3.5 Exercises

1–12 Use implicit differentiation to determine dy/dx for the given equation. Then check your answer by expressing y explicitly and using differentiation rules.

1. $x + y^2 = 2$

2. $xy = 3$

3. $x^2 - y^2 = 1$

4. $4x^2 + 25y^2 = 100$

5. $3xy^2 = x - 5$

6. $y^2\sqrt{x} = 2x^2 + 1$

7. $y\sqrt{x+2} = xy - 2$

8. $2x^2y - 3y - x - 1 = 0$

9. $2y\cos x - xy = x + 3$

10. $ye^x + 2y - 1 = 0$

11. $\frac{2}{x} - \frac{3}{y} = 4$

12. $x^2\sqrt{y} - x^2 - 1 = e^2$

13–20 Find dx/dy by implicit differentiation. Then check your answer by expressing x explicitly in terms of y and differentiating with respect to y using differentiation rules.

13. $x - y^2 = 0$

14. $xy - y^3 = 3y$

15. $x^3 + y^3 = 1$

16. $-5xy^2 + 4xy - 3y^2 - y - 2 = 0$

17. $xy + 3\sin y = e^y$

18. $xy = \sqrt{y^2+1} - 5x$

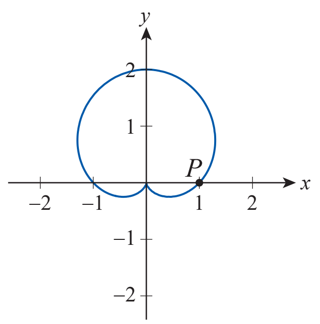
19. $\sqrt[3]{8x^3 - 5y^4} = 3$

20. $(y+2)\sqrt{x+3} = \sqrt{y}$

21–28 Use implicit differentiation to find the equations of the tangent and normal lines at point P for the well-known curve.

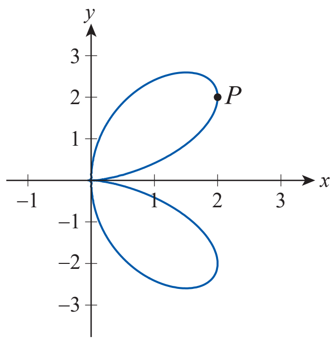
21. $x^2 + y^2 - (x^2 + y^2 - y)^2 = 0;\quad P(1,0)$

Cardioid



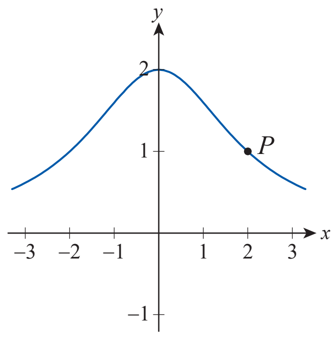
22. $(x^2 + y^2)^2 - 8y^2x = 0;\quad P(2,2)$

Bifolium



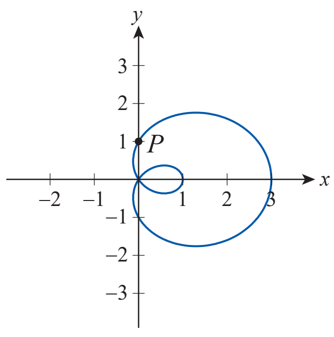
23. $(x^2 + 4)y = 8;\quad P(2,1)$

Witch of Agnesi



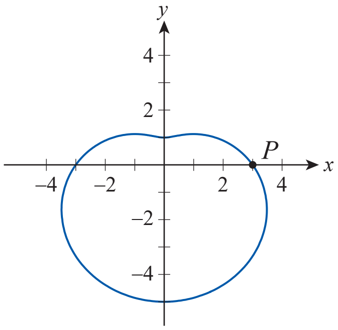
24. $x^2 + y^2 = (x^2 + y^2 - 2x)^2;\quad P(0,1)$

Limaçon



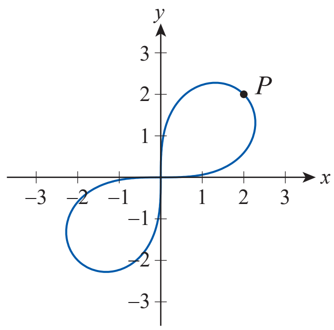
25. $9\left(x^2+y^2\right)=\left(x^2+y^2+2y\right)^2$; $P(3,0)$

Dimpled Limaçon



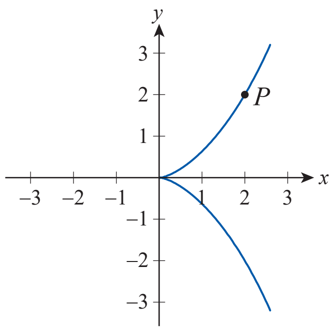
26. $\left(x^2+y^2\right)^2=16xy$; $P(2,2)$

Lemniscate



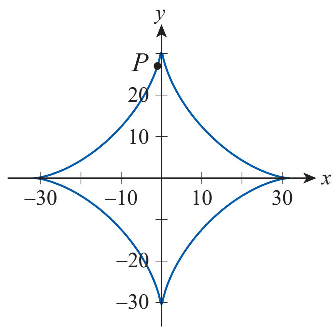
27. $(6-x)y^2=2x^3$; $P(2,2)$

Cissoid



28. $x^{2/3}+y^{2/3}=10$; $P(-1,27)$

Astroid



29–44 Find dy/dx by implicit differentiation.

