Example 4 Finding the Area Bounded by the Graphs of Equations

Find the area of the region bounded by the graphs of the equations $y = 0$, $3x - 5y = 12$, and $y = \sqrt{x}$.

Solution

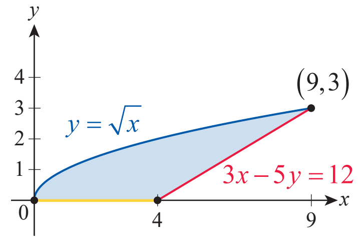


Figure 7

Two edges of the described region are lines, and the remaining edge is the graph of the function $y = \sqrt{x}$, which we can also think of as the upper half of the parabola $x = y^2$. As always, if it's possible to sketch a picture of what we are doing, such a sketch is bound to be helpful—in this case, it's easy to graph the region we're discussing (see Figure 7).

Note that we need two integrals if we want to express the area of the region as an integral in x, since the lower function changes at $x = 4$. On the interval $[0,4]$, the lower function is $g(x) = 0$ (corresponding to the equation $y = 0$), and on the interval $[4,9]$ the lower function is $g(x) = \frac{3}{5}(x-4)$, which we obtain by solving $3x - 5y = 12$ for y. The upper function is $f(x) = \sqrt{x}$ for the entire interval $[0,9]$. So the total area of the described region is

$$A = \int_0^4 \sqrt{x}\, dx + \int_4^9 \left[\sqrt{x} - \frac{3}{5}(x-4)\right] dx.$$

This is certainly doable, and you are asked to evaluate the above integrals in Exercise 52. But if we think of the region as being composed of horizontal strips, we see that the left edge can be described as a single function of y, and the same is true for the right edge—we don't have to divide the interval of integration into subintervals. Specifically, the left edge of the region is the function $g(y) = y^2$ (corresponding to the equation $x = y^2$) and the right edge is the function $f(y) = \frac{5}{3}y + 4$ (obtained by solving $3x - 5y = 12$ for x). So, each horizontal differential element of area can be written as

$$dA = \left[f(y) - g(y)\right]dy = \left(\frac{5}{3}y + 4 - y^2\right)dy$$

and thus the area is calculated as follows.

$$\begin{aligned} A = \int_0^3 dA &= \int_0^3 \left(\frac{5}{3}y + 4 - y^2\right)dy \\ &= \left[\frac{5}{6}y^2 + 4y - \frac{1}{3}y^3\right]_0^3 \\ &= \frac{15}{2} + 12 - 9 = \frac{21}{2} \end{aligned}$$

5.5 Exercises

1–51 Evaluate the definite integral. Whenever possible, take advantage of symmetry.

