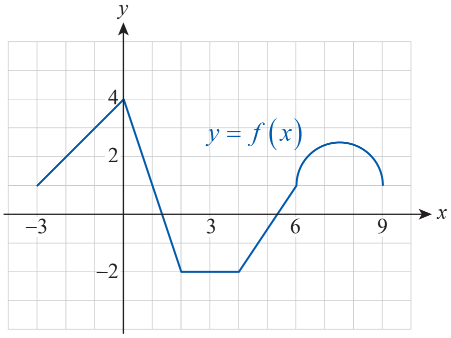
5.2 Exercises

1. Use the given graph along with appropriate formulas from geometry to evaluate each of the indicated definite integrals. (Note that the graph of f consists of linear pieces and a semicircle.)

a. $\int_{-3}^{1} f(x)\,dx$ **b.** $\int_{-3}^{9} f(x)\,dx$

c. $\int_{0}^{6} |f(x)|\,dx$ **d.** $\int_{0}^{9} [f(x)-2]\,dx$



2. Calculate the total distance traveled by the stone of Example 1 by evaluating $\int_0^4 |v(t)|\,dt$.

3. Suppose that in Example 1, George shoots the stone upward while standing near the edge of a deep canyon, and this time pulls the slingshot a bit harder, achieving a velocity function of $v(t)=80-32t$ ft/s. What is the height of the stone, relative to its initial height, at $t=4$ seconds? How about at $t=6$ seconds? 10 seconds?

4–13 Use the given partition and sample points to approximate the definite integral of $f(x)$ on the indicated interval. (Note that the subintervals do not always have to be of equal width, and the sample points may be unevenly spaced.)

4. $f(x)=\frac{1}{3}x+1, \quad x_0=0<1<2<3<4<5<6=x_6, \quad x_i^*=x_i$

5. $f(x)=x^2+x+2, \quad x_0=-1<0<1<2<3=x_4, \quad x_i^*=x_{i-1}$

6. $f(x)=-x-\frac{3}{2}, \quad x_0=-2<-1.5<-0.9<0<1=x_4, \quad x_1^*=-1.8, x_2^*=-1, x_3^*=-0.4, x_4^*=0.5$

7. $f(x)=\frac{1}{x^2}, \quad x_0=1<2<3<4=x_3, \quad x_i^*=\frac{x_{i-1}+x_i}{2}$

8. $f(x)=\frac{1}{1+x^2}, \quad x_0=-3<-2<-1<0<1<2<3=x_6, \quad x_i^*=\frac{x_{i-1}+x_i}{2}$

9. $f(x)=x^3-x, \quad x_0=0<0.3<0.5<1<1.5=x_4, \quad x_1^*=0.25, x_2^*=0.5, x_3^*=1, x_4^*=1.2$

10. $f(x)=\sin x, \quad x_0=0<\frac{\pi}{6}<\frac{\pi}{4}<\frac{\pi}{3}<\frac{\pi}{2}<\frac{2\pi}{3}<\frac{3\pi}{4}<\frac{5\pi}{6}<\pi=x_8, \quad x_i^*=x_{i-1}$

11. $f(x)=\ln(x+1), \quad x_0=-0.5<1<2<2.5=x_3, \quad x_1^*=0, x_2^*=e-1, x_3^*=2$

12. $f(x)=10^{-x}, \quad x_0=0<0.05<0.15<1=x_3, \quad x_1^*=0.01, x_2^*=0.1, x_3^*=1$

13. $f(x)=\sqrt{x}, \quad x_0=0<\frac{1}{25}<\frac{4}{25}<\frac{9}{25}<\frac{16}{25}<1=x_5, \quad x_i^*=x_i$

14–27 Use the concept of the definite integral to find the total area between the graph of $f(x)$ and the x–axis, by taking limits of the associated Riemann sums. When setting up the Riemann sums, make your choice between the left-endpoint, right-endpoint, and midpoint strategies. (**Hint:** Extra care is needed on those intervals where $f(x)<0$. Remember that the definite integral represents a signed area.)

