This chapter uses the Cartesian coordinate system to develop relationships between algebra and geometry. These relationships have proven extremely important to the development of mathematics and mathematical problem-solving. In particular, you will develop graphing skills for linear equations and linear inequalities in two variables. The graphs of linear equations are straight lines, hence the term linear. The graphs of linear inequalities are half-planes separated by straight lines.

Three useful forms of linear equations are the **standard form**, the **slope-intercept form**, and the **point-slope form**. Each form is equally important and useful depending on the information given and the application of the equation. Your algebraic skills should allow you to change from one form to another and to recognize when the same equation is in a different form.

The concept of slope underlies all our work with straight lines. Horizontal lines (with slope 0) and vertical lines (with undefined slope) can be related to slope. Relationships between two or more lines can be discussed in terms of their slopes. Parallel lines have the same slope, and perpendicular lines have slopes that are negative reciprocals of each other.

Slope occurs as a part of our daily lives in many ways. The slope of a roof is particularly important in areas with lots of snow. Large trucks must be very careful on mountain roads with steep slopes. Airplanes must deal with their rate of descent or the slope of their paths in landing and taking off. We will see that slope can be related to the rate of change in prices of products that we purchase.

Cartesian Coordinate System and Linear Equations: $Ax + By = C$

2.1

- *Graph and label ordered pairs of real numbers as points on a plane.*
- *Find ordered pairs that satisfy a given linear equation.*
- *Recognize the **standard form** of a linear equation*: $Ax + By = C$.
- *Graph linear equations.*
- *Locate the **x-intercept** and the **y-intercept** of a linear equation.*

The linear (or first-degree) equations and inequalities we have discussed so far in this text have involved only one variable. In this chapter, we will discuss linear equations and inequalities that contain two variables. The context of the material should indicate clearly which type of linear equation is being discussed. As we will see in Section 2.4, except for special cases, linear equations in two variables are also called **linear functions**.

Cartesian Coordinate System

We begin with the concept of **ordered pairs** of real numbers. We write these in the form (x, y) and show how they are related to points on a graph and solutions of equations in two variables. For example, consider the equation $y = 3x + 1$ and the ordered pair $(2, 7)$. Substituting $x = 2$ in the equation gives $y = 3 \cdot 2 + 1 = 7$. We say that $(2, 7)$ is **a solution of** (or **satisfies**) the equation. The first number, 2, is called the **first coordinate** (or **x-coordinate**), and the second number, 7, is called the **second coordinate** (or **y-coordinate**). The ordered pair $(2, 7)$ is not the same as the ordered pair $(7, 2)$ because the order of the numbers is not the same. Note that $(7, 2)$ does not satisfy the equation because $3 \cdot 7 + 1 = 22 \neq 2$.

René Descartes

The **Cartesian coordinate system** (named after seventeenth century French mathematician René Descartes (1596-1650)) relates algebraic equations and ordered pairs of real numbers to geometry in a plane. In this system, two number lines intersect at a right angle and separate the plane into four **quadrants**. The **origin**, designated by the ordered pair $(0, 0)$, is the point of intersection of the two lines. The horizontal number line is called the **horizontal axis** or **x-axis**. The vertical number line is called the **vertical axis** or **y-axis**. All points that lie on the x-axis have a y-coordinate of 0 and all points that lie on the y-axis have an x-coordinate of 0. Points that lie on either axis are not in a quadrant. They are simply on an axis (Figure 2.1).
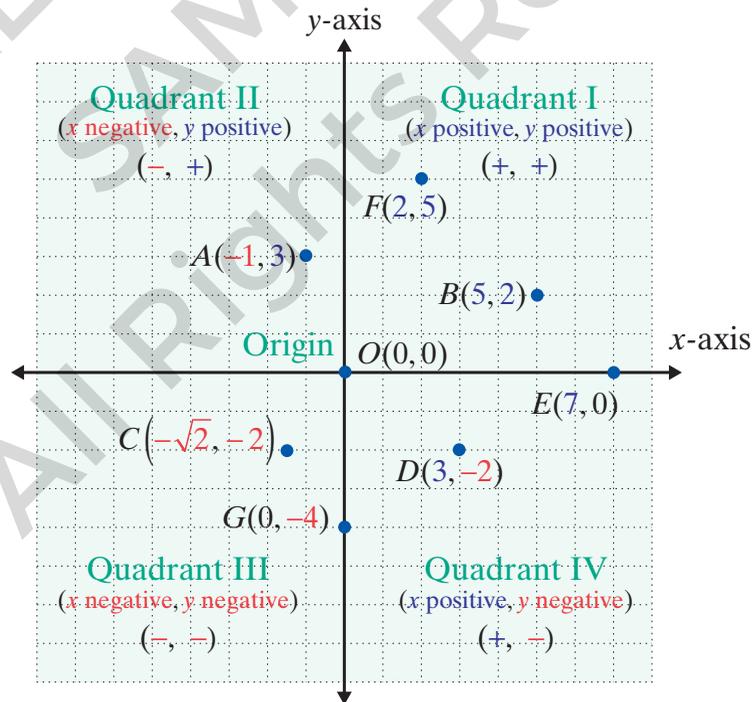

