10.9 Exercises

1–5 Use Euler's formula to express the complex number in the form $x+iy$.

1. $e^{i\pi/2}$

2. $e^{-3i\pi/4}$

3. $5e^{2+i\pi/3}$

4. e^{e^i}

5. $\dfrac{e^{2i}-e^{-2i}}{3i}$

6–10 Use Euler's formula to express the complex number in the form $e^xe^{iy}=re^{iy}$.

6. $\dfrac{1-i}{\sqrt{2}}$

7. $-2\left(1+\sqrt{3}i\right)$

8. $\dfrac{3+3i}{-\sqrt{3}-i}$

9. $(1+i)^4$

10. $\dfrac{3i}{2e^{4+i}}$

11. Use Euler's formula to verify the following formulas.

a. $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ **b.** $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$

12. Use the Maclaurin series to verify the following formulas.

a. $\sinh i\theta=i\sin\theta$ **b.** $\cosh i\theta=\cos\theta$

13–21 Find the first six nonzero terms of the binomial series expansion of the indicated function.

13. $f(x)=\left(1+x^2\right)^5$

14. $f(x)=\dfrac{1}{\sqrt{1+x}}$

15. $f(x)=\sqrt{1-x}$

16. $f(x)=(1-x)^{3/2}$

17. $f(x)=\sqrt[3]{1+x^2}$

18. $f(x)=\left(1+\dfrac{x}{2}\right)^{-3}$

19. $f(x)=\left(1-2x^2\right)^{-2/3}$

20. $f(x)=\left(1-\dfrac{x}{3}\right)^{-3/2}$

21. $f(x)=\dfrac{5}{\sqrt[4]{1+2x}}$

22. Show that if $m\in\mathbb{N}$, the Maclaurin series of $(1+x)^m$ is finite, and use this observation to provide a "calculus-based" proof for the Binomial Theorem, which says $(A+B)^n=\sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^k$, for any positive integer n and any two expressions A and B.

23. Use the binomial series of $f(x)=1/\sqrt{1-x^2}$ to find the series expansion of $g(x)=\sin^{-1}x$ about 0. What is the radius of convergence? (**Hint:** Use the equality $\binom{-1/2}{n}=(-1)^n\binom{2n}{n}\Big/4^n$. Compare your result to that of Exercise 68 of Section 10.8.)

24–25 Use the method of Exercise 23 to find the Maclaurin series expansion of the indicated function along with the radius of convergence.

24. $h(x)=\cos^{-1}x$

25. $h(x)=\sinh^{-1}x$

26. By following the outline below, prove that the binomial series

$$\sum_{n=0}^{\infty}\binom{m}{n}x^n=\sum_{n=0}^{\infty}\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n$$

converges to the function $(1+x)^m$ on $(-1,1)$.

a. Using the Ratio Test, it is straightforward to check that the above series converges for $|x|<1$. Let us denote its sum by $f(x)$ and differentiate the series term by term to find the series expansion of $f'(x)$.

b. Multiply the series expansion of $f'(x)$ by $(1+x)$ and verify that the result is the series for $m\cdot f(x)$. (**Note:** Since $|x|<1$, the series is absolutely convergent, so termwise differentiation and rearranging terms is possible without changing the sum.)

c. Conclude that $f(x)$ is a solution of the separable differential equation

$$\frac{y'}{y}=\frac{m}{1+x}.$$

d. Solve the above equation by conventional means to conclude that $f(x)=C(1+x)^m$ for some constant C.

e. Finally, use the initial condition $f(0)=1$ to conclude that $C=1$.

27–28 The period of a swinging pendulum of length L released from rest can be well approximated by simple harmonic motion and its period is approximately $T \approx 2\pi\sqrt{L/g}$, where g is the acceleration caused by gravity. However, this model is inaccurate for larger starting angles. In physics, it is shown that if the pendulum is released at angle θ_0 from vertical, the actual formula is

$$T = 4\sqrt{\frac{L}{g}}K(k),$$

where $k = \sin(\theta_0/2)$ and

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is a *complete elliptic integral of the first kind*.

In Exercises 27 and 28, use a Taylor series approximation to derive an estimate for the period of the swinging pendulum (keep in mind that we don't have a closed formula for this integral).

