If $r = 20$ cm and $dV/dt = 200\ \text{cm}^3/\text{s}$, then we have

$$\frac{dS}{dt} = \left(\frac{2}{20\ \text{cm}}\right)\left(200\ \text{cm}^3/\text{s}\right) = 20\ \text{cm}^2/\text{s},$$

the answer we obtained before.

But we might try to attack the problem in a different manner. If we *could* express S in terms of V, we could differentiate both sides with respect to t and again arrive at a relation between dS/dt and dV/dt. In pursuit of this, note that

$$\frac{r}{3}S = \frac{r}{3}\left(4\pi r^2\right) = \frac{4}{3}\pi r^3 = V,$$

so

$$S = \frac{3}{r}V.$$

But then this seems to imply that

$$\frac{dS}{dt} = \frac{dS}{dV}\cdot\frac{dV}{dt} = \frac{3}{r}\cdot\frac{dV}{dt},$$

which is not the result of $\frac{2}{r}\cdot\frac{dV}{dt}$ that we found above! What has gone wrong?

The answer is that just as V is a function of r, r can also be said to be a function of V; that is, as V changes, it certainly implies a change in r. In fact,

$$r = cV^{1/3},$$

where $c = \left[3/\left(4\pi\right)\right]^{1/3}$ (this comes from solving $V = \frac{4}{3}\pi r^3$ for r). So,

$$S = \frac{3}{r}V = \frac{3}{cV^{1/3}}V = \frac{3}{c}V^{2/3}.$$

We leave it as an exercise (Exercise 15) to show that if this last equation is differentiated with respect to t and simplified, the result is again $\frac{dS}{dt} = \frac{2}{r}\cdot\frac{dV}{dt}$.

Moral: Watch out for hidden functional relationships between the variables in your equations!

3.8 Exercises

1. A theme park ride is descending on a parabolic path that can be approximated by the equation $y = -\frac{1}{90}x^2 + 90$ (distance is measured in feet). If the horizontal component of its velocity is a constant $6\ \text{ft/s}$, find the rate of change of its elevation when $x = 22.5$.

2. Adapt your solution from Exercise 1 to find dx/dt at $x = 30$ feet if the equation of the ride's path is $y = 0.01\left(x - 95\right)^2 - 2.25$ and $dy/dt = -20\ \text{ft/s}$.

3–10 Find the rate of change using the given information.

3. $\frac{dy}{dt}$ at $y = 3$, if $y = \sqrt{x+2}$ and $\frac{dx}{dt} = 1$

4. $\frac{dy}{dt}$ at $x = 2$, if $x^2 + y^2 = 5$, $y > 0$, and $\frac{dx}{dt} = 3$

5. $\frac{dx}{dt}$ at $y = 0.5$, if $y = \frac{1}{x}$ and $\frac{dy}{dt} = -2$

6. $\frac{dx}{dt}$ at $x = 1$, if $xy^2 = \frac{1}{4}$ and $\frac{dy}{dt} = -0.25$

7. $\frac{dy}{dt}$ at $x = 0$, if $y = \frac{x+2}{x^2+1}$ and $\frac{dx}{dt} = -5.2$

8. $\frac{dy}{dt}$ at $y = \frac{1}{2}$, if $y = \frac{1}{2}e^{-x}$ and $\frac{dx}{dt} = 25$

9. $\frac{dy}{dt}$ at $x = -\frac{3\pi}{4}$, if $y = 2\sin\left(x + \frac{\pi}{4}\right)$ and $\frac{dx}{dt} = 7.4$

10. $\frac{dx}{dt}$ at $y = \frac{\pi}{4}$, if $x = \cot y$ and $\frac{dy}{dt} = -3.35$

11. The length of a rectangle is increasing at a rate of 5 in./s, while its width is decreasing at 2 in./s. Find the rate of change of its area when its length is 45 in. and its width is 25 in.

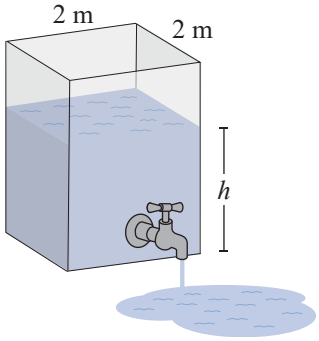
12. Find a formula for the rate of change of the distance from the origin of a point moving on the graph of $f(x) = x^2$ when $x = 2$ and $dx/dt = 3$ units per second.