Figure 2.1

The following important relationship between ordered pairs of real numbers and points in a plane is the cornerstone of the Cartesian coordinate system.

One-to-One Correspondence

*There is a **one-to-one correspondence** between points in a plane and ordered pairs of real numbers.*

In other words, for each point in a plane there is one and only one corresponding ordered pair of real numbers, and for each ordered pair of real numbers there is one and only one corresponding point in the plane.

Example 1: Graphing Ordered Pairs

Graph the set of ordered pairs $\{A(-3,1), B(-1,-3), C(0,2), D(1,5), E(2,-4)\}$.
(**Note:** The listing of ordered pairs within the braces can be in any order.)

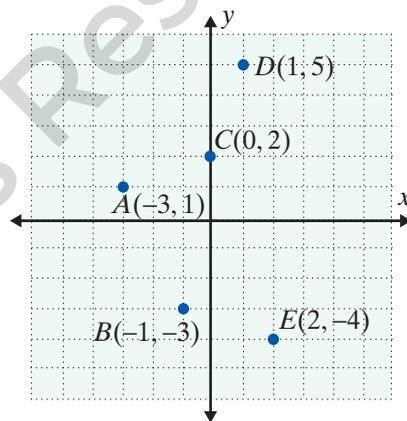**Solution:** To locate each point, **start at the origin**, and:

For $A(-3,1)$, move 3 units left and 1 unit up.

For $B(-1,-3)$, move 1 unit left and 3 units down.

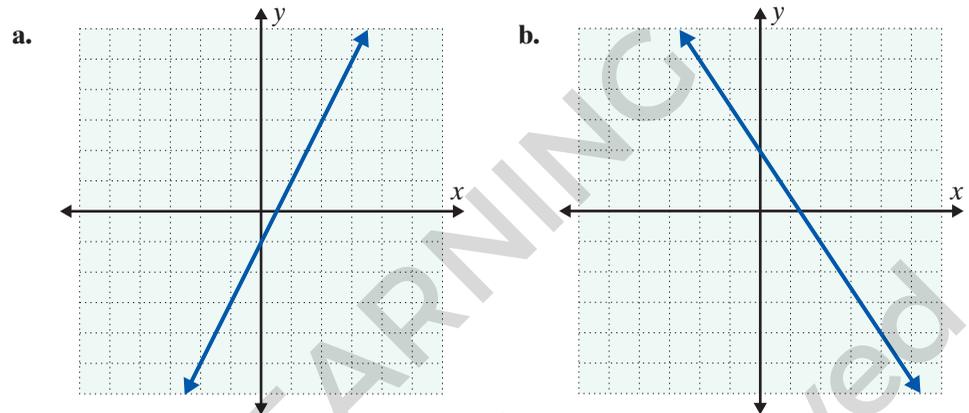For $C(0,2)$, move 2 units up (do not move any units left or right.)

For $D(1,5)$, move 1 unit right and 5 units up.

For $E(2,-4)$, move 2 units right and 4 units down.

Example 2: Reading Points on a Graph

The graphs of two straight lines are given. Each line has an infinite number of points. Use the grid to help you locate (or estimate) three points on each line.

a.

b.

Solution:

a.   Three points on this graph are $(-1,-3)$, $(0,-1)$, $(3,5)$. (Of course there are many different correct answers to this type of question. Use your own judgment.)

b.   Three points on this graph are $(-2,5)$, $(2,-1)$, $(4,-4)$. (You may estimate with fractions. For example, one point appears to be $\left(\frac{1}{2}, \frac{5}{4}\right)$.)