1. $\int_0^1 2(2x+1)^5\, dx$

2. $\int_0^2 2x\sqrt{4-x^2}\, dx$

3. $\int_0^2 (x^2-1)(x^3-3x)^8\, dx$

4. $\int_0^4 (x-2)(2x^2-8x)^{49}\, dx$

5. $\int_1^2 w^2(w^3+4)^{99}\, dw$

6. $\int_1^{10} 2x^3(x^4-1)^{49}\, dx$

7. $\int_1^3 (2-x)^6\, dx$

8. $\int_0^2 x\sqrt{x^2+1}\, dx$

9. $\int_0^1 x^3\sqrt[3]{x^4+1}\, dx$

10. $\int_0^3 \sqrt{2x+1}\,dx$

11. $\int_1^4 \frac{2x+1}{\left(x^2+x+1\right)^2}\,dx$

12. $\int_0^1 \frac{z^2}{z^3+2}\,dz$

13. $\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx$

14. $\int_1^3 \frac{8x^3+20x}{x^4+5x^2+6}\,dx$

15. $\int_1^2 \left(1+\frac{1}{t^2}\right)^2 \frac{1}{t^3}\,dt$

16. $\int_1^2 \frac{6x+10.5}{\sqrt{2x^2+7x}}\,dx$

17. $\int_e^{e^2} \frac{1}{s\ln\left(s^3\right)}\,ds$

18. $\int_{10}^{100} \frac{\log x}{x\ln 10}\,dx$

19. $\int_1^{\sqrt{2}} (\ln 2)\,x\cdot 2^{x^2-1}\,dx$

20. $\int_0^{\sqrt{\pi}} 4x\cos\frac{x^2}{2}\,dx$

21. $\int_0^1 \sin \pi x\,dx$

22. $\int_{\sqrt{\pi/4}}^{\sqrt{3\pi/4}} \frac{-x}{\sin^2\left(x^2\right)}\,dx$

23. $\int_0^{\pi/4} \sin^2 2x\cos 2x\,dx$

24. $\int_0^1 x\sec^2\left(2x^2-1\right)dx$

25. $\int_{\pi^2/16}^{9\pi^2/16} \frac{\cot\sqrt{x}\csc\sqrt{x}}{\sqrt{x}}\,dx$

26. $\int_{-\pi}^{\pi} \sin x\sin(\cos x)\,dx$

27. $\int_1^e \frac{\ln\left(2x^2\right)}{x}\,dx$

28. $\int_0^1 \frac{e^{2t}}{e^t+1}\,dt$

29. $\int_{-2}^2 \frac{t^3}{t^2+1}\,dt$

30. $\int_0^{\pi/4} \frac{\csc^2\theta}{e^{\cot\theta}}\,d\theta$

31. $\int_{\pi^2/16}^{\pi^2/4} \frac{\csc^2\left(\frac{\pi}{4}+\sqrt{t}\right)}{\sqrt{t}}\,dt$

32. $\int_0^2 \frac{e^{\sqrt{2x}}}{\sqrt{2x}}\,dx$

33. $\int_0^1 \frac{\sqrt{v}}{\sqrt{v^{3/2}+1}}\,dv$

34. $\int_0^2 \frac{1}{\sqrt{x}\left(2+\sqrt{x}\right)^2}\,dx$

35. $\int_0^{\sqrt{3}} \frac{e^{\arctan x}}{1+x^2}\,dx$

36. $\int_0^1 \frac{1}{x^{2/3}\left(1+x^{2/3}\right)}\,dx$

37. $\int_0^{\pi/2} \frac{\sin 2x}{\cos^2 x+1}\,dx$

38. $\int_0^1 -e^{-x}\sec\left(e^{-x}-1\right)\tan\left(e^{-x}-1\right)dx$

39. $\int_4^9 \frac{\sqrt{2+\sqrt{t}}}{\sqrt{t}}\,dt$

40. $\int_0^4 \sqrt{2+\sqrt{t}}\,dt$

41. $\int_{-1}^1 \frac{x^2+x+1}{x-2}\,dx$

42. $\int_1^e \frac{(\ln x+1)(2\ln x+3)^2}{x}\,dx$

43. $\int_2^3 (x+3)(x-2)^7\,dx$

44. $\int_{-1}^0 x\sqrt[3]{x-1}\,dx$

45. $\int_{-1}^1 x\sqrt{4-x^2}\,dx$

46. $\int_4^{e+3} \frac{2x+5}{x-3}\,dx$

47. $\int_{-4}^0 \frac{x^2-x+3}{x+5}\,dx$

48. $\int_1^{e^2} \frac{(2\ln x+3)(\ln x-1)^2}{x}\,dx$

49. $\int_1^9 \frac{\left(2\sqrt{x}+1\right)\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

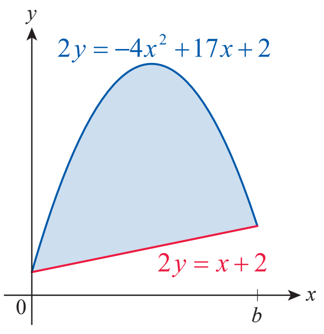
50. $\int_0^{\pi^{2/3}} \sqrt{x}\sin^2\left(x^{3/2}\right)\cos^3\left(x^{3/2}\right)dx$

51. $\int_0^{\pi/4} \tan x\sec^3 x\,dx$

52. Evaluate $A=\int_0^4 \sqrt{x}\,dx+\int_4^9\left[\sqrt{x}-\frac{3}{5}(x-4)\right]dx$ from Example 4.

53–56 Find the area of the region bounded by the graphs of the given equations, as shown.

53.