29. $x^4+y^4=1$

30. $\sqrt{x}+\sqrt{y}=4$

31. $x^3y^4-x^4y^3=1$

32. $y=\cos(x-2y)$

33. $(x+y)^3+3=x+y$

34. $e^{xy}=e^x+e^y$

35. $\sin^2 x+\cos^2 y=\tan\left(x^2+y^2\right)$

36. $\sqrt{x^2+y^2}=2x$

37. $\dfrac{x+3y^2}{y-x^2}=2x+1$

38. $\dfrac{y}{x^3}-\dfrac{x}{y^3}=x^3y^3$

39. $\sqrt{2xy}=3y-5x$

40. $y-x=x^4y^4$

41. $\tan x=\sin y-2xy$

42. $e^x\tan x=y+\cos y$

43. $\sqrt{\sin x+\cos x}=\sec(x+y)$

44. $(\tan x+\cot y)^2=1+x$

45. Find ds/dt by implicit differentiation: $s^2t^3-2t=\sqrt{s}$.

46. Find dt/ds by implicit differentiation: $s\sin t=t\cos s$.

47–52 Find d^2y/dx^2 by implicit differentiation.

47. $4y^2-x^2=4$

48. $y-x=xy-2$

49. $xy^2+5=x$

50. $y^3=xy+1$

51. $x^3+y^3=3$

52. $\sqrt{x}+\sqrt{y}=2$

53. Notice that for a circle centered at the origin, any line tangent to the curve in the first quadrant has negative slope; this is consistent with our observation that $dy/dx<0$ when $x>0$ and $y>0$ (see Example 5). Verify that the sign of dy/dx in each of the quadrants is what we would expect.

54. Verify that the sign of the second derivative d^2y/dx^2 of the circle in Example 5 is what we would expect in each quadrant. (**Hint:** Traverse the circle from left to right and examine whether the first derivative is increasing or decreasing; then draw a conclusion regarding the sign of the second derivative.)

55–58 Find all points on the given curve where it has horizontal or vertical tangent lines.

55. $xy^2 - x^2y = \frac{1}{4}$

56. $x^2 - xy + y^2 = \frac{1}{4}$ (Rotated ellipse)

57. $(x^2 - 2x + 5)y = 5$ **58.** $xy + y^2x^2 = 1$

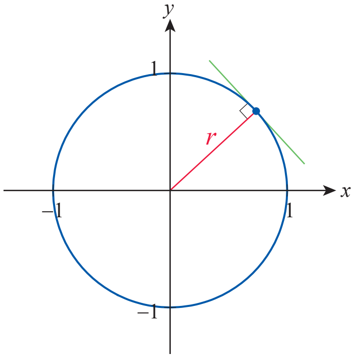
59. Two graphs are called *orthogonal* if their respective tangent lines are perpendicular at their point(s) of intersection. Show that the graphs of $x^2 - y^2 = 5$ and $xy = 6$ are orthogonal.

60. Generalizing Exercise 59, show that the families of curves $x^2 - y^2 = a$ and $xy = b$ are orthogonal for a, $b \in \mathbb{R}$. (Such families of curves are called *orthogonal trajectories*.)

61. Repeat Exercise 60 for the families $x^2 + y^2 = a$ and $y - bx = 0$.

62. Repeat Exercise 60 for the families $x^2 + y^2 = ax$ and $x^2 + y^2 = by$.

63. Use implicit differentiation to prove that a tangent line to a circle is always perpendicular to the radius connecting the center and the point of tangency. (**Hint:** We can assume without loss of generality that the circle is a unit circle in the xy-coordinate system, centered at the origin.)



64. Use implicit differentiation to prove that the equation of the line tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y = 1$.

65. Use implicit differentiation to find the equations of the two lines tangent to the ellipse $2x^2 + y^2 = 2$ that pass through the point $(0, 2)$.

66. An object of mass m is attached to a spring and is moving along the x-axis so that its position and velocity satisfy the equation $m(v - v_0)^2 = -kx^2$, where v_0 represents the initial velocity. Use implicit differentiation to verify Hooke's Law; that is, prove that the restoring force exerted by the spring satisfies $F = -kx$. (**Hint:** Differentiate and use Newton's Second Law of Motion, which states that $F = ma$.)

3.5 Technology Exercises

67–70. Use the implicit graphing capabilities of a graphing utility to graph the curves along with the tangent lines you found in Exercises 55–58 and visually verify that your answers are correct.

71–73. Use a graphing utility to graph the families of curves in Exercises 60–62 for several different values of the parameters a and b. Visually verify that they are orthogonal.

74. Beautiful, "irregular" curves can be created by using a graphing utility to plot graphs of equations such as the following.

$$(x^2 - 1)(x - 2)(x - 3) = (y - 1)(y - 2)(y - 3)$$

Graph the above equation and explain why this graph cannot be that of a function. Then try experimenting by slightly modifying the above equation and thus creating your own curves. (Answers will vary.)

75. Repeat Exercise 74 starting with the equation $x^5 - 3x^3 - x^2 = -y^5 + 3y^3 - y^2$.