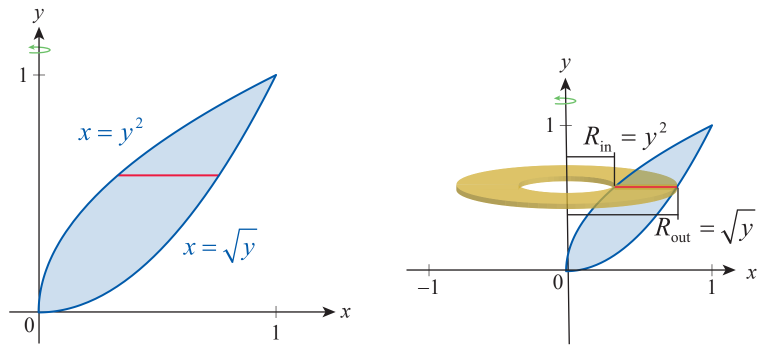
Method 2: The Washer Method

In this example, the washer method applies just as well. The setup differs, of course, as we need to describe the inner and outer radii of a sequence of vertically stacked washers. Using this formulation, the outer radius R_{out} is the function $\sqrt{y}$ and the inner radius R_{in} is the function y^2. Note that our integration will be with respect to y over the interval $0 \le y \le 1$; Figure 13 illustrates the region and a typical washer cross-sectional cut of the region. The volume computation is as follows.

$$V = \pi\int_0^1 \left(R_{\text{out}}^2 - R_{\text{in}}^2\right)dy = \pi\int_0^1 \left(y - y^4\right)dy$$
$$= \pi\left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 = \frac{3\pi}{10}$$



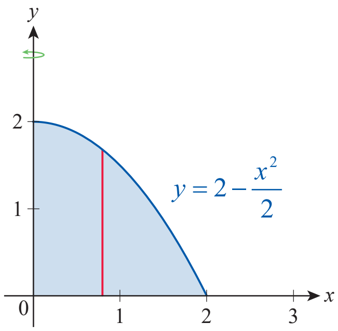
(a) (b)

Figure 13 The Washer Method

6.2 Exercises

1–6 Use the shell method to find the volume of the solid obtained by revolving the shaded region about the indicated line.

1.



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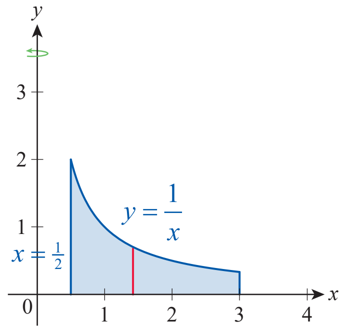

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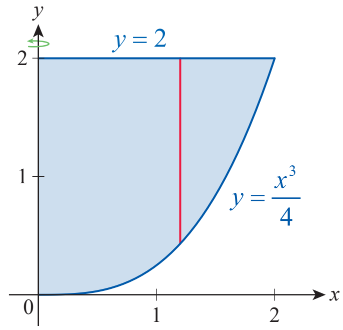

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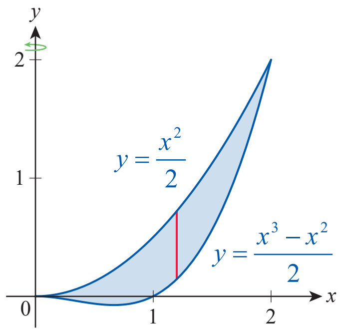

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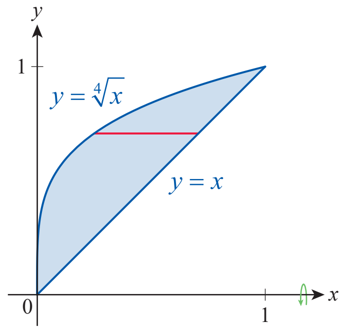
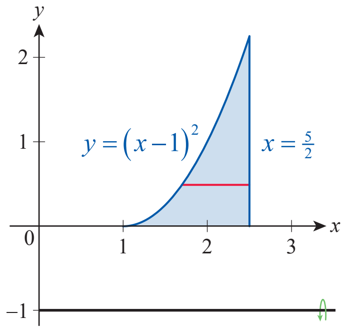
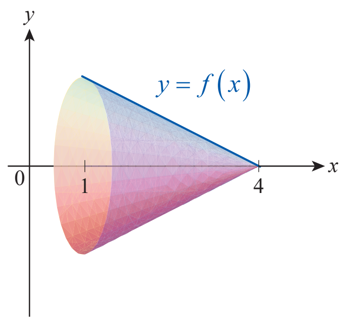


6.

