6.4 Exercises

Basic Concepts

1. What is the Fundamental Counting Principle?

2. What is a factorial and how is it calculated?

3. Describe the difference between permutations and combinations.

4. Give an example of a situation in which you would need to determine the number of distinguishable permutations.

Exercises

5. The blue plate lunch at a local cafeteria consists of an entrée, a side item, and a dessert. If there are 6 choices for an entrée, 5 choices for a side item, and 4 choices for a dessert, how many different lunches are available?

6. You are interested in buying a home in a new subdivision. The builder offers 3 basic floor plans, each with 4 possible arrangements for the garage, and siding in 6 different colors. How many different homes can be built?

7. Compute each of the following.

a. 1!

b. 3!

c. 5!

d. 7!

8. A wedding planner needs to select 6 songs from a playlist containing 12 songs to play at the reception. How many different sequences are possible?

9. In how many ways can 11 kids be picked for the 9 positions on a baseball team?

10. How many distinguishable permutations can be made from the word STATISTICS?

11. How many distinguishable permutations can be made from the word SASSAFRAS?

12. A person tosses a coin 11 times. In how many ways can he get 9 heads?

13. How many 5 card hands can be dealt from a deck of 52 cards?

14. There are eight people hosting a party. Three people are needed to decorate for the party. How many ways can the decorating crew be chosen?

15. In how many ways can a graduate student fulfill their degree requirements in statistics if 10 classes are needed from a choice of 15 classes?

16. The Johnson family is planning their vacation. Each of the five family members is allowed to nominate three places they would like to visit. If they want to visit four different places during the trip, in how many ways can they plan their trip, assuming that no family members choose the same place?

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17. Kara was born on 11/21/1992. She would like to make an eight-digit password using all of the digits in her birth date. How many different eight-digit passwords could she create?

18. Employees at a local software company need a unique seven-digit code to access the building. The manager wants to make each person's code from the company's phone number, 555-8212.

- **a.** If there are 509 employees who need codes, will the manager have enough unique codes using only the digits in the phone number?
- **b.** Would there be enough ten-digit codes if he used the area code, 516, as well?

19. Which of the following words would produce the greatest number of different five-letter arrangements? (**Hint:** Think before you calculate!)

- **a.** TEARS
- **b.** STOPS
- **c.** TESTS
- **d.** ROOST

20. Receipts often show the last four digits of your credit card. Assume American Express offers 15-digit cards starting with 34 or 37. If a thief has the last four digits of your American Express credit card, what is the probability of them correctly guessing the first 11 digits? Express your answer as a fraction.

6.5 Bayes' Theorem

We discussed conditional probability and independence in Section 6.3. **Bayes' Theorem** (also referred to as **Bayes' Rule** or **Bayes' Law**) is a clever way of obtaining a conditional probability given new information. The additional information is obtained for a subsequent event and is used to revise the initial probability. We begin with an example.

Example 6.5.1

Using Bayes' Theorem to Determine a Probability

Suppose that 85% of all passengers in an airport fly on a major airline, while the remaining 15% fly on a small airline. Of those passengers traveling on a major airline, suppose we know that 65% are traveling for business. Of those passengers traveling on a small airline, 25% are traveling for business. (Notice that even though we only talk about business passengers, there are also implied non-business passengers as well.) Now a business passenger is selected at random. What is the probability that the business passenger traveled on a major airline?

Solution

Let's first define the events associated with this problem.

$$M = \text{Major Airline}$$
$$S = \text{Small Airline}$$
$$B = \text{Business Passenger}$$

Here are the probabilities given to us in the problem.

$$P(M) = 0.85$$
$$P(S) = 0.15$$
$$P(B \mid M) = 0.65$$
$$P(B \mid S) = 0.25$$

To determine the probability that the selected business passenger traveled on a major airline, we need to find the probability $P(M \mid B)$. Using the four probabilities above and Bayes' Theorem, we proceed as follows.

$$P(M \mid B) = \frac{P(M \cap B)}{P(B)}$$

By the definition of a conditional probability.

$$= \frac{P(M \cap B)}{P(B \cap M) + P(B \cap S)}$$

The denominator says that all business passengers travel either on a major airline or on a small airline; those are the only two alternatives and they are mutually exclusive. Thus, $P(B)$ is equivalent to the denominator.

$$= \frac{P(M) \cdot P(B \mid M)}{P(M) \cdot P(B \mid M) + P(S) \cdot P(B \mid S)}$$

The numerator and denominator result from rearranging the conditional probability formula and solving for the probability of the intersection of two events. Note that $P(M \cap B)$ is equal to both $P(M) \cdot P(B \mid M)$ and $P(B) \cdot P(M \mid B)$ by the definition of conditional probability.

$$= \frac{0.85 \cdot 0.65}{0.85 \cdot 0.65 + 0.15 \cdot 0.25}$$

Substitute the probability values given in the problem.

$$= \frac{0.5525}{0.59} = 0.936 \approx 94\%$$

Therefore, we know that if the passenger was traveling for business, there is about a 94% chance that he or she will be traveling on a major airline.

