Example 5: Continuous Compounding Interest

The formula $A = Pe^{rt}$ represents the amount accumulated when a principal is compounded continuously (as discussed in Section 5.1). How long will it take a principal P to double if interest is compounded continuously at 10 percent?

Solution

If P is doubled, then the amount A is $2P$.

$$Pe^{0.10t} = 2P \qquad r = 10\% = 0.10$$

$$e^{0.10t} = 2$$

$$0.10t = \ln 2$$

$$t = \frac{\ln 2}{0.10} \approx \frac{0.6931}{0.10} = 6.931$$

The principal will double in approximately 6.9 years.

So far we discussed the natural logarithms and we saw that $y = e^x$ and $y = \ln x$ are inverse functions.

Now, suppose B is a positive number and $B > 1$. The exponential function with base B, specifically $f(x) = B^x$, has an inverse function, and we denote this inverse by $f^{-1}(x) = \log_B x$. The inverse is the **logarithmic function with base B**. It follows that $B^{\log_B x} = x$ and $\log_B(B^x) = x$.

Note that $\log_{10} x$ is called the **common logarithm**, and is usually written $\log x$.

Historically, the natural logarithms (base e) and common logarithms (base 10) have proved sufficient for computations and most theoretical purposes. For this reason, tables for base e and base 10 are built into all graphing calculators. Such calculators normally have a button labeled "ln" for the natural logarithm and a button labeled "log" for the common logarithm.

5.2 EXERCISES

PRACTICE

In Exercises 1–6, write each exponential equation as a logarithmic equation.

1. $e^{0.5} \approx 1.648721$

2. $e^{1.6} \approx 4.953032$

3. $e^{-2.1} \approx 0.122456$

4. $e^{-0.3} \approx 0.740818$

5. $e^{0.25} \approx 1.284025$

6. $e^{-0.42} \approx 0.657047$

In Exercises 7–12, write each logarithmic equation as an exponential equation.

7. $\ln(1.822119) \approx 0.6$

8. $\ln(12.182494) \approx 2.5$

9. $\ln(23.6) \approx 3.161247$

10. $\ln(0.697676) \approx -0.36$

11. $\ln(0.069460) \approx -2.6670$

12. $\ln 3 \approx 1.098612$

In Exercises 13–22, use the inverse relationship between $f(x) = e^x$ and $g(x) = \ln x$ to simplify each expression.

13. $e^{\ln(4.6)}$

14. $e^{\ln(0.7)}$

15. $\ln\left(e^3\right)$

16. $\ln\left(e^{-0.86}\right)$

17. $\ln \sqrt[3]{e}$

18. $\ln \dfrac{1}{e^2}$

19. $e^{\ln(5x)}$

20. $e^{\ln(x+4)}$

21. $\ln\left(e^{2t-1}\right)$

22. $\ln\left(e^{1-0.2t}\right)$

In Exercises 23–34, use the properties of logarithms to rewrite each expression as the logarithm of a single expression. Be sure to use positive exponents and avoid radicals.

23. $\ln x + 3\ln y$

24. $\dfrac{1}{2}(\ln x + \ln y)$

25. $\ln x - \ln y + \ln z$

26. $\ln x + \ln y - \dfrac{1}{2}\ln z$

27. $\ln x + \ln(x+3)$

28. $\ln(x+4) + \ln(x-1)$

29. $2\ln x - \ln(x+1)$

30. $\ln\left(x^2+2\right) - \dfrac{1}{2}\ln x$

31. $\ln\left(x^2-2x+1\right) - \ln(x-1)$

32. $3\ln x + \dfrac{1}{2}\ln(x+5)$

33. $\ln(x+3) + \ln(x+1) - \dfrac{1}{2}\ln x$

34. $\ln\left(x^2-3x-4\right) - \ln(x-4)$

In Exercises 35–46, use the properties of logarithms to rewrite each expression as a sum, difference, or constant multiple of logarithms. Replace all radicals with exponents.

