

$$\begin{aligned}f(4) + 2 &= 4f'(c) \\f(4) &= 4f'(c) - 2 \leq 4 \cdot 3 - 2 \\f(4) &\leq 10\end{aligned}$$

Hence the largest possible value of $f(4)$ is 10.

Example 6 Using Corollary 2 of the Mean Value Theorem

Find the unique function f whose derivative is $3x^2$ and whose graph passes through $(1, 5)$.

Solution

By now, we have differentiated enough polynomials to recognize $3x^2$ as the derivative of x^3 . If we let $g(x) = x^3$, then g and f have the same derivative; and by Corollary 2, it must be the case that $f(x) = g(x) + C$ for some constant C . We can determine C as follows.

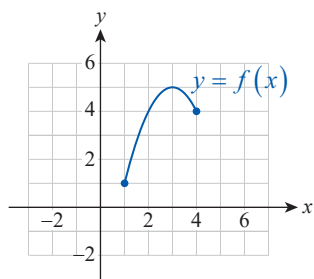
$$\begin{aligned}f(1) &= 5 && \text{The graph of } f \text{ passes through } (1, 5). \\g(1) + C &= 5 \\1^3 + C &= 5 \\C &= 4\end{aligned}$$

Now it can be easily verified that $f(x) = x^3 + 4$ satisfies the given criteria, and by Corollary 2 it is the unique function to do so.

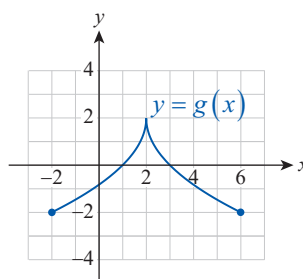
4.2 Exercises

1–4 Use the graph of the function to visually estimate the value of c in the given interval that satisfies the conclusion of the Mean Value (or Rolle's) Theorem; then check your guess by calculation. If such a c doesn't exist, explain why.

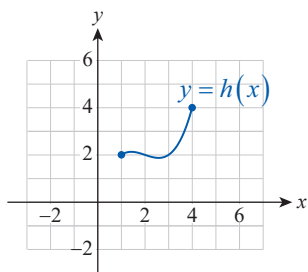
1. $f(x) = -x^2 + 6x - 4$ on $[1, 4]$



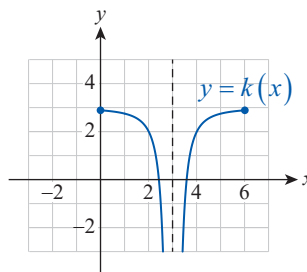
2. $g(x) = -2\sqrt{|x-2|} + 2$ on $[-2, 6]$



3. $h(x) = \frac{x^3}{3} - 2x^2 + \frac{11x}{3}$ on $[1, 4]$



4. $k(x) = -\frac{1}{(x-3)^2} + 3$ on $[0, 6]$



5–8 Prove that the equation has exactly one real solution on the given interval.

5. $x^5 - 3x^2 = 25$ on $[2, 3]$
6. $5x^3 + 7x = 9$ on \mathbb{R}
7. $\arctan x = 3 - x$ on $[0, 3]$
8. $\tan x = \cos x$ on $\left(0, \frac{\pi}{2}\right)$

9–20 Determine whether Rolle's Theorem applies to the function on the given interval. If so, find all possible values of c as in the conclusion of the theorem. If the theorem does not apply, state the reason.

9. $f(x) = -x^2 + 4x - 3$ on $[1, 3]$
10. $g(x) = x^3 - 5x^2 + 2x + 10$ on $[-1, 4]$
11. $h(x) = 2x^4 - x^3 + 6x^2 + x - 8$ on $[1, 3]$
12. $F(x) = \frac{3}{(x-1)^2}$ on $[0, 2]$
13. $G(x) = \frac{1}{x^2 + 1}$ on $[-3, 3]$
14. $k(x) = \frac{x}{x^2 - 1}$ on $[-3, 3]$
15. $H(x) = x^{4/3} - 10$ on $[-10, 10]$
16. $m(x) = -\cos x$ on $[0, 4\pi]$
17. $T(z) = \cot z$ on $[0, 5\pi]$
18. $F(x) = \sec x$ on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$
19. $w(t) = \csc t$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
20. $A(x) = |x+1| - 5$ on $[-6, 4]$

21–32 Determine whether the Mean Value Theorem applies to the function on the given interval. If so, find all possible values of c as in the conclusion of the theorem. If the theorem does not apply, state the reason.