Graphing Straight Lines

Now consider the equation in two variables

$$3x + y = 3$$

or, solved for y,

$$y = 3 - 3x.$$

The **solution set** to this equation consists of an infinite set of ordered pairs in the form (x, y). The variable x is called the **independent variable**, and the variable y is called the **dependent variable**.

Solution Set of an Equation in Two Variables

*The **solution set** of an equation in two variables, x and y, consists of all ordered pairs of real numbers (x, y) that satisfy the equation.*

To find some of the solutions of the equation $y = 3 - 3x$, we form a table by:

1. choosing arbitrary values for *x,* and
2. finding the corresponding values for *y* by substituting into the equation.

We say that these ordered pairs **satisfy** the equation. In Figure 2.2 we have found five ordered pairs that satisfy the equation and graphed the corresponding points.

Choices	Substitutions	Results
x	**3 − 3x = y**	**(x, y)**
−1	$3 - 3(-1) = 6$	$(-1, 6)$
0	$3 - 3 \cdot 0 = 3$	$(0, 3)$
$\frac{2}{3}$	$3 - 3\left(\frac{2}{3}\right) = 1$	$\left(\frac{2}{3}, 1\right)$
2	$3 - 3 \cdot 2 = -3$	$(2, -3)$
3	$3 - 3 \cdot 3 = -6$	$(3, -6)$
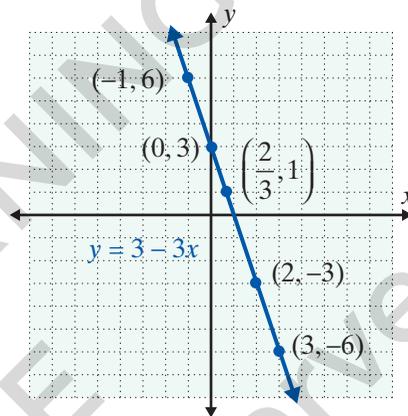
Figure 2.2

The five points in Figure 2.2 appear to lie on a straight line. They in fact do lie on a straight line, and any ordered pair that satisfies the equation $y = 3 - 3x$ will also lie on that same line.

Just as we use the terms **ordered pair** and **point** (the graph of an ordered pair) interchangeably, we use the terms **equation** and **graph of an equation** interchangeably. The equations

$$2x + 3y = 4, \quad y = -5, \quad x = 1.4, \quad \text{and} \quad y = 3x + 2$$

are called **linear equations**, and their graphs are straight lines on the Cartesian plane.

Standard Form of a Linear Equation

Any equation of the form

$$Ax + By = C,$$

where A, B, and C are real numbers and A and B are not both equal to 0, is called the **standard form** *of a* **linear equation***.*

NOTES   Note that in the standard form $Ax + By = C$, A and B may be positive, negative, or 0, but A and B **cannot both be 0**.

Every straight line corresponds to some linear equation, and the graph of every linear equation is a straight line. We know from geometry that **two points determine a line**. This means that the graph of a linear equation can be found by locating any two points that satisfy the equation.

To Graph a Linear Equation in Two Variables

1. *Locate any two points that satisfy the equation. (Choose values for x and y that lead to simple solutions. Remember that there are an infinite number of choices for either x or y. But, once a value for x or y is chosen, the corresponding value for the other variable is found by substituting into the equation.)*

2. *Plot these two points on a Cartesian coordinate system.*

3. *Draw a straight line through these two points. [**Note:** Every point on that line will satisfy the equation.]*

4. ***To check:*** *Locate a third point that satisfies the equation and check to see that it does indeed lie on the line.*

Example 3: Graphing a Linear Equation in Two Variables

Graph each of the following linear equations.

a. $2x + 3y = 6$

Solution: Make a table with headings *x* and *y* and, whenever possible, **choose values for *x* or *y* that lead to simple solutions for the other variable**. In this example, we have found four ordered pairs that satisfy this equation by choosing two *x* values and two *y* values that we felt would result in simple solutions.

