Example 5 Using Euler's Method

Use Euler's method to approximate a solution to the IVP $y' = y - 2x$ with the condition $y(0) = 1$.

Solution

We have $x_0 = 0$, $y_0 = 1$, and $f(x, y) = y - 2x$. We need to choose a *step size h* in order to apply the iterative approximations $x_n = x_0 + nh$ and $y_n = y_{n-1} + h(y_{n-1} - 2x_{n-1})$. We will compare the results with $h = 0.5$ and $h = 0.1$.

$h = 0.5$

n	x_n	y_n
0	0	1
1	0.5	1.5
2	1.0	1.75
3	1.5	1.625
4	2.0	0.9375

Table 1

$h = 0.1$

n	x_n	y_n	n	x_n	y_n
0	0	1	11	1.1	1.3470
1	0.1	1.1	12	1.2	1.2617
2	0.2	1.19	13	1.3	1.1479
3	0.3	1.269	14	1.4	1.0027
4	0.4	1.3359	15	1.5	0.8230
5	0.5	1.3895	16	1.6	0.6053
6	0.6	1.4285	17	1.7	0.3458
7	0.7	1.4514	18	1.8	0.0404
8	0.8	1.4565	19	1.9	–0.3156
9	0.9	1.4422	20	2.0	–0.7272
10	1.0	1.4064			

Table 2

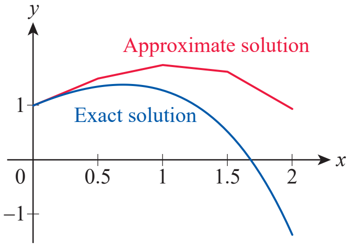


Figure 8 $h = 0.5$

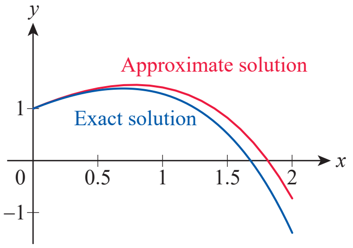


Figure 9 $h = 0.1$

The tables show the results of calculating (x_n, y_n) up to $x = 2$ with the two different step sizes, and Figures 8 and 9 compare the graphs of these estimated points (in red) with the exact solution $y = 2x + 2 - e^x$ (which you will determine when you solve the first-order linear equation $y' = y - 2x$ in Exercise 25).

8.3 Exercises

1–6 Match the differential equation with its slope field (labeled A–F).

1. $y' = x$

2. $y' = 1 - yx$

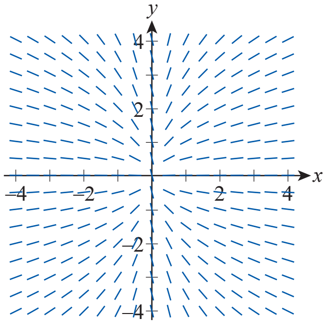
3. $y' = \dfrac{y}{2x}$

4. $y' = \dfrac{x^2 y}{3}$

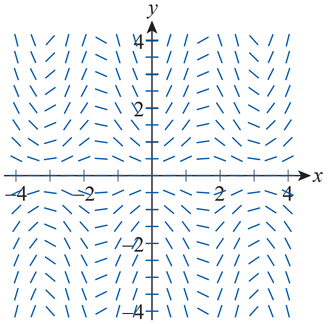
5. $y' = \sqrt{x^2 + y^2}$

6. $y' = y \sin 2x$

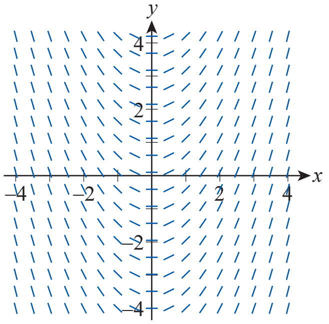
A.



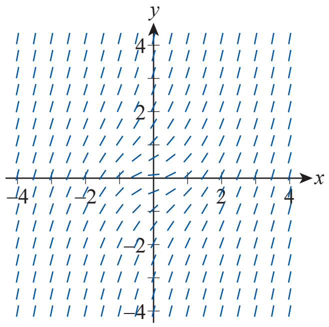
B.



