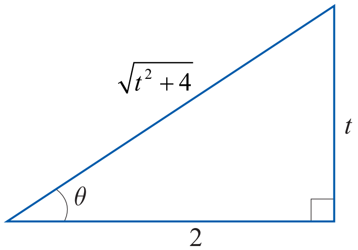


Figure 6 $\tan\theta = t/2$

Note that our antiderivative is a function of θ, but the limits of integration are still in terms of t. In order to evaluate the expression, we sketch the triangle in Figure 6 and write everything in terms of t.

$$\frac{1}{12}\sin\theta\Big]_{t=0}^{t=3} = \frac{1}{12}\cdot\frac{t}{\sqrt{t^2+4}}\Big]_{t=0}^{t=3}$$

$$= \frac{1}{12}\cdot\frac{3}{\sqrt{13}} = \frac{1}{4\sqrt{13}}$$

7.4 Exercises

1–6 Choose the substitution(s) that are helpful in evaluating the integral. (Do not actually evaluate the integral. There may be more than one correct answer.)

1. $\int x\sqrt{x-1}\,dx$

a. $x = \tan\theta$ **b.** $x = \sec\theta$

c. $x = \sin\theta$ **d.** $\theta = x - 1$

2. $\int \frac{dx}{\sqrt{x^2+1}}$

a. $x = \sin\theta$ **b.** $\tan x = \theta$

c. $x = \tan\theta$ **d.** $x = \sec\theta$

3. $\int x\sqrt{9-x^2}\,dx$

a. $x = 3\sec\theta$ **b.** $\theta = 3\sin x$

c. $x = 3\sin\theta$ **d.** $\theta = 9 - x^2$

4. $\int \frac{dz}{z^2\sqrt{z^2-9}}$

a. $z = \sin t$ **b.** $z = \sec t$

c. $z = 3\sec t$ **d.** $z = 3\cosh t$

5. $\int \frac{dx}{x^2\sqrt{4x^2+9}}$

a. $x = 2\tan t$ **b.** $x = \frac{3}{2}\tan t$

c. $x = 3\tan t$ **d.** $x = \frac{3}{2}\sec t$

6. $\int \frac{\sqrt{9-(x-1)^2}}{x-1}\,dx$

a. $x = 3\sin\theta + 1$ **b.** $x = 3\sin\theta$

c. $x - 1 = 3\sec t$ **d.** $x - 1 = 3\tan t$

7–12 Choose the correct answer. (You should be able to identify the correct answer without actually evaluating the integral.)

7. $\int \frac{dx}{\sqrt{x^2+1}}$

a. $\sqrt{\arctan x} + C$

b. $2\sqrt{x^2+1} + C$

c. $\ln\left|\sqrt{x^2+1} + x\right| + C$

d. $\ln\sqrt{x^2+1} + C$

8. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

a. $\frac{x^2}{2}\arcsin\frac{x}{2} + C$

b. $\frac{2x^3}{3}\sqrt{9-x^2} + C$

c. $-\sqrt{9-x^2} + C$

d. $\frac{9}{2}\arcsin\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$

9. $\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

a. $\frac{t}{2}\sqrt{1-t^2} - \frac{\tan^{-1} t}{2} + C$

b. $\frac{\sin^{-1} t}{2} - \frac{t}{2}\sqrt{1-t^2} + C$

c. $-\sqrt{1-t^2} + C$

d. $\frac{1}{2}\left(t\sqrt{1-t^2} + \ln\left|t + \sqrt{t^2-1}\right|\right) + C$

10. $\int \frac{\sqrt{1-x^2}}{x}\,dx$

a. $\frac{1}{x\arcsin x}+C$

b. $\sqrt{1-x^2}+\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+C$

c. $\frac{2\left(\sqrt{1-x^2}\right)^{3/2}}{3x}+C$

d. $\sqrt{x^2-1}+\arctan\frac{1}{\sqrt{x^2-1}}+C$

11. $\int \frac{dx}{\sqrt{x^2-4}}$

a. $\frac{1}{2}\arcsin\frac{x}{2}+C$ b. $2\sqrt{x^2-4}+C$

c. $\ln\left|x+\sqrt{x^2-4}\right|+C$ d. $\frac{2\left(\sqrt{x^2-4}\right)^{3/2}}{3}+C$

12. $\int \frac{dx}{\left(x^2+9\right)^{3/2}}$

a. $\frac{x}{9\sqrt{x^2+9}}+C$ b. $\frac{-2}{\sqrt{x^2+9}}+C$

c. $\frac{1}{3}\left(\arctan\frac{x}{3}\right)^{3/2}+C$ d. $\ln\left(x+9\right)^{3/2}+C$

13–48 Use the three trigonometric substitutions discussed in this section to evaluate the given indefinite or definite integral. (**Note:** Not all integrals require trigonometric substitution.)

