

Example 7: Properties of Limits

Find $\lim_{x \rightarrow -\infty} \left(\frac{x^3 + x^2 - x + 1}{x^2 - 4} \right)$, if it exists.

Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left(\frac{x^3 + x^2 - x + 1}{x^2 - 4} \right) &= \lim_{x \rightarrow -\infty} \left(\frac{\frac{x^3}{x^3} + \frac{x^2}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}}{\frac{x^2}{x^3} - \frac{4}{x^3}} \right) \\ &= \lim_{x \rightarrow -\infty} \left(\frac{1 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}}{\frac{1}{x} - \frac{4}{x^3}} \right) \\ &= \frac{1}{0} \end{aligned}$$

Divide each term by the highest power of x present, which in this function is x^3 .

Simplify.

Since $\frac{1}{0}$ is undefined, the limit is either $+\infty$ or $-\infty$. (See Example 3.)

Investigating the expression shows that the highest power is x^3 . This term will dominate for very large values of x and will be negative for negative values of x . Thus

$$\lim_{x \rightarrow -\infty} \left(\frac{x^3 + x^2 - x + 1}{x^2 - 4} \right) = -\infty.$$

Summary of Limits for Rational Functions as $x \rightarrow +\infty$ (or $x \rightarrow -\infty$)

Consider the function

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

Case 1: For $m = n$, $\lim_{x \rightarrow +\infty} f(x) = \frac{a_n}{b_m}$.

Case 2: For $m > n$, $\lim_{x \rightarrow +\infty} f(x) = 0$.

Case 3: For $m < n$, $\lim_{x \rightarrow +\infty} f(x) = +\infty$. (Or $-\infty$ depending on the signs of a_n and b_m .)

10.3 EXERCISES

 PRACTICE

In Exercises 1–28, find the indicated limit, if it exists.

1. $\lim_{x \rightarrow -2} 6$

2. $\lim_{x \rightarrow 4} 2x$

3. $\lim_{x \rightarrow 3^-} (x^2 + 1)$

4. $\lim_{x \rightarrow -3^+} (5 - 2x^2)$

5. $\lim_{x \rightarrow 1^+} \left(\frac{x+2}{x-1} \right)$

6. $\lim_{x \rightarrow 1^-} \left(\frac{x+2}{x-1} \right)$

7. $\lim_{x \rightarrow \frac{1}{3}} \left(\frac{3x+1}{x+2} \right)$

8. $\lim_{x \rightarrow 0^+} \left(\frac{x+4}{x-4} \right)$

9. $\lim_{x \rightarrow 0^-} \left(\frac{x}{x^2 + 2x} \right)$

10. $\lim_{x \rightarrow 0^-} \left(\frac{2x^2 + x}{x} \right)$

11. $\lim_{x \rightarrow +\infty} \left(\frac{x}{x^2 + 3} \right)$

12. $\lim_{x \rightarrow +\infty} \left(\frac{2x^2 + 7}{3x^2 - 2} \right)$

13. $\lim_{x \rightarrow -\infty} \left(\frac{x^3 + 64}{x^2 - 2x + 1} \right)$

14. $\lim_{x \rightarrow -\infty} \left(\frac{4x - x^3}{x^2 + 2x - 7} \right)$

15. $\lim_{x \rightarrow 2} \left(\frac{x^2 - x - 2}{x^2 - 4} \right)$

16. $\lim_{x \rightarrow -3} \left(\frac{x^2 - 9}{x^2 + 2x - 3} \right)$

17. $\lim_{x \rightarrow 0^+} \left(4 - \frac{3}{x} \right)$

18. $\lim_{x \rightarrow 1^-} \left(2x + \frac{5}{x-1} \right)$

19. $\lim_{x \rightarrow +\infty} \left(8 + \frac{1}{x} \right)$

20. $\lim_{x \rightarrow -\infty} \left(11 - \frac{2}{x^2} \right)$

21. $\lim_{x \rightarrow 4} \sqrt{x+5}$

22. $\lim_{x \rightarrow 2} \sqrt{3x+10}$

23. $\lim_{x \rightarrow 1} (\sqrt{x} - 3)$

24. $\lim_{x \rightarrow 4} (\sqrt{x} + 6)$

25. a. $\lim_{x \rightarrow 4} \left(\frac{\sqrt{x} - 2}{x - 4} \right)$ [Hint: $x - 4 = (\sqrt{x} + 2)(\sqrt{x} - 2)$]

b. $\lim_{x \rightarrow 9} \left(\frac{x - 9}{\sqrt{x} - 3} \right)$ [Hint: $x - 9 = (\sqrt{x} + 3)(\sqrt{x} - 3)$]

