Example 5: Maximizing Profits

A company finds that its profit in dollars for producing x units of a product in one week is given by $P(x) = -2x^2 + 1600x$. If the company is set up so that no more than 500 units can be manufactured in any one week, how many units should the company produce to maximize profit?

Solution

The production restrictions indicate that x is in the closed interval [0, 500]. Find the critical values by setting $P'(x) = 0$ and solving for x (Step 1).

$$\begin{aligned} P'(x) &= -4x + 1600 \\ 0 &= -4x + 1600 \\ 4x &= 1600 \\ x &= 400 \end{aligned}$$

Note that P' is defined for all x in [0, 500].

Now evaluate $P(x)$ for $x = 0$, $x = 400$, and $x = 500$ (Step 2).

$x = 0$	$P(0) = -2(0)^2 + 1600 \cdot 0 = 0$	
$x = 400$	$P(400) = -2(400)^2 + 1600 \cdot 400 = 320{,}000$	Absolute max
$x = 500$	$P(500) = -2(500)^2 + 1600 \cdot 500 = 300{,}000$	

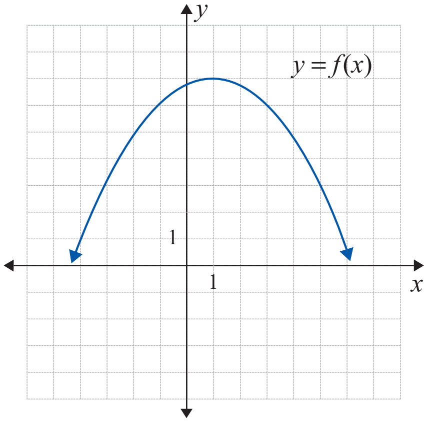
The profit is maximized at \$320,000 when 400 units are produced (Step 3).

11.6 EXERCISES

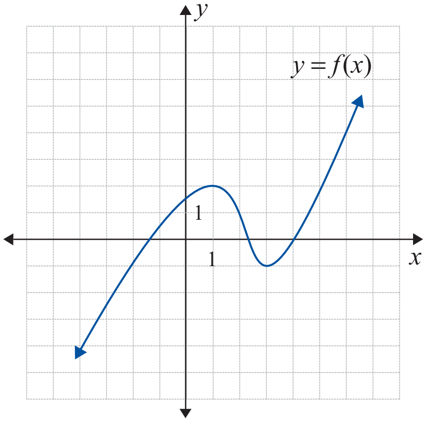
PRACTICE

In Exercises 1–8, find the absolute extrema for each graph of $f(x)$ on the given interval.

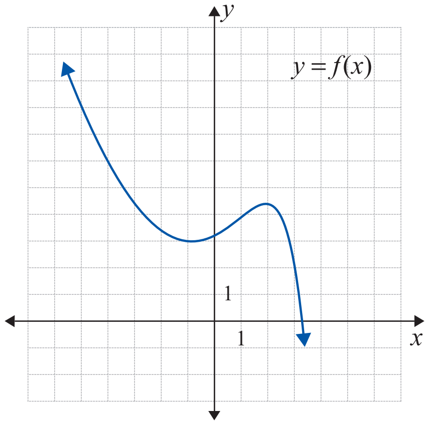
1. [3, 5]



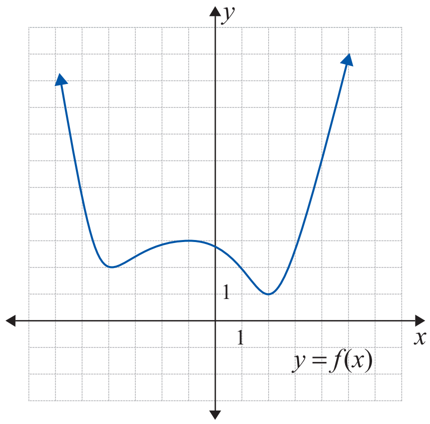
2. [−1, 4]



