

Solving $f''(x) = 0$ gives us the possible inflection points 0 and 1 (note that 0 is both a critical point and a potential inflection point).

We proceed to evaluate the signs of f' and f'' on the intervals of interest.

Interval	$(-\infty, 0)$	$(0, \frac{3}{2})$	$(\frac{3}{2}, \infty)$
Sign of f'	$f'(-1) = -10 < 0$	$f'(1) = -2 < 0$	$f'(2) = 8 > 0$
Monotonicity of f	Decreasing	Decreasing	Increasing

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f''	$f''(-1) = 24 > 0$	$f''(\frac{1}{2}) = -3 < 0$	$f''(2) = 24 > 0$
Concavity of f	Concave up	Concave down	Concave up

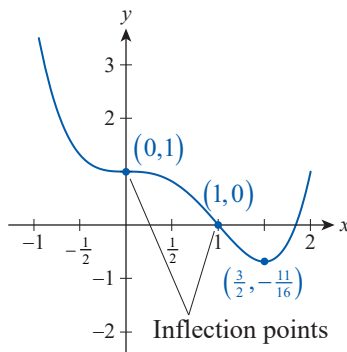


Figure 10

$$f(x) = x^4 - 2x^3 + 1$$

We now have a wealth of information about f at our disposal, and can easily answer all of the questions about its graph.

First, the two tables above identify the intervals of constant monotonicity and concavity and whether f is increasing, decreasing, concave up, or concave down on each. Note how those intervals correspond to the graph of f shown in Figure 10.

Second, we can use the First Derivative Test to realize that f does not have a relative extremum at the critical point 0, but does have a relative minimum at $\frac{3}{2}$. And since the concavity changes from up to down at 0 and from down to up at 1, both 0 and 1 are inflection points.

We could also have used the Second Derivative Test to determine that f has a relative minimum at the critical point $\frac{3}{2}$, even though we never actually evaluated f'' at the point. The reason is that we know f'' exists on the open interval $(1, \infty)$ containing $\frac{3}{2}$ (indeed, f'' exists everywhere), and that it is positive on the interval. In particular, it is positive at $\frac{3}{2}$.

The last step is to evaluate f at the two inflection points and the one relative extremum—those values are shown in Figure 10. Note that f has no absolute maximum value, but the relative minimum at $\frac{3}{2}$ is also its absolute minimum.

Example 4 illustrates that the tools of calculus can answer questions about particular points of interest on the graph of a function. We will explore this idea further in Section 4.5, but we will acquire one additional tool before doing so.

4.3 Exercises

1–22 Determine the intervals of monotonicity of the given function.

1. $f(x) = x^2 - 4x + 1$

2. $g(x) = \frac{3}{2}x + 5$

3. $h(x) = \frac{2}{3}x^3 + 4x^2 - 10x + \frac{5}{3}$

4. $F(x) = 0.75x^4 + x^3 - 15x^2 + 24x + 7$

5. $G(x) = -x^4 - 2x^3 + 8x^2 - 6x + 1$

6. $k(x) = (x^2 - 4)(x^2 - 3)$

7. $m(x) = -\frac{x^6}{3} - \frac{2x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x$

