Example 9: L'Hôpital's Rule and the Indeterminte Form ∞^0

Determine $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Solution

The base has a limit of ∞ and the exponent has a limit of 0. We proceed as in the last two examples.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x} \qquad \text{Indeterminate form } \frac{\infty}{\infty}$$

Applying l'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

and therefore $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^0 = 1$.

13.5 EXERCISES

PRACTICE

Evaluate the limit using previous techniques. Then decide whether l'Hôpital's Rule is applicable and, if so, use it to check your answer.

1. $\lim_{x\to 3} \frac{2x^2-18}{x-3}$

2. $\lim_{x\to -2} \frac{x^3+8}{x+2}$

3. $\lim_{x\to\infty} \frac{6x^2-x+7}{x-3x^2}$

4. $\lim_{x\to -\infty} \frac{5x^2-2x+1}{2.5x^3-3x^2+6}$

5. $\lim_{x\to 0} \frac{2x}{\sqrt{x+3}-\sqrt{3}}$

6. $\lim_{x\to 0^+} \left(\sqrt{x}\right)^{1/x}$

7. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x\sqrt{x+1}}\right)$

Two functions are in competition to determine the indicated limit. Identify the type of the indeterminate form, and fill out the table to decide which function dominates.

8. $\lim_{x\to\infty} f(x)$, where $f(x) = \frac{\sqrt{5x^3+7}}{0.2x^2+1}$

x	1	10	100	1000	10,000	100,000
$f(x)$						

9. $\lim_{x\to\infty} g(x)$, where $g(x)=\dfrac{0.5\sqrt{x}}{\ln(x+1)}$

x	1	10	100	1000	10,000	100,000
$g(x)$						

10. $\lim_{x\to\infty} h(x)$, where $h(x)=x^{100}e^{-x}$

x	1	10	100	1000	10,000	100,000
$h(x)$						

Check whether l'Hôpital's Rule applies to the given limit. If it does, use it to determine the value of the limit. If it does not, find the limit some other way. (When necessary, apply l'Hôpital's Rule several times.)

11. $\lim_{x\to\infty}\dfrac{2x+5}{x^2-7}$

12. $\lim_{x\to\infty}\dfrac{4-2.5x}{x+3}$

13. $\lim_{x\to-\infty}\dfrac{1.5x^3-2x^2+x+9}{x^2+2.1x-4}$

14. $\lim_{x\to-\infty}\dfrac{4.5x^4+x^3-2}{3-1.5x^4}$

15. $\lim_{x\to0^+}\dfrac{\sqrt{x}}{\ln x}$

16. $\lim_{x\to\infty}\dfrac{\frac{1}{x}+2}{2x+1}$

17. $\lim_{t\to0}\dfrac{t}{\sqrt{2t+9}-3}$

18. $\lim_{x\to\infty}\dfrac{\ln x}{\ln(x^2+3x)}$

19. $\lim_{x\to0}\dfrac{\log_{10}(x^2+2x+1)}{\log_{10}(x+1)}$

20. $\lim_{x\to0}\dfrac{x}{3^{x/2}-1}$

21. $\lim_{x\to\infty}\dfrac{2^x}{x^2-3x+4}$

22. $\lim_{x\to\infty}\dfrac{x+2^x}{5^x-x}$

23. $\lim_{x\to\infty}\dfrac{4^x+x^2}{3^x-x}$

24. $\lim_{x\to\infty}\dfrac{\ln(\ln x)}{x\ln x}$

25. $\lim_{x\to\infty}\dfrac{\log_4(2x+1)}{\log_5(x-4)}$

26. $\lim_{x\to0^+}\dfrac{\log_4(x+1)}{\log_3 x}$

27. $\lim_{x\to0}\dfrac{3^x-1}{x3^x}$

Identify the indeterminate product, quotient, difference, or power, and use l'Hôpital's Rule to find the limit. If the limit is not of indeterminate form, say so and find it by other means.

28. $\lim_{x\to0^+} x\ln x$

29. $\lim_{x\to\infty}\dfrac{\sqrt{2x^2+1}}{x+3}$

30. $\lim_{x\to\infty}(\ln x)^{-1/x}$

31. $\lim_{x\to0^+}\left(\dfrac{1}{x}\right)^x$

32. $\lim_{x\to0^+}(-\ln x)^x$

33. $\lim_{x\to1^+}\left(\dfrac{1}{\ln x}-\dfrac{2}{x-1}\right)$

34. $\lim_{x\to0^+}\left(\dfrac{1}{x}+\ln x\right)$

35. $\lim_{x\to4^+}\left(\dfrac{32}{x^2-16}-\dfrac{x}{x-4}\right)$

36. $\lim_{x\to0^+} x^{(x^2)}$

37. $\lim_{x\to0^+}(2^x-x)^{1/x}$

38. $\lim_{x\to0^+}(1-x)^{1/x}$

39. $\lim_{x\to\infty}\left(\sqrt{x^2-3x}-\dfrac{3}{x^2+1}\right)$

40. $\lim_{x\to\infty}(x-1)^{1/x}$ **41.** $\lim_{x\to\infty}\frac{\ln x}{x^{7/5}}$ **42.** $\lim_{x\to\infty}\frac{x^{100}}{3^x}$

43. $\lim_{x\to\infty}\frac{\ln(100x^2+e^x)}{100x}$ **44.** $\lim_{x\to 0}(1+2x)^{1/x}$ **45.** $\lim_{x\to 1}x^{1/(1-x)}$

Find the limit. If applicable, use l'Hôpital's Rule (as many times as appropriate).

