Example 7: Properties of Limits

Find $\lim_{x\to-\infty}\left(\frac{x^3+x^2-x+1}{x^2-4}\right)$, if it exists.

Solution

$$\lim_{x\to-\infty}\left(\frac{x^3+x^2-x+1}{x^2-4}\right)=\lim_{x\to-\infty}\left(\frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}}{\frac{x^2}{x^3}-\frac{4}{x^3}}\right)$$

Divide each term by the highest power of x present, which in this function is x^3.

$$=\lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}{\frac{1}{x}-\frac{4}{x^3}}\right)$$

Simplify.

$$=\frac{1}{0}$$

Since $\frac{1}{0}$ is undefined, the limit is either $+\infty$ or $-\infty$. (See Example 3.)

Investigating the expression shows that the highest power is x^3. This term will dominate for very large values of x and will be negative for negative values of x. Thus

$$\lim_{x\to-\infty}\left(\frac{x^3+x^2-x+1}{x^2-4}\right)=-\infty.$$

Summary of Limits for Rational Functions as $x\to+\infty$ (or $x\to-\infty$)

Consider the function

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0},$$

where $a_n\neq 0$ and $b_m\neq 0$.

Case 1: For $m=n$, $\lim_{x\to+\infty}f(x)=\frac{a_n}{b_m}$.

Case 2: For $m>n$, $\lim_{x\to+\infty}f(x)=0$.

Case 3: For $m<n$, $\lim_{x\to+\infty}f(x)=+\infty$. (Or $-\infty$ depending on the signs of a_n and b_m.)

2.3 EXERCISES

PRACTICE

In Exercises 1–28, find the indicated limit, if it exists.

1. $\lim_{x\to-2}6$

2. $\lim_{x\to4}2x$

3. $\lim_{x\to3^-}\left(x^2+1\right)$

4. $\lim_{x\to -3^+}\left(5-2x^2\right)$

5. $\lim_{x\to 1^+}\left(\frac{x+2}{x-1}\right)$

6. $\lim_{x\to 1^-}\left(\frac{x+2}{x-1}\right)$

7. $\lim_{x\to \frac{1}{3}}\left(\frac{3x+1}{x+2}\right)$

8. $\lim_{x\to 0^+}\left(\frac{x+4}{x-4}\right)$

9. $\lim_{x\to 0^-}\left(\frac{x}{x^2+2x}\right)$

10. $\lim_{x\to 0^-}\left(\frac{2x^2+x}{x}\right)$

11. $\lim_{x\to +\infty}\left(\frac{x}{x^2+3}\right)$

12. $\lim_{x\to +\infty}\left(\frac{2x^2+7}{3x^2-2}\right)$

13. $\lim_{x\to -\infty}\left(\frac{x^3+64}{x^2-2x+1}\right)$

14. $\lim_{x\to -\infty}\left(\frac{4x-x^3}{x^2+2x-7}\right)$

15. $\lim_{x\to 2}\left(\frac{x^2-x-2}{x^2-4}\right)$

16. $\lim_{x\to -3}\left(\frac{x^2-9}{x^2+2x-3}\right)$

17. $\lim_{x\to 0^+}\left(4-\frac{3}{x}\right)$

18. $\lim_{x\to 1^-}\left(2x+\frac{5}{x-1}\right)$

19. $\lim_{x\to +\infty}\left(8+\frac{1}{x}\right)$

20. $\lim_{x\to -\infty}\left(11-\frac{2}{x^2}\right)$

21. $\lim_{x\to 4}\sqrt{x+5}$

22. $\lim_{x\to 2}\sqrt{3x+10}$

23. $\lim_{x\to 1}\left(\sqrt{x}-3\right)$

24. $\lim_{x\to 4}\left(\sqrt{x}+6\right)$

25. a. $\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right)$ [**Hint:** $x-4=\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)$]

b. $\lim_{x\to 9}\left(\frac{x-9}{\sqrt{x}-3}\right)$ [**Hint:** $x-9=\left(\sqrt{x}+3\right)\left(\sqrt{x}-3\right)$]