14. $f(x)=2x+4$ on $[0,2]$

15. $f(x)=x-1$ **on** $[0,5]$

16. $f(x)=\frac{3-x}{2}$ on $[0,5]$

17. $f(x)=x^2$ **on** $[1,3]$

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18. $f(x)=x^2-1$ on $[-1,1]$

19. $f(x)=x^2-4x$ on $[0,5]$

20. $f(x)=\frac{x^2}{2}+2$ on $[-2,2]$

21. $f(x)=3x^2-3$ on $[-1,1]$

22. $f(x)=x^2-2x-3$ on $[-1,4]$

23. $f(x)=x^3$ on $[0,1]$

24. $f(x)=4x^3-32$ on $[0,2]$

25. $f(x)=x^3+3x^2+1$ on $[0,3]$

26. $f(x)=\begin{cases}1-(x-1)^2 & \text{if } 0\le x\le 3\\ x-6 & \text{if } 3\le x\le 4\end{cases}$

27. $f(x)=\begin{cases}x^3 & \text{if } 0\le x\le 2\\ 8x-2x^2 & \text{if } 2\le x\le 4\end{cases}$

28. Generalize Exercise 13 to n subintervals and find the definite integral $\int_0^1\sqrt{x}\,dx$ by letting $n\to\infty$. (**Hint:** Let $x_i^*=i^2/n^2$.)

29. Use the same approach as in Exercise 28 to find $\int_0^2\sqrt[3]{x}\,dx$. (**Hint:** Let $x_i^*=2i^3/n^3$.)

30–33 Express the integral as a limit of Riemann sums. (Do not attempt to evaluate the limit.)

30. $\int_1^3\frac{1}{x}\,dx$

31. $\int_0^4\left(x^2-\log_2 x\right)dx$

32. $\int_{-a}^{a}\frac{1}{x^2+1}\,dx$

33. $\int_2^b\sqrt[4]{x}\,dx$

34–45 Sketch the region whose (signed) area is represented by the definite integral, and then use appropriate formulas from geometry to evaluate the integral.

34. $\int_{-1}^5 3\,dx$

35. $\int_{2.5}^{12}(-2)\,dx$

36. $\int_4^2(1-x)\,dx$

37. $\int_8^3\left(4-\frac{1}{2}x\right)dx$

38. $\int_0^4|2x-3|\,dx$

39. $\int_{-1}^5\left(5-|2x|\right)dx$

40. $\int_0^{10}\left(|x-2|-|7-x|\right)dx$

41. $\int_{-5}^0\sqrt{25-x^2}\,dx$

42. $\int_{-a/2}^{a}\sqrt{a^2-x^2}\,dx,\quad a>0$

43. $\int_{-2}^5\left(2-[\![x]\!]\right)dx$

44. $\int_{-3}^8[\![3x-1]\!]\,dx$

45. $\int_{-4}^6\left(x-[\![x]\!]\right)dx$

46. Use Riemann sums resulting from midpoint estimates to prove $\int_a^b x\,dx=\left(b^2-a^2\right)/2$. (**Hint:** Notice that after using $(b+a)(b-a)=b^2-a^2$, each Riemann sum becomes a collapsing sum.)

47. Provide an alternate proof for Exercise 46 by making a sketch and using areas of triangles.

48.* Mimic the argument used in Exercise 46, but using $x_i^*=\sqrt{\left(x_{i-1}^2+x_{i-1}x_i+x_i^2\right)/3}$, to prove the formula $\int_a^b x^2\,dx=\left(b^3-a^3\right)/3$.

49. The Dirichlet function is defined as follows.

$$\xi(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

Prove that $\xi(x)$ is not integrable. (**Hint:** For a given n, form a Riemann sum by choosing each sample point x_i^* to be rational, then see what happens if each x_i^* is irrational. Use your observation to argue that $\lim_{n\to\infty}\sum_{i=1}^{n}f\left(x_i^*\right)\Delta x$ does not exist.)

50. Prove that the function $f(x)=\begin{cases}1/x^2 & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ is not integrable on $[0,1]$.

(**Hint:** By examining the first term of each R_n, show that $\lim_{n\to\infty}R_n$ does not exist.)

