

Simplex Method for Maximization vs. the Duality Principle for Minimization

The following table gives a summary and a comparison of the techniques for maximization and minimization for solving optimization problems.

Simplex Method: Maximization	Duality Principle: Minimization
1. Maximize $f(x_1, x_2, \dots, x_n)$ subject to constraints of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq c$.	1. Minimize $f(x_1, x_2, \dots, x_n)$ subject to constraints of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n \geq c$.
2. Introduce slack variables s_1, s_2, \dots, s_m , and form the initial simplex tableau.	2. Write down the matrix for minimization with constraints, A .
3. Repeat the pivoting process until an optimal solution is reached.	3. Write down the matrix for the dual maximization problem, A^T .
4. Look at the final simplex tableau. The maximum value of f is in the lower-right corner. The nonzero coordinates at which the maximum is attained are in the last column in the rows corresponding to the 1 entry of each x_i that is basic, and each x_i that is nonbasic is 0.	4. Introduce slack variables s_1, s_2, \dots, s_m , and form the initial simplex tableau.
	5. Repeat the pivoting process until an optimal solution to the dual problem is reached.
	6. Look at the final simplex tableau. The minimum value of f is in the lower-right corner. The coordinates at which the minimum is attained are in the last row of the columns corresponding to the slack variables.

7.4 EXERCISES

PRACTICE

For each given matrix A , find the transpose A^T .

$$1. A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 6 & 2 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

$$3. A = \begin{bmatrix} 1 & 3 \\ 6 & -5 \\ 7 & 2 \\ -1 & 0 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 6 & 10 & -3 & 8 \\ 4 & -1 & 3 & 7 \end{bmatrix}$$

For each given (primal) minimization problem, state the dual maximization problem.

5. Minimize $f(x_1, x_2) = 7x_1 + 2x_2$
subject to the following constraints.

$$\begin{cases} 3x_1 + 5x_2 \geq 18 \\ 2x_1 + 6x_2 \geq 15 \end{cases}$$

6. Minimize $f(x_1, x_2) = 3x_1 + 8x_2$
subject to the following constraints.

$$\begin{cases} x_1 \geq 3 \\ 2x_1 + x_2 \geq 5 \end{cases}$$

7. Minimize
 $f(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3$ subject
to the following constraints.

$$\begin{cases} 2x_1 + 3x_2 \geq 16 \\ 5x_1 + 4x_3 \geq 20 \end{cases}$$

8. Minimize
 $f(x_1, x_2, x_3) = 3x_1 + 4x_2 + 5x_3$
subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \geq 8 \\ 2x_1 + 2x_2 + x_3 \geq 14 \\ x_1 + 3x_2 + x_3 \geq 12 \end{cases}$$

Solve each given minimization problem. See Example 1.

9. Minimize $f(x_1, x_2) = 3x_1 + x_2$
subject to the following constraints.

$$\begin{cases} 4x_1 + 5x_2 \geq 13 \\ 2x_1 + x_2 \geq 5 \end{cases}$$

10. Minimize $f(x_1, x_2) = 2x_1 + 4x_2$
subject to the following constraints.

$$\begin{cases} 3x_1 + x_2 \geq 6 \\ x_1 + x_2 \geq 4 \end{cases}$$

11. Minimize
 $f(x_1, x_2, x_3) = x_1 + 2x_2 + x_3$ subject
to the following constraints.

$$\begin{cases} x_1 + 3x_2 \geq 6 \\ 2x_1 + x_3 \geq 10 \end{cases}$$

12. Minimize
 $f(x_1, x_2, x_3) = 11x_1 + 8x_2 + 7x_3$
subject to the following constraints.

$$\begin{cases} x_1 + 2x_2 + 2x_3 \geq 8 \\ x_1 + 2x_2 + x_3 \geq 6 \\ 4x_1 + x_2 + x_3 \geq 4 \end{cases}$$

13. Minimize
 $f(x_1, x_2, x_3) = 7x_1 + 10x_2 + 7x_3$
subject to the following constraints.

$$\begin{cases} x_1 + x_2 + 2x_3 \geq 5 \\ x_1 + 3x_2 \geq 6 \\ 2x_1 + x_2 + x_3 \geq 4 \end{cases}$$