Notice how the conditional probability of 94% is not intuitive. It is called the **posterior probability**. The **prior probability** of a passenger traveling on a major airline of 85% has been increased to 94%, given the information that the passenger was traveling for business purposes.

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The following is a formal statement of Bayes' Theorem.

Bayes' Theorem

Let A be an event and B_1, B_2, ..., B_N be N mutually exclusive and collectively exhaustive events. Then Bayes' Theorem states,

$$\begin{aligned} P(B_i \mid A) &= \frac{P(B_i \cap A)}{P(A)} \\ &= \frac{P(B_i \cap A)}{P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_N)} \\ &= \frac{P(B_i) \cdot P(A \mid B_i)}{P(B_1) \cdot P(A \mid B_1) + P(B_2) \cdot P(A \mid B_2) + \cdots + P(B_N) \cdot P(A \mid B_N)} \\ &= \frac{P(B_i) \cdot P(A \mid B_i)}{\sum_{i=1}^{N} P(B_i) \cdot P(A \mid B_i)}. \end{aligned}$$

THEOREM

Let's look at another example using Bayes' Theorem.

Example 6.5.2

Using Bayes' Theorem to Determine the Probability of Having a Disease

Let D be the event that a person has a rare disease. Suppose that the rare disease has an incidence rate of 1% in the population, $P(D) = 0.01$. $\bar{D}$ is the event that a person does not have the rare disease (i.e., the complement of D). Suppose a machine is used to diagnose the disease. Let C be the event that the disease is confirmed as the diagnosis. Suppose that the probability of the machine falsely confirming the disease when one doesn't have it is $P\left(C \mid \bar{D}\right) = 0.15$, called a *false positive*; while $P(C \mid D) = 0.95$, which says that the machine correctly confirms the disease with an accuracy of 95%. Now, suppose that the machine confirms that a person has the disease. What is the probability that the person actually has the disease? In other words, what is $P(D \mid C)$?

Solution

Here are the probabilities given to us in the problem.

$$\begin{aligned} P(D) &= 0.01 \\ P(\bar{D}) &= 0.99 \\ P(C \mid D) &= 0.95 \\ P(C \mid \bar{D}) &= 0.15 \end{aligned}$$

To find the probability that a person with a positive diagnostic result actually has the disease we proceed as follows.

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$$
\begin{aligned}
P(D \mid C) &= \frac{P(D \cap C)}{P(C)} \\
&= \frac{P(D \cap C)}{P(C \cap D) + P(C \cap \overline{D})} \\
&= \frac{P(D) \cdot P(C \mid D)}{P(D) \cdot P(C \mid D) + P(\overline{D}) \cdot P(C \mid \overline{D})} \\
&= \frac{0.01 \cdot 0.95}{0.01 \cdot 0.95 + 0.99 \cdot 0.15} \\
&= \frac{0.0095}{0.0095 + 0.1485} = \frac{0.0095}{0.1580} \approx 0.06
\end{aligned}
$$

This is somewhat of an assuring result in that you have only a 6% chance of having the disease even though the machine yielded a positive diagnostic.

6.5 Exercises

Basic Concepts

1. Briefly explain the relationship between conditional probability and Bayes' Theorem.
2. What is the difference between prior and posterior probabilities?
3. What is Bayes' Theorem?
4. How is Bayes' Theorem used to "revise" a probability based on additional information?

Excercises

5. In a production line, 8% of all items produced are defective. 75% of all defective items are fully inspected, while 10% of all non-defective items go through a complete inspection. Given that an item is completely inspected, what is the probability that it is defective?

6. The issue of Corporate Tax Reform has been cause for much debate in the United States, especially in the House Ways and Means Committee as well as the Senate Finance Committee. Among those in the legislature, 45% are Republicans and 55% are Democrats. It is reported that 30% of the Republicans and 70% of the Democrats favor some type of Corporate Tax Reform to prevent American companies from operating in foreign countries. Suppose a member of Congress is randomly selected and they are found to favor some type of corporate tax reform. What is the probability that this person is a Democrat?

Randomness and Statistics: Part 1

Henri Poincaré was a French mathematician, theoretical physicist, and philosopher of science and is considered one of the greatest mathematicians of all time. Concerning the game of roulette, he remarked that the game appears random because we are ignorant of what causes the various results (numbers). That is, if we had a complete mathematical model of the physics of the roulette wheel and the initial conditions (speed of the ball, speed of the wheel, and position on the wheel when the ball was thrown), we could predict with certainty what the outcome would be. However, since we don't have such a model, the whole process appears random because we are ignorant of the initial conditions and the physical model.

One of the reasons we study probability distributions is that they provide a means of describing our state of ignorance about the outcome of some random phenomena. One of the reasons we study statistics is that it can teach us how to develop statistical models that can reduce our ignorance of seemingly random phenomena. Statistical prediction models have been developed for climate modeling, weather forecasting, ecological modeling, sports modeling, crime prediction, stock market prediction, economic forecasting, credit scoring, disease prediction, epidemiology, horse racing, natural language processing, and autonomous systems, among others.

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