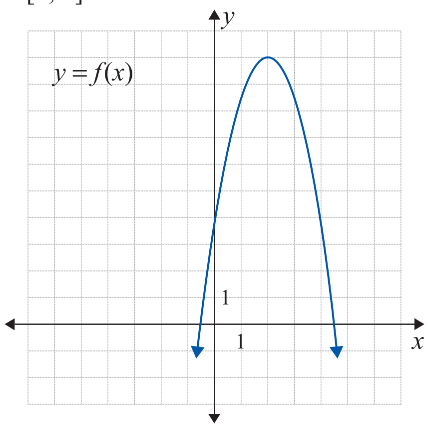
3. $[-5, 3]$



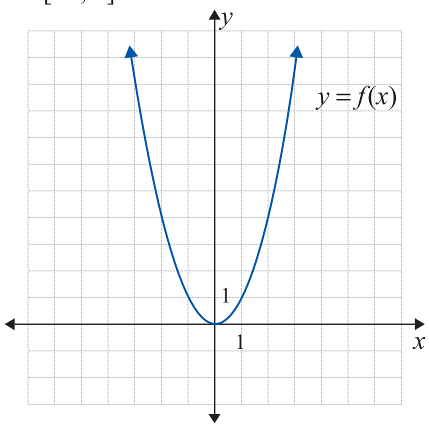
4. $[-4, 4]$



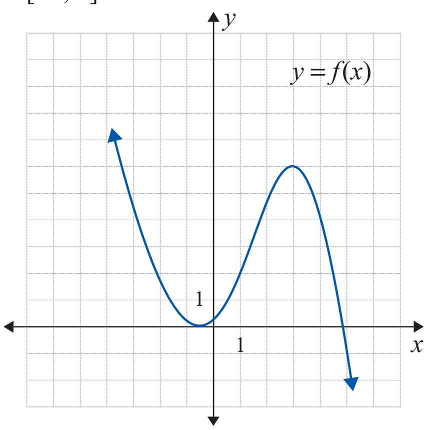
5. $[1, 4]$



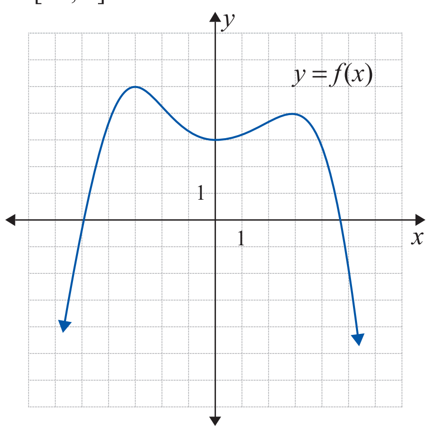
6. $[-2, 2]$



7. $[-3, 5]$



8. $[-4, 5]$



In Exercises 9–38, find the absolute extrema for each function on the given interval.

9. $f(x) = x^2 - 8x;\ [0, 5]$

10. $f(x) = 3x^2 - 12x;\ [0, 4]$

11. $f(x) = 6 + 10x - x^2;\ [3, 6]$

12. $f(x) = 11 - 4x - x^2;\ [-3, 0]$

13. $f(x) = 14 - 3x;\ [0, 4]$

14. $f(x) = 7 + \frac{1}{2}x;\ [-2, 4]$

15. $f(x) = 8 - x^3;\ [-1, 3]$

16. $f(x) = x^3 + 4;\ [-2, 2]$

17. $f(x)=x^3-12x;\ [-3, 4]$

18. $f(x)=x^3-3x;\ [-2, 3]$

19. $f(x)=2x^3-x^2;\ [0, 2]$

20. $f(x)=x^3+2x^2;\ [0, 3]$

21. $f(x)=9x-3x^2-x^3;\ [-4, 2]$

22. $f(x)=x^3-3x^2-24x;\ [-3, 3]$

23. $f(x)=2x^3-3x^2-12x-10;\ [0, 4]$

24. $f(x)=x^3+3x^2-24x;\ [0, 3]$

25. $f(x)=x^{\frac{2}{3}}-4;\ [-1, 8]$

26. $f(x)=3x^{\frac{2}{3}}+2;\ [-1, 4]$

27. $f(x)=3x^{\frac{1}{3}}-4x;\ [0, 1]$

28. $f(x)=3x^{\frac{2}{3}}+x;\ [-9, 1]$

29. $f(x)=\sqrt{x^2+4};\ [-1, 2]$

30. $f(x)=\sqrt{9-x^2};\ [-1, 2]$

31. $f(x)=\sqrt[3]{x^2-1};\ [-2, 2]$

32. $f(x)=(x^2-1)^{\frac{2}{3}};\ [-2, 2]$

33. $f(x)=x+\frac{4}{x};\ \left[\frac{1}{2}, 3\right]$

34. $f(x)=2x+\frac{18}{x};\ [1, 4]$

35. $f(x)=4x+\frac{9}{x};\ [1, 3]$

36. $f(x)=9x+\frac{16}{x};\ [1, 2]$

37. $f(x)=x^2+\frac{16}{x};\ \left[\frac{1}{2}, 4\right]$

38. $f(x)=x^2+\frac{2}{x};\ [1, 4]$

APPLICATIONS

39. Revenue: The weekly revenue from the sale of x units of a product is given by $R(x)=24x-0.5x^2$ dollars. If the company is set up so that it can produce no more than 40 units per week, how many units should the company produce to maximize revenue?