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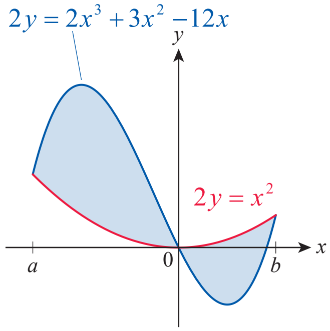

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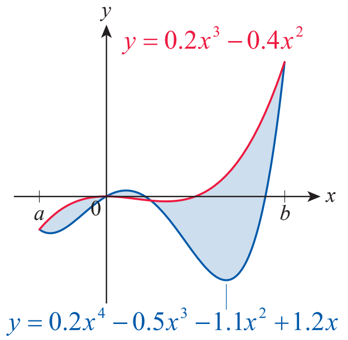

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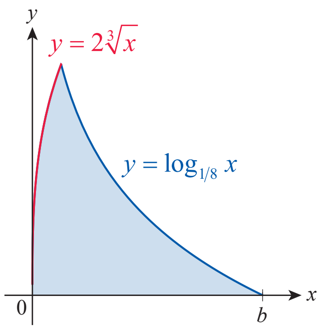


57–96 Find the area of the region bounded by the graphs of the given equations. Be careful to find intersection points, if applicable, and to identify the upper and lower functions on each interval. If convenient or necessary, divide the region into horizontal rather than vertical strips and integrate with respect to *y*. Whenever possible, take advantage of symmetry.

57. $y = x^2, \quad y = 2x$

58. $y = x^2, \quad y = 2$

59. $y = 4x - x^2, \quad y = x$

60. $y = 1 - x^4, \quad y = |x| - 1$

61. $y = |x^3 - 1|, \quad 3y = 5x + 11$

62. $y = x^2, \quad y = x^4$

63. $y = 2x - x^2, \quad y = x^3$

64. $y = \sqrt{x}, \quad y = 2 - x$

65. $x = y^2, \quad y = x^3, \quad x \geq 0$

66. $y = \sqrt[3]{x}, \quad y = \sqrt[7]{x}$

67. $2xy - y = 3 - 2x, \quad y = x, \quad y = 0$

68. $x + 30y = 2y^3 + 5, \quad 9y - 31 = x + y^3$

69. $y = \sqrt{x+1}, \quad y = x^2 - 1$

70. $y = x^3, \quad y = x$

71. $y = \dfrac{1}{x}, \quad y = \dfrac{1}{x^2}, \quad x = e$

72. $y = x^3, \quad y = \dfrac{3}{2} - \dfrac{x}{2}, \quad y = 0$

73. $y = 4x^2 - x^4, \quad y = -5x^2$

74. $y = 3x^2 - x - 4, \quad y = x^2 + 3x + 2$

75. $y^2 - x = 2, \quad y = x$

76. $1 - y^2 = x, \quad (1-y)^2 = x$

77. $3y - x = 3y^2, \quad 6y^3 = x + 6y^2$

78. $y = x^3 - 3x^2 + 2x, \quad y = x^2 - x$

79. $y = x^3 - 6x^2 + 5, \quad y = -x^3 + 12x - 11$

80. $y = x^4 - \dfrac{x^3}{2} - 4x^2, \quad y = -\dfrac{x^3}{2}$

81. $y = 2\sqrt[3]{x+1}, \quad y = 2 - x, \quad y = 0$

82. $y = \sqrt{x}, \quad y = \dfrac{1}{x^2}, \quad x = 4$

83. $x = y^3 - 10y^2 + 20y, \quad x = 4y^2 - 33y + 40$

84. $x = y^4 - 3y^3 - y^2, \quad x = 5y^2 - 8y$

85. $y = \sqrt[4]{x}, \quad y = -\dfrac{1}{16}x^2 + 18, \quad y = 0$

86. $(y-1)^2 = \dfrac{1}{x}, \quad x - 2 = 4y, \quad y = 0$

87. $y = \cos x, \quad y = \sin 2x, \quad x = 0, \quad x = \pi$

88. $y = \dfrac{2}{x^2+1}, \quad y = x^2$

89. $y = \sin x, \quad y = \sqrt{2} - \sin x, \quad x = \dfrac{\pi}{4}, \quad x = \dfrac{3\pi}{4}$