51. Repeat Exercise 50 for $g(x)=\begin{cases}1/x & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ on $[0,1]$.

(**Hint:** Show that arbitrarily large Riemann sums can be constructed by choosing appropriate x_i^*'s.)

52–59 Decide whether the function is integrable on the indicated interval. If not, say why. (Do not evaluate the integral.)

52. $f(x)=\frac{1}{\sqrt{x+2}}$ on $[-1,1]$

53. $g(x)=\frac{2}{x}$ on $[-2,2]$

54. $h(x)=\frac{-3}{x-1}$ on $[0,5]$

55. $F(x)=\frac{x}{|x|}$ on $[-3,4]$

56. $G(x)=x\cdot[\![x]\!]$ on $[-2,2]$

57. $H(x)=\begin{cases}\frac{\sin x}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ on $[-1,1]$

58. $u(x)=\begin{cases}\cos\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ on $[-1,1]$

59. $v(x)=\begin{cases}3.14 & \text{if } x \text{ is rational}\\ \pi & \text{if } x \text{ is irrational}\end{cases}$ on $[0,2]$

60–65 Match the given property of the definite integral to the relevant illustration (labeled A–F).

60. $\int_a^b k\,dx=k(b-a)$ (Property 3)

61. $\int_a^b kf(x)\,dx=k\int_a^b f(x)\,dx$ (Property 4)

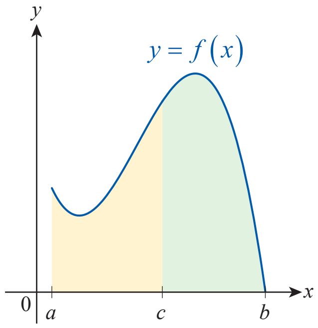
62. $\int_a^b [f(x)\pm g(x)]\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx$ (Property 5)

63. $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx$ (Property 6)

64. If $f(x)\le g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx$. (Property 7)

65. If $m=\min\limits_{a\le x\le b} f(x)$ and $M=\max\limits_{a\le x\le b} f(x)$, then $m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$. (Property 8)

A.



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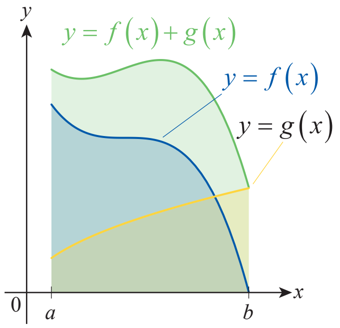

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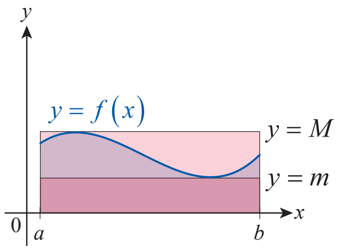

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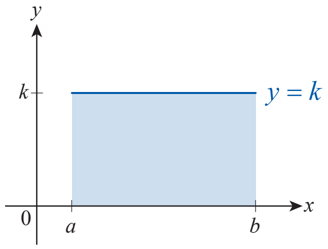

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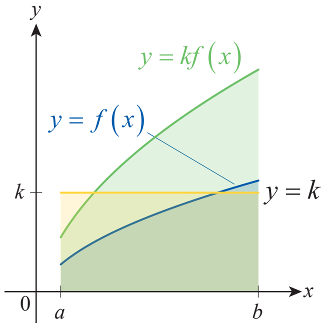

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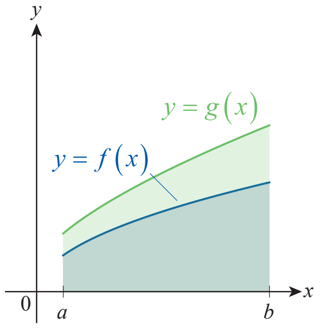

66–75 Use the properties of the definite integral to find the given integral, if possible, given that $\int_a^b f(x)\,dx=3$, $\int_c^b f(x)\,dx=-1$, and $\int_a^b g(x)\,dx=-5$.

