$$\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} 2\pi r\, ds \\
&= 2\pi \int_{-\pi/2}^{\pi/2} (R\cos t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= 2\pi R \int_{-\pi/2}^{\pi/2} \cos t \sqrt{R^2 \sin^2 t + R^2 \cos^2 t}\, dt \\
&= 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos t\, dt \\
&= 2\pi R^2 \left[\sin t\right]_{-\pi/2}^{\pi/2} = 4\pi R^2
\end{aligned}$$

9.2 Exercises

1–4 Identify the curve as differentiable, smooth, or neither.

1. $x = t + 3, \quad y = 2|t - 1|, \quad 0 \le t \le 2$
2. $x = t - 1, \quad y = 2t^2, \quad t \in \theta$
3. $x = -2t^2, \quad y = t^3 + 1, \quad -5 \le t \le 5$
4. $x = 3\cos\theta, \quad y = 2\sin\theta, \quad -\pi < \theta \le \pi$
5. Show that (t, t) is a smooth parametrization of the (smooth) curve $y = x$, while (t^3, t^3) is not.

6–14 Find the equations of any horizontal or vertical tangent lines to the curve.

6. $x = 5t - 2, \quad y = 6t^2 + 1$
7. $x = t^2 - t, \quad y = 1 + 2t^2$
8. $x = 2t^2 + 1, \quad y = (t - 2)^2$
9. $x = t^3 + 1, \quad y = t$
10. $x = 3t^2, \quad y = \dfrac{1}{t + 3}, \quad t > -3$
11. $x = \sqrt{t}, \quad y = \ln t, \quad t \ge 1$
12. $x = \sqrt{t}, \quad y = t^2 - 2$
13. $x = 2\cos t, \quad y = 5\sin t$
14. $x = \tan t, \quad y = \sec t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

15–24 Find the values of dy/dx and d^2y/dx^2 for the curve at the given point.

15. $x = 2t - 1, \quad y = 2t^2 + 1; \quad \left(0, \dfrac{3}{2}\right)$
16. $x = 2t^3 + 1, \quad y = (t + 1)^2; \quad (3, 4)$
17. $x = \dfrac{1}{t + 1}, \quad y = t^2 - 1; \quad (1, -1)$
18. $x = \sqrt{t}, \quad y = 2t + 1; \quad (1, 3)$
19. $x = te^t, \quad y = e^{-t}; \quad (0, 1)$
20. $x = \cos t, \quad y = \ln(\cos t); \quad \left(\dfrac{1}{2}, -\ln 2\right)$
21. $x = \sin t, \quad y = \sin 2t; \quad \left(\dfrac{\sqrt{2}}{2}, 1\right)$
22. $x = \dfrac{\sin^3 t}{3}, \quad y = \dfrac{\cos^3 t}{3}; \quad \left(\dfrac{\sqrt{2}}{12}, \dfrac{\sqrt{2}}{12}\right)$
23. $x = \ln t, \quad y = 2\sqrt{t}; \quad (0, 2)$
24. $x = \cos 2t, \quad y = \sin 3t; \quad \left(\dfrac{1}{2}, 1\right)$

25–28 Find the value(s) of the parameter for any inflection point(s) of the given curve.

25. $x = t + 2, \quad y = t^3 - 10t^2$
26. $x = t^2(t^2 - 4), \quad y = 2t + 3$
27. $x = t^4 - t, \quad y = t^2$
28. $x = \sqrt{t}, \quad y = t^2 - 5t$

29–34 Find the area enclosed by the given curve.

29. $x = \sin t, \quad y = \sin\dfrac{t}{2}, \quad 0 \le t \le 2\pi$
30. $x = 2\cos t, \quad y = \sin 2t, \quad -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$
31. $x = 2t\cos t, \quad y = 2t\sin t, \quad -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$
32. $x = \dfrac{3}{4}\cos t + \dfrac{1}{4}\cos 3t, \ y = \dfrac{3}{4}\sin t - \dfrac{1}{4}\sin 3t, \ 0 \le t \le 2\pi$

33. $x = \sin t, \quad y = e^{-t/2}, \quad 0 \le t \le \pi$ and the y-axis

34. $x = t^4 - 4, \quad y = t^3 - 2t, \quad -\sqrt{2} \le t \le \sqrt{2}$

35. Use the parametric representation of the ellipse from Exercise 35 of Section 9.1 to arrive at its area formula of $A = \pi ab$.

36. Prove that the area A under one arch of the cycloid generated by a circle of radius a is three times the area of the circle, $A = 3\pi a^2$. (This result was first proved by the French mathematician Gilles de Roberval ca. 1630.)