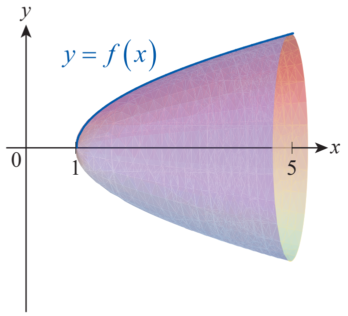


7–8 Use the shell method to find the volume of the solid that is generated by revolving the region bounded by $y = 0$ and the graph of f about the x-axis over the indicated interval.

7. $f(x) = 2 - 0.5x; \quad 1 \le x \le 4$



8. $f(x) = \sqrt{x-1}; \quad 1 \le x \le 5$



9–26 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the y-axis.

9. $y = -2x + 6, \quad x = 0, \quad y = 0$

10. $y - x = 0, \quad y = 0, \quad y = 6$

11. $2y - 5x + 8 = 0, \quad y = 0, \quad x = \frac{8}{5}, \quad x = 3$

12. $3y - x = 0, \quad x = 0, \quad x = 3, \quad 3y + 2x - 9 = 0$

13. $y = x^2, \quad x = 0, \quad y = 9$

14. $y = x^2 - 1, \quad y = 2x - 1$

15. $y = x^2 - 2x + 3, \quad x = 0, \quad x = 3, \quad 3y = 2x + 12$

16. $y = \sqrt{4-x}, \quad x = 0, \quad y = 0$

17. $y = x^{3/5}, \quad y = 0, \quad x = 1$

18. $y = \sqrt{4-x^2}, \quad x = 0, \quad y = 0$

19. $y = (x-2)^3, \quad x = 2, \quad y = 8$

20. $y = \sqrt[3]{x} + 1, \quad x = 0, \quad x = 1, \quad y = 0$

21. $y = \cos^2(x^2), \quad y = \sin^2(x^2), \quad x = 0, \quad x = \frac{\sqrt{\pi}}{2}$

22. $y = \sec^2(x^2), \quad x = 0, \quad x = \frac{\sqrt{\pi}}{2}, \quad y = 0$

23. $y = \frac{1}{(x+1)^2}, \quad x = 0, \quad x = 1, \quad y = 0$

24. $y = \frac{1}{3\sqrt{x}}, \quad x = 1, \quad x = 4, \quad y = 0$

25. $y = \frac{e^x}{\pi x}, \quad x = 1, \quad x = 2, \quad y = 0$

26. $y = \begin{cases} \dfrac{1-\cos x}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = 2\pi$

27–38 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the x-axis.

27. $7y + 4x = 28, \quad x = 0, \quad y = 0$

28. $y - x + 1 = 0, \quad y + 2x - 8 = 0, \quad y = 0$

29. $y = \sqrt{x}, \quad y = x - 2, \quad y = 0$

30. $y = 2x^2 - 8, \quad x = 0, \quad y = 0$

31. $y = \sqrt[3]{3-x}, \quad y = \frac{1}{2}x, \quad y = 0$

32. $x = \frac{1}{y+2}, \quad x = 0, \quad y = 0, \quad y = 2$

33. $2x\sqrt{y+1} = 1, \quad x = 0, \quad y = 0, \quad y = 3$

34. $y\sqrt{y^3+1} = x, \quad x = 0, \quad y = 0, \quad y = 2$

35. $\frac{1}{y^2+1} = x, \quad x = 0, \quad y = 0, \quad y = 4$

36. $x = \frac{1}{y(y^2+1)}, \quad x = 0, \quad y = 1, \quad y = \sqrt{3}$

37. $y = \sqrt{\ln(1-x)+1}, \quad x = 0, \quad y = 0$

38. $y = \sqrt{\arcsin x}, \quad x = 0, \quad y = \sqrt{\frac{\pi}{2}}$

39–44 Revolve the region bounded by the graphs of the equations about the given line and use the shell method to find the volume of the resulting solid.

