint:** If the final mix needs at least 35% water, that means

$$41\%(\text{mix in tank})+35\%(\text{mix I})+40\%(\text{mix II})+15\%(\text{mix III}) \geq 35\%(\text{mix in tank}+\text{mix I}+\text{mix II}+\text{mix III}).$$

30. An eccentric barbecue restaurant produces and sells three kinds of bratwursts well-known to the locals in the area: Plain Jane, Spicy Samantha, and Cheezy Chelsea. All of these bratwursts consist of pork, spices, and cheese. The following table summarizes how much of each ingredient are in each kind of bratwurst.

	Plain Jane	Spicy Samantha	Cheezy Chelsea
Pork	8 oz	6 oz	7 oz
Spices	1 oz	3 oz	1 oz
Cheese	1 oz	1 oz	2 oz

Suppose the restaurant has 200 pounds (3200 ounces) of pork available, and they want to use at least 800 ounces of spices. Suppose they also have 600 ounces of cheese and must use all of it due to an obscure health code regulation preventing the restaurant from having any traces of unused cheese. Finally, suppose each Plain Jane brings in a profit of \$0.70, each Spicy Samantha brings in a profit of \$0.50, and each Cheezy Chelsea brings in a profit of \$0.60. How many of each kind of sausage should the restaurant produce and sell to maximize its profit?

WRITING & THINKING

31. Use the big M method to try and maximize $f(x_1,x_2)=3x_1+9x_2$ subject to the following constraints.

$$\begin{cases} x_1+3x_2=7 \\ 2x_1+x_2\le 5 \end{cases}$$

What do you find?

32. Use the big M method to try and maximize $f(x_1,x_2)=x_1+2x_2$ subject to the following constraints.

$$\begin{cases} x_1-x_2\le 5 \\ 2x_1-x_2=20 \end{cases}$$

What do you find?

33. Consider the problem of minimizing $f(x_1,x_2)=2x_1+4x_2$ subject to the following constraints.

$$\begin{cases} 3x_1+x_2=6 \\ x_1+x_2\ge 4 \end{cases}$$

a. Use the big M method to solve.

b. According to the first constraint, $3x_1+x_2=6$ holds, and therefore the inequality $3x_1+x_2\ge 6$ must also hold. It seems intuitive that we could use the Theorem of Duality (instead of the big M method) to solve this problem if we replace the first constraint with $3x_1+x_2\ge 6$. Try it. Do you obtain the same solution as in part a.? If you obtain a different solution, can you explain why?