14. Minimize $f(x_1, x_2) = 3x_1 + 4x_2$
subject to the following constraints.

$$\begin{cases} 2x_1 + 2x_2 \geq 7 \\ 2x_1 + 3x_2 \geq 10 \end{cases}$$

15. Minimize
 $f(x_1, x_2, x_3) = 8x_1 + 3x_2 + 7x_3$
subject to the following constraints.

$$\begin{cases} x_1 + x_2 + x_3 \geq 9 \\ 2x_1 + x_2 + 2x_3 \geq 11 \\ x_1 + 3x_2 + 3x_3 \geq 17 \end{cases}$$

16. Minimize $f(x_1, x_2) = 6x_1 + 8x_2$
subject to the following constraints.

$$\begin{cases} x_1 + x_2 \geq 8 \\ 3x_2 \geq 20 \\ -x_1 + x_2 \geq 1 \end{cases}$$

 **APPLICATIONS**

Use the simplex method to solve each given application. See Examples 2 and 3.

17. A tool company produces wrenches and screwdrivers at factories in two locations (plant A and plant B). It has orders to produce 12,000 wrenches and 10,000 screwdrivers. For each hour, plant A can produce 400 wrenches and 200 screwdrivers, and the operation costs are \$1200 per hour. At plant B, 200 wrenches and 500 screwdrivers can be produced in an hour, and the operation costs are \$1500 per hour. How many hours should the company run each plant to fill the orders at a minimum cost? What is the minimum cost?
18. An electronics factory produces televisions and DVD players on two assembly lines. Line 1 can produce 50 televisions and 75 DVD players at a cost of \$225 per hour. Line 2 can produce 60 televisions and 60 DVD players at a cost of \$210 per hour. The factory needs to produce at least 300 televisions and 375 DVD players to fill an order. How many hours should each line be run to fill the order at a minimum cost? What is the minimum cost?
19. An athlete drinks two kinds of dietary drinks to help maintain a healthy diet. Drink I has 3 units of protein, 3 units of carbohydrates, and 2 units of vitamin D per one-quart (32 oz) bottle. Drink II has 2 units of protein, 3 units of carbohydrates, and 5 units of vitamin D per bottle. Suppose each bottle of drink I costs \$8 and each bottle of drink II costs \$12. Finally, suppose the athlete wants to consume a minimum of 7 units of protein per day, 9 units of carbohydrates per day, and 12 units of vitamin D per day. Find the combination of drinks that the athlete should purchase and consume per day in order to minimize the cost while meeting the dietary requirements.
20. A swine farmer is considering two different types of feed for fattening his pigs. Brand 1 feed costs 40 cents per pound and brand 2 feed costs 35 cents per pound. Brand 1 feed contains 10 units of fiber and 4 units of protein per pound. Brand 2 feed has 12 units of fiber and 3 units of protein per pound. Suppose the farmer would like his pigs to consume at least 160 units of fiber and 55 units of protein for each feeding. How many pounds of each feed should the farmer buy per feeding to satisfy the nutritional requirements at a minimum cost?
21. A petroleum company owns 3 refineries (A, B, and C), and each one produces 3 grades of oil (5W30, 10W30, and 10W40). The following table summarizes how many barrels of each kind of oil per day each refinery produces for the petroleum company.

	5W30	10W30	10W40
Refinery A	300	300	200
Refinery B	200	400	250
Refinery C	250	300	450

Suppose the daily costs to operate the refineries are \$25,000 (A), \$30,000 (B), and \$40,000 (C). Also, suppose the petroleum company has orders totaling 5300 barrels of 5W30, 6900 barrels of 10W30, and 6500 barrels of 10W40. How many days should the petroleum company run each of the three refineries so that the costs of operation are minimized and yet the orders are fulfilled?

22. Three factories dump waste water containing three kinds of pollutants into a lagoon. In order to reduce the pollution levels, the factories must treat the waste water they dump. The following table shows the possible percent reduction of each pollutant at each factory.

	Factory 1	Factory 2	Factory 3
Pollutant 1	40%	10%	30%
Pollutant 2	50%	15%	20%
Pollutant 3	30%	25%	35%

Suppose the costs to dump each ton of waste at each factory is as follows: \$100 (factory 1), \$40 (factory 2), and \$60 (factory 3). Moreover, suppose the state requires a reduction of at least 110 tons per day of pollutant 1, at least 120 tons per day of pollutant 2, and at least 122 tons per day of pollutant 3. Find the number of tons of waste that must be treated each day in order to minimize the cost of treatment. What is the minimum cost?

