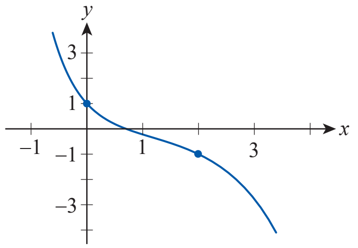

Figure 4
Boundary Value Problem Solution

$$c_1 = \frac{-e^4 - 1}{e^6 - 1} \quad \text{and} \quad c_2 = \frac{e^6 + e^4}{e^6 - 1}$$

The graph of $y(x) = c_1 e^x + c_2 e^{-2x}$ for these particular choices of c_1 and c_2 is shown in Figure 4. Note that the graph does indeed satisfy the boundary conditions.

b. The boundary conditions $y(0) = 0$ and $y(2) = 0$ can only be imposed on the general solution $y(x) = c_1 e^x + c_2 e^{-2x}$ by letting $c_1 = 0$ and $c_2 = 0$, resulting in the trivial solution $y(x) = 0$.

8.4 **Exercises**

1–18 Find the general solution of the given differential equation.

1. $y'' + y' - 2y = 0$

2. $y'' - 4y = 0$

3. $3y'' - 5y' - 2y = 0$

4. $2y'' + 9y' - 5y = 0$

5. $y'' + 2y' + y = 0$

6. $2y'' - 12y' + 18y = 0$

7. $y'' + y' + \frac{y}{4} = 0$

8. $y'' + y = 0$

9. $y'' - 4y' + 13y = 0$

10. $y'' - 2y' + 5y = 0$

11. $25y'' - 10y' + y = 0$

12. $3y'' - \sqrt{3}y' + y = 0$

13. $3\frac{d^2y}{dx^2} - \sqrt{13}\frac{dy}{dx} + y = 0$

14. $\frac{d^2S}{dt^2} - 12S = 0$

15. $2\frac{d^2w}{dt^2} + 2\frac{dw}{dt} + w = 0$

16. $y'' + k^2 y = 0$

17. $y'' - k^2 y = 0$

18. $y'' - k^2 y' = 0$

19–26 Solve the given second-order initial value problem.

19. $y'' - 3y = 0; \quad y(0) = 5; \quad y'(0) = 0$

20. $y'' - 3y' - 4y = 0; \quad y(0) = 1; \quad y'(0) = -1$

21. $y'' - y' - 2y = 0; \quad y(0) = 4; \quad y'(0) = 5$

22. $y'' - 2y' + y = 0; \quad y(0) = -1; \quad y'(0) = 0$

23. $y'' - 2y' + 5y = 0; \quad y(0) = 1; \quad y'(0) = -3$

24. $y'' - 6y' + 18y = 0; \quad y(0) = 0; \quad y'(0) = 6$

25. $4y'' - 12y' = -9y; \quad y(0) = -1; \quad y'(0) = \frac{1}{2}$

26. $y'' - 10y' + 26y = 0; \quad y(0) = -2; \quad y'(0) = -7$

27–34 Solve the boundary value problem, if possible.

27. $y'' - 5y' = 0; \quad y(0) = 5 - 5e^5; \quad y(1) = 0$

28. $9y'' - 6y' + y = 0; \quad y(0) = -1; \quad y(1) = 0$

29. $y'' - y = 0; \quad y(0) = 1; \quad y(1) = e$

30. $y'' + 0.2y' + 0.01y = 0; \quad y(0) = 0; \quad y(2) = 4e^{-1/5}$

31. $y'' + 9y = 0; \quad y(0) = 0; \quad y\left(\frac{2\pi}{3}\right) = 1$

32. $y'' + 4y' = 0; \quad y(0) = -5; \quad y(3) = -5$

33. $y'' - 2y' + 17y = 0; \quad y(0) = 2; \quad y\left(\frac{\pi}{8}\right) = 1$

34. $4y'' - 4y' + 5y = 0; \quad y(0) = 1; \quad y\left(\frac{\pi}{2}\right) = 1$

35–38 Our techniques from this section easily generalize to higher-order homogeneous linear equations with constant coefficients. For example, the characteristic equation of

$$y''' - 3y'' + y' - 3y = 0$$

is the cubic polynomial

$$r^3 - 3r^2 + r - 3 = 0$$

with characteristic roots $r_1 = 3$ and $r_{2,3} = \pm i$. These give rise to the following general solution.

$$y = c_1 e^{3x} + c_2 \cos x + c_3 \sin x$$

In Exercises 35–38, use this generalized technique to find the general solution of the differential equation.