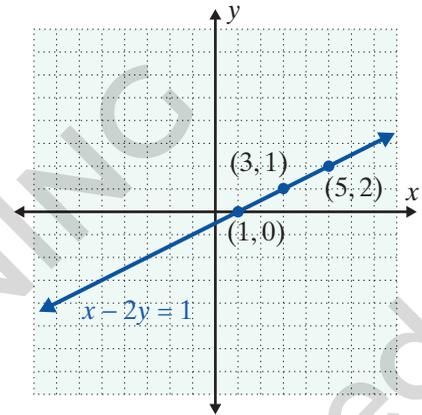| x	2x + 3y = 6	y
0	$2(0) + 3y = 6$	2
−3	$2(-3) + 3y = 6$	4
3	$2x + 3(0) = 6$	0
$\dfrac{5}{2}$	$2x + 3\left(\dfrac{1}{3}\right) = 6$	$\dfrac{1}{3}$
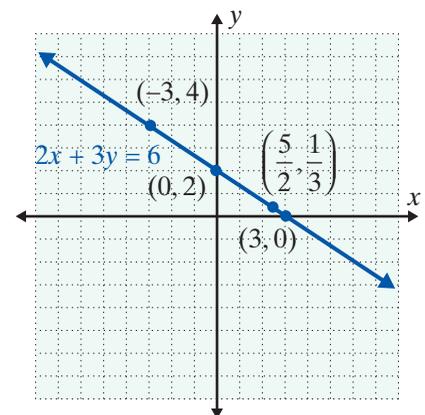
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b. $x - 2y = 1$

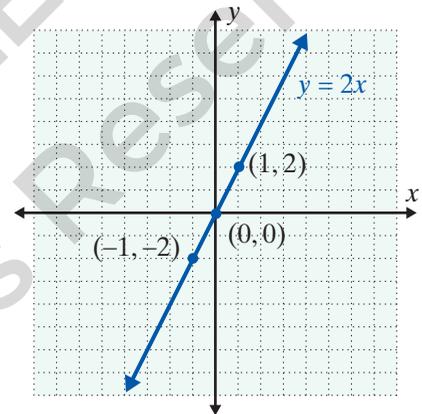> **Solution:** Solve for x and substitute 0, 1, and 2 for y: $x = 2y + 1$.

Results	Substitutions	Choices
x	**x = 2y + 1**	**y**
1	$x = 2(0) + 1$	0
3	$x = 2(1) + 1$	1
5	$x = 2(2) + 1$	2

c. $y = 2x$

> **Solution:** Substitute -1, 0, and 1 for x.

Choices	Substitutions	Results
x	**y = 2x**	**y**
−1	$y = 2(-1)$	−2
0	$y = 2(0)$	0
1	$y = 2(1)$	2

Locating the *y*-intercept and *x*-intercept

While the choice of the values for x or y can be arbitrary, letting $x = 0$ will locate the point on the graph where the line crosses (or intercepts) the y-axis. This point is called the **y-intercept** and is of the form $(0, y)$. The **x-intercept** is the point found by letting $y = 0$. This is the point where the line crosses (or intercepts) the x-axis and is of the form $(x, 0)$. These two points are generally easy to locate and are frequently used as the two points for drawing the graph of a linear equation. If the line passes through the point $(0, 0)$, then the y-intercept and the x-intercept are the same point, namely the origin. In this case you will need to locate some other point to draw the graph.

Intercepts

1. To find the **y-intercept** *(where the line crosses the y-axis), substitute x = 0 and solve for y.*

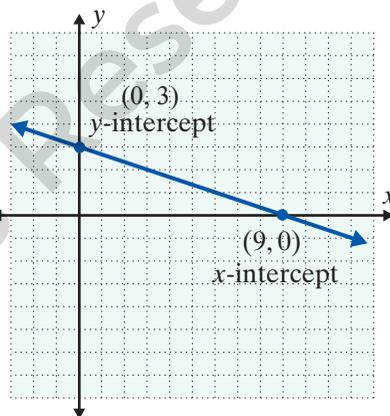2. To find the **x-intercept** *(where the line crosses the x-axis), substitute y = 0 and solve for x.*
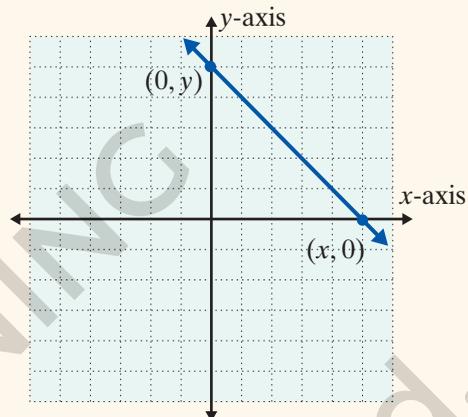
Example 4: *x*- and *y*-Intercepts

