3.3 Exercises

1–12 Use the results of this section to find the indicated limit.

1. $\lim_{x\to 0}\frac{\sin 3x}{2x}$
2. $\lim_{x\to 0}\frac{-\sin\frac{x}{2}}{5x}$
3. $\lim_{x\to 0}\frac{\sin \pi x}{x}$
4. $\lim_{x\to 0}\frac{\tan 4x}{5x}$
5. $\lim_{x\to 0}\frac{\cos 5x-1}{2x}$
6. $\lim_{x\to 0}\frac{\cos x-1}{\sin x}$
7. $\lim_{x\to 0}\frac{x+\tan x}{\sin x}$
8. $\lim_{\beta\to 0}\frac{\csc\beta-\cot\beta}{\beta\csc\beta}$
9. $\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}$
10. $\lim_{\alpha\to 0}\frac{\tan(\alpha^2)}{\alpha}$
11. $\lim_{t\to 0}\frac{2t+3\tan t}{\sin t}$
12. $\lim_{\theta\to 0}\frac{\cos\theta\sin\theta-\sin\theta}{\theta^2}$

13–30 Differentiate the given function.

13. $f(x)=2\sin x-5\cos x$
14. $g(x)=3x^2+2\tan x$
15. $h(x)=x\cos x$
16. $F(x)=2.5x(1-\cot x)$
17. $G(x)=2\sqrt{x}\sec x$
18. $k(x)=\pi x\sin x+\pi x$
19. $L(x)=-3e^x(\csc x+\cot x)$
20. $f(x)=2\cos 2x-2\cos x$
21. $g(x)=\cot^2 x$
22. $h(x)=\frac{\tan x}{x}$
23. $F(t)=\frac{1-\cos t}{t^2}$
24. $W(x)=\frac{1+\cos x}{1+\sin x}$
25. $R(z)=\frac{e^z+\sin z}{z}$
26. $N(w)=\frac{2\sqrt{w}-\sec w}{\sqrt{w}}$
27. $B(x)=\frac{\frac{1}{\sin x}-\sin x}{\cos x}$
28. $G(y)=y\cot y\csc y$
29. $T(s)=s^2e^s\cot s$
30. $r(t)=\frac{1}{t\sin t\cos t}$

31–36 Find all points where the function has a horizontal tangent line.

31. $f(x)=\frac{1}{2}x+\sin x$
32. $g(x)=x+\sin 2x$
33. $h(x)=\sec^2 x$
34. $T(s)=\tan s-s$
35. $K(u)=\tan u+\cot u$
36. $F(t)=\frac{1-\sin t}{1-\cos t}$

37–40 Find all x-values where the tangent line to the graph of the function is parallel to the given line.

37. $f(x)=\sin x+\frac{3}{2};\quad y-x=\frac{3}{2}$
38. $g(x)=\cot x;\quad y+2x=\pi$
39. $G(x)=\frac{x}{3}-\tan x;\quad x+y=5$
40. $F(x)=\sin x\cos x;\quad 2x+2y=7$

41–44 Find the equation of the tangent line to the graph of the given function at the indicated point.

41. $f(x)=2x\cos x;\quad (0,0)$
42. $g(x)=\tan x-\sec x;\quad (0,-1)$
43. $h(x)=2\csc x-\sin x;\quad \left(\frac{\pi}{2},1\right)$
44. $k(x)=\frac{\cot x}{x};\quad \left(\frac{\pi}{4},\frac{4}{\pi}\right)$
45. Let us assume that for some function f, we have $f(0)=1$ and $f'(0)=2$. Let $F(x)=f(x)\tan x$, $G(x)=f(x)/\cos x$, and $H(x)=f(x)\sin x\cos x$. Find $F'(0)$, $G'(0)$, and $H'(0)$.

46–48 Verify the trigonometric identity by differentiating both sides of the equation. (**Hint:** If $f'(x)=g'(x)$, it doesn't necessarily follow that $f(x)=g(x)$. In general, we can only conclude that $f(x)=g(x)+c$ for some constant c.)

46. $\tan x\cot x=1$
47. $(1-\cos\theta)(1+\cos\theta)=\sin^2\theta$
48. $\frac{1}{1-\cos x}+\frac{1}{1+\cos x}=2\csc^2 x$

49–52 Find $f'(x)$, $f''(x)$, and $f'''(x)$. Observing a pattern, find a formula for $f^{(n)}(x)$.

49. $f(x)=\sin x$
50. $f(x)=\cos x$
51. $f(x)=e^x\sin x$
52. $f(x)=e^x\cos x$

53. Provide a second proof of the limit statement $\lim_{\theta\to 0}(\cos\theta-1)/\theta=0$ by multiplying both the numerator and denominator by $\cos\theta + 1$ to obtain

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}.$$

Then by the Pythagorean identity $\sin^2 x+\cos^2 x=1$, you obtain

$$\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$
$$=\lim_{\theta\to 0}\frac{\sin\theta(-\sin\theta)}{\theta(\cos\theta+1)}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1}.$$

Conclude the argument by using the first limit statement in the lemma at the beginning of this section.

54. Prove that $\frac{d}{dx}(\cos x)=-\sin x$ by mimicking the proof of the theorem "Derivative of Sine." (**Hint:** You will need the angle sum identity $\cos(u+v)=\cos u\cos v-\sin u\sin v$.)