13. $\int \frac{3}{\sqrt{x^2+9}}\,dx$

14. $\int x\sqrt{x-1}\,dx$

15. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

16. $\int \frac{\sqrt{9-x^2}}{2x}\,dx$

17. $\int \frac{t^2}{\sqrt{25-t^2}}\,dt$

18. $\int \frac{ds}{s\sqrt{s^2-4}}$

19. $\int \frac{dx}{x^2\sqrt{x^2-36}}$

20. $\int \frac{\sqrt{x^2-2}}{x^3}\,dx$

21. $\int \frac{-x}{\sqrt{1-x^2}}\,dx$

22. $\int \frac{x^2}{\sqrt{x^2+25}}\,dx$

23. $\int \frac{2-x}{\sqrt{4+x^2}}\,dx$

24. $\int \frac{x^2-2x+5}{\sqrt{1-x^2}}\,dx$

25. $\int_{5/2}^{3} \frac{dz}{2z^2\sqrt{z^2-4}}$

26. $\int_0^5 \frac{z}{\sqrt{z^2+4}}\,dz$

27. $\int_0^{\sqrt{5}} w^2\sqrt{5-w^2}\,dw$

28. $\int_3^5 \frac{\sqrt{x^2-9}}{x}\,dx$

29. $\int \frac{dt}{t^2\sqrt{t^2+9}}$

30. $\int \frac{s^2}{\sqrt{s^2-25}}\,ds$

31. $\int \frac{\sqrt{4-9x^2}}{x}\,dx$

32. $\int \frac{dy}{y\sqrt{25+9y^2}}$

33. $\int \frac{\left(4-w^2\right)^{3/2}}{w^6}\,dw$

34. $\int_1^2 \frac{dx}{4x^2\sqrt{4x^2+1}}$

35. $\int_{1/2}^{2} \frac{dy}{y^2\sqrt{16-y^2}}$

36. $\int \sqrt{1-x^2}\,dx$

37. $\int \sqrt{9x^2+16}\,dx$

38. $\int x^2\sqrt{9-x^2}\,dx$

39. $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

40. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

41. $\int \frac{x}{\sqrt{3x^2+1}}\,dx$

42. $\int \frac{x^2}{\left(x^2+16\right)^{3/2}}\,dx$

43. $\int \frac{-x}{\left(1-x^2\right)^{3/2}}\,dx$

44. $\int \frac{dt}{\sqrt{9t^2-4}}$

45. $\int \frac{dx}{x^2\sqrt{25-9x^2}}$

46. $\int \frac{dx}{\sqrt{9x^2+4}}$

47. $\int_{-2}^{2} \frac{dt}{\left(t^2+5\right)^{3/2}}$

48. $\int \frac{dx}{\left(x^2+1\right)^2}$

49–60 Complete the square and use applicable substitutions from this section to evaluate the given integral.

49. $\int \frac{dx}{\sqrt{x^2+x+3}}$

50. $\int \frac{dv}{\left(9v^2-18v+5\right)^{3/2}}$

51. $\int \frac{x^2}{\sqrt{6x-x^2}}\,dx$

52. $\int_1^3 \frac{2}{\sqrt{4x-x^2}}\,dx$

53. $\int_2^3 \frac{dx}{\sqrt{-2x^2+8x-4}}$

54. $\int \frac{dx}{\sqrt{x^2-6x+10}}$

55. $\int_2^3 \frac{dx}{\left(4x^2-8x+3\right)^{3/2}}$

56. $\int \frac{v^2}{\sqrt{3+2v-v^2}}\,dv$

57. $\int \sqrt{7+6x-x^2}\,dx$

58. $\int \sqrt{4x^2-16x+25}\,dx$

59. $\int \frac{2}{\left(x^2-10x+29\right)^2}\,dx$

60. $\int \frac{ds}{\left(s^2-8s+17\right)^{3/2}}$

61–63 Use an appropriate substitution followed by a trigonometric substitution to evaluate the integral.

61. $\int e^x\sqrt{1-e^{2x}}\,dx$

62. $\int \frac{\sqrt{x}}{1+x}\,dx$

63. $\int \frac{\cot x\csc x}{\sqrt{\sin^2 x+1}}\,dx$

64–71 Use an appropriate trigonometric substitution to find a general formula for the expression. (Assume $a > 0$.)