37. Sometimes it may be necessary or convenient to divide a region between a curve and the y-axis into horizontal strips when approximating its area. Modify our discussion preceding Example 4 to arrive at the formula $\int_{y=g(a)}^{y=g(b)} x\,dy = \int_{t=a}^{t=b} f(t)g'(t)\,dt$.

38–39 Use Exercise 37 to determine the area between the given curve and the y-axis.

38. $x = 2t - 1, \quad y = t^2 - 2, \quad \frac{1}{2} \le t \le 3$

39. $x = 2\cos t, \quad y = 1 + \text{sin}t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

40–47 Find the arc length of the given curve over the indicated interval.

40. $x = t^2, \quad y = t^3 - \frac{t}{3}, \quad 1 \le t \le 3$

41. $x = \sqrt{t}, \quad y = \frac{t^2}{8} + \frac{1}{4t}, \quad 1 \le t \le 9$

42. $x = e^t, \quad y = \frac{e^{2t}}{8} - t, \quad 0 \le t \le 2$

43. $x = \frac{1}{t}, \quad y = \frac{t}{4} + \frac{1}{3t^3}, \quad \frac{1}{2} \le t \le 1$

44. $x = \sin^3 t, \quad y = \cos^3 t, \quad 0 \le t \le 2\pi$

45. $x = \sqrt{2}e^t \sin t, \quad y = \sqrt{2}e^t \cos t, \quad 0 \le t \le \frac{\pi}{2}$

46. $x = t^3, \quad y = t\left(t^4 - \frac{9}{20}\right), \quad 0 \le t \le 1$

47. $x = -\ln t, \quad y = t + \frac{1}{4t}, \quad \frac{1}{2e} \le t \le 1$

48–57 Find the area of the surface generated by revolving the parametric curve about the indicated axis.

48. $x = t + 1, \quad y = \frac{t-1}{2}, \quad 1 \le t \le 5,$

a. about the x-axis **b.** about the y-axis

49. $x = 2t - 3, \quad y = 8 - 2t, \quad 2 \le t \le 4,$

a. about the x-axis **b.** about the y-axis

50. $x = 4t^2 - 4t + 4, \quad y = 2t - 1, \quad 1 \le t \le 2,$

about the x-axis

51. $x = t, \quad y = t^3 + \frac{1}{12t}, \quad 1 \le t \le 2,$

a. about the x-axis **b.** about the y-axis

52. $x = t^2, \quad y = 2t^3 - \frac{t}{6}, \quad 0 \le t \le 1,$

a. about the x-axis **b.** about the y-axis

53. $x = \ln t, \quad y = t + \frac{1}{4t}, \quad 1 \le t \le e,$

a. about the x-axis **b.** about the y-axis

54. $x = t^5 - \frac{9t}{20}, \quad y = t^3, \quad 0 \le t \le 2,$

about the x-axis

55. $x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \le t \le \pi,$

about the y-axis

56. $x = 2\cos t, \quad y = \sin t, \quad 0 \le t \le \pi,$

about the x-axis

57. $x = 2t + 1, \quad y = \cosh 2t, \quad 0 \le t \le 1,$

a. about the x-axis **b.** about the y-axis

58. Suppose one arch of the cycloid generated by a circle of radius 1 rolling along the x-axis is rotated about the x-axis. Find the area of the resulting surface.

59. Revisit Example 5 by using the more general parametrization found in Exercise 34 of Section 9.1. Do you obtain the same answer?

60. Evaluate the integral in Example 6 by writing it as

$$\int \frac{1+t^2}{\sqrt{1+t^2}}\,dt = \int \frac{1}{\sqrt{1+t^2}}\,dt + \int \frac{t^2}{\sqrt{1+t^2}}\,dt;$$

then show

$$\int \frac{t^2}{\sqrt{1+t^2}}\,dt = t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt,$$

and finally proceed to solve for $\int \sqrt{1+t^2}\,dt$.

61–64 Consider a particle moving along a curve in the *xy*-plane such that its coordinates at time *t* are $x = f(t)$, $y = g(t)$, $a \le t \le b$. Since the distance traveled by the particle at time *t* can be calculated from the arc length

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du,$$

by Part I of the Fundamental Theorem of Calculus (Section 5.3) we obtain

$$\begin{aligned}\text{speed} = \frac{ds}{dt} &= \frac{d}{dt}\int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du \\ &= \sqrt{[x'(t)]^2 + [y'(t)]^2}.\end{aligned}$$

In Exercises 61–64, use the above formula to determine the speed of the particle traveling along the given curve at the specified time. (Distance is measured in meters, time in seconds.)