21. $f(x) = |x-1| + 3$ on $[-1, 2]$
22. $g(x) = -\frac{1}{2}x + 4$ on $[0, 8]$
23. $h(x) = -x^2 + 4x + 4$ on $[-1, 4]$
24. $F(x) = \frac{1}{2}x^3 - x^2$ on $[-2, 2]$

25. $G(x) = \frac{2x}{x+1}$ on $[-2, 3]$
26. $k(x) = \frac{x+1}{2x-5}$ on $[-1, 2]$
27. $H(x) = \frac{5}{x^2+5}$ on $[-5, 5]$
28. $m(x) = (x-4)^{2/3} + 2$ on $[1, 5]$
29. $C(x) = \sqrt{1-x^2}$ on $[-1, 0]$
30. $L(t) = \ln|t+1|$ on $[-3, 1]$
31. $u(z) = \tan z$ on $[0, \pi]$
32. $v(s) = \arctan s$ on $[0, 1]$
33. If $f(-2) = 2$ and $f'(x) \leq 2$ for all x , what is the largest possible value of $f(2)$?
34. If $g(1) = 4$ and $g'(x) \geq -3.5$ for all x , what is the smallest possible value of $g(3)$?
35. One of your classmates claims that he found a function f such that $f(-5) = -1$, $f(5) = 1$, and $f'(x) \leq 0.1$ for all x . Explain how you know that he made an error in his calculations.
36. If $|F'(x)| \leq 2.5$ for all x , prove that $|F(7) - F(3)| \leq 10$.
37. Find the function f that passes through $(3, 1)$ and whose derivative is $3x^2 - 2x + 1$.
38. Find the function g that passes through $(\pi/2, 0)$ and whose derivative is $2 \cos x$.
39. Suppose that the velocity function of a moving object is $v(t) = -2t + 5$ and that its position at $t = 1$ is -8 units from the origin. Find a formula for the position function.
40. Find the velocity and position functions of an object thrown vertically upward from an initial height of 20 ft with initial velocity $v(0) = 42$ ft/s. (The acceleration caused by gravity is $a = -32$ ft/s². Ignore air resistance.)
41. Generalize Example 1 by proving the following statement: if $f(x)$ is differentiable on $[a, b]$, $f'(x) \neq 0$ on (a, b) , and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has exactly one real solution.