23. Biologists in a lab feed two different foods (I and II) to mice; each food contains three ingredients (A, B, and C). Food I has 6 units of ingredient A per gram, 2 units of ingredient B per gram, and 1 unit of ingredient C per gram. Food II has 4 units of ingredient A per gram, 8 units of ingredient B per gram, and 3 units of ingredient C per gram. Suppose that the lab requires the mice to have consumed a total of at least 60 grams of ingredient A and 50 grams of ingredient B. Also suppose that ingredient C can be harmful to the mice. How many grams of each type of food should be given to the mice to meet the requirements for ingredients A and B, while at the same time minimizing the amount of ingredient C?
24. A nutritionist advises an individual who is suffering from iron and vitamin B deficiency to take at least 3000 milligrams (mg) of iron, 2000 mg of vitamin B1, and 1500 mg of vitamin B2 over a period of time. Two vitamin pills are suitable, brand A and brand B. Each brand A pill costs 10 cents and contains 50 mg of iron, 15 mg of vitamin B1, and 10 mg of vitamin B2. Each brand B pill costs 15 cents and contains 10 mg of iron, 28 mg of vitamin B1, and 22 mg of vitamin B2. What combination of pills should the individual purchase in order to meet the minimum iron and vitamin requirements at the lowest cost?
25. A candidate wishes to use a combination of radio and television advertisements in his campaign. Suppose that every one-minute spot on the radio reaches 10 thousand voters, and every one-minute spot on television reaches 70 thousand voters. The candidate feels that he needs to reach at least 3 million voters, and he must buy at least 90 minutes of advertisements. If a radio ad costs \$75 per minute, and a television ad costs \$450 per minute, how many minutes of each medium should the candidate use to minimize the cost of advertisements?
26. Biologists at a university have a research pond on their grounds that is stocked with bass and catfish. Suppose each bass consumes 8 units of food per day while each catfish consumes 20 units of food per day. In addition, suppose each bass produces 6 units of waste per day while each catfish produces 14 units of waste per day. Suppose the biologists require that the total amount of food the fish eat in this pond is at least 220 units. In addition, the biologists want to keep at least 20 fish in this pond. How many bass and catfish should the biologists stock in this pond to minimize the amount of waste produced?

27. A dieting company offers three types of packaged meals (I, II, and III), and it groups its customers into three groups (A, B, and C) depending on the diet. The following table gives the percent of the daily nutritional requirements that a serving of each meal provides and the number of ounces of detrimental substances in each serving of food.

	Meal I	Meal II	Meal III
Group A	20% per serving	30% per serving	10% per serving
Group B	30% per serving	10% per serving	10% per serving
Group C	10% per serving	20% per serving	20% per serving
Detrimental Substances	0.4 oz per serving	0.5 oz per serving	0.2 oz per serving

Determine the combination of food types that will provide at least 100% of the daily requirements for each group and will minimize the amount of detrimental substances.

WRITING & THINKING

28. Suppose we want to minimize $f(x_1, x_2) = 4x_1 + 6x_2$ subject to the following constraints.

$$\begin{cases} 2x_1 + 2x_2 \geq 5 \\ x_1 - 3x_2 \leq -6 \end{cases}$$

- Notice that this optimization problem is different because one of the constraints has the \leq inequality. Rewrite the second constraint so that it has the \geq inequality.
 - Use the methods of this section to solve this problem.
29. Suppose we want to minimize $f(x_1, x_2) = 4x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} -3x_1 + 6x_2 \geq 1 \\ x_1 - 2x_2 \geq 3 \\ x_1 + x_2 \geq 2 \end{cases}$$

- Try using the methods of this section to solve this problem. What do you find?
 - Since there are only two decision variables, draw a graph of the feasible region. Then use the graphical approach to solve this problem.
30. Suppose we want to minimize $f(x_1, x_2) = 6x_1 + 8x_2$ subject to the following constraints.

$$\begin{cases} 3x_1 + 4x_2 \geq 7 \\ 2x_1 + x_2 \geq 3 \end{cases}$$

Find two solutions to this problem. (**Hint:** At some point, you should obtain a tableau in which you can either choose row 1 or row 2 as the pivot row. Try each one.)