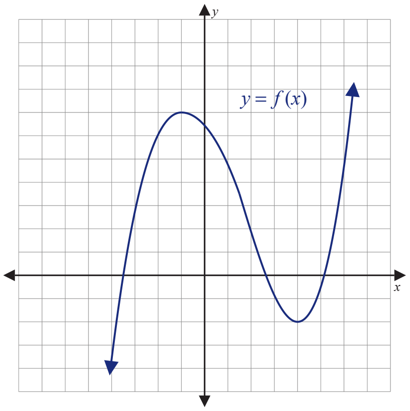
Example 5: Graphing the Derivative

Consider the given graph of a function.

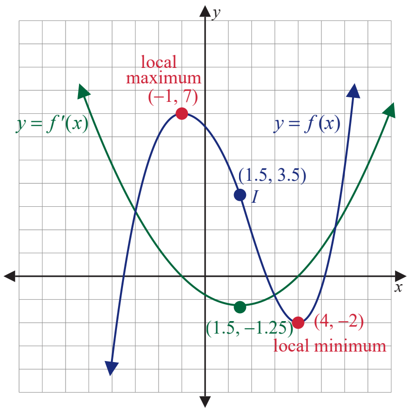


a. Identify the local extrema and locate the point(s) of inflection.

b. Determine the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing, and identify the intervals on which $f(x)$ is concave upward and concave downward.

c. Sketch on the same coordinate plane a possible graph of $f'(x)$.

Solution



a. A local maximum is located at $(-1, 7)$, and a local minimum is located at $(4, -2)$. There is a point of inflection at about $(1.5, 3.5)$.

b. The function is increasing on the intervals $(-\infty, -1)$ and $(4, +\infty)$. It is decreasing on the interval $(-1, 4)$. The function is concave downward on the interval $(-\infty, 1.5)$ and concave upward on $(1.5, +\infty)$.

c. The slope of $f(x)$ is positive but decreasing from $-\infty < x < -1$, and is 0 at $x = -1$. The slope then becomes negative and continues decreasing until $x = 1.5$ at which point slope is a minimum. Then the slope starts to increase from some negative value (which we estimate to be -1.25) to 0 (at $x = 4$). It then becomes positive and continues increasing.

12.3 EXERCISES

PRACTICE

In Exercises 1–16, sketch the graph of a continuous function that satisfies all the given conditions.

1. **a.** $f(-1) = 2$
b. $f'(-1) = 0$
c. $f'(x) < 0$ if $x < -1$
d. $f'(x) > 0$ if $x > -1$
e. $f''(x) > 0$ for all x

2. **a.** $f(3) = 4$
b. $f'(3) = 0$
c. $f'(x) < 0$ if $x > 3$
d. $f'(x) > 0$ if $x < 3$
e. $f''(x) < 0$ for all x

3. **a.** $f(-2) = 4, f(-1) = 1, f(1) = -1$
b. $f'(-2) = 0, f'(1) = 0$
c. $f'(x) < 0$ if $-2 < x < 1$
d. $f'(x) > 0$ if $x < -2$ or $x > 1$
e. $f''(-1) = 0$
f. $f''(x) < 0$ if $x < -1$
g. $f''(x) > 0$ if $x > -1$

4. **a.** $f(0) = -2, f(2) = 0, f(3) = 3$
b. $f'(0) = 0, f'(3) = 0$
c. $f'(x) < 0$ if $x < 0$ or $x > 3$
d. $f'(x) > 0$ if $0 < x < 3$
e. $f''(2) = 0$
f. $f''(x) < 0$ if $x > 2$
g. $f''(x) > 0$ if $x < 2$

5. **a.** $f(-3)=5, f(-1)=2, f(0)=-1$
b. $f'(-3)=0, f'(0)=0$
c. $f'(x)<0$ if $x<0$ and $x\neq -3$
d. $f'(x)>0$ if $x>0$
e. $f''(-3)=0, f''(-1)=0$
f. $f''(x)<0$ if $-3<x<-1$
g. $f''(x)>0$ if $x<-3$ or $x>-1$

6. **a.** $f(1)=2, f(2)=3, f(4)=4, f(6)=2$
b. $f'(1)=0, f'(4)=0$
c. $f'(x)<0$ if $x>4, x<1$
d. $f'(x)>0$ if $1<x<4$

7. **a.** $f(x)=ax^3+bx+c$
b. $f(0)=0$
c. $f(1)=15$
d. $f'(-1)=0$ and $x=-1$ is a local max
e. $f''(x)>0$ if $x<10$

8. **a.** $f(10)=5$
b. $f'(5)=0$
c. $f''(5)=10$
d. $f''(x)<0$ if $x>10$

9. **a.** $f''(x)>0$ if $x<5$
b. $f''(5)=0$
c. $f''(x)<0$ if $x>5$
d. $f'(x)>0$ for all x