26. a. $\lim_{h \rightarrow 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$ [Hint: Multiply by $\frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}}$, simplify the numerator, and calculate the limit.]

b. $\lim_{h \rightarrow 0} \frac{\sqrt{5x+h} - \sqrt{5x}}{h}$

27. $\lim_{x \rightarrow +\infty} \left(\frac{x^2 + 3x - 4}{x^2 + 5x - 9} \right)$

28. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x + 1}{2x^3 + 3x^2 + x - 2} \right)$

29. For the function $f(x) = \begin{cases} -3 & \text{if } x \leq 1 \\ x - 4 & \text{if } x > 1 \end{cases}$, find the following:

a. $\lim_{x \rightarrow 1^-} f(x)$ b. $\lim_{x \rightarrow 1^+} f(x)$ c. $\lim_{x \rightarrow 1} f(x)$ d. $\lim_{x \rightarrow 2} f(x)$

30. For the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$, find the following:

a. $\lim_{x \rightarrow 0^-} f(x)$ b. $\lim_{x \rightarrow 0^+} f(x)$ c. $\lim_{x \rightarrow 0} f(x)$ d. $\lim_{x \rightarrow -2} f(x)$

31. For the function $f(x) = \begin{cases} 5-x & \text{if } x \leq 2 \\ x^2-1 & \text{if } x > 2 \end{cases}$, find the following:

a. $\lim_{x \rightarrow 2^-} f(x)$ b. $\lim_{x \rightarrow 2^+} f(x)$ c. $\lim_{x \rightarrow 2} f(x)$ d. $\lim_{x \rightarrow 0} f(x)$

32. For the function $f(x) = \begin{cases} x^3+4 & \text{if } x \leq -2 \\ \sqrt{x^2+5} & \text{if } x > -2 \end{cases}$, find the following:

a. $\lim_{x \rightarrow -2^-} f(x)$ b. $\lim_{x \rightarrow -2^+} f(x)$ c. $\lim_{x \rightarrow -2} f(x)$ d. $\lim_{x \rightarrow -1} f(x)$

APPLICATIONS

33. **Utility costs:** The Municipal Gas Company uses the following function for computing their customers' monthly gas bills:

$$C(x) = \begin{cases} 0.37x + 3.00 & \text{if } 0 < x \leq 24 \\ 0.78x - 6.84 & \text{if } x > 24 \end{cases},$$

where x is the number of therms (thermal units) used and $C(x)$ is the cost in dollars.

a. Find $\lim_{x \rightarrow 24^-} C(x)$. b. Find $\lim_{x \rightarrow 24^+} C(x)$. c. Find $\lim_{x \rightarrow 24} C(x)$.

34. **Income tax:** A federal income tax schedule can be given by the function

$$T(x) = \begin{cases} 0.15x & \text{if } 0 < x \leq 23,900 \\ 0.28x - 3107 & \text{if } 23,900 < x \leq 61,650 \\ 0.33x - 6189.50 & \text{if } 61,650 < x \leq 123,790 \end{cases},$$

where x is the taxable income in dollars, $0 < x \leq 123,790$, and $T(x)$ is in dollars.

a. Find $\lim_{x \rightarrow 23,900^-} T(x)$. b. Find $\lim_{x \rightarrow 23,900^+} T(x)$.

c. Find $\lim_{x \rightarrow 23,900} T(x)$. d. Find $\lim_{x \rightarrow 61,650} T(x)$.

35. **Average cost:** A manufacturer of golf clubs estimates that if x sets of golf clubs are produced, then the average cost of producing each set is $A(x) = 73 + \frac{5780}{x}$ dollars. What will be the average cost of producing each set in the long run $\left(\lim_{x \rightarrow +\infty} A(x)\right)$?

36. **Dictation rate:** It has been determined that after t weeks of class, a certain student in an intermediate shorthand class can take dictation at a rate of $W(t) = 60 + \frac{70t^2}{t^2+15}$ words per minute. What will be this student's rate of taking dictation in the long run $\left(\lim_{t \rightarrow +\infty} W(t)\right)$?