8. $n(x) = \frac{-2}{x+1}$

9. $H(x) = \frac{x^2 + 3}{x+3}$

10. $R(t) = \frac{t+2}{t^2-9}$

11. $C(x) = \frac{3x^2 + 1}{x - 2}$

13. $f(t) = 2.5 - |t - 3.125|$

15. $w(s) = \sqrt{s}(s - 1)$

17. $G(x) = \sin^2 x + 1$

19. $g(t) = -2\sqrt{t}e^{-2t}$

21. $A(x) = 0.5x^{1/5}(x^2 - 4)$

12. $A(x) = |x + 4| - 1$

14. $t(x) = x^{2/3} + 2$

16. $F(x) = 6 - (x - 3)^{3/5}$

18. $m(x) = 2^{-x} + 2^{2x}$

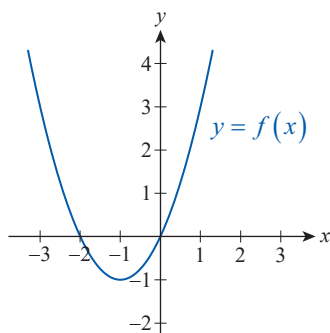
20. $L(x) = x \ln x$

22. $U(s) = s\sqrt{3 - s^2}$

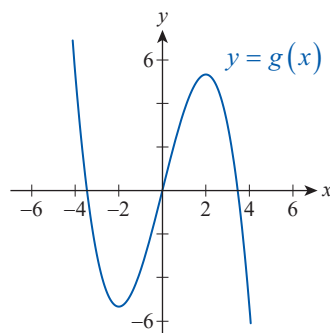
23–42. Use the First Derivative Test to classify the relative extrema, if any, of the functions given in Exercises 3–22.

43–50 Identify all intervals of monotonicity as well as intervals of concavity for the graphed function. Find all local extrema and inflection points, if any.

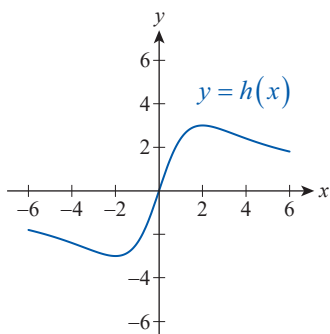
43. $f(x) = x^2 + 2x$



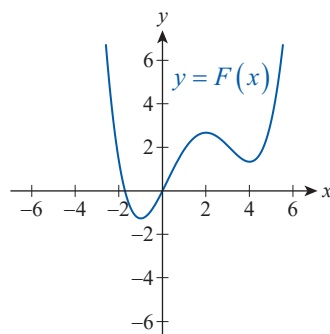
44. $g(x) = -\frac{1}{3}x^3 + 4x$



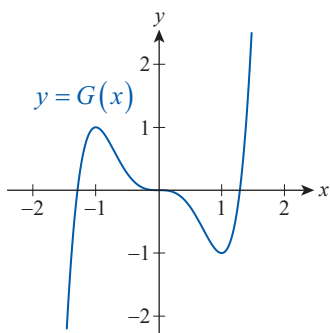
45. $h(x) = \frac{12x}{x^2 + 4}$ (Serpentine)



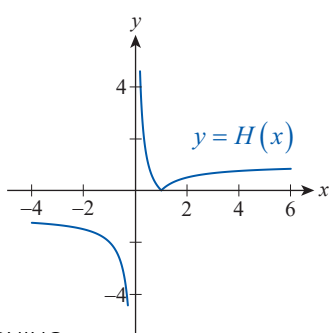
46. $F(x) = \frac{1}{16}x^4 - \frac{5}{12}x^3 + \frac{1}{4}x^2 + 2x$



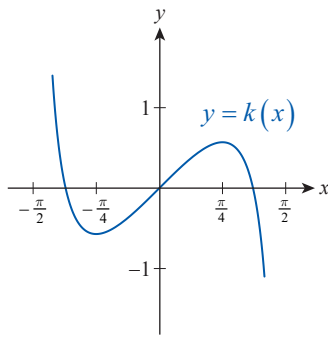
47. $G(x) = 1.5x^5 - 2.5x^3$



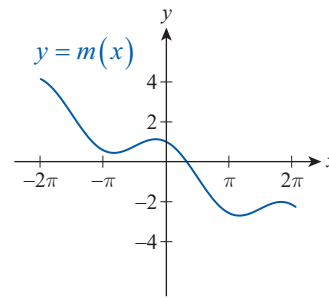
48. $H(x) = \frac{|1 - x|}{x}$



49. $k(x) = 2x - \tan x$, $|x| < \frac{\pi}{2}$



50. $m(x) = \cos x - \frac{x}{2}$, $|x| \leq 2\pi$



51–62 Determine the intervals of concavity of the given function.