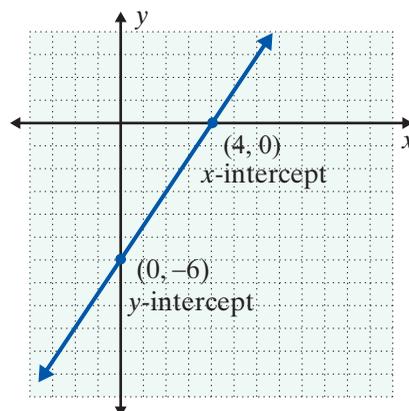Graph the following linear equations by locating the *y*-intercept and the *x*-intercept.

a. $x + 3y = 9$

Solution: $x = 0 \rightarrow$ $(0) + 3y = 9$
$$3y = 9$$
$$y = 3$$
$(0, 3)$ is the *y*-intercept.
$y = 0 \rightarrow$ $x + 3(0) = 9$
$$x = 9$$
$(9, 0)$ is the *x*-intercept.

Plot the two intercepts and draw the line that contains them.

b. $3x - 2y = 12$

Solution: $x = 0 \rightarrow 3(0) - 2y = 12$
$$-2y = 12$$
$$y = -6$$
$(0, -6)$ is the *y*-intercept.
$y = 0 \rightarrow 3x - 2(0) = 12$
$$3x = 12$$
$$x = 4$$
$(4, 0)$ is the *x*-intercept.

Plot the two intercepts and draw the line that contains them.

> **NOTES**
>
> In general, the intercepts are easy to find because substituting 0 for x or y leads to an easy solution for the other variable. However, when the intercepts result in a point with fractional (or decimal) coordinates and estimation is involved, then a third point that satisfies the equation should be found to verify that the line is graphed correctly.

Practice Problems

1. For x in the set $\{-1,\ 2,\ 3\}$, find the corresponding ordered pairs that satisfy the equation $x - 2y = 3$.

2. Find the missing coordinate of each ordered pair so that it belongs to the solution set of the equation $2x + y = 4$:
    $(0,\quad),\ (\quad, 0),\ (\quad, 8),\ (-1,\quad)$.

3. Does the ordered pair $\left(1, \dfrac{3}{2}\right)$ satisfy the equation $3x + 2y = 6$?

4. Find the x-intercept and y-intercept of the equation $-3x + y = 9$.

2.1 Exercises

List the sets of ordered pairs that correspond to the graphs in Exercises 1 – 6. Assume that the grid lines are marked one unit apart.

1.

2.
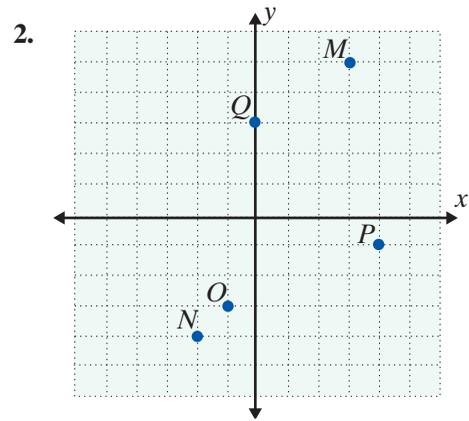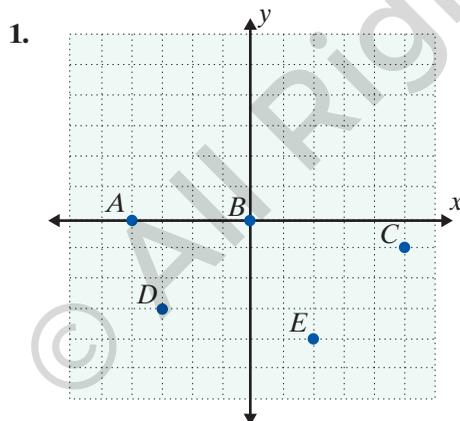
Answers to Practice Problems: 1. $(-1, -2),\ \left(2, -\dfrac{1}{2}\right),\ (3, 0)$ **2.** $(0, 4), (2, 0), (-2, 8), (-1, 6)$ **3.** Yes

4. x-intercept $= (-3, 0)$, y-intercept $= (0, 9)$

3.

4.

5.