13. Find the rate of separation between the moving points (x_1, y_1) and (x_2, y_2) on the graph of $y = \sin x$ when $x_1 = \pi/2$ if they start at the origin at the same time, and the horizontal components of their velocities are $dx_1/dt = \frac{1}{2}$ units per second and $dx_2/dt = -\frac{1}{2}$ units per second, respectively.

14. A Ferris wheel of radius 34 feet needs 3 minutes to complete a full revolution. At what rate is a rider descending when she is 51 feet above ground level?

15. Using the notation of Example 5, use the Chain Rule to differentiate the equation $S = (3/c)V^{2/3}$ with respect to time to obtain $\frac{dS}{dt} = \frac{2}{r} \cdot \frac{dV}{dt}$. (**Hint:** After differentiating, make use of the equation $r = cV^{1/3}$ again.)

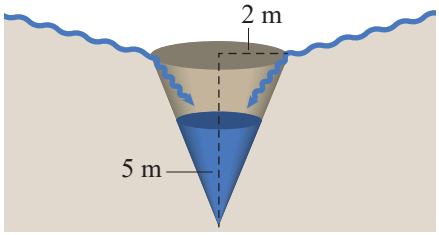
16. Rework Example 1 assuming that the tank is a rectangular prism with a 2 m by 2 m square base.



17. Rework Example 1 again, this time assuming that the tank is an inverted, right square pyramid with height 4 m and a 2 m by 2 m base. How fast is the level of water falling when its depth is 2 m?

18. A spectator is tracking a stunt plane at an air show with his camera. If the plane is on a near-vertical path, rising at a speed of 100 feet per second, and the camera is 400 feet from the point on the ground directly below the plane, how fast is the camera angle (measured with respect to the ground) changing when the plane's altitude is 400 feet? How fast is the distance between the camera and the plane increasing at that instant?

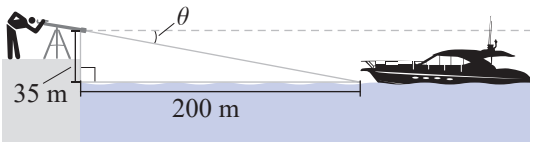
19. A cistern in the form of an inverted circular cone is being filled with water at the rate of 75 liters per minute. If the cistern is 5 meters deep, and the radius of its opening is 2 meters, find the rate at which the water level is rising in the cistern half an hour after the filling process began. (**Hint:** 1 m^3 = 1000 L)



20. Repeat Exercise 19, this time assuming that the cistern is in the form of a pyramid with a 4-by-4-meter square opening.

21. A ship passes a lighthouse at 3:15 p.m., sailing to the east at 10 mph, while another ship sailing due south at 12 mph passes the same point half an hour later. How fast will they be separating at 5:45 p.m.?

22. A tourist at scenic Point Loma, California uses a telescope to track a boat approaching the shore. If the boat moves at a rate of 10 meters per second, and the lens of the telescope is 35 meters above water level, how fast is the angle of depression of the telescope (θ) changing when the boat is 200 meters from shore?

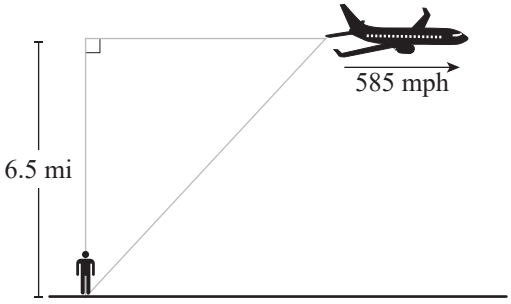


23. When preparing cereal for her child, a mother is pouring milk into a bowl, the shape of which can be approximated by a hemisphere with a radius of 6 in. If milk is being poured at a rate of $4\ \text{in.}^3/\text{s}$, how fast is the level of milk rising in the bowl when it is 1.5 inches deep? (**Hint:** The volume of fluid of height h in a hemispherical bowl of radius r is $V = \pi h^2\left(r - \frac{1}{3}h\right)$.)

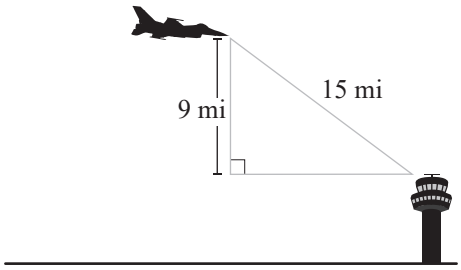
24. Suppose that in Exercise 29 of Section 3.7, the sand is being poured at a rate of 8 cubic inches per second. Find the rate of change of the height of the cone when it is 4 inches tall.