51. $f(x) = 4x - x^2$

52. $g(x) = \frac{x^3}{3} - \frac{x^2}{2} + x + 1$

53. $h(x) = 4x^3 - 3x^2 - 36x + 5$

54. $k(x) = -x^4 + 5x + 1$

55. $F(x) = x^4 + 4x^3 + 72x$

56. $G(x) = -x^4 + 6x^3 + 24x^2 - 4x + 2$

57. $k(x) = 0.3x^5 + x^4 - 3x^3 + 12x + 4$

58. $v(x) = \frac{5}{x-2}$

59. $m(x) = \frac{x^2 + 5}{x-5}$

60. $F(x) = \frac{2^x}{x}$

61. $H(x) = 9^x + 3^{-x}$

62. $u(x) = (x-2)^{5/7}$

63–82 Use the first and second derivatives to identify the intervals of monotonicity, extrema, intervals of concavity, and inflection points of the given function.

63. $f(x) = \frac{1}{2}x + 5$

64. $g(x) = x^2 - 8x + 3.5$

65. $h(x) = -\frac{1}{2}x^2 + 5x + \frac{8}{3}$

66. $F(x) = 2x^3 + 3x^2 - 7$

67. $G(x) = -4x^3 - 3x^2 + 18x + 10$

68. $K(x) = 0.5x^4 + 2x^3 - 6x^2 - 16x + 19.5$

69. $L(x) = -x^4 + 12x^2 - 20x + 3$

70. $m(x) = -\frac{3}{x^2}$

71. $n(x) = \frac{x+1}{x-2}$

72. $H(x) = \frac{2x}{x^2 - 4}$

73. $r(x) = \frac{2x^2 + 1}{x-4}$

74. $t(x) = \frac{5}{4} \left(x - \frac{4}{5} \right)^{4/5}$

75. $F(x) = \frac{(x-1)^2}{x^2 - 1}$

76. $f(x) = \sqrt[3]{x} - x$

77. $g(x) = x^{2/3} \left(\frac{2}{3} - x \right)$

78. $h(x) = x\sqrt{9-x^2}$

79. $u(x) = \sqrt{x}e^{-x}$

80. $v(x) = \sin^2 x - 2\cos x$

81. $k(x) = \cos x - \sin x$

82. $L(x) = -x^2 \ln|x|$

83–90 Sketch a graph of a function satisfying the given conditions. (Answers will vary.)

83. f is differentiable on \mathbb{R} , has both a local maximum and minimum, but no global extrema.

84. f is differentiable on \mathbb{R} , f' has a zero, and f has no local extrema.

85. f is differentiable on \mathbb{R} , f is an odd function, $f'(x) > 0$ on $(-4, 4)$, $f'(x) \leq 0$ elsewhere, and $\lim_{|x| \rightarrow \infty} f(x) = 0$.

86. $\lim_{|x| \rightarrow \infty} f(x) = 1$ and $f(0) = 0$ is a global minimum.

87. $f(x)$ is everywhere positive on \mathbb{R} , but its derivative is everywhere negative.

88. $f'(x) < 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$.

89. f has vertical asymptotes at $x = \pm 2$, a horizontal asymptote at $y = 1$, f is an even function, $f'(x) < 0$ on $(0, 2)$ and $(2, \infty)$, f has a local maximum at 0, $f''(x) < 0$ on $(0, 2)$, $f''(x) > 0$ on $(2, \infty)$, and f has no absolute extrema.

90. $f(0) = f(2) = 0$, $f'(1) = f'(3) = f'(4) = 0$, $f''(1) < 0$, $f''(2) = 0$, and $\lim_{x \rightarrow \infty} f(x) = 0$.

91–96 The function $p(t)$ gives the position, relative to its starting point, of an object moving along a straight line. Identify the time intervals when the object is moving in the positive versus negative direction, as well as those intervals when it is accelerating or slowing down. Find the times when the object changes direction as well as when its acceleration is zero.