42. Two highway patrol cars are stationed 7 miles apart along a straight highway where the speed limit is 65 mph. The first patrol car clocks a red Porsche at 64 mph, and five minutes later, the second police car clocks it at 61.5 mph. Explain why the driver is pulled over and issued a ticket.
43. A plane leaves London Heathrow Airport to arrive at Houston Intercontinental Airport ten and a half hours later. The distance between the two airports is 4830 miles. Explain why the plane must have reached a speed of 450 mph at least twice during the trip.
44. Two MotoGP riders finish a race in a tie. Show that there was at least one moment during the race when the two riders had the exact same speed. (**Hint:** Ignoring the difference in their starting positions, consider the difference of the two position functions and use Rolle's Theorem.)
45. Suppose that $f'(x) = x$ for all $x \in \mathbb{R}$. Prove that there exists a constant C such that $f(x) = \frac{1}{2}x^2 + C$. (**Hint:** Use Corollary 2 from the text.)
46. Use Corollary 2 of this section to prove the well-known trigonometric identity $\cos^2 x + \sin^2 x = 1$. (**Hint:** Use the corollary for the functions $f(x) = \cos^2 x + \sin^2 x$ and $g(x) = 1$, and argue that necessarily $C = 0$.)
47. Follow the hint given in Exercise 46 to prove the identity $\sin^{-1} x + \cos^{-1} x = \pi/2$.
48. Let $f(x) = \cot \pi x$ and $g(x) = \cot \pi x + \lfloor x \rfloor$. Show that $f'(x) = g'(x)$, but $f(x) - g(x)$ is not constant. Why does this not contradict Corollary 2 of this section?
49. Suppose that for a function $f(x)$, the second derivative $f''(x)$ exists for all x on an interval I . Prove that if f has three zeros on I , then its second derivative f'' also has a zero. (**Hint:** Apply Rolle's Theorem on f and then on f' .)
50. Suppose that f is twice differentiable on \mathbb{R} and a and b are two successive zeros of f' . Prove that f can have at most one zero on (a, b) . (**Hint:** Start by assuming that f has at least two zeros, and apply Rolle's Theorem.)
51. Suppose that f is continuous on $[a, b]$, differentiable on (a, b) , and $c \in (a, b)$ with $f'(c) = 0$. Does it follow that $f(a) = f(b)$? Explain.
- 52.* Suppose that f is twice differentiable on \mathbb{R} and $f'(c) = 0$ for some $c \in \mathbb{R}$. If $f''(c) \neq 0$, prove that there exist $a, b \in \mathbb{R}$ such that $c \in (a, b)$ and $f(a) = f(b)$.
53. Suppose that the rabbit population at a game preserve is observed monthly and is found to have increased from 550 to 1150 rabbits in a year's time. Explain why there must exist a time during the year when the population is increasing at a rate of 50 rabbits per month.
54. Show that if f is a quadratic function, that is, $f(x) = Ax^2 + Bx + C$, then on any interval $[a, b]$, the point c satisfying the conclusion of the Mean Value Theorem is the midpoint of the interval.
- 55.* Prove that for all $0 < \alpha < \beta < \pi/2$, $(\beta - \alpha)\cos \beta \leq \sin \beta - \sin \alpha \leq (\beta - \alpha)\cos \alpha$. (**Hint:** Apply the Mean Value Theorem for the function $f(x) = \sin x$ on the interval $[\alpha, \beta]$.)
56. Use the Mean Value Theorem to prove the inequality $\ln(1+x) \leq x$ for all $x \geq 0$. (**Hint:** For $x > 0$, consider $f(t) = \ln(1+t)$ on the interval $[0, x]$ and apply the Mean Value Theorem.)
57. Prove the inequality $\sqrt[3]{1+x} \leq 1 + \frac{1}{3}x$ for $x \geq 0$. (Adapt and follow the hint given in Exercise 56.)
- 58.* Use the Mean Value Theorem to show that $\lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-5}) = 0$. (**Hint:** Apply the theorem to the function $f(x) = \sqrt{x}$ on the interval $[x-5, x]$ for a fixed x -value. Then let x approach infinity.)
59. Use the Mean Value Theorem to prove the following inequality for all $\alpha, \beta \in \mathbb{R}$.
- $$|\arctan \alpha - \arctan \beta| \leq |\alpha - \beta|$$
- 60.* Generalizing Exercise 59, prove that if f is differentiable and $|f'(x)| \leq M$ for some $M > 0$, then for all x, y in the domain of f , $|f(x) - f(y)| \leq M|x - y|$. (Such a function f is said to have the Lipschitz property with constant M .)
61. Use Exercise 60 to prove that $f(x) = \cos 3x$ has the Lipschitz property with constant 3.
- 62.* We say that the function $f(x)$ has a *fixed point* $c \in \mathbb{R}$ (or that it leaves c fixed) if $f(c) = c$. Prove that if f is differentiable on \mathbb{R} and $f'(x) < 1$ for all x , then f can have no more than one fixed point.

63.* Prove Cauchy's Mean Value Theorem, which is stated as follows: If f and g are both continuous on the closed interval $[a, b]$ and differentiable on (a, b) , and if $g'(x) \neq 0$ on (a, b) and $g(a) \neq g(b)$, then there is a point $c \in (a, b)$ for which $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

(**Hint:** Apply Rolle's Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Concept Check

64–68 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

- 64.** There are situations when the Mean Value Theorem applies but Rolle's Theorem doesn't.
- 65.** There are situations when Rolle's Theorem applies but the Mean Value Theorem doesn't.
- 66.** If a and b are zeros of the function $f(x)$, then there exists a point c in (a, b) such that $f'(c) = 0$.
- 67.** If a and b are zeros of the polynomial $p(x)$, then there exists a point c in (a, b) such that $p'(c) = 0$.
- 68.** If $f'(x) = 0$ for all x in the domain of f , then $f(x)$ is a constant function.

4.2 Technology Exercises

69–72 Use a graphing utility to graph the function on the interval $[a, b]$ along with its secant line through the points $(a, f(a))$ and $(b, f(b))$ on the same screen. Then find and graph the line(s) tangent to the graph that are parallel to the secant line.

69. $f(x) = \sqrt{x}$ on $[0, 9]$

70. $f(x) = \frac{x^3}{2} - x^2 + x + 7.5$ on $[1, 3]$

71. $f(x) = \frac{3x}{x+2}$ on $[-1, 1]$

72. $f(x) = \sin^2 x + 3x$ on $[0, \pi]$