25. When finished playing in the sand, the child of Exercise 24 takes advantage of a nice wind and starts flying his kite on the beach. When the kite reaches an altitude of 60 feet the wind starts blowing it horizontally away from the child at a rate of 15 feet per second while maintaining the altitude of the kite. How fast does the child have to be letting out the string when 100 feet are already out?

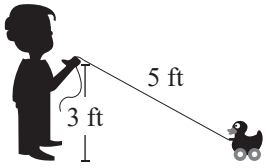
26. A passenger airplane, flying at an altitude of 6.5 miles at a ground speed of 585 miles per hour, passes directly over an observer who is on the ground. How fast is the distance between the observer and the plane increasing 3 minutes later?



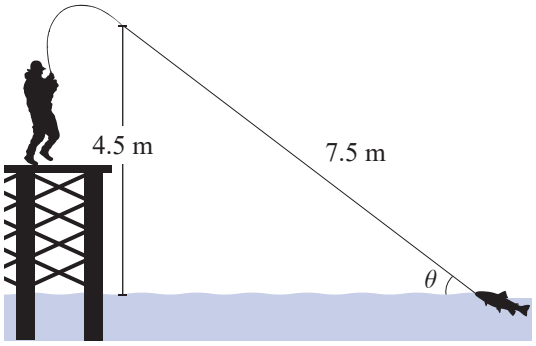
27. A military plane is flying directly toward an air traffic control tower, maintaining an altitude of 9 miles above the tower. The radar detects that the distance between the plane and the tower is 15 miles and that it is decreasing at a rate of 950 miles per hour. What is the ground speed of the plane?



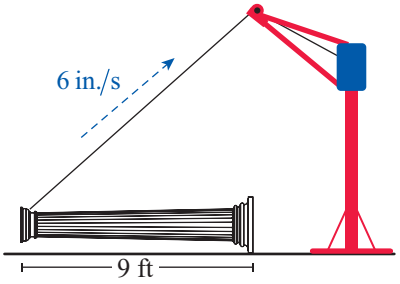
28. A child is retrieving a wheeled toy that is attached to a string by pulling in the string at a rate of 1 foot per second. If the child's hands are 3 feet from the ground, at what rate is the toy approaching when 5 feet of the string are still out?



29. A fisherman is reeling in a fish at a rate of 20 centimeters per second. If the tip of his fishing rod is 4.5 meters above the water, and we are assuming that the fish is near the water surface throughout the process, how fast is it approaching when 7.5 meters of fishing line are still out? How fast is the angle θ between the fishing line and the water increasing at that instant?



30. A construction worker is using a winch to pull a 9-foot column to a vertical position. If the winch is in the exact position where the top of the installed column is supposed to be, and the rope is being pulled at the rate of 6 inches per second, at what rate is the angle between the column and the ground changing when it is $\pi/6$ radians? At what rate is the top of the column rising vertically at that instant? (Assume the base of the column doesn't slip during lifting.)



31. The volume of a cube is decreasing at a rate of $150\ \text{mm}^3/\text{s}$. What is the rate of change of the cube's surface area when its edges are 30 mm long?

32. The acute angles of a rhombus are increasing at a rate of 0.25 radians per second. If the sides of the rhombus are 20 cm, at what rate is the area of the rhombus increasing when the acute angles are $\pi/3$ radians?

33. A 35-foot-by-18-foot rectangular pool, whose depth increases uniformly from 3 feet to 8 feet (along the 35-foot side), is being filled with water at the rate of 4.5 cubic feet per minute. You observe that water appears to be "creeping up" on the angled bottom much faster than it rises along the vertical walls. Find the rate at which the water rises along the angled bottom at the instant when the water level is 2 feet at the deep end of the pool.

34. Considering again the pool of Exercise 33, suppose that it is measured that the water is climbing upward along the angled bottom at a rate of $3.02\ \text{in./min}$ when the water level is 1 foot at the deep end. Assuming that the pump is working at the same rate of 4.5 cubic feet per minute, use this information to prove that the pool has a leak, and find the rate at which water is leaking out of the pool.

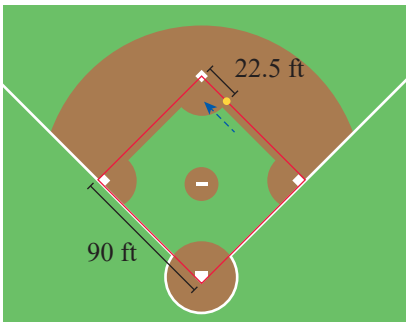
35. A trough that is 5 meters long and 1 meter across at the top has a cross-section in the form of an isosceles trapezoid and both of its endplates are vertical. The altitude of the trapezoid is 40 centimeters, and the shorter base is 20 centimeters long. If the trough is being filled at the rate of 30 liters per minute, how fast is the water level rising at the instant when the water's depth is 20 centimeters?