66. $\int_a^b [f(x)-g(x)]\,dx$

67. $\int_a^c [2f(x)+1]\,dx$

68. $\int_c^a 10f(x)\,dx$

69. $\int_a^a f(x)g(x)\,dx$

70. $\int_a^b \left[4f(x)+\dfrac{g(x)}{10}\right]dx$

71. $\int_b^a \dfrac{\sqrt{2}}{2}g(x)\,dx$

72. $\int_a^b [f(x)+2g(x)-2]\,dx$

73. $\int_a^b [f(x)]^2\,dx$

74. $\int_a^b \dfrac{5}{g(x)}\,dx$

75. $\int_a^b \left[\dfrac{f(x)}{3}-\pi g(x)\right]dx$

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76–83 Use the results from Exercises 46 and 48, along with the properties of the definite integral and formulas from geometry, to evaluate the given integral.

76. $\int_0^2 (3x-1)\,dx$

77. $\int_{\sqrt{2}}^{-1}\left(1-\frac{\sqrt{2}}{2}x\right)dx$

78. $\int_{-1}^4 \left(x^2+5\right)dx$

79. $\int_1^4 \left(2x^2-x\right)dx$

80. $\int_0^3 \left(t^2+\frac{t}{4}+4\right)dt$

81. $\int_0^1 \left(2\sqrt{x}+x\right)dx$

82. $\int_2^0 \left(\frac{\sqrt[3]{x}}{4}-x^2\right)dx$

83. $\int_{-2}^2 \left(u-3\sqrt[3]{u}\right)du$

84. Suppose that f is an even function, g is odd, and both are integrable on $[-a,a]$. Use the properties of the definite integral to prove the following statements.

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{and} \quad \int_{-a}^a g(x)\,dx = 0$$

85–90 Suppose that f is an even function, g is odd, both are integrable on $[-2,2]$, and we know that $\int_0^2 f(x)\,dx = 1$, while $\int_0^2 g(x)\,dx = 2.5$. If possible, find the integral.

85. $\int_{-2}^2 \left[f(x)+g(x)\right]dx$

86. $\int_{-2}^2 \left[2f(x)-3g(x)\right]dx$

87. $\int_{-2}^2 g(x)\,dx$

88. $\int_{-2}^2 f(x)g(x)\,dx$

89. $\int_{-1}^1 \left[f(x)\right]^2 dx$

90. $\int_0^2 \left|g(x)\right|dx$

91. Use Property 8 of the definite integral to prove the validity of the following upper and lower estimates: $12 \le \int_0^4 \sqrt{x^2+9}\,dx \le 20.$

92–96 Use an argument similar to the one you gave in Exercise 91 to give upper and lower estimates for the given definite integral.

92. $\int_{-1}^4 \sqrt{5+x}\,dx$

93. $\int_2^3 \sqrt{3-x}\,dx$

94. $\int_4^5 \frac{1}{x-2}\,dx$

95. $\int_0^6 \left(\frac{x^2}{32}-\frac{x}{4}+\frac{3}{2}\right)dx$

96. $\int_1^{\sqrt{3}} \arctan x\,dx$

97. Use Property 7 of the definite integral to prove the following inequalities.

a. $\int_0^1 \sqrt{1-x}\,dx \le \int_0^1 \sqrt{1-x^2}\,dx$

b. $\int_0^{\pi/2} \cos x\,dx \le \int_0^{\pi/2} \frac{\sin x}{x}\,dx$

98. Prove Property 4 of the definite integral. (**Hint:** Write a typical Riemann sum for f on $[a,b]$; use the Constant Multiple Rule for Finite Sums, followed by properties of limits.)

99. Prove Property 6 of the definite integral in general; that is, prove that the property $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ holds irrespective of the order of the points a, b, and c. (**Hint:** The standard case of $a < c < b$ is discussed in the text. To start you off with the remaining cases, assume, for example, that $a < b < c$. By an argument analogous to the one given in the text, we see that $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$. Observe by Property 2 that $\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx$, and rearrange the terms. Handle the remaining cases in a similar fashion.)

100. Prove Property 7 of the definite integral. (**Hint:** For a particular partition of $[a,b]$ and choice of sample points, argue that $\sum_{i=1}^n f(x_i^*)\Delta x \le \sum_{i=1}^n g(x_i^*)\Delta x$, and take the limits as $n \to \infty$.)

