3.9 Exercises

1–12 Find the linearization of the function at the given value.

1. $f(x)=x^3-x;\quad x=1$
2. $g(x)=\sqrt{x-3};\quad x=4$
3. $h(x)=\left(x^4-5x^2+1\right)^7;\quad x=0$
4. $k(x)=\left(x^2+1\right)^{-2};\quad x=2$
5. $C(\theta)=\cos\theta;\quad \theta=0$
6. $T(\theta)=\tan\theta;\quad \theta=0$
7. $F(t)=\left(t^2+5t-6\right)^{-1/3};\quad t=2$
8. $r(x)=\dfrac{1}{x+4};\quad x=-3$
9. $t(u)=\dfrac{u+2}{u^2-15};\quad u=-4$
10. $v(x)=\sin \pi x;\quad x=\dfrac{1}{6}$
11. $G(z)=e^z;\quad z=0$
12. $U(s)=\ln\left(s^4+1\right);\quad s=1$

13–24 Find the value of the differential dy for the given values of x and dx.

13. $y=3x^2+x;\quad x=1,\quad dx=0.2$
14. $y=x\sqrt{x-5};\quad x=6,\quad dx=0.01$
15. $y=\dfrac{4x+1}{x-3};\quad x=2,\quad dx=0.1$
16. $y=\sec x;\quad x=\dfrac{\pi}{4},\quad dx=\dfrac{1}{8}$
17. $y=x^{3/2}+x^{-3/2};\quad x=4,\quad dx=\dfrac{1}{16}$
18. $y=\ln x+\dfrac{1}{\ln x};\quad x=e,\quad dx=0.01$
19. $y=x\tan x;\quad x=-\dfrac{\pi}{4},\quad dx=\dfrac{1}{4}$
20. $y=e^{\sqrt{x^2+3}};\quad x=1,\quad dx=0.001$
21. $y=\sqrt{\ln(x+1)};\quad x=e-1,\quad dx=\dfrac{-1}{e^2}$
22. $y=\arctan x;\quad x=-1,\quad dx=\dfrac{-1}{2^5}$
23. $y=\dfrac{\tan x}{x^2+1};\quad x=\dfrac{\pi}{3},\quad dx=-0.1$
24. $y=\cos(\arcsin x);\quad x=0.6,\quad dx=-0.16$

25–28 Calculate the values of dy and Δy and then use graph paper to draw the curve near the given point, indicating all three of the line segments dx, dy, and Δy.

25. $y=\dfrac{1}{2}x^2;\quad x=1,\quad dx=\dfrac{1}{2}$
26. $y=\tan x;\quad x=0,\quad dx=\dfrac{\pi}{6}$
27. $y=2^x;\quad x=1,\quad dx=\dfrac{1}{4}$
28. $y=\dfrac{1}{x^2};\quad x=1,\quad dx=-\dfrac{1}{4}$

29–40. Find the values of Δy and compare them with dy at the indicated points for the curves given in Exercises 13–24.

41–48 Use linear approximation to approximate the given number. Compare this approximation to the actual value obtained using a calculator. Round your answer to four decimal places. (**Hint:** First identify $f(x)$ and c; then find and appropriately evaluate $L(x)$.)

41. $\sqrt{9.1}$
42. $(1.01)^3$
43. $(7.9)^{2/3}$
44. $\dfrac{1}{10.1}$
45. $\sqrt[5]{31}$
46. $\cos 1$
47. $\ln 2.7$
48. $e^{1.05}$

49. Prove the Power and Quotient Rules for differentials.
    a. $d\left(x^n\right)=nx^{n-1}dx$
    b. $d\left(\dfrac{u}{v}\right)=\dfrac{v\,du-u\,dv}{v^2}$

50. Use the equations for V and dV from Example 4 to prove that the propagated error in the calculated volume of a sphere, in percentage terms, is three times larger than the margin of error in the measured radius; that is,
$$\frac{dV}{V}=3\cdot\frac{dr}{r}.$$

51. Prove or disprove that an analogous equation to that obtained in Exercise 50 is true for a cube; that is, if the measured side length of a cube is a units with a margin of error of da, then
$$\frac{dV}{V}=3\cdot\frac{da}{a}.$$

52–71 Use differentials or linearization to provide the requested approximations.