36. Rework Example 3, this time assuming that, after turning right, Susan drove up a short ramp to reach an elevated highway 20 feet above the other drivers. She is now proceeding levelly at a distance of 40 feet, as measured along the ground, from the intersection.

37. An electrician is working on top of a 15 ft ladder that is leaning against the wall when its bottom starts sliding at a rate of $1\ \text{ft/s}$. Fortunately, a fellow worker catches it when the ladder's bottom is 5 ft from the wall. How fast is the top of the ladder (along with the electrician) sliding down the wall at that instant?

38. Adam is arriving home one evening in his SUV and is slowly approaching his garage door at a rate of $5\ \text{ft/s}$ when the sensor lights come on. If the lights are mounted directly above the door at a height of 15 ft from the ground and Adam's SUV is 6 ft tall, at what rate is the length of the car's shadow shrinking when it is 25 ft from the garage door? What is the speed of the tip of the car's shadow?

39. A baseball player is running from first base to second base at 25 feet per second. At what rate is his distance increasing from home plate when he is 22.5 feet from second base? (**Hint:** The baseball diamond is a 90-foot-by-90-foot square.)



40.* A container in the shape of a cone, standing on its circular base, is being filled with water at the rate of 1.5 cubic feet per minute. If the radius of the base is 2 feet and the height of the cone is $2\sqrt{3}$ feet, how fast is the water level rising when it is 2 feet deep? (**Hint:** The volume of liquid in a partially filled conical tank is $V = \frac{1}{3}\pi d\left(R^2 + Rr + r^2\right)$, where R is the radius of the base, r is the radius of the top of the liquid, and d is its depth.)

41.* Italian police are chasing a criminal down a narrow street at a speed of 90 kilometers per hour. If the blue light on the top of the car is rotating counterclockwise at a rate of 1 rotation per second, and the buildings are only 3 meters from the car on the right, how fast is the beam moving on the wall at the instant when it is already 6 meters ahead of its source?

42.* When studying for a calculus test, Roger accidentally pushes his book over the edge of his 2.5 ft high desk. If his 6 ft tall lamp is standing 3 ft from where the textbook fell down, how fast was the book's shadow moving when the text hit the ground? (Ignore air resistance. Use $g \approx 32 \text{ ft/s}^2$.)

43.* A wall clock has a 10 in. minute hand and a 6 in. hour hand. At what rate are the tips of the hands approaching each other at 3 o'clock?

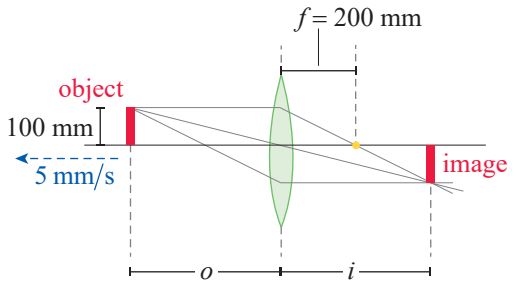
44.* The *lens equation*, easily derivable from geometric similarity for a thin converging lens, is

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f},$$

where o (the *object distance*) and i (the *image distance*) are the respective distances of the object and the image from the lens, and f is the *focal length* of the lens. Suppose a 100 mm high object is being slowly moved away from a lens at a speed of 5 mm/s. The focal length of the lens is 200 mm.

a. Find the rate at which the image changes its location when the object distance is 600 mm.

b. Find the rate at which the image changes its size at the same instant.



45. Suppose that the torque output of an automobile engine, as a function of engine speed, is approximated by

$$T(s)=\left(-0.001/150^4\right)(s-3000)^4+160 \text{ lb-ft},$$

where s is measured in revolutions per minute (rpm), and that the engine revs up from 0 to 5000 rpm (assume no gear shift takes place).

a. Use a graphing utility to graph the torque as a function of s on the interval $[0,5000]$ (this is called the engine's *torque curve*).

b. If the power output of the engine, measured in horsepower (hp), is calculated by $P=\frac{1}{5252}sT(s) \text{ hp}$, and the engine is revving up according to the function $s(t)=1000t$ (t is measured in seconds), find the rate of change of the power output at $t = 3$ seconds.