6.
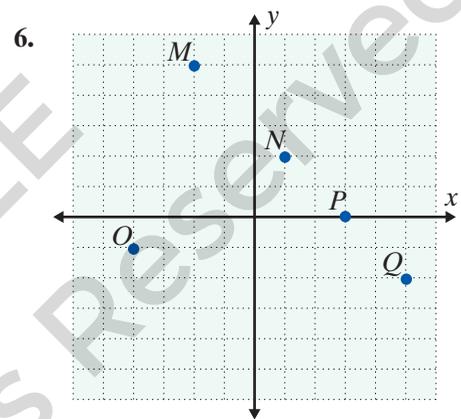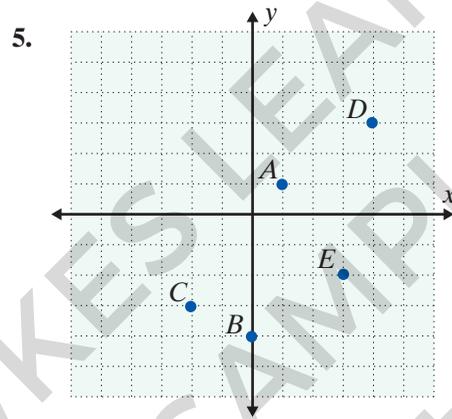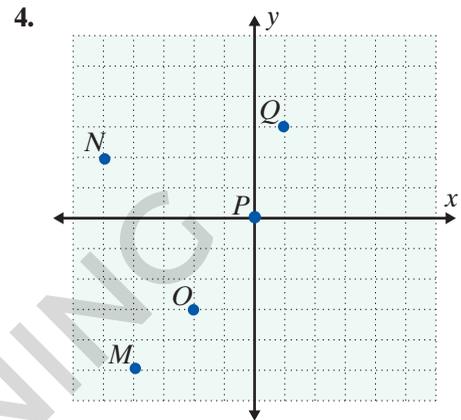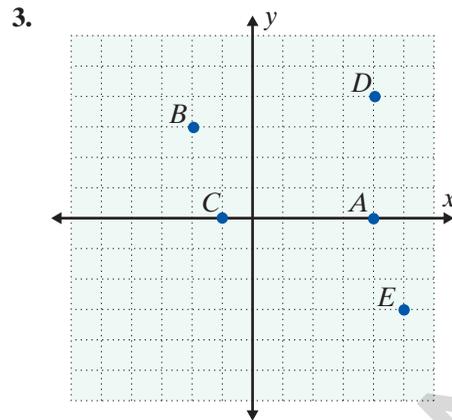
Graph the sets of ordered pairs and label the points in Exercises 7 – 12.

7. $\left\{A(-5, 1), B(-3, 4), C(-1, 1), D(2, 2), E(2, -2)\right\}$

8. $\left\{M(-4, 1), N(-2, 5), O(0, 3), P(1, 6), Q(3, 2)\right\}$

9. $\left\{P(-3, 2), Q(-1, -1), R(1, 5), S(3, -2), T(6, 5)\right\}$

10. $\left\{C(-5, -5), D(-3, 1), E(0, -2), F(3, 1), G(5, 0)\right\}$

11. $\left\{A(-3, 4), B(-2, -1), C(-1, 6), D(2, 0), E(3, -3)\right\}$

12. $\left\{M(-7, 2), N(-4, 5), O(0, -4), P(1, -2), Q(4, -4)\right\}$

In Exercises 13 – 18, the graph of a straight line is shown. Each line has an infinite number of points. List any three points on each line. (There is more than one correct answer.)

13.

14.

15.

16.

17.