35. $\ln\left(x^3y^2\right)$

36. $\ln\left((xy)^{-2}\right)$

37. $\ln \sqrt[3]{\dfrac{x}{y}}$

38. $\ln \sqrt[3]{xy^2}$

39. $\ln\left(\dfrac{15x}{\sqrt{y}}\right)$

40. $\ln\left(\dfrac{\sqrt{3x}}{y^4}\right)$

41. $\ln \sqrt{4x+1}$

42. $\ln\left(x\sqrt{2x+3}\right)$

43. $\ln \dfrac{e}{(x+2)^2}$

44. $\ln\left(\left(\dfrac{x-5}{e^x}\right)^2\right)$

45. $\ln\left(x^2-16\right)$

46. $\ln\left(x^2-3x-10\right)$

In Exercises 47–60, use the properties of logarithms to solve each equation.

47. $2e^{2x+1} = 26$

48. $e^{2x+3} = 3.5$

49. $e^{-0.6t} = 0.8$

50. $6e^{-0.4t} = 48$

51. $3 \cdot 5^x = 54$

52. $4^{2x} = 0.016$

53. $10^{3x-1} = 8.6$

54. $10^{1-2x} = 72$

55. $\ln x + \ln(2x) = \ln 18$

56. $\ln(x+4) + \ln(x-4) = \ln 9$

57. $\ln(x+1) + \ln(x-1) = 0$

58. $\ln(x+5) + \ln(x-5) = 0$

59. $\ln(x^2 + 4x - 5) - \ln(x+5) = 1$

60. $\ln(x^2 + 2x - 3) - \ln(x-1) = 2$

APPLICATIONS

61. Compound interest: A deposit of \$5000 is made into an account that pays interest at an annual rate of 9.6 percent. Find the time required for the investment to double if the interest is compounded as indicated. Round to the nearest hundredth of a year.

a. annually **b.** quarterly **c.** monthly

62. Compound interest: A deposit of \$2000 is made into an account that pays 8.4 percent annual interest. Find the time required for the account to be worth \$3600 if the interest is compounded as indicated. Round to the nearest hundredth of a year.

a. annually **b.** quarterly **c.** monthly

63. Interest compounded continuously: A deposit of \$5000 is made into an account that pays interest at a rate of 8 percent compounded continuously. Find the time required for the amount to double. Round to the nearest hundredth of a year.

64. Interest compounded continuously: Anna deposited \$6000 into an account that pays 7.6 percent annual interest compounded continuously. How long will it take for the balance to reach \$10,000? Round to the nearest hundredth of a year.

65. Depreciation: The value V of a machine t years after it was purchased is given by $V = C(1-r)^t$, where C is the original cost and r is the rate of depreciation. A machine is purchased for \$12,000. If $r = 0.16$, in how many years will the machine be worth only \$6000? Round to the nearest hundredth of a year.

66. Depreciation: The value V of a machine t years after it was purchased is given by $V = C(1-r)^t$, where C is the original cost and r is the rate of depreciation. The owner of a furniture shop purchased a special saw for \$8600. She plans to depreciate the saw at a rate of 12 percent per year. In how many years will the value of the saw be \$2380? Round to the nearest hundredth of a year.

67. Printing costs: The total cost in dollars of printing x thousand copies of a book is given by $C(x) = 9000 + 40{,}000\ln(0.5x+1)$.

a. Find $C(2)$, $C(8)$, and $C(10)$.
b. Find the cost of printing 12,000 copies.
c. If the total cost of printing is \$100,000, how many books are printed?

68. City planning: To plan for the future needs of the city, the Department of Public Works uses the function $P(t) = 36{,}000 + 8400\ln(0.8t+1)$ to estimate the population t years from now.

a. What is the city's current population?
b. Find $P(2)$, $P(6)$, and $P(10)$.
c. In how many years will the population reach 60,000? Round to the nearest hundredth of a year.