39. $y = 5 - 2x, \quad x = 0, \quad y = 0$

a. About $x = -2$ **b.** About $y = -1$

c. About $x = 2.5$ **d.** About $y = 7$

40. $y=\frac{3}{2}x, \quad x=2, \quad y=0$

a. About $x=-1$ **b.** About $y=-2$

c. About $x=2$ **d.** About $y=3$

41. $y=\sqrt{x}, \quad y=0, \quad x=4$

a. About $x=-1$ **b.** About $y=-\frac{1}{2}$

c. About $x=6$ **d.** About $y=2$

42. $y=x, \quad y=x^3, \quad x=0, \quad x=1$

a. About $x=-3$ **b.** About $y=-\frac{1}{2}$

c. About $x=\frac{3}{2}$ **d.** About $y=2$

43. $y=4x-x^2, \quad y=0$

a. About $x=-1$ **b.** About the x-axis

c. About $x=2$ **d.** About $y=-1$

44. $y=\sqrt{4x-x^2}, \quad y=0$

a. About the x-axis **b.** About $y=-1$

c. About $x=2$ **d.** About $y=2$

45–50 Using the shell method, find a formula for the volume of the solid that results when the region bounded by the graphs of the equations is revolved about the indicated axis. Do not evaluate the integral.

45. $y=2e^{-x}-1, \quad x=0, \quad x=\ln 2, \quad y=0;$ about the y-axis

46. $y=2e^{-x}-1, \quad x=0, \quad x=\ln 2, \quad y=0;$ about the x-axis

47. $y=\arcsin x, \quad x=0, \quad x=1, \quad y=0;$ about the y-axis

48. $y=\arcsin x, \quad x=0, \quad x=1, \quad y=0;$ about the x-axis

49. $y=\tan x, \quad x=0, \quad y=\sqrt{3};$ about the y-axis

50. $y=\tan x, \quad x=0, \quad y=\sqrt{3};$ about the x-axis

51–59 The given integral represents the volume of a solid of revolution. Describe the solid. (Do not evaluate the integral.)

51. $4\pi\int_0^2 \left(2x-x^2\right)dx$

52. $2\pi\int_0^{\sqrt{3}} \left(3y-y^3\right)dy$

53. $4\pi\int_0^1 y\sqrt{1-y^2}\,dy$

54. $4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx$

55. $2\pi\int_0^1 (1+x)\left(1-x^4\right)dx$

56. $2\pi\int_0^1 (y+2)\left(e-e^y\right)dy$

57. $\pi\int_{\pi/2}^{\pi} (2x-\pi)\sin x\,dx$

58. $2\pi\int_0^{1/2} 3y\,dy+2\pi\int_{1/2}^1 y\left(\frac{1}{y^2}-1\right)dy$

59. $\pi\int_{\pi/4}^{3\pi/4} (2x+2)\csc x\,dx$

60. Use the washer method to solve the problem posed in Example 3. (**Hint:** Completing the square for $f(y)=-y^2+y$ yields $f(y)=x=\frac{1}{4}-\left(y-\frac{1}{2}\right)^2$. Express y in terms of x to obtain the graphs bounding the given region. Since the axis of rotation is $y=2$, you will obtain $R_{\text{in}}=2-\left(\frac{1+\sqrt{1-4x}}{2}\right)$ and $R_{\text{out}}=2-\left(\frac{1-\sqrt{1-4x}}{2}\right)$. Simplify the integral $V=\pi\int_0^{1/4}\left(R_{\text{out}}^2-R_{\text{in}}^2\right)dx$ and verify that it equals $\pi/2$.)

61–66 Use the shell method or the disk/washer method to find the volume of the solid obtained by revolving the region bounded by the graphs of the equations about the given axis. Choose the method that seems to work best.