64. $\int \sqrt{x^2+a^2}\,dx$

65. $\int \sqrt{x^2-a^2}\,dx$

66. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

67. $\int \frac{dx}{x\sqrt{x^2-a^2}}$

68. $\int \sqrt{a^2-x^2}\,dx$

69. $\int \frac{dx}{\sqrt{x^2+a^2}}$

70. $\int \frac{-dx}{x\sqrt{a^2-x^2}}$

71. $\int \frac{-dx}{x\sqrt{x^2+a^2}}$ $(x > 0)$

72–74. Use hyperbolic substitutions to find alternative general formulas for Exercises 69–71.

75. Evaluate Exercise 42 by using the hyperbolic substitution $x = 4\sinh t$, and then generalize your result to arrive at both a trigonometric and a hyperbolic formula for

$$\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx.$$

76–77 Combine integration by parts and trigonometric substitution to evaluate the integral.

76. $\int t \arcsin t\,dt$

77. $\int t \arccos t\,dt$

78. Find the area of the region between the graph of $y = \frac{1}{\left(x^2+2\right)^{3/2}}$ and the x-axis from $x=-1$ to $x=1$.

79. Repeat Exercise 78 for the curve $y = \frac{1}{x^2\sqrt{x^2-3}}$ on the interval $[2,3]$.

80. Find the area enclosed by the unit circle $x^2+y^2=1$ and the parabola $y=\sqrt{2}x^2$.

81. Rotate the region bounded by the graph of $y=\frac{\sqrt{x^2-9}}{x^3}$, $3 \le x \le 5$, about the y-axis. Use the shell method to find the volume of the resulting solid.

82. Repeat Exercise 81 for the curve $y=\frac{27x^2}{\left(9x^2+4\right)^{3/2}}$ on the interval $[0,1]$.

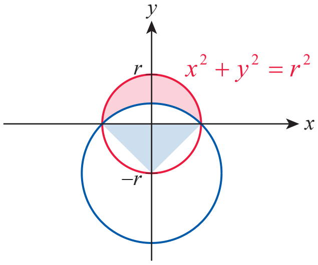
83. Use the method of disks to determine the volume of the solid obtained by revolving the graph of the curve $y=\frac{\sqrt[4]{16-4x^2}}{x^2}$, $1 \le x \le 2$, about the x-axis.

84. Find the arc length of the prototypical parabola $y=x^2$ between the origin and the point $(2,4)$.

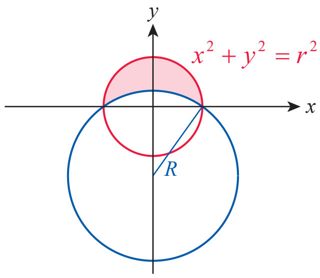
85. Find the arc length for the graph of $y=\ln x$ between $x=\sqrt{3}$ and $x=2\sqrt{2}$.

86. A cylindrical fuel tank of radius 10 in. is positioned so its axis is horizontal. Find the fluid force acting on one end of the tank if it is partially filled with diesel fuel so that the top 4 in. of the tank are empty. Use $55\ \text{lb/ft}^3$ for the weight density of diesel fuel.

87. In an attempt to square the circle, Hippocrates of Chios showed about 2500 years ago that the area of the red shaded region (called a *lune*) in the figure below is equal to the area of the shaded triangle (which in turn is half of a square). Given that the bigger circle is centered at $(0,-r)$, use calculus to prove Hippocrates' result.



88.* Find a more general formula for the area of the lune in the case where the radius of the bigger circle is R.



Concept Check

89–92 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

89. According to the text, the best substitution for $\int \frac{1}{\sqrt{x^2-1}}\,dx$ is $x=\sin\theta$.

90. Substituting $x=\sin\theta$, we obtain $\int 2x\sqrt{1-x^2}\,dx = \int 2\sin\theta\cos\theta\,d\theta = \int \sin 2\theta\,d\theta$.

91. A straightforward way to evaluate $\int_0^{1/2} \frac{1}{1-x^2}\,dx$ is to find $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$ by substituting $x=\sin\theta$ and then squaring the answer.

92. According to this textbook, the only way to evaluate the integral $\int \frac{1}{x^2 - 4}\,dx$ is to substitute $x = 2\sec\theta$.

7.4 Technology Exercises

93–96 Use a graphing utility to revisit the given exercise. Do you get the same answer that you obtained by hand?

93. Exercise 22 **94.** Exercise 37 **95.** Exercise 54 **96.** Exercise 58