18.
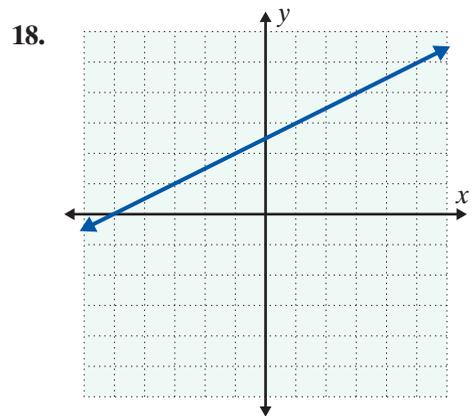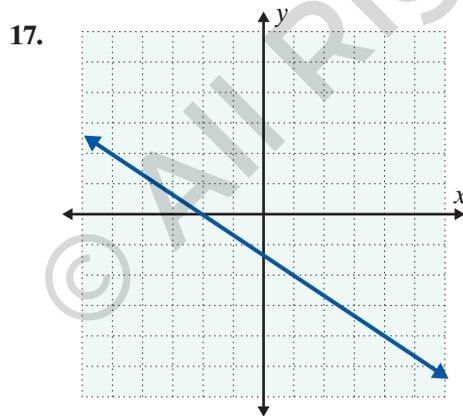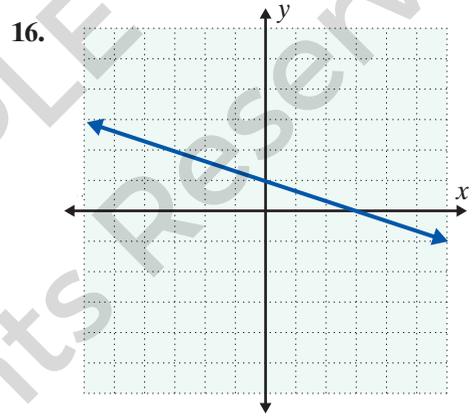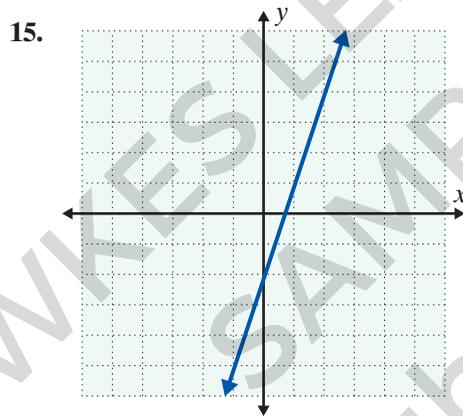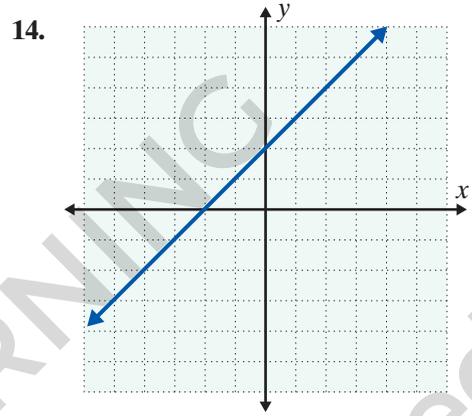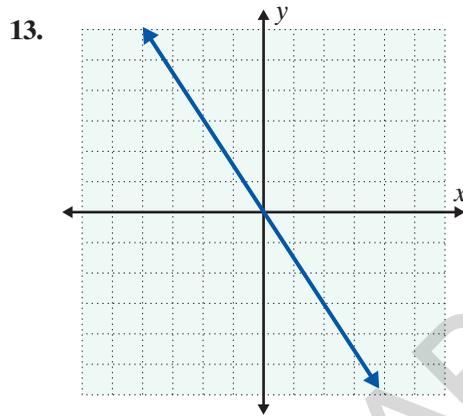
In Exercises 19 – 26, find the missing coordinate of each ordered pair so that the ordered pair belongs to the solution set of the given equation.

19. $2x + y = 5$

x	y
0	
	0
−2	
	3

20. $x + 2y = 6$

x	y
0	
	0
4	
	−2

21. $3x − y = 4$

x	y
0	
	0
2	
	5

22. $x − 3y = 9$

x	y
0	
	0
−3	
	−1

23. $y = 5 − 2x$
   a. $(0, \quad)$
   b. $(\quad, 0)$
   c. $(2, \quad)$
   d. $(\quad, 7)$

24. $y = 5x − 3$
   a. $(0, \quad)$
   b. $(\quad, 0)$
   c. $(−1, \quad)$
   d. $(\quad, 7)$

25. $3x − 2y = 6$
   a. $(0, \quad)$
   b. $(\quad, 0)$
   c. $(−2, \quad)$
   d. $(\quad, 3)$

26. $5x + 2y = 10$
   a. $(0, \quad)$
   b. $(\quad, 0)$
   c. $(4, \quad)$
   d. $(\quad, 10)$

Locate at least two ordered pairs of real numbers that satisfy each of the linear equations in Exercises 27 – 52 and graph the corresponding line in the Cartesian coordinate system.