61. $x = t^2, \quad y = 2t + 1; \quad t = 2$

62. $x = \ln(t+1), \quad y = t^2 - 2; \quad t = 1$

63. $x = 3\cos 3t, \quad y = \sin 3t; \quad t = \frac{\pi}{2}$

64. $x = t^2, \quad y = \sqrt{t}; \quad t = 4$

65. Suppose that the position of a particle in the *xy*-coordinate system is given by $x = t^2 - 1, y = t^3 - 6t$, $t \ge 0$. When does the particle reach its minimum speed?

66. Using the discussion preceding Example 4, prove the following: If $x = f(t)$ and $y = g(t)$ define a parametric curve over $[a, b]$ such that *g* is continuous, *f* is continuously differentiable, and $y(x)$ is a continuous function of *x*, then

$$\int_{f(a)}^{f(b)} y\, dx = \int_a^b g(t) f'(t)\, dt.$$

67–70 Use the result of Exercise 66 to solve the exercise.

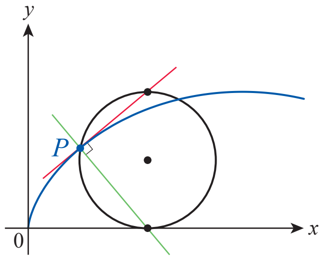
67. Find the volume of the solid generated by rotating the parametrically defined curve $x = 2t - 1, y = 1 - t^2$, $-1 \le t \le 1$, about the *x*-axis.

68. Repeat Exercise 67 for the parametric curve $x = \sin^3 t$, $y = \cos^3 t$, $-\pi/2 \le t \le \pi/2$.

69. Find the centroid of the region bounded by the parametric curve $x = 9 - t$, $y = \sqrt{t}$, $0 \le t \le 9$, and the coordinate axes.

70. Repeat Exercise 69 for the parametric curve $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le \pi/2$.

71. Prove the following property of cycloids: If *P* is the intersection point of the rolling circle and the cycloid, then the line tangent to the cycloid at *P* passes through the highest point of the circle, while the normal line intersects the circle at its lowest point.



72.* Prove that if the curve *C* defined by $x = f(t)$ and $y = g(t)$, $t \in [a, b]$, is differentiable with both f' and g' bounded (meaning there are constants K_1 and K_2 such that $|f'(t)| \le K_1$ and $|g'(t)| \le K_2$ for all $t \in [a, b]$), then *C* has finite length (such curves of finite length are called *rectifiable*). (**Hint:** As a first step, use the boundedness of the derivatives and the Mean Value Theorem to prove that there is a constant *M* such that whenever $P_1(f(t_1), g(t_1))$ and $P_2(f(t_2), g(t_2))$ are two points on *C*, $|f(t_2) - f(t_1)| \le M|t_2 - t_1|$, and $|g(t_2) - g(t_1)| \le M|t_2 - t_1|$. As a consequence, $|P_1P_2| \le 2M|t_2 - t_1|$. Now you can use a Riemann-sum argument to finish your proof.)

73. Use Exercise 72 to prove that if the parametric curve *C* is continuously differentiable on $[a, b]$ (i.e., both f' and g' have continuous derivatives on $[a, b]$), then *C* is rectifiable.

Concept Check

74–78 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. Every smooth parametric curve is differentiable.

75. Every continuous, differentiable parametric curve is smooth.

76. If *C* is smooth at (x_0, y_0), then for every parametrization such that $x_0 = f(t_0)$ and $y_0 = g(t_0)$, both f' and g' are continuous at t_0 and at least one of $f'(t_0)$ and $g'(t_0)$ is nonzero.

77. If the graph of a parametric curve is a continuously differentiable function $y = f(x)$, then the curve is smooth.

78. If a parametric curve is defined by $x = f(t)$ and $y = g(t)$, then $\dfrac{d^2y}{dx^2} = \dfrac{d^2y/dt^2}{d^2x/dt^2}$.

9.2 Technology Exercises

79. Use a computer algebra system to approximate the length of the ellipse parametrized by $x = 5\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$.

80–81 Use a computer algebra system to approximate the length of the curve with the given parametrization.

80. $\left(\sqrt{t}, \sin t\right), \quad 0 \le t \le 2\pi$

81. $\left(\ln(t+1), \sqrt{t}\right), \quad 0 \le t \le 9$

82–83 Use a computer algebra system to approximate the area of the surface obtained by rotating the given parametric curve about the **a.** x-axis and **b.** y-axis.

82. $\left(3t-2, \sqrt{1+\dfrac{1}{t}}\right), \quad 1 \le t \le 2$

83. $\left(4t+1, \sqrt{t^2+1}\right), \quad -\dfrac{1}{4} \le t \le 2$