90. $y = \cot x, \quad y = 2\cos x, \quad 0 < x < \pi$

91. $y = x + 1, \quad y = \sqrt{18 - (x+1)^2}, \quad y = 0$

92. $y = \arctan x, \quad y + x = 1 + \dfrac{\pi}{4}, \quad x = 0$

93. $y = \cos\left(\dfrac{\pi}{2}x\right), \quad y = x^4 - 1$

94. $y = 2\sqrt{2}\sin x, \quad y = \csc^2 x, \quad 0 < x < \pi$

95. $y = \sin^2 x, \quad y = \cos^2 x, \quad x = 0, \quad x = \pi$

96. $y = \tan^2 x, \quad y = 0, \quad x = 0, \quad x = \dfrac{\pi}{4}$

97. Use the Substitution Rule to prove the following property of the definite integral.

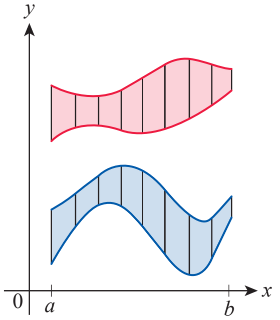
$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

Note that the above is often referred to as the *translation invariant property* of the definite integral. Using a generic $f(x)$, make a sketch of both integrals and explain the reason for the name of this property.

98. Use Exercise 97 to explain why the following definite integrals are equal.

$$\int_{-1}^{2} 4x\sqrt[3]{x+2}\,dx = \int_{1}^{4} (4x-8)\sqrt[3]{x}\,dx$$

99. Italian mathematician Bonaventura Cavalieri (1598–1647), who can be considered as one of the early forerunners of modern calculus, discovered what we today call *Cavalieri's Principle*. Use our discussion preceding Example 3 to prove the following version of Cavalieri's Principle: Suppose two plane regions are included between the lines $x = a$ and $x = b$, and are bounded by graphs of integrable functions. If they have the property that any vertical line intersects both regions in line segments of the same length, then the regions have equal areas.



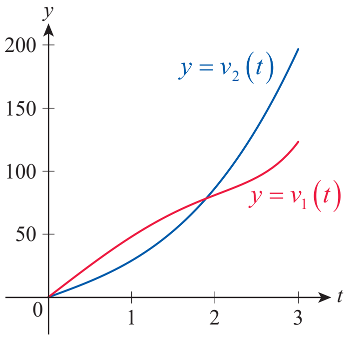
100. Consider the region bounded by the graphs of the equations $y = \sqrt{x}$, $x = 9$, and the x-axis. Find the vertical line $x = a$ that bisects the region in two subregions of equal area.

101. The graphs below show the velocities of two bikes at a motorcycle race right after the start (velocity is measured in km/h). Use the figure to answer the following questions.

a. Which bike is ahead initially?

b. What happens at the instant when the curves intersect?

c. Do the curves suggest that a pass happened, and if so, approximately when?



102. Suppose that the function $B(t) = 85\cdot(1.1163)^t$ approximates the birth rate of a rabbit population on an isolated island, while the death rate is $D(t) = 21\cdot(1.0811)^t$ (t is measured in months). Find the area between the graphs of these two functions on the interval $[0,12]$. Use your own words to give a real-life interpretation to this number.

5.5 Technology Exercises

103–107 Use a graphing utility to plot the graphs of $f(x)$ and $g(x)$ on the same screen. After choosing the appropriate viewing window, identifying intersection points, and finding the region bounded by the curves, use the integration features of your technology to find the area of the region. (**Hint:** As in Examples 3 and 4, be sure to identify the upper and lower functions on each subinterval and integrate accordingly. As a final step, you may want to check your answer by evaluating $\int_a^b |f(x)-g(x)|\,dx$, where a and b are the first and last of the intersection points. Do you obtain the same answer?)

103. $f(x) = 35x - 9$,
$g(x) = 6x^3 - 4.95x^2 - 3.04x - 22.2525$

104. $f(x) = 3\sin x$, $g(x) = 0.3x$

105. $f(x) = e^x$, $g(x) = \frac{1}{2}x + 2$

106. $f(x) = 2x^4 - 8x^3 - 6.5x^2 + 29x - 12$,
$g(x) = 2x^3 - 4x^2 - 3.5x + 2.5$

107. $f(x) = \dfrac{0.8^x \sin 2x}{2}$, $g(x) = \dfrac{1}{2}\sqrt{x}$