27. $x + y = 3$

28. $x + y = 4$

29. $y = x$

30. $2y = x$

31. $2x + y = 0$

32. $3x + 2y = 0$

33. $2x + 3y = 7$

34. $4x + 3y = 11$

35. $3x − 4y = 12$

36. $2x − 5y = 10$

37. $−4x + y = 4$

38. $−3x + 3y = 6$

39. $3y = 2x − 4$

40. $4x = 3y + 8$

41. $3x + 5y = 6$

42. $2x + 7y = −4$

43. $2x + 3y = 1$

44. $5x − 3y = −1$

45. $5x − 2y = 7$

46. $3x + 4y = 7$

47. $\dfrac{2}{3}x − y = 4$

48. $x + \dfrac{3}{4}y = 6$

49. $2x + \dfrac{1}{2}y = 3$

50. $\dfrac{2}{5}x − 3y = 5$

51. $5x = y + 2$

52. $4x = 3y − 5$

Graph the linear equations in Exercises 53 – 60 by locating the y-intercept and the x-intercept.

53. $x - 2y = 8$

54. $x + y = 6$

55. $2x + 3y = 12$

56. $3x - 7y = -21$

57. $4x - y = 10$

58. $\frac{1}{2}x + 2y = 3$
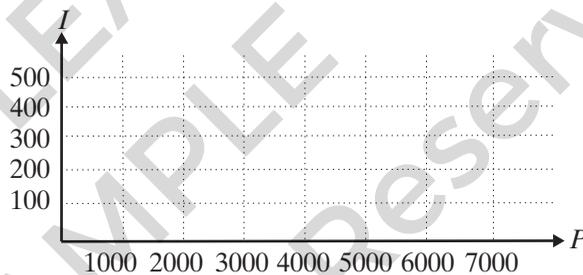
59. $3x + 2y = 15$

60. $x - 4y = -6$

61. Interest rate: Given the equation $I = 0.06P$, where I is the interest earned on a principal P at the rate of 6%:

    a. Make a table of ordered pairs for the values of P and I if P has the values $1000, $2000, $4000, $5000, and $7000.

    b. Graph the points corresponding to the ordered pairs.

P	I
1000	
2000	
4000	
5000	
7000	

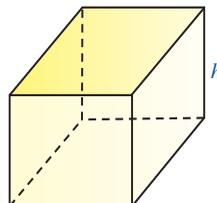**62. Volume of a box:** Given the equation $V = 25h$, where V is the volume (in cm^3) of a box with a variable height h in cm and a fixed base of area 25 cm^2:
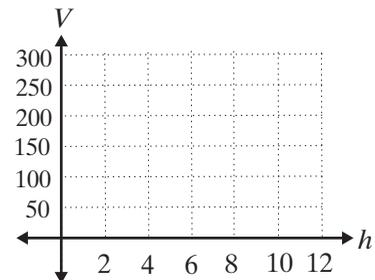
    a. Make a table of ordered pairs for the values of h and V with h as the values 2 cm, 3 cm, 8 cm, 10 cm, and 12 cm.

    b. Graph the points corresponding to the ordered pairs.

h	V
2	
3	
8	
10	
12	

Area of base = 25 cm^2

Writing and Thinking About Mathematics

In statistics, data is sometimes given in the form of ordered pairs where each ordered pair represents two pieces of information about one person. For example, ordered pairs might represent the height and weight of a person or the person's number of years of education and that person's annual income. The ordered pairs are plotted on a graph and the graph is called a **scatter diagram** (or **scatter plot**). Such scatter diagrams are used to see if there is any pattern to the data and, if there is, then the diagram is used to predict the value for one of the variables if the value of the other is known. For example, if you know that a person's height is 5 ft 6 in., then his or her weight might be predicted from information indicated in a scatter diagram that has several points of known information about height and weight.

63. Exercise: The following table of values gives the number of push-ups and pull-ups completed by ten students in a physical education class.
   a. Plot these points on a scatter diagram.

Person	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
Push-ups	20	15	25	23	35	30	42	40	25	35
Pull-ups	5	2	9	8	10	11	15	14	7	12

   b. Does there seem to be a pattern in the relationship between push-ups and pull-ups? If so, what is this pattern?
   c. Using the scatter diagram in Part **a**, predict the number of pull-ups that a student might be able to do if he or she has just done each of the following numbers of push-ups: 22, 32, 35, and 45. (**Note:** In each case, there is no one correct answer. The answers are only estimates based on the diagram.)

64. Explain in your own words why it is sufficient to find the *x*-intercept and *y*-intercept to graph a line (assuming that they are not the same point).

65. Explain in your own words how you can determine if an ordered pair is a solution to an equation.