91. $p(t) = 2t^2 - 3t + 2.5$

92. $p(t) = 5t - \frac{1}{2}t^2$

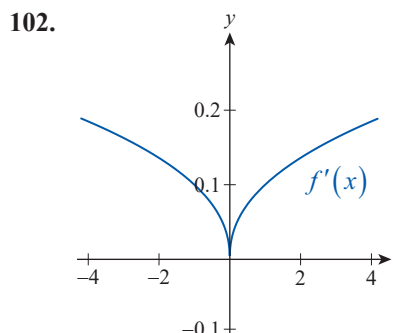
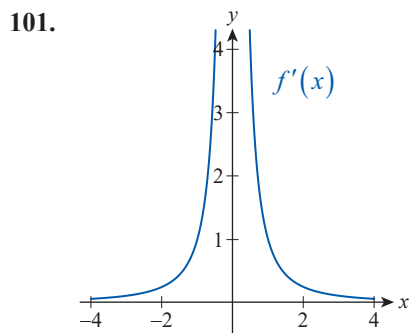
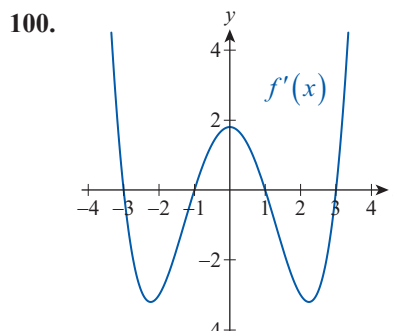
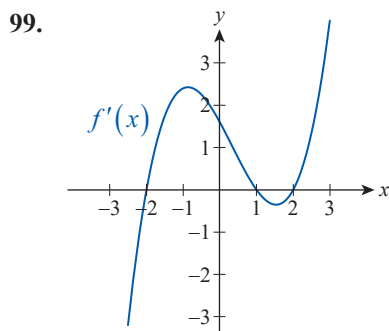
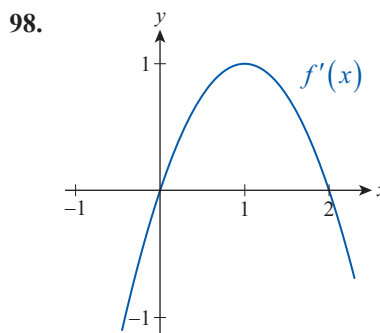
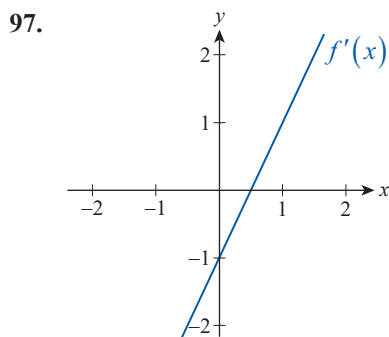
93. $p(t) = 2t^3 - 15t^2 + 24t$

94. $p(t) = -2t^3 + 22.5t^2 - 66t$

95. $p(t) = e^{-t} \sin t$

96. $p(t) = \frac{1-t}{2t+2}$

97–102 The graph of the derivative $f'(x)$ of the function $f(x)$ is given. Use it to sketch an approximate graph of $f''(x)$. Then try to sketch a possible graph of $f(x)$ as well. (**Note:** There are many possible correct answers for the graph of f . Can you see why?)



103. Suppose $T(t)$ is the outside temperature, over a 24-hour period on a typical spring day where you live (t is measured in hours, with $t = 0$ corresponding to midnight). Given the following data, what time(s) of day might c_i represent ($i = 1, 2, \dots, 6$)? Explain your choice(s).