40. Revenue: The revenue from the sale of x units of a product is given by $R(x)=12x-0.04x^2$ thousand dollars, where $0 \le x \le 250$. How many units should be sold to maximize the revenue?

41. Profit: A manufacturer of telescopes has determined that the revenue from the production and sale of x telescopes is $R(x)=140x-0.5x^2$ dollars. The cost function is given by $C(x)=x^2+20x+1050$ dollars. Find the level of production and sales that will maximize the profit if $0 \le x \le 70$.

42. Profit: A marketing analyst for a company that produces skateboards has determined that if the company sells x units of the deluxe model, the revenue function is $R(x)=79.9x-0.03x^2$ dollars and the cost function is $C(x)=0.08x^2+5.1x+5800$ dollars. Find the sales level that will yield maximum profit if $0 \le x \le 380$.

43. Average cost: The cost of producing x compact refrigerators is given by $C(x)=2880+35x+0.2x^2$ dollars. Find the value of x that minimizes the average cost function if $0 \le x \le 150$.

44. Average cost: The cost of producing x electronic games is given by $C(x)=1080+42x+0.3x^2$ dollars. Find the value of x that minimizes the average cost function if $0 \le x \le 90$.

45. **Air quality:** The Air Quality Management District monitors the level of pollution in the air. On a good day, the level is approximately $P(t)=35+\dfrac{126t}{0.5t^2+18}$ PSI (Pollution Standard Index), where t is the number of hours after 7:00 a.m. and $0\le t\le 11$. At what time will the pollution level be a maximum?

46. **Air quality:** On a moderately smoggy day, the level of nitrogen dioxide in the air is approximately $N(t)=0.126+\dfrac{0.36t}{2t^2+40.5}$ ppm (parts per million), where t is the number of hours after 8:00 a.m. and $0\le t\le 10$. At what time will the level of nitrogen dioxide reach its maximum?

47. **Bacteria:** It is estimated that t hours after a particular bacterium is introduced into a culture, the population of bacteria in the culture will be $P(t)=\dfrac{4800}{\sqrt{12-0.5t}}$, where $0\le t\le 6$. What will be the absolute maximum population? At what time t will this occur?

48. **Population:** It is estimated that t years from now the population of a small community will be $P(t)=\dfrac{5000}{\sqrt{25+0.4t}}$ people, where $0\le t\le 10$. What will be the maximum population?

49. **Altitude:** The altitude of an airplane following a certain flight path is given by the function $F(t)=\dfrac{13t}{20}-\dfrac{3t^2}{200}$, where t is in minutes, and $F(t)$ is in thousands of feet. Find the absolute maximum of the function over the interval [0, 43].

50. **Velocity of a car:** The velocity of a car in miles per hour varies according to the function $F(x)=\dfrac{x^2}{100}-\dfrac{19x}{25}+\dfrac{5349}{100}$, where x denotes time in seconds. Find the absolute maximum and minimum velocities over the interval [0, 100]. Find the car's velocity at each endpoint.

51. **Tire distortion:** When a car accelerates, its tires are distorted by the force exerted on them by the motor. For a certain model car, the distortion after the driver floors the accelerator is modeled by the function $F(t)=0.00026t^3-0.0533t^2+2.74014t$, where $0<t<100$ is the number of milliseconds after acceleration begins, and $F(t)$ is a percentage of the tire's maximum flexibility (at $F(t)=100$, the tire tears into pieces). Find the time when the maximum distortion occurs, and find the percentage that the tires are distorted. Do the tires survive the acceleration?

52. **Pollution:** The amount of pollution (measured in parts per million) in a small river is given by the function $F(t)=\dfrac{450t^3}{169}-\dfrac{18{,}225t^2}{169}+\dfrac{23{,}625t}{23}+\dfrac{260{,}675}{169}$, where t is the number of years after 1980. Sometime after 1980, an environmental protection law was enacted to reduce the amount of pollution in the river, and in the same year the pollution levels began to decrease. Find the maximum pollution for $0\le t\le 20$, and determine the year the law was enacted. (The largest integer less than the t-value specifies the year of enactment.)

53. **Wind resistance:** An aeronautics company is testing a new airframe. The company wishes to determine an ideal cruising speed, and one step in the process is to find the speed at which the airframe experiences the least wind resistance (other than when it is not moving). After performing a number of tests, the engineers determine that the wind resistance can be modeled by a function $G(v)=\frac{v^3}{3,645,000}-\frac{8v^2}{6075}+\frac{350v}{243}+\frac{614,500}{729}$, where $300 \le v \le 4000$ is the wind velocity in feet per second and $G(v)$ is in newtons. Determine the ideal cruising speed for the airframe based on this model.