46. $\lim_{x\to\infty}\frac{2x^5+x^3-4}{e^x}$ **47.** $\lim_{x\to\infty}x^{1/x^3}$ **48.** $\lim_{x\to 0^+}x^{x^x}$

49. $\lim_{x\to 0^+}(x^x)^x$ **50.** $\lim_{x\to\infty}x^{1/x^n},\quad n\in\mathbb{Z}^+$ **51.** $\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}$

52. $\lim_{x\to\infty}\frac{x^2+1}{2^x}$ **53.** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ **54.** $\lim_{x\to\infty}\sqrt[x]{x}$

55. $\lim_{x\to\infty}\frac{2^x+5^x}{6^x}$

Convince yourself that the initial use of l'Hôpital's Rule is not helpful in finding the limit. If possible, try to find a way to make use of the theorem, or evaluate the limit in some other way.

56. $\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x}}$ **57.** $\lim_{x\to\infty}\frac{\sqrt[3]{x+1}-2}{\sqrt{x^2+2}}$ **58.** $\lim_{x\to\infty}\frac{2^x+3^x}{5^x}$

59. $\lim_{x\to\infty}\frac{5^x-6^x}{7^x+8^x}$ **60.** $\lim_{x\to\infty}\frac{2^{-x}}{x^{-1}}$ **61.** $\lim_{x\to\infty}\left(\frac{1}{x+1}\right)^{-x^3}$

62. $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{e^{-x}}$ **63.** $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

64. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$ **65.** $\lim_{x\to\infty}2^{-x}x\ln x$

WRITING & THINKING

Find the error(s) in the limit calculation.

66. $\lim_{x\to 2}\frac{x^2-2}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4\quad$ (Incorrect!)

67. $\lim_{x\to -\infty}\frac{5^x+1}{5^x}=\lim_{x\to -\infty}\frac{(\ln 5)5^x}{(\ln 5)5^x}=1\quad$ (Incorrect!)

Use l'Hôpital's Rule to prove the assertion.

68. $\lim_{x\to\infty}\frac{p(x)}{e^{kx}}=0\quad (p(x)$ is a polynomial, $k>0)$

69. $\lim_{x\to\infty}\frac{(\ln x)^n}{x^k}=0\quad (n\in\mathbb{N},\ k>0)$ **70.** $\lim_{x\to\infty}\frac{a^x}{x^n}=\infty\quad (a>1,\ n\in\mathbb{N})$

Find the value(s) of c satisfying the conclusion of Cauchy's Mean Value Theorem. If the theorem doesn't apply, explain why.

71. $f(x)=x, \quad g(x)=x^2+1; \quad [0,1]$

72. $f(x)=x^3-1, \quad g(x)=x^2+2x; \quad [-1,1]$

73. $f(x)=x^3-x, \quad g(x)=-x^2+2x+3; \quad [-1,3]$

74. $f(x)=x^3, \quad g(x)=-x^2; \quad [-2,3]$

75. $f(x)=x^2+3x, \quad g(x)=3x^2-5x+3; \quad [-1,3]$

76. $f(x)=\frac{1}{x}, \quad g(x)=\ln x; \quad [1,2]$

77. $f(x)=x^2-5x-9, \quad g(x)=x^3+x+10; \quad [-3,2]$

78. Recall the compound interest formula for the value of an investment of P dollars after t years, compounded n times a year at an annual interest rate of r:

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

Use l'Hôpital's Rule to prove that if we let $n\to\infty$, we obtain the continuous compounding formula $A=Pe^{rt}$.

79. The strength of an electric field due to a disk charge is obtained from the formula

$$E(x)=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

where σ is the electric charge per unit area (in $\mathrm{C/m^2}$), $\varepsilon_0=8.85\cdot10^{-12}\ \mathrm{C^2/Nm^2}$, R is the radius of the ring, and x is the distance to the charge in meters. Use l'Hôpital's Rule to confirm that $E(x)\to 0$ as $x\to\infty$. How is E affected by σ and R at a given distance? What happens to the rate of change of E as x increases? (**Hint:** Apply l'Hôpital's Rule to dE/dx as $x\to\infty$.)

80. Marquis de l'Hôpital first illustrated the rule named after him in his 1696 textbook, *Analyse des Infiniment Petits*. He used an example where the objective was to find

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

for $a>0$. Determine the above limit.

TECHNOLOGY

Use a graphing utility to graph the function for different values of the parameter c. Examine how the values of the parameter affect the indicated limit.

81. $\lim_{x\to\infty}\left(1+\frac{1}{cx}\right)^x$; What happens to the limit when $|c|\to\infty$?

82. $\lim_{x\to0^+}\frac{1-c^x}{cx}$; What happens to the limit when $c\to\infty$?