35. $y''' - 2y'' - 3y' = 0$

36. $y''' - 4y'' + 5y' - 2y = 0$

37. $y''' - y'' + 2y' - 2y = 0$

38. $y^{(4)} - 3y'' - 4y = 0$

39–42 The following exercises offer a glimpse into one way of handling certain nonhomogeneous linear equations. A more general version of the theorem below is proved in differential equations texts.

Theorem: If y_p is any given particular solution of the nonhomogeneous linear equation

$$ay'' + by' + cy = F(x) \qquad \text{(1a)}$$

on an interval I and y_c (also called the complementary function) is the general solution of the associated homogeneous equation

$$ay'' + by' + cy = 0 \qquad \text{(1b)}$$

on the same interval, then the general solution of (1a) on I can be written as follows.

$$y = y_c + y_p$$

In other words, the general solution of (1a) is the sum of any one of its particular solutions and the general solution of the associated homogeneous equation (1b).

For example, you can easily verify that $y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x$ is a particular solution of the equation

$$y'' + y' = \cos x \qquad \text{(2)}$$

while the general solution of its associated homogeneous equation $y'' + y' = 0$ is $y_c = c_1 + c_2 e^{-x}$. Thus, the general solution of (2) is as follows.

$$y = y_c + y_p = c_1 + c_2 e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$

But how can we find a particular solution y_p in order to put the theorem to work? One method that works well for certain equations is one we have already seen in Exercises 99–102 of Section 7.1—the method of undetermined coefficients. In the above case, knowing that all derivatives of the sine and cosine functions are again of the same type, we might guess that a particular solution of (2) has the following form.

$$y_p = A\cos x + B\sin x$$

Substituting this into (2), we obtain

$$(B - A)\cos x - (A + B)\sin x = \cos x,$$

which yields the coefficients $A = -\frac{1}{2}$ and $B = \frac{1}{2}$, and thus,

$$y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x.$$

In Exercises 39–42, use this theorem and the method of undetermined coefficients to find the general solution of the equation with the indicated initial "guess" for y_p.

39. $y'' + y' - 2y = x$; guess $y_p = Ax + B$

40. $y'' + 2y = x^2 + 1$; guess $y_p = Ax^2 + Bx + C$

41. $y'' + 4y' - 5y = \sin x$; guess $y_p = A\cos x + B\sin x$

42. $y'' + 4y = e^{3x}$; guess $y_p = Ae^{3x}$

43–44 Use the method described in the directions for Exercises 39–42 to solve the given initial value problem.

43. $y'' + y = x^2$; $y(0) = -1$; $y'(0) = 1$; guess $y_p = Ax^2 + Bx + C$

44. $y'' - 2y' = 4\sin 2x$; $y(0) = 1$; $y'(0) = -2$; guess $y_p = A\cos 2x + B\sin 2x$

45. Show that the only solution of the boundary value problem

$$y'' + 4y = 0; \quad y(0) = 0; \quad y\left(\frac{\pi}{4}\right) = 0$$

is the trivial solution $y = 0$.

46. Show that the boundary value problem

$$y'' + 9y = 0; \quad y(0) = 0; \quad y\left(\frac{2\pi}{3}\right) = 0$$

has infinitely many solutions. (Contrast this with Exercise 31.)