a. $T'(c_1) > 0, T''(c_1) < 0$

b. $T'(c_2) > 0, T''(c_2) > 0$

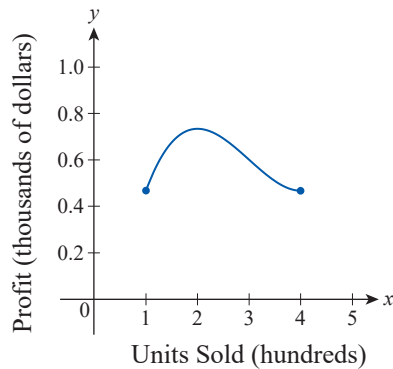
c. $T'(c_3) = 0, T''(c_3) > 0$

d. $T'(c_4) < 0, T''(c_4) < 0$

e. $T'(c_5) < 0, T''(c_5) > 0$

f. $T'(c_6) = 0, T''(c_6) < 0$

104. The graph below shows the profit (in thousands of dollars) from a product per hundreds of units sold. Use the graph to visually estimate the production level at which the marginal profit starts increasing.



105. An aftermarket auto accessories company manufactures StopTheMess trunk liners and organizers. The overhead cost of operating the plant is \$5000 per month and the cost of manufacturing each item is \$20. The company estimates that 200 liners can be sold monthly for \$50 apiece, and that sales will increase by 10 liners per month for each dollar decrease in price.
- Find a formula for the profit function $P(n)$, where n is the number of trunk liners manufactured (suppose $200 \leq n \leq 350$).
 - Identify the intervals on which P is increasing or decreasing, and find the production level that maximizes profits.
106. The strength of an electric field due to a charged ring obeys the equation
- $$E = \frac{kqx}{(x^2 + R^2)^{3/2}}$$
- where q is the electric charge measured in coulombs (C), $k \approx 8.99 \cdot 10^9 \text{ Nm}^2/\text{C}^2$, R is the radius of the ring and x is the distance to the charge in meters. Find a formula for the rate of change of E as x increases. What happens to E and dE/dx as $x \rightarrow \infty$?
107. Suppose that $f'(x) = (x+4)(x+1)^2(x-3)^3$. By examining the zeros of $f'(x)$, identify the x -coordinates of the local maxima and minima of $f(x)$. (**Hint:** Recall what you learned about multiplicities of zeros and sign changes of polynomials.)
108. Repeat Exercise 107 for $g(x)$ if $g'(x) = 3x(x-2)(x+5)^4(x-\frac{1}{3})^2(x+1)^3$.
109. Use derivatives to prove that if $x \in (-\infty, 1)$, then $1/(1-x) \geq 1+x$. (**Hint:** Start by assuming that $x \geq 0$. Rewriting the inequality as $f(x) \geq g(x)$, show that $D(x) = f(x) - g(x)$ is increasing, while $D(0) = 0$. To handle the case of $x < 0$, use the fact that $D'(x) < 0$ along with $D(0) = 0$.)
110. Prove that quadratic functions cannot have any inflection points, while cubic functions have exactly one. What can you say about fourth-degree polynomials?
111. Suppose we know that the derivative of a function f is $f'(x) = a/(x^2 + 1)$ for some nonzero $a \in \mathbb{R}$. Prove that f is increasing or decreasing everywhere on \mathbb{R} .
112. Generalize Exercise 111 by proving the following statement: If f is differentiable on an interval I and $f'(x) \neq 0$ in the interior of I , then f is increasing or decreasing everywhere on I . (**Hint:** Indirectly assume that f' changes signs, and use the Darboux property of derivative functions.)
113. Use Exercise 112 to find conditions under which the general cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ is decreasing everywhere on \mathbb{R} .
114. Prove that a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one inflection point. Find its first coordinate, assuming that the three real roots of $p(x)$ are x_1, x_2 , and x_3 .
115. Determine the conditions on the coefficients of the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ in order for $p(x)$ to have a local maximum at $x = 0$ and a local minimum at $x = 4$.
- 116.* Consider the following function.
- $$f(x) = \begin{cases} 2x^2 & \text{if } x \text{ is irrational} \\ 4x^2 & \text{if } x \text{ is rational} \end{cases}$$
- Use f to explain why the changing of signs of the derivative is not necessary for a function to have a local extremum.
117. By considering $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 3-x & \text{if } x > 1 \end{cases}$, explain why the First Derivative Test can't be used for a discontinuous function.