27.* Prove that if $|k| < 1$, the Maclaurin series expansion for $K(k)$ is

$$K(k) = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\frac{(1)(3)\cdots(2n-1)}{(2)(4)\cdots(2n)}\right]^2 k^{2n}.$$

(**Hint:** Substitute $k\sin\theta$ in the Maclaurin expansion of $1/\sqrt{1-x^2}$.)

28. Use the series from Exercise 27 to establish the following approximation for the period of the swinging pendulum, which is much more accurate for larger angles than the simple model we have seen previously.

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{\theta_0^2}{16}\right)$$

(**Hint:** In addition to the Maclaurin series, also use the approximation $\sin x \approx x$.)

29–30 Perhaps surprisingly, an explicit formula for the circumference of an ellipse is not known, for the problem leads to an integral that is (perhaps unsurprisingly) called the *complete elliptic integral of the second kind*. (By now you are probably expecting the fact that it cannot be evaluated in closed form, which is indeed the case.)

Use the Taylor series of this integral to approximate the circumference of an ellipse.

29.* The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta.$$

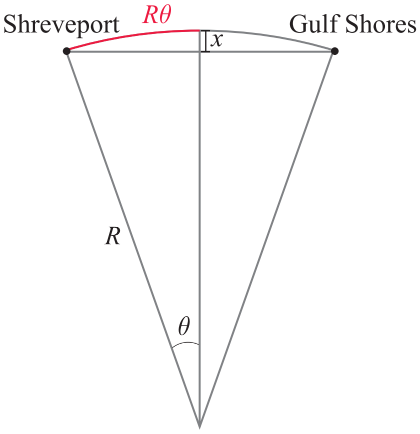
Prove that if $|k| < 1$, the Maclaurin series expansion for $E(k)$ is

$$E(k) = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\frac{(1)(3)\cdots(2n-1)}{(2)(4)\cdots(2n)}\right]^2 \frac{k^{2n}}{2n-1}.$$

(See the hint given in Exercise 27.)

30. Given that the circumference of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a<b$, is $C = 4bE(k)$ where $k=\sqrt{1-(a^2/b^2)}$, use the first four terms of the series expansion from Exercise 29 to approximate the circumference of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

31. The arc of a great circle of Earth connecting Shreveport, Louisiana, with Gulf Shores, Alabama, is approximately 430 miles long. Use Taylor series to estimate how much the arc will recede from its chord between these two cities. Use the first three nonzero terms of the series of $\cos\theta$. Approximate the radius of Earth with $R \approx 4000$ miles.



32. Use the Maclaurin series expansion of e^x to prove that e is irrational. (**Hint:** Expressing e as a power series, you can argue by contradiction as follows. Assuming p and q are positive integers with $e = p/q$ and multiplying the power series by $q!$, we obtain

$$p(q-1)! - 2q! - (3\cdot4\cdot\cdots\cdot q) - (4\cdot5\cdot\cdots\cdot q) - \cdots - q - 1$$
$$= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots$$

where the assumptions imply that the left-hand side must be an integer. Finally, argue that the expression on the right-hand side must be between 0 and 1, an apparent contradiction, finishing your proof.)

33. Recall from Section 8.2 that the differential equation obeyed by the current in an RL circuit is

$$L\frac{dI}{dt}+RI=V.$$

Solve this linear equation for $I(t)$ and use the series expansion of your solution to show that when t is small,

$$I(t)\approx\frac{V}{L}t.$$

34. The weight of an object at a height h above the surface of Earth is

$$W(h)=\frac{R^2W_0}{(R+h)^2},$$

where W_0 is the object's weight on Earth's surface. Use the first four nonzero terms of the Maclaurin expansion of $W(h)$ to approximate the height required for an object to lose 5% of its weight. (Use $R\approx 4000$ miles.)

35. According to Einstein's theory of special relativity, if a particle is moving with velocity v, the mass of the particle is given by

$$m(v)=\frac{m_r}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_r is the rest mass of the particle and c is the speed of light in a vacuum. Use the first three nonzero terms of the Maclaurin polynomial of $m(v)$ to estimate the speed necessary to increase the mass of the particle by 1%. (Use $c\approx 3\times 10^8$ m/s.)