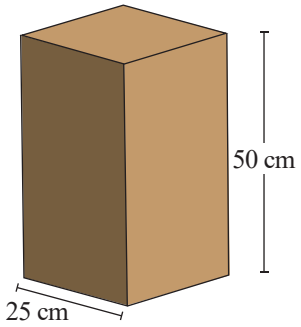
52. The side of a square was measured to be 9.5 cm with a possible error of 0.5 mm. Approximate the propagated error in the calculated area of the square. Express your answer as a percentage error.

53. The radius of a circular disk was measured to be $10\frac{1}{8}$ inches. Estimate the maximum allowable error in the measurement of the radius if the percentage error in the calculated area of the disk cannot exceed 2.5 percent.

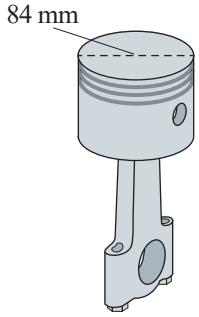
54. The base and altitude of a triangle were measured to be 7 and 9 inches, respectively. If the possible error in both cases is $\frac{1}{16}$ inches, approximate the propagated error when computing the area of the triangle.

55. Two sides of a triangle were measured to be 60 and 80 mm, respectively, while the included angle is 60 degrees. If the margin of error of the linear measurements is 0.1 mm, while that of the angle measurement is 0.1 degrees, find the possible propagated error in the calculated area of the triangle.

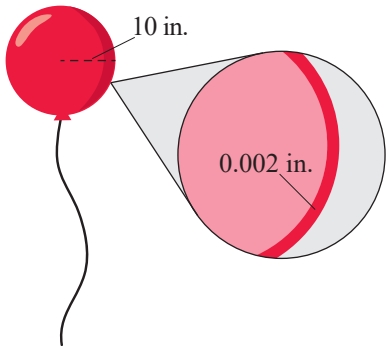
56. A box in the shape of a rectangular prism has a square base. If the edge of the base is 25 cm and the height is 50 cm, both with a possible measurement error of 0.2 mm, estimate the propagated errors in both the computed volume and surface area of the box. Express both answers as percentage errors.



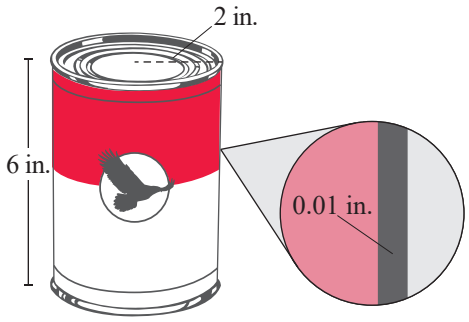
57. A cylindrical piston of diameter 84 mm is being manufactured for an automobile engine. If the maximum percentage error in the measurement of the diameter is 0.05%, estimate the greatest possible value of the propagated error in the computed cross-sectional area of the piston. Express your answer as a percentage error.



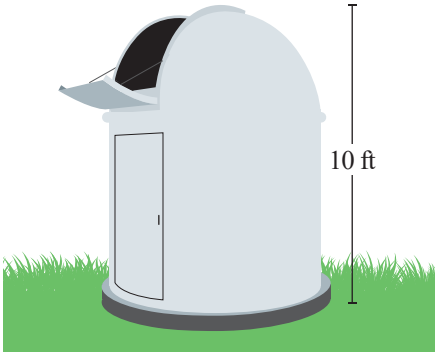
58. If the radius of an inflated balloon is 10 inches and the thickness of its wall is 0.002 inches, estimate the volume of the material it is made of. (Assume the balloon is perfectly spherical.)



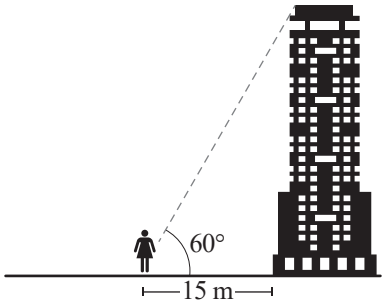
59. A tin can has a circular base of radius 2 inches and a height of 6 inches. If the thickness of its walls is 0.01 inches, estimate the volume of the material it is made of.



60. The exterior of a small private observatory needs to be painted. The building is approximately a circular cylinder with a hemisphere on top. The radius of the base is 3.5 feet and the height of the entire structure is 10 feet. Express the volume as a function of the radius of the base and use linearization to estimate the amount of paint that will provide a coat that is $\frac{1}{32}$ inches thick.



