

10.3 EXERCISES

 PRACTICE

- Use the Taylor series for e^x , centered at $x = 0$, and substitution to obtain a power series for $y = x^2e^x$.
- Use the Taylor series for e^x , centered at $x = 0$, and substitution to obtain a power series for $y = \frac{e^x - 1}{x}$.
- Use the geometric series $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots + r^n + \dots$ to obtain a power series for each of the following functions.
 - $\frac{1}{1+t}$
 - $\frac{1}{1+t^2}$
 - $\frac{t}{1+t}$
 - $\frac{t}{1+t^2}$
- Find a Taylor series centered at $x = 0$ for each of the following functions.
 - $y = e^{-x}$
 - $y = e^{-x^2}$
 - $y = e^{7x}$
 - $y = \frac{x}{e^x}$
- Find the first four derivatives of $g(x) = \frac{1}{1-x^2}$.
 - Use your answer to part a. to construct the Taylor polynomial of degree 4 for $g(x)$ centered at $x = 0$.
 - Use the geometric series for $g(x)$ with $a = 1$ and $r = x^2$ and compare results.
- In engineering, the hyperbolic sine function, abbreviated \sinh , is defined by $\sinh t = \frac{e^t - e^{-t}}{2}$.
 - Use the first four derivatives of $\sinh t$ to determine the Taylor polynomial of degree 4, centered at $t = 0$, for $y = \sinh t$.
 - Combine the known Taylor series for e^t and e^{-t} and get a Taylor series centered at $t = 0$ for $\sinh t$. Compare your results with your answer to part a.
- In engineering, the hyperbolic cosine function, abbreviated \cosh , is defined by $\cosh t = \frac{e^t + e^{-t}}{2}$.
 - Use the first four derivatives of $\cosh t$ to determine the Taylor polynomial of degree 4, centered at $t = 0$, for $y = \cosh t$.
 - Combine the known Taylor series for e^t and e^{-t} and get a Taylor series centered at $t = 0$ for $\cosh t$. Compare your results with your answer to part a.
- Find a power series representation for the function $y = \frac{1}{\sqrt{1-t}}$ by using the binomial series. Let $p = -\frac{1}{2}$ and replace x with $-t$.

9. Suppose $f(x) = \frac{1}{1-x}$.
- Find $f'(x)$.
 - Replace $\frac{1}{1-x}$ with a power series centered at $x=0$, and differentiate term by term.
 - Equate your answers to parts a. and b. Express your result in the following form.

$$f'(x) = ? = \sum_{n=?}^{\infty} a_n x^n$$

10. a. Get a power series representation for $y = \frac{1}{\sqrt{1-u^2}}$, centered at $u=0$. **Hint:** In Exercise 8, replace t with u^2 .
- b. Since $\frac{d}{du} \arcsin u = \frac{1}{\sqrt{1-u^2}}$, we can say $\arcsin x = \int_0^x \frac{1}{\sqrt{1-u^2}} du$. In this equation for $\arcsin x$, substitute the series from part a. for the integrand, integrate term by term, and get a power series representation for $\arcsin x$.
11. Calculate the limit $\lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \right)$ as follows: replace $\sin t$ with its Taylor series centered at $t=0$, simplify the quotient, and then evaluate the limit as t approaches 0.

12. Evaluate $\int_0^1 \frac{\sin t}{t} dt$ in two ways.
- Use a graphing calculator. (Remember to set your calculator to radian mode.)
 - Replace $\frac{\sin t}{t}$ with a Taylor polynomial of degree 6, centered at $t=0$, and then integrate term by term. Compare the answers.
13. Find the functions whose Taylor series coefficients, for the series centered at $x=0$, are specified by the given sequences.
- $\{0, 0, 0, 1, 1, 1, 1, \dots, 1, \dots\}$
 - $\{1, 2, 3, 4, 5, \dots\}$
 - $\{1, 1, 1, 0, 1, 1, \dots, 1, \dots\}$
 - $\{1, 6, 15, 20, 15, 6, 1, 0, 0, 0, \dots\}$

14. Find the functions whose Taylor series coefficients, for the series centered at $x=0$, are specified by the given sequences.
- $\left\{0, 1, 0, \frac{-1}{3!}, 0, \frac{1}{7!}, 0, \frac{-1}{9!}, 0, \frac{1}{11!}, 0, \frac{-1}{13!}, 0, \dots\right\}$
 - $\{2, 2, 2, 2, \dots\}$
 - $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$
 - $\{1, 5, 10, 10, 5, 1, 0, 0, 0, \dots\}$

15. Find all seven complex number solutions of the equation $x^7 = 1$.

In Exercises 16–19, find $y = f(x)$, a polynomial of degree 4 that is an approximate solution of the boundary value problem with the given conditions.

16. $f'(x) = y - x + 1, \quad f(0) = 1$

17. $f'(x) = y + 1, \quad f(0) = 1$

18. $f'(x) = (1 + x)y, \quad f(0) = 1$

19. $f''(x) = -f(x) + 1, \quad f(0) = 1, \quad f'(0) = 1$

WRITING & THINKING

20. Determine a calculus relationship between the hyperbolic functions $\sinh t$ and $\cosh t$ and their derivatives. (See Exercises 6 and 7.)

21. A function is called *odd* if it satisfies $f(x) = -f(-x)$ for all x in its domain. For example $y = x^3 - x$ is odd. Make a conjecture about the Taylor series of odd functions.

22. A function is called *even* if $f(x) = f(-x)$ for all values of x in its domain. For example $y = x^2 + 3$ is even. Make a conjecture about the Taylor series of even functions.

23. a. Integrate $\int_0^{\frac{1}{2}} \frac{1}{1-t} dt$ using a graphing calculator.

b. Integrate $\int_0^{\frac{1}{2}} \frac{1}{1-t} dt$ exactly.

c. Integrate $\int_0^{\frac{1}{2}} \frac{1}{1-t} dt$ by replacing the integrand with a Taylor series centered at $t = 0$ and integrate term by term.

d. Rewrite your answer to part c. using summation notation and equate it to your answer to part b.

24. Replace the integral in Exercise 23 with $\int_0^1 \frac{1}{1+t} dt$ and repeat the four steps.

25. Since $\frac{d}{dt} \arctan t = \frac{1}{1+t^2}$, we have $\arctan x = \int_0^x \frac{1}{1+t^2} dt$.

a. Substitute your answer to Exercise 3b into the integral and integrate term by term to get a power series expansion for the arctangent function (first discovered in the 14th century by the Indian mathematician Madhava of Sangamagrama).

b. The interval of convergence for a geometric series is $(-1, 1)$. This means the interval of convergence for the answer to part a. is also expected to be $(-1, 1)$. However, in this case, the interval includes the endpoint $x = 1$. Since the angle whose tangent is 1 is the angle $\frac{\pi}{4}$, $\arctan 1 = \frac{\pi}{4}$. Thus $\pi = 4 \arctan 1$. Substitute $x = 1$ into your power series from part a. and get a numerical representation for π as an infinite series (originally published by Leibniz around 1676).