54. **Bacteria:** A bacteriologist doing research on antibiotics has discovered that a certain type of disease-causing bacteria can be effectively treated using a cocktail of two different antibiotics administered at specific intervals. The population changes in response to the antibiotics according to the function $F(t)=-0.0224t^3+4.5676t^2-252.4610t+5000$, where t is in the interval [0, 200]. The maximum values of the function correspond to the times when the antibiotics were administered. At what times, t, are the antibiotics administered and what is the population of bacteria at those times?

55. **Gravitational pull:** A rocket traveling to the moon is affected by the gravity of Earth and of the moon. The total gravitational force exerted on the rocket is approximated by the function $F(h)=\frac{43,750h^2}{3}-\frac{70,000h}{3}+10,000$, where $0<h<1$ is the height of the object, given in percentage of the distance between Earth and the moon, and $F(h)$ is measured in newtons. Find the height at which the minimum occurs and the gravitational force at that altitude. Additionally, there is an onboard experiment which can only be performed if the gravitational force falls below 1000 newtons. Can this experiment be performed?

56. **Thrown object:** A cell phone is thrown into the air. The position of the phone is given by the function $F(t)=-\frac{49t^2}{10}+\frac{297t}{10}$, where $0<t<10$ is the number of seconds after the phone is thrown, and $F(t)$ is measured in feet. Find the maximum height the phone attains. Also find the time when the phone hits the ground. The phone will shatter if it hits the ground with a speed greater than 32 fps (feet per second). Does the phone shatter?

57. **Acceleration:** Due to many variable factors in car engines, acceleration is never constant. The acceleration of a particular car in a particular test is approximated by the function $F(t)=-\frac{260t^3}{137}+\frac{2418t^2}{137}-\frac{7371t}{137}+75$, where $0\le t\le 5$ is the number of seconds after acceleration begins. What is the maximum acceleration of the car during this test? When does it occur? What is the minimum acceleration, and when does it occur?

58. **Power consumption:** A certain piece of equipment in a chemistry lab draws power according to the function

$$G(m)=\left(-6.087\times10^{-6}\right)m^3+\left(8.641\times10^{-3}\right)m^2-1.751m+1000,$$

where $0<m<1000$ is the mass of the sample to be analyzed in grams and G is power consumption in watts. Find the sample mass which causes the equipment to draw the most power. How much power does the equipment draw for a sample of this mass?

59. **Chemistry:** A certain chemical procedure requires the addition of reactants and catalysts at precise times to maintain reaction rates. The rate of reaction is given by the function

$$H(t)=\left(9.423\times10^{-7}\right)t^3-\left(1.230\times10^{-4}\right)t^2+\left(4.384\times10^{-3}\right)t+0.001,$$

where $0<t<100$ is the number of seconds after beginning the reaction, and $H(t)$ is measured in number of moles formed per second. Find the time at which a relative maximum reaction rate is reached and the number of moles per second being formed at that time. At what time was the second reactant/catalyst mixture added? (The reactant/catalyst mixture is added at a relative minimum.) Round to two decimal places.

60. **Phone traffic:** Phone traffic varies greatly over the course of a day. A phone provider estimates that the number of international calls active per minute on a certain holiday is given by the function $P(t)=-0.001t^3+t^2+50t+150{,}000$, where $0<t<1000$ is the number of minutes after 6.00 a.m. and $P(t)$ is the number of active international calls. For what t does maximum phone traffic occur? How many calls are active at this time?

61. **Stored energy:** While a rubber ball is moving, it has kinetic energy. When the ball impacts a hard surface, the kinetic energy is converted to potential energy, and then back to kinetic energy. This happens quite rapidly. The amount of potential energy after impact is approximated by the function $E(t)=-\frac{3t^2}{80}+\frac{3t}{2}$, where t is the number of nanoseconds after impact and t is in [0, 40]. When does the ball have maximum potential energy? How much potential energy does it have at its maximum?

62. **Photosynthesis:** Plants absorb different amounts of light, depending on the wavelength of the light. A study of Rhododendron bushes show that they absorb light according to the formula $L(w)=\frac{-2.22w^3}{10^{10}}+\frac{4.44w^2}{10^6}-\frac{2w}{10^3}+0.444$, where $100\le w\le 650$ is the wavelength of light in nanometers, and $L(w)$ is the proportion of the light shone on the bush. What wavelength does Rhododendron absorb the best? What proportion of light of this wavelength does Rhododendron absorb?

63. **Package delivery:** A study says that the package flow in the Southeast USA during the month of August follows the function $D(t)=\frac{7t^3}{9300}-\frac{7t^2}{248}+\frac{7t}{31}+1$, where $1\le t\le 31$ is the day of the month, and $D(t)$ is given in millions of packages. On which day are the most packages delivered? How many packages are delivered on this day?