101. Use Property 7 to prove that the definite integral of a nonnegative function is nonnegative: If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$. Then state and prove the analogous statement for nonpositive functions.

102. Prove Property 8 of the definite integral. (**Hint:** Use Property 7 with the constant function $g(x) = M$. The other inequality can be handled in a similar manner.)

103. Use Properties 4 and 7 to prove the following statement: If f is integrable on $[a,b]$, then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$. (**Hint:** Let $k = 1$ or $k = -1$ so that $\left|\int_a^b f(x)\,dx\right| = k \cdot \int_a^b f(x)\,dx$. Use the fact that $k \cdot f(x) \le \left|f(x)\right|$, along with Properties 4 and 7.)

104–115 Find the average value of the function over the given interval. (**Hint:** Instead of using Riemann sums, try using the results from Exercises 46 and 48 along with formulas from geometry and the properties of the definite integral.)

104. $f(x) = 3x-1$ on $[0,4]$

105. $g(x) = -1-\frac{1}{2}x$ on $[-2,2]$

106. $h(x) = x^2-2$ on $[-1,5]$

107. $F(x) = -3x^2+7x+12$ on $[-2,3]$

108. $G(x) = 9x-x^3$ on $[-4,4]$

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109. $H(x) = x^3 - 2x^2 - 1$ on $[0,2]$

110. $k(x) = |x-4| - 2$ on $[0,7]$

111. $m(x) = |x| + |x+1|$ on $[-3,2]$

112. $u(x) = \sqrt{1-(x-1)^2}$ on $[0,2]$

113. $v(x) = [\![x]\!]$ on $\left[\frac{1}{2}, 3\right]$

114. $t(x) = \sqrt{x} - 1$ on $[0,5]$

115. $w(x) = \sqrt[3]{x} - x$ on $[-1,8]$

116–119 Recognize the given limit as a Riemann sum of a function over an interval and then use geometry to evaluate it.

116. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n}\left(2 - \frac{i}{n}\right)$

117. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{3}{n}\left(\frac{2i}{3n} + 4\right)$

118. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n-1} \left(\frac{2}{n} + \frac{4i}{n^2}\right)$

119. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{2}{n}\sqrt{4 - \left(\frac{2i}{n}\right)^2}$

120. Prove that if $f(x)$ is an increasing nonnegative function on $[a,b]$, then for every n, $L_n \le \int_a^b f(x)\,dx \le R_n$. Then state and prove the analogous statement for a decreasing function $g(x)$ on the same interval.

121. Prove that L_n corresponding to $f(x)$ of Exercise 120 is increasing as $n \to \infty$, while R_n is decreasing. Then state and prove the analogous statement for $g(x)$.

122. Use geometry and a fundamental trigonometric identity to find $\int_0^\pi \sin^2 x\,dx$. (**Hint:** Start out by comparing the given integral with $\int_0^\pi \cos^2 x\,dx$.)

123. Use the result of Exercise 122 to evaluate $\int_0^\pi \left(2\sin^2 x + x^2 - 3x\right)dx$.

124.* Suppose that the nonnegative function $R(x)$ has the property that $R(x) = 0$ whenever x is rational. If R is integrable on the interval $[a,b]$, prove that $\int_a^b R(x)\,dx = 0$.

Concept Check

125–130 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

125. If f and g are both integrable on $[a,b]$, then $\int_a^b f(x)\cdot g(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$.

126. The integral $\int_a^b f(x)\,dx$ is numerically equal to the area between the graph of $f(x)$ and the x-axis.

127. A Riemann sum for $f(x)$ on $[a,b]$ can be based upon a division of $[a,b]$ into subintervals of unequal width.

128. If $|f(x)|$ is integrable on $[a,b]$, then so is $f(x)$.

129. If $f(x)$ is positive and increasing on $[a,b]$, then $\int_a^b f(x)\,dx \ge f(a)(b-a)$.

130. If $\int_a^b f(x)\,dx < 0$, then $f(x) \le 0$ on $[a,b]$.

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