118. Suppose that $f(x)$ and $g(x)$ are at least twice differentiable and that both their first and second derivatives are positive everywhere on an interval I . Which of the following statements can you prove from these conditions? Prove those statements that are true, and provide counterexamples for the rest.

- $f(x) + g(x)$ is increasing on I .
- $f(x) + g(x)$ is concave up on I .
- $f(x) \cdot g(x)$ is increasing on I .
- $f(x) \cdot g(x)$ is concave up on I .
- $f(g(x))$ is increasing on I .
- $f(g(x))$ is concave up on I .

119.* Use mathematical induction to prove the following generalization of the Second Derivative Test: Suppose that the derivatives of all orders of the function f exist at c , up to $f^{(2k)}(c)$, and that $f'(c) = f''(c) = \dots = f^{(2k-1)}(c) = 0$, but $f^{(2k)}(c) \neq 0$. Then if $f^{(2k)}(c) < 0$, f has a relative maximum at c ; if $f^{(2k)}(c) > 0$, f has a relative minimum at c .

120.* Use mathematical induction to prove that if f is $(2k+1)$ -times differentiable at c , $f'(c) = f''(c) = \dots = f^{(2k)}(c) = 0$, but $f^{(2k+1)}(c) \neq 0$, then f has a point of inflection at c .

Concept Check

121–128 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

- Not all fourth-degree polynomials have inflection points.
- A function with no inflection points cannot change concavity.
- If $f'(x)$ is negative on $(-\infty, c)$ and positive on (c, ∞) , then f has a minimum at c .
- If $f(x)$ and $g(x)$ are decreasing, then so is $(f+g)(x)$.
- If $f(x)$ and $g(x)$ are decreasing, then so is $(f \cdot g)(x)$.
- A polynomial of degree n cannot have more than $n-1$ extrema on \mathbb{R} .
- If $c \in \mathbb{R}$ is a critical point, then the function has a local minimum or a local maximum at c .

128. If $f''(c) = 0$, then c is an inflection point.

4.3 Technology Exercises

129–130 Use a graphing utility to graph the given function along with its first and second derivatives on the same screen. Use your graphs to explain the behavior of the function with regard to the signs and values of its derivatives.

129. $f(x) = (x^2 + 1)\sqrt{9 - x^2}$ on $[-3, 3]$

130. $g(x) = \sqrt{x} \cos x - \sin 2x$ on $[0, 4\pi]$

131. The table below shows the temperature of a pediatric patient over a 24-hour period (measurements were taken every two hours, starting at midnight). Use the regression capabilities of a graphing utility to approximate the data by a fourth-degree polynomial. When do you estimate the patient's temperature to have been the highest? The lowest? When did the highest rates of increase and decrease occur?

Patient's Temperature

Time	12 a.m.	2 a.m.	4 a.m.	6 a.m.	8 a.m.	10 a.m.	12 p.m.
Temp (°F)	99.9	99.5	99.1	98.9	98.7	98.8	99.4
Time	2 p.m.	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 a.m.	
Temp (°F)	100.0	102.1	101.9	101.3	101.0	99.9	

- 132.** In the first few months following the launch of a new product, monthly sales were given by the function $S(t) = 300t^2 / (t^2 + 2)$, where t is measured in months.
- Use a graphing utility to graph the function over the first year, and estimate when the rate of growth in sales was greatest.
 - Use the differentiation capabilities of a computer algebra system to check your estimate in part a.

133–136 Use a graphing utility to graph the given function for different values of the parameter(s). Examine how the values of the parameter(s) affect the number of local extrema. How about inflection points?

133. $f(x) = x^4 + cx^3$; $1 \leq c \leq 3$

134. $g(x) = 0.5x^5 + cx^4 - dx$; $0 \leq c, d \leq 3$

135. $h(x) = \cos x - \sin(cx)$; $0 \leq c \leq 4$

136. $k(x) = \sin^2(cx)\cos(dx)$; $0 \leq c, d \leq 5$