36.* The fastest average qualifying speed at the Indy 500 race, 236.986 mph (approximately 106 m/s), was reached by Arie Luyendyk in 1996. Using the error term for the series of E_k found in Example 3, estimate the magnitude of the maximum error one makes when calculating the race car's kinetic energy at this speed using the classic Newtonian formula of $E_k\approx mv^2/2$. (**Hint:** The series of E_k can be rewritten as

$$E_k=mc^2\left[\frac{1}{2}\left(\frac{v^2}{c^2}\right)+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\frac{5}{16}\left(\frac{v^2}{c^2}\right)^3+\cdots\right].$$

Substituting $x=-v^2/c^2$, notice that the series in the brackets on the right-hand side is "almost" the binomial expansion of $1/\sqrt{1+x}$. Next, use Taylor's Theorem with $r_1(x)$, the remainder of order 1, note that $|v|\le 106$, and give an upper bound for $|r_1(v/c)|$.)

37. The phase speed at which a surface wave propagates on water of depth d is well approximated by the expression

$$s=\sqrt{\frac{\lambda g}{2\pi}\tanh\frac{2\pi d}{\lambda}},$$

where λ is the wavelength.

a. Explain why the following "rule of thumb" is valid: If the water is deeper than three times the wavelength, then $s\approx\sqrt{\lambda g/(2\pi)}$. (Note that this formula shows that the speed of propagation depends only on wavelength in deep water; in case of large wavelengths such as tidal waves, this speed can be enormous. In March 2011, just before the catastrophic Japan tsunami, s was about the same speed as that of a passenger jet!)

b. Use Taylor series to show that in shallow water, $s\approx\sqrt{gd}$. (In contrast to the previous case, the speed of propagation in shallow water depends only on water depth, rather than wavelength.)

38–41 Find a power series solution of the equation. In each case, use the initial conditions $y(0)=a$ and $y'(0)=b$.

38. $y''-2xy=0$

39. $y''+9y=0$

40. $y''=\dfrac{y'}{1-x}$

41. $(1-x^2)y''=4xy'+2y$

42. Explain why in our discussion preceding Example 5, we did not attempt to find a Fourier coefficient b_k for the index $k=0$.

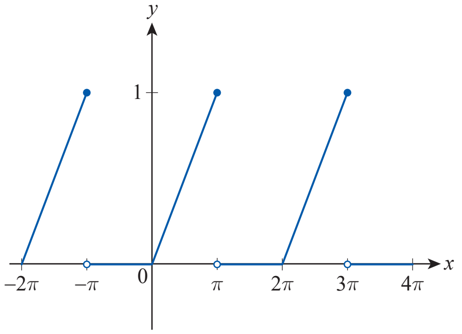
43. Verify the Fourier coefficients of Example 5.

a. $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}x\,dx=0$

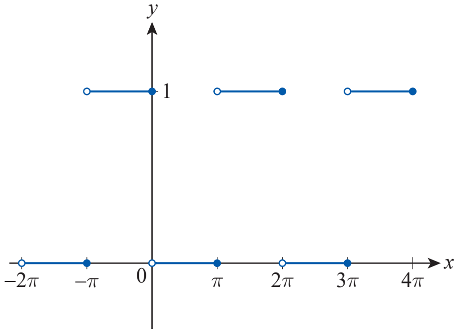
b. $a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}x\cos kx\,dx=0 \quad (k\ge 1)$

c.
$$\begin{aligned}b_k&=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kx\,dx\\&=-\frac{2}{k}\cos k\pi+\frac{2}{k^2\pi}\sin k\pi\\&=(-1)^{k+1}\left(\frac{2}{k}\right)\quad (k\ge 1)\end{aligned}$$

44. Consider the function given by its graph below. After extending it to $\mathbb{R}$ in a 2π-periodic manner, find its Fourier coefficients.



45. Repeat Exercise 44 for the function graphed below.



46. Let $\hat{f}(x) = 3x^2$ be defined on the interval $[-\pi, \pi]$, and let $f(x)$ be the function obtained by extending $\hat{f}(x)$ to the entire real line in a 2π-periodic fashion. Find the Fourier series expansion of $f(x)$.

10.9 Technology Exercises

47–50 Use a graphing utility to graph the function along with its Maclaurin polynomial of order 5 on the same screen (use the polynomial of order 9 in Exercise 50). What is the largest interval on which the approximation is acceptable? (Answers will vary.)

47. $f(x) = \sqrt[5]{(2x-1)^7}$

48. $f(x) = \dfrac{x^2 - 1}{\sqrt[3]{(x+5)^2}}$

49. $f(x) = \dfrac{x+1}{\sqrt{1+x^2}}$

50. $f(x) = (x^3 - x)\sqrt[4]{x^4 + 1}$