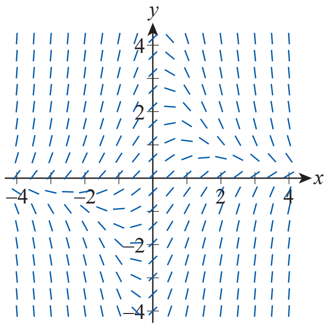
C.



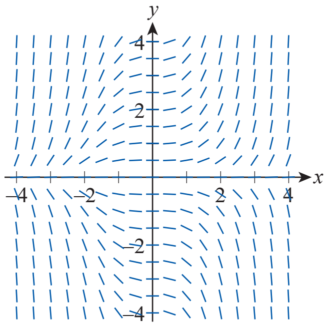
D.



E.

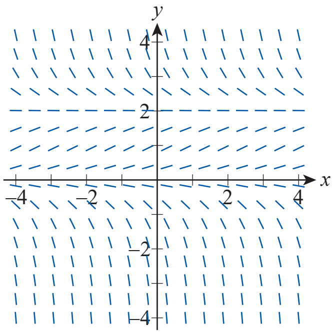


F.

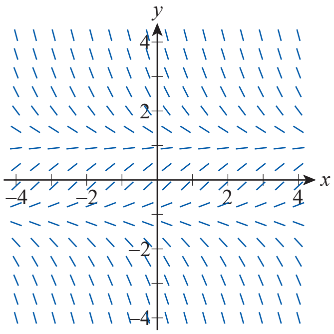


7–10 An autonomous equation and its slope field are given. Find any equilibrium solutions and classify them as stable or unstable.

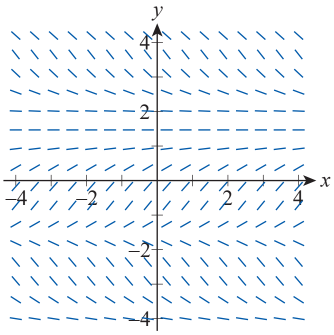
7. $y' = y - \frac{y^2}{2}$



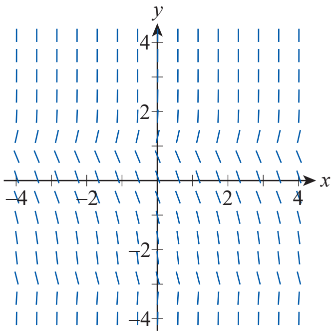
8. $y' = \frac{1-y^2}{\sqrt{1+y^2}}$



9. $y' = \cos y(1-\sin y), \quad -4 \le y \le 4$



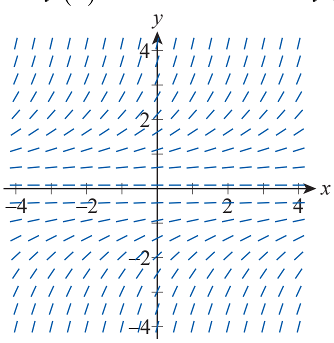
10. $y' = (y+3)(y^3-1)$



11–12 An autonomous equation and its slope field are given. Sketch the graphs of the particular solutions satisfying the specified initial conditions.

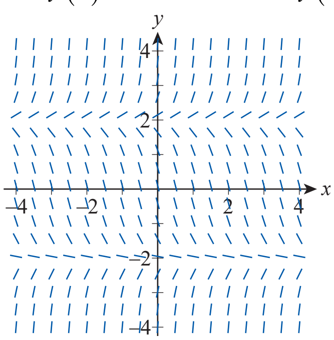
11. $y' = \frac{1}{4}y^2$

a. $y(0) = -1$ **b.** $y(1) = 1$



12. $y' = y^2 - 4$

a. $y(0) = 0$ **b.** $y(4) = 4$



13–16 Graph by hand the slope field of the given differential equation. If applicable, find and classify each equilibrium solution as stable or unstable.

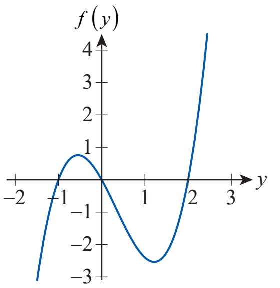
13. $y' = \frac{1}{3}y$

14. $y' = \left(2 - \frac{1}{2}y^2\right)y$

15. $y' = -y(1-y)(2-y)$

16. $y' = x - 3y$

17. Create a rough sketch of the slope field of the differential equation $y' = f(y)$, where the graph of f is given below. Classify equilibria as stable or unstable.