61. A trigonometry student stands 15 meters from a building and measures the angle of elevation to the top of the building as 60°. How accurate does her angle measurement have to be if she wants her propagated percentage error in estimating the height of the building to be no more than 5%?



62. Referring to Exercise 44 of Section 3.8, estimate the change in image distance when the object distance increases from 60 cm to 61 cm.

63. The magnetic force experienced by a wire carrying a current I in an external magnetic field of uniform strength B is found from the equation

$$F = BIL\sin\theta,$$

where L is the length of the wire (measured in meters), and θ is the angle between the directions of B and I.

a. Find the magnetic force on a 50 cm wire if $B = 0.03\ \mathrm{N/(A\cdot m)}$, $I = 25$ amperes (A), and $\theta = 30°$.

b. Estimate the change in force if θ is increased to 33°.

c. Calculate the true value of the change and compare it with your approximation.

64. Estimate the change in the force in Exercise 63 if θ is increased to 33°, I is increased to 27 A, and B is decreased to $0.025\ \mathrm{N/(A\cdot m)}$.

65. The kinetic energy (in J) of a moving object is found from the equation $E_{kin} = \frac{1}{2}mv^2$, where m is the mass (in kg) of the object and v is its velocity (in $\mathrm{m/s}$). Estimate the change in kinetic energy of a 1400 kg car that is accelerating from $100\ \mathrm{km/h}$ (approx. 62 mph) to $112\ \mathrm{km/h}$ (approx. 70 mph). What is the estimated percentage change?

66. When air resistance is negligible, the speed of impact of an object falling from height h is $v_i = \sqrt{2hg}$. Suppose that a rock is dropped from a height of 5 meters.

a. Find the speed of impact as the rock hits the ground.

b. Approximate the height from which the rock has to be dropped in order to increase the speed of impact by 10 percent. Express the height difference in both absolute and relative (percentage) terms.

c. Find the true value of the above height and compare it with your approximation.

67.* The volume of a cube of side length a is being determined by immersing the cube into a container of water and measuring the volume of the displaced water, and then the surface area is calculated. Estimate the percentage error we can allow in the measurement of the volume if the calculated surface area cannot differ from the true value by more than 2%. Can you generalize the result?

68. The profit function for a company is found to be $P(x) = -1.2x^2 + 500x - 2600$, where x is the number of units manufactured. If the current production level is 100 units, estimate the percentage change in profit if production is raised to 110 units.

69. For the company in Exercise 68, estimate how much the company has to increase production from 100 units in order to achieve a 10 percent profit increase.

70.* The diameter of the bottom of a 4.5-inch-tall paper cup is 2.5 inches, while the diameter of its opening is 3.5 inches. If the cup is filled with iced soda to a depth of 4.3 inches and an additional 1-cubic-inch ice cube is dropped in, predict whether the cup will overflow. (**Hint:** See Exercise 40 of Section 3.8 for help in finding the volume of soda in the cup.)

71. Suppose the velocity function of a moving object is $v(t) = 1/(1+t^2)$, and that it is moving in the positive direction along the x-axis. If you know that its location at $t = 2$ is $x = 5$, estimate its position half a second later.

72. The actual error in measurement is sometimes called absolute error, while the percentage error is referred to as relative. Write a short paragraph comparing absolute, relative, and propagated errors. Illustrate with a concrete example.

73. Examine the answer you obtained for Exercise 57. Can you state and prove a result, analogous to the one in Exercise 50, for the radius and cross-sectional area of a right circular cylinder? Explain.

Concept Check

74–80 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. Since the differential dx is an increment, its value is always positive.

75. If $f(x) = k$, then $df = 0$.

76. If f is linear, then $\Delta f / \Delta x = df / dx$.

77. If f is differentiable at c, then $\lim_{\Delta x \to 0} (\Delta f / \Delta x) = df / dx$.

78. Propagated error is also called percentage error.

79. The differential dy is always a bit less than Δy.

80. If f is increasing and $dx < 0$, then $dy > \Delta y$.

3.9 Technology Exercises

81–92. Use a graphing utility to graph the functions given in Exercises 13–24 in the same viewing window along with their linear approximations at the specified x-values. Use the **Zoom** and **Trace** features to find the maximum value for dx so that the approximation is accurate to 0.01.