55. Provide an alternative proof of the fact that $\frac{d}{dx}(\sin x)=\cos x$ by using the identity

$$\sin x-\sin c=2\sin\frac{x-c}{2}\cos\frac{x+c}{2}.$$

(**Hint:** Rewrite the difference quotient $\frac{\sin x-\sin c}{x-c}$ as $\frac{2\sin\frac{x-c}{2}\cos\frac{x+c}{2}}{x-c}$. Let $c\to x$, and use the lemma from the beginning of this section.)

56. Use the definition of the derivative and the lemma from the beginning of this section to show that $(\sin 3x)'=3\cos 3x$. Generalize to obtain that if $k\in\mathbb{R}$, $(\sin(kx))'=k\cos(kx)$.

57. Repeat Exercise 56 with $f(x)=\cos(kx)$.

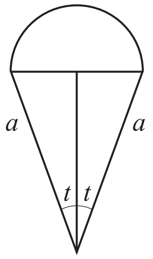
58. Find a constant a such that the graphs of $f(x)=a\sin x$ and $g(x)=a\cos x$ intersect at right angles, that is, their respective tangent lines are perpendicular at their point(s) of intersection.

59. Prove the remaining two cases of the theorem "Derivatives of Tangent, Cotangent, Secant, and Cosecant," namely, the statements

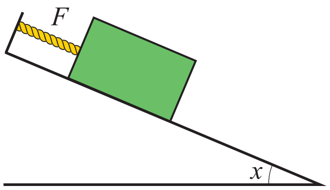
$$\frac{d}{dx}(\cot x)=-\csc^2 x \quad\text{and}\quad \frac{d}{dx}(\sec x)=\sec x\tan x.$$

(**Hint:** Mimic the proof presented in the text, using the derivatives of sine and cosine along with appropriate differentiation rules.)

60. The cross-section of an ice cream cone is an isosceles triangle, with the angular opening at the bottom being $2t$ (radians). Assuming that the ice cream sits on top of the cone in the shape of a perfect hemisphere, let V_i = volume of the ice cream, V_c = volume of the cone. Express both of these volumes in terms of t, and then compute $\lim_{t\to 0^+}\frac{V_i}{V_c}$.



61. An object is tied to the top of an inclined surface of variable angle of elevation so that the rope is parallel to the surface. The tension in the rope is given by $F=mg(\sin x-\mu\cos x)$, where m is the mass of the object, g is the gravity constant, and μ is the coefficient of friction (assume all units are metric units).



a. What is the rate of change of F with respect to x?

b. For what x-value (if any) is this rate of change equal to 0?

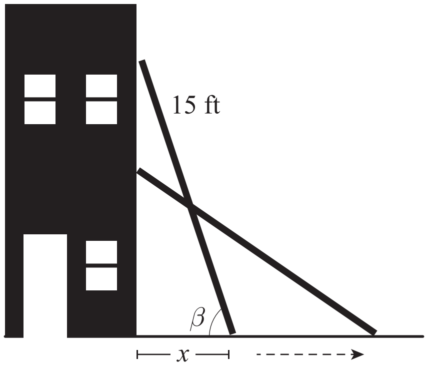
62. Suppose an object oscillating in fluid obeys the position function $y=10e^{-0.2t}\cos(2\pi t)$, where y is the distance from equilibrium, measured in centimeters with upward displacement considered positive, and t is measured in seconds. Such motion is called *damped harmonic motion*. Can you see why?

a. Find the position, velocity, and acceleration at $t=3.5$ seconds.

b. What is the maximum displacement of the object and when does it occur?

(**Hint:** Use the definition of the derivative to find the derivatives of $e^{-0.2t}$ and $\cos(2\pi t)$. You may also want to review Exercise 57 for the latter.)

63. A 15 ft ladder is leaning against a wall, making an angle of β with the horizontal, when it starts sliding. If x denotes the distance of the bottom of the ladder from the wall, find the rate of change of x with respect to β when $\beta = \pi/6$ (or 30°). Interpret the result.



64. A man is pulling his child on a sled at a constant rate, via a rope that makes an angle of α with the horizontal. Since there is no acceleration, the pulling force satisfies the equation $F\cos\alpha = \mu(mg - F\sin\alpha)$, where μ is the coefficient of friction, m is the total mass of the sled and child, and g is the gravity constant.

a. Express F as a function of α.

b. Find the rate of change of F with respect to α.

c. What is the above rate when $\alpha = 60°$?

d. When (if ever) is this rate of change 0?

3.3 Technology Exercises

65–70 Use a graphing utility to find the derivative of $f(x)$. Then graph f along with its derivative on the same screen. By zooming in, if necessary, find at least two x-values where the graph of f has a horizontal tangent line. What can you say about f' at such points? (Answers will vary.)

65. $f(x) = \dfrac{x}{1+\cos x}$

66. $f(x) = \dfrac{1-\sec x}{1+\sec x}$

67. $f(x) = \dfrac{\csc x}{x}$

68. $f(x) = \dfrac{\sin x}{\cos x + \tan x}$

69. $f(x) = \cos x(\cot x + \tan x)$

70. $f(x) = \dfrac{\cot x}{\sec x + x\cos x}$

71. Find the maximum velocity and acceleration values in Exercise 62 by using a graphing utility to graph the velocity and acceleration functions of the oscillating object.