18. A can of soda that was forgotten on the kitchen counter and warmed up to 22 °C was put back in the refrigerator whose interior temperature is kept at a constant 3 °C. If the soda's temperature after 5 minutes is 17 °C, what will its temperature be after 20 minutes? Sketch the slope field resulting from your model; include the equilibrium and the soda's temperature curve.

19. A cake is removed from a 320 °F oven and is kept at a room temperature of 72 °F to cool down. The cake's temperature after 4 minutes is 190 °F. Use Newton's Law of Cooling to model the cooling process, and make a rough sketch of the slope field for your model, highlighting the equilibrium and the cake's temperature curve.

20. Recall from Exercise 79 in Section 7.2 that the spread of a disease in a community of N people can be modeled by the logistic differential equation $dI/dt = kI(N - I)$, where $I(t)$ stands for the number of persons already infected. Sketch the slope field and graph of this model, assuming a community of 200 people with one sick person initially, and five more catching the disease three days later.

21. An owl population grows logistically with a carrying capacity of 500 owls and constant of proportionality $k = 0.3$ per year.

a. Find the population size $P(t)$ as a function of time if initially 100 owls are present in the ecosystem.

b. How long does it take for the owl population to reach 300?

22. Use the method of Example 4 to answer the question in Exercise 63b of Section 8.1.

23. Repeat Exercise 22, this time assuming that the air resistance is proportional to $\sqrt{v}$.

24. Find the terminal velocity in Example 4 by solving the equation $\frac{dv}{dt} = g - \frac{k}{m}v^2$ and finding $\lim_{t\to\infty} v(t)$. (**Hint:** Begin with the two steps below.)

$$\frac{dv}{dt} = g\left(1 - \frac{k}{mg}v^2\right)$$

$$\frac{mg}{k}\left(\frac{dv}{\frac{mg}{k} - v^2}\right) = g\,dt$$

25. Verify that $y = 2x + 2 - e^x$ is the exact solution of the initial value problem of Example 5, that is, $y' = y - 2x$ with the condition $y(0) = 1$.

26–31 For the initial value problem, **a.** use Euler's method with the indicated step sizes to approximate the given value of y and **b.** solve the IVP by conventional methods and compare your approximations with the exact answer.

26. $y' = 3y;\quad y(0) = 2;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.125$

27. $y' = 2y + x;\quad y(0) = 1;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.1$

28. $y' = xy;\quad y(0) = 1;$
approximate $y(2)$ with (i) $h = 0.4$ (ii) $h = 0.2$

29. $y' = x^2 - y;\quad y(0) = 3;$
approximate $y(1.5)$ with (i) $h = 0.3$ (ii) $h = 0.15$

30. $y' = 2x - 2y + 1;\quad y(0) = -1;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.1$

31. $y' = 1 + y^2;\quad y(0) = 0;$
approximate $y\left(\frac{\pi}{3}\right)$ with (i) $h = \frac{\pi}{12}$ (ii) $h = \frac{\pi}{24}$

32–33. The following formula is called the improved Euler's method, or Heun's method.

$$x_n = x_{n-1} + h = x_0 + nh \text{ and}$$

$$y_n = y_{n-1} + h\frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2}, \text{ where}$$

$$y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

(Note that the quotient appearing in this formula can be interpreted as the average slope between x_{n-1} and x_n. The derivation of Heun's method is left for a textbook on differential equations or numerical methods.)

Use Heun's method to redo Exercises 26–27 with $h = 0.25$ and compare your results with your earlier answers to illustrate the accuracy of the improved Euler's method.

8.3 Technology Exercises

34–37. Use a graphing utility to create the slope fields you sketched in Exercises 13–16. If applicable, visually check the location of, and classify, any equilibria.

38–43. Use a graphing utility to improve your approximations in Exercises 26–31, using Euler's method with 20 equal increments. Then graph the results along with the exact solutions to visually check the accuracy of the method.