47. Suppose that

$$2\frac{d^2y}{dt^2} + c\frac{dy}{dt} + 5y = 0$$

is the equation of a damped oscillating motion. Find a value of c such that the motion is **a.** underdamped, **b.** critically damped, and **c.** overdamped. Using the case analysis of Example 5, explain why these terms are appropriate for each type of motion. (Answers to parts a. and c. will vary.)

48. The viscosity of hydraulic fluid in automobile shock absorbers will determine the value of c in an equation of damped oscillating motion (see Example 5 or Exercise 47). Discuss which of the three cases is appropriate for the design of automobile shock absorbers and why. (Shock absorbers are designed for cars to prevent or "smooth out" wheel oscillations caused by an uneven road surface.)

49. When suspended at one end of a spring, an object of mass 0.1 kg stretches the spring by 5 cm. Find the value of c such that the resulting motion is critically damped. (This value is called the *critical damping constant*. For a refresher on how to determine the spring constant, see Example 2 in Section 6.5.)

50. Suppose that a 5 kg object attached to a spring with spring constant $k = 13$ is pulled down 25 cm below equilibrium and released. Find and graph the displacement function if the surrounding medium offers resistance with a damping constant of $c = 2$ kg/s.

51. A 4 lb weight stretches a spring by 6 in., while the damping constant is $\frac{1}{4}$ slug/s. The weight is pulled down 9 in. below equilibrium and released with an upward velocity of 2 ft/s. Find the equation of motion, solve it for the displacement function, and graph your result. Use 1 slug ≈ 32 lb.

52. A 4 kg object stretches a spring by 12 cm. The object is then pushed upward from equilibrium by 20 cm and released. Find and graph the displacement function if the damping constant is $c = 3$ kg/s.

53. If we place the system of Exercise 52 in a high-viscosity fluid with a damping constant of 75 kg/s, the motion will become overdamped. Find and graph the displacement function in this case.

54. Show that if the mass-spring model of Example 5 is critically damped or overdamped, that is, if $c^2 - 4mk \geq 0$, then the oscillating object cannot pass through the equilibrium more than once. (**Hint:** Show that the equation $y(t) = 0$ cannot have two or more solutions.)

55. Prove that $y = c_1 \cosh(kx) + c_2 \sinh(kx)$ is a general solution of the equation in Exercise 17. Show that the above family of functions is the same as the one you obtained in Exercise 17. (**Hint:** Start by showing that both $y_1(x) = \sinh(kx)$ and $y_2(x) = \cosh(kx)$ satisfy the differential equation, and argue that they are also linearly independent. Compare this answer to the one from Exercise 16.)

56. Show that if $y(t)$ is the solution of a damped mass-spring model

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0,$$

then

$$\lim_{t\to\infty} y(t) = 0.$$

(**Hint:** Handle the critically, under-, and overdamped cases separately, using the fact that all constants m, c, and k are greater than 0 in the equation of motion. Note that the conclusion of this exercise is consistent with our everyday experience of damped oscillations "dying down" over time.)

57.* At an amusement park, a boat slides down a ramp and splashes into the water at a speed of 15 m/s. The resistance offered by the water is proportional to the boat's speed with a coefficient of 270 kg/s. If the combined mass of the boat and passengers is 300 kg, how long does it take for it to come to a complete stop? What distance will it travel in the water while slowing down? Consider the boat stopped if your model predicts a velocity less than 1 cm/s. (**Hint:** See Example 5.)

58.* An RLC circuit is a simple electric circuit with inductance L (in henries, H), resistance R (in ohms, Ω), and capacitance C (in farads, F). The differential equation describing an RLC circuit is as follows.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V$$

(Note that $I(t) = dq(t)/dt$. See Exercise 62 in Section 8.1 and Example 5 in Section 8.2.)

Suppose that in an RLC circuit a switch is open (i.e., there is no current), $V = 0$, and the capacitor has an initial charge of 3 coulombs. Then, at time $t = 0$, the switch is flipped closed. Find and graph the current $I(t)$ if the capacitance is $C = 10^{-2}$ F, the resistance is 1.5 Ω, and the inductance is 0.1 H. What happens to $I(t)$ as $t \to \infty$?