61. $y(x+1)=1, \quad y=0, \quad x=0, \quad x=1;$ about the x-axis

62. $y=1-(x-1)^4, \quad x=0, \quad y=0;$ about the y-axis

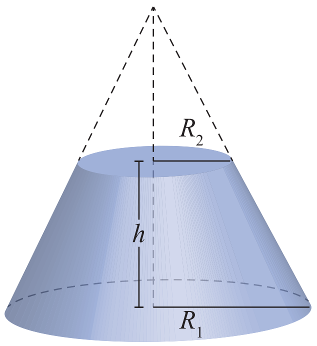
63. $y=x^3-6x^2+8x, \quad y=0, \quad x=0, \quad x=2;$ about the x-axis

64. $y=2x+\frac{1}{x^2}, \quad y=0, \quad x=1, \quad x=2;$ about the y-axis

65. $y=3-\frac{x^2}{3}, \quad y=0;$ about the x-axis

66. $y=\begin{cases}\frac{\sin x}{x} & \text{if } x>0\\ 1 & \text{if } x=0\end{cases}, \quad y=0, \quad x=0, \quad x=\pi;$ about the y-axis

67. The solid that remains after "chopping off" the upper part of a right circular cone by a plane parallel to its base is called a frustum of a cone. Suppose that the radii of the base and top are R_1 and R_2, respectively, while the height is h.



a. Use the shell method to prove that the volume of the cone frustum is $V = \frac{\pi h}{3}\left(R_1^2 + R_1 R_2 + R_2^2\right)$. (**Hint:** One approach is to rotate the region between the line segment $y = \frac{R_1 - R_2}{h}x,\ 0 \le x \le h$, and the line $y = -R_2$ about the said line and use the volume of the resulting solid to find that of the frustum.)

b. Now use the method of disks to establish the above formula. Which method do you prefer?

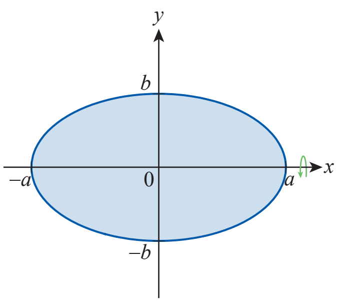
68. Use the shell method to prove the volume formula for a sphere of radius R: $V = \frac{4}{3}\pi R^3$.

69. Use the shell method to find the volume of the wooden toy piece of Exercise 17 of Section 6.1.

70. How deep is the water in a bowl-shaped hemispherical tank of radius r when the tank is filled to exactly $\frac{14}{27}$ of its full capacity?

71.* Use the shell method to find the formula for the volume of the torus of Exercise 84 of Section 6.1.

72.* The graph of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse centered at the origin, with its axes of symmetry of lengths $2a$ and $2b$, coinciding with the coordinate axes. By rotating the region bounded by this ellipse about the x-axis, use the shell method to find a formula for the volume of the resulting ellipsoid.



Concept Check

73–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

73. If V is the volume of the solid obtained by revolving the region bounded by $y = f(x)$ and the x-axis $(a \le x \le b)$ about the x-axis and k is a positive constant, then the volume of the solid generated by $y = kf(x)$ over the same interval is kV.

74. The volume of the solid generated by revolving about the y-axis the region bounded by $y = x^2$, the y-axis, and $y = b$ is directly proportional to b.

75. The volume of a solid of revolution can be interpreted as the limit of Riemann sums.

76. The shell method always results in a less complicated integral than the disk method.

77. If R is the region bounded by $y = f(x)$ and $y = g(x)$, while R_1 is bounded by $y = f(x) + 1$ and $y = g(x) + 1$, then the solids obtained by revolving about the x-axis the regions R and R_1, respectively, have equal volume.

6.2 Technology Exercises

78–81 Use a graphing utility to sketch the region bounded by the graphs of the equations. Then use the shell method, along with the integration capabilities of your technology, to find the volume of the solid generated by rotating the region about the given line.

78. $y = x^2 \cos^2 x, \quad y = 0, \quad x = 0, \quad x = \pi/2;$ about the y-axis

79. $x^{2/3} + y^{5/3} = 1, \quad x = 0, \quad y = 0;$ about the y-axis

80. $x = \left(\frac{\pi}{2} - y\right)^3 \cos y, \quad y = 0, \quad x = 0;$ about $y = -1$

81. $x^3 = y(y-2)^2, \quad y = 0, \quad x = 0;$ about $y = -2$