26. a. $\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}$ [**Hint:** Multiply by $\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$, simplify the numerator, and calculate the limit.]

b. $\lim_{h\to 0}\frac{\sqrt{5x+h}-\sqrt{5x}}{h}$

27. $\lim_{x\to +\infty}\left(\frac{x^2+3x-4}{x^2+5x-9}\right)$

28. $\lim_{x\to -\infty}\left(\frac{x^2+x+1}{2x^3+3x^2+x-2}\right)$

29. For the function $f(x)=\begin{cases}-3 & \text{if } x\le 1\\ x-4 & \text{if } x>1\end{cases}$, find the following limits.

a. $\lim_{x\to 1^-}f(x)$ **b.** $\lim_{x\to 1^+}f(x)$ **c.** $\lim_{x\to 1}f(x)$ **d.** $\lim_{x\to 2}f(x)$

30. For the function $f(x)=\begin{cases}2x+1 & \text{if } x<0\\ x^2+1 & \text{if } x\ge 0\end{cases}$, find the following limits.

a. $\lim_{x\to 0^-}f(x)$ **b.** $\lim_{x\to 0^+}f(x)$ **c.** $\lim_{x\to 0}f(x)$ **d.** $\lim_{x\to -2}f(x)$

31. For the function $f(x)=\begin{cases}5-x & \text{if } x\le 2\\ x^2-1 & \text{if } x>2\end{cases}$, find the following limits.

a. $\lim\limits_{x\to 2^-} f(x)$ **b.** $\lim\limits_{x\to 2^+} f(x)$ **c.** $\lim\limits_{x\to 2} f(x)$ **d.** $\lim\limits_{x\to 0} f(x)$

32. For the function $f(x)=\begin{cases}x^3+4 & \text{if } x\le -2\\ \sqrt{x^2+5} & \text{if } x>-2\end{cases}$, find the following limits.

a. $\lim\limits_{x\to -2^-} f(x)$ **b.** $\lim\limits_{x\to -2^+} f(x)$ **c.** $\lim\limits_{x\to -2} f(x)$ **d.** $\lim\limits_{x\to -1} f(x)$

APPLICATIONS

33. Utility costs: The Municipal Gas Company uses the following function for computing their customers' monthly gas bills:

$$C(x)=\begin{cases}0.37x+3.00 & \text{if } 0<x\le 24\\ 0.78x-6.84 & \text{if } x>24\end{cases},$$

where x is the number of therms (thermal units) used and $C(x)$ is the cost in dollars.

a. Find $\lim\limits_{x\to 24^-} C(x)$. **b.** Find $\lim\limits_{x\to 24^+} C(x)$. **c.** Find $\lim\limits_{x\to 24} C(x)$.

34. Income tax: A federal income tax schedule can be given by the function

$$T(x)=\begin{cases}0.15x & \text{if } 0<x\le 23{,}900\\ 0.28x-3107 & \text{if } 23{,}900<x\le 61{,}650\\ 0.33x-6189.50 & \text{if } 61{,}650<x\le 123{,}790\end{cases},$$

where x is the taxable income in dollars, $0<x\le 123{,}790$, and $T(x)$ is in dollars.

a. Find $\lim\limits_{x\to 23{,}900^-} T(x)$. **b.** Find $\lim\limits_{x\to 23{,}900^+} T(x)$.

c. Find $\lim\limits_{x\to 23{,}900} T(x)$. **d.** Find $\lim\limits_{x\to 61{,}650} T(x)$.

35. Average cost: A manufacturer of golf clubs estimates that if x sets of golf clubs are produced, then the average cost of producing each set is $A(x)=73+\dfrac{5780}{x}$ dollars. What will be the average cost of producing each set in the long run $\left(\lim\limits_{x\to +\infty} A(x)\right)$?

36. Dictation rate: It has been determined that after t weeks of class, a certain student in an intermediate shorthand class can take dictation at a rate of $W(t)=60+\dfrac{70t^2}{t^2+15}$ words per minute. What will be this student's rate of taking dictation in the long run $\left(\lim\limits_{t\to +\infty} W(t)\right)$?