10. **a.** $f(4)=8$
b. $f'(4)=0$
c. $f''(4)=8$

11. **a.** $f(-5)=4$
b. $f'(-5)=0$
c. $f''(-5)=-2$

12. **a.** $f(x)=ax^2+bx+c$
b. $f'(-3)=0$
c. $f''(-3)=2$

13. **a.** $f(x)=ax^3+bx^2+cx+d$
b. $f(0)=25$
c. $f'(4)=0, f'(-4)=0$
d. $f''(4)=48, f''(-4)=-48$

14. **a.** $f(x)=ax^2+bx+c$
b. $f(0)=79$
c. $f'(5)=0$
d. $f''(x)=6$

15. **a.** $f(x)=ax^3+bx^2+cx+d$
b. $f(0)=2$
c. $f'(0)=5$
d. $f''(0)=4$
e. $f''(1)=12$

16. **a.** $f(x)=ax^3+bx$
b. $f'(0)=-12$
c. $f'(2)=0$

For each of the functions in Exercises 17–36, determine $f'(x)$ and $f''(x)$. Then complete a summary table like those in Examples 3 and 4. Use this table to sketch the graph of the function. (If available, use a graphing utility or calculator to obtain a suitable window and confirm the accuracy of your calculations.)

17. $f(x)=x^2-4x+7$

18. $f(x)=x^2+6x-8$

19. $f(x)=6+5x-x^2$

20. $f(x)=2+3x-2x^2$

21. $f(x)=x^3+3x^2-6$

22. $f(x)=\frac{1}{3}x^3-4x+3$

23. $f(x)=\frac{1}{3}x^3+x^2-3x+5$

24. $f(x)=2x^3-3x^2-12x+5$

25. $f(x)=x^4-2x^2+4$

26. $f(x)=x^4-8x^2-3$

27. $f(x)=\frac{1}{4}x^4-x^3+5$

28. $f(x)=x^4+4x^3+12$

29. $f(x)=x^4-4x+7$

30. $f(x)=3x^4-4x^3+3$

31. $f(x)=(x+5)(x-3)^2$

32. $f(x)=(x+1)^2(x-10)^2$

33. $f(x)=(2x+1)(x-8)^3$

34. $f(x)=(x-5)(x-10)(x+3)$

35. $f(x)=2x(5x+8)^3$

36. $f(x)=16x(21+x)^3$

37. Suppose that $f(x)=mx^2+6x+4$. Determine a value of m so that $f(x)$ has a minimum at $x=-1$.

38. Given $y=4x^2+nx+8$, determine a value for n so that y has a minimum at $x=2$.

39. Determine a value for m such that at $x=1$ the tangent to the function $f(x)=mx^2+6x+1$ has an equation of $y=12x-2$.

40. Determine a value for m so that $y=4x^3+mx^2$ has an inflection point at $x=-10$.

APPLICATIONS

41. In an action movie, the hero is seen fighting the villain inside a plane, which has a large hole in its side. The hero (actually a movie stunt man) is then thrown from the plane. He falls quickly and soon reaches a constant velocity. The hero opens his parachute but it deploys slowly, as if he is having trouble, but finally, in triumph, all is well and he drifts steadily and slowly to the ground.

a. Draw a graph of the hero's vertical distance to the ground, represented by y, versus time t (in seconds).

b. Describe any interesting points on the graph with points in the movie narrative.

42. The effectiveness of a certain medical injection is modeled by $E(t)=0.01t(100-t)$, where t is time in minutes and E is a measure of concentration in the bloodstream. Effectiveness readings above 9.0 are satisfactory and readings above 30 are dangerous.

a. If an injection is given at midnight, when are the readings satisfactory? When do they become too low?

b. How high do the effectiveness readings get?

c. The supervising nurse and the resident pharmacologist must assign a schedule for injections. For the next week, assuming injected dosages are additive, give a reasonable schedule for injections so that the patient's E-reading stays at or above 9 but never exceeds 30.

43. The productivity rating of an individual worker at the Cruz Corporation assembly line is based on the number of tasks accomplished, mistakes made, and responsiveness to difficulties encountered. The average of all scores allows the company to use a simple model based on time on the floor given by $\text{PR}=-0.4x^3+2x^2+10x+5$, where x is in hours at work. A PR score of 20 is acceptable and a score of 40 is highly unusual.

a. When are workers' scores the highest?

b. Design an 8-hour day where workers do the most demanding jobs for about 6 hours and have 2 hours for less stressful work. Explain your reasoning.

WRITING & THINKING

44. Is it possible for a polynomial $y = ax^2 + bx + c$ to have an inflection point?