10.7 Exercises

1–32 Determine the radius and interval of convergence for the power series. Be sure to check for convergence at the endpoints.

1. $\sum_{n=0}^{\infty}(3x)^n$

2. $\sum_{n=0}^{\infty}3x^n$

3. $\sum_{n=1}^{\infty}\frac{(4x)^n}{n^4}$

4. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$

5. $\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$

6. $\sum_{n=0}^{\infty}\frac{(5x)^n}{2n!}$

7. $\sum_{n=0}^{\infty}\frac{(2n)!(5x)^n}{2n!}$

8. $\sum_{n=0}^{\infty}\frac{2n!(5x)^n}{(2n)!}$

9. $\sum_{n=1}^{\infty}\frac{(2x)^n}{(n-1)!}$

10. $\sum_{n=1}^{\infty}nx^n$

11. $\sum_{n=1}^{\infty}\frac{n^2(x+3)^n}{2^n}$

12. $\sum_{n=0}^{\infty}\frac{3x^n}{k^n},\quad k\neq 0$

13. $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n n!$

14. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n3^n}$

15. $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n^3}$

16. $\sum_{n=0}^{\infty}\frac{(x+1)^n}{(n+2)3^n}$

17. $\sum_{n=0}^{\infty}\frac{(x-5)^n n!}{5^n}$

18. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(-1)^n n}$

19. $\sum_{n=0}^{\infty}\frac{(x+2)^n}{(n+2)2^n}$

20. $\sum_{n=0}^{\infty}\frac{(x-k)^n}{k^n},\quad k>0$

21. $\sum_{n=4}^{\infty}\frac{(x-1)^n}{(n-2)(n-3)}$

22. $\sum_{n=1}^{\infty}\frac{(x+2)^n}{\sqrt{n}}$

23. $\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{(\ln n)^n}$

24. $\sum_{n=1}^{\infty}\frac{n^2(2x-3)^n}{n!}$

25. $\sum_{n=1}^{\infty}n3^n(x-1)^n$

26. $\sum_{n=2}^{\infty}\frac{x^n\ln n}{n!}$

27. $\sum_{n=2}^{\infty}\frac{x^{2n+5}}{\ln\sqrt{n}}$

28. $\sum_{n=0}^{\infty}x^{4^n}$

29. $\sum_{n=2}^{\infty}\frac{3^n(x+2)^n}{n\ln n}$

30. $\sum_{n=0}^{\infty}2^n(x+4)^{3n+1}$

31. $\sum_{n=0}^{\infty}\pi(x-3)^n$

32. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-3)^n$

33–40 Determine the interval of convergence for the given series and the limiting function of the series on that interval.

33. $\sum_{n=0}^{\infty}(3x-1)^n$

34. $\sum_{n=0}^{\infty}(2x+3)^n$

35. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^n$

36. $\sum_{n=0}^{\infty}2\left(\frac{x}{5}\right)^n$

37. $\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n$

38. $\sum_{n=0}^{\infty}\frac{(2x+3)^n}{3^n}$

39. $\sum_{n=0}^{\infty}\frac{(5x-6)^{2n}}{9^n}$

40. $\sum_{n=0}^{\infty}\frac{(2x-3)^n}{3^n}$

41–48 By "reversing" the process used to find the limiting function in Exercises 33–40, we can find the power series about 0 for certain functions. For example, if $f(x)=1/(1-3x)$, we can recognize it as the sum of the geometric series $\sum_{n=0}^{\infty}(3x)^n=\sum_{n=0}^{\infty}3^n x^n$. (This is valid only if x is in the interval of convergence, in this case, $-\frac{1}{3}<x<\frac{1}{3}$.)

Use this approach to find the power series representation of the given function. What is the radius of convergence? (**Hint:** In Exercises 47 and 48, use partial fractions.)

41. $f(x)=\frac{1}{1-2x}$

42. $f(x)=\frac{1}{1+4x}$

43. $f(x)=\frac{1}{1+x^2}$

44. $f(x)=\frac{1}{1-x^2}$

45. $f(x)=\frac{3}{3-4x}$

46. $f(x)=\frac{2}{3+8x^3}$

47. $f(x)=\frac{x+3}{1-x^2}$

48. $f(x)=\frac{2x}{1-4x^2}$

49. Find the power series representation of $f(x)=1/x$ centered at $a=1$. What is the radius of convergence? (**Hint:** Start by rewriting $1/x$ as $\frac{1}{1-(1-x)}$, and proceed along the lines of Exercises 41–48.)

50. Use series multiplication to find a second solution to Exercise 48. (**Hint:** Multiply the series expansions of $1/(1-2x)$ and $1/(1+2x)$, and then multiply the result by $2x$.)

51. Differentiating the result of Exercise 49, find the power series representation of $g(x)=1/x^2$ centered at $a=1$. What is the radius of convergence?

52. Use the result of Exercise 49 to find the power series representation about $a=1$ of $h(x)=\ln x$. What is the radius of convergence?

53. Use Exercise 52 to prove that $\ln 2=\sum_{n=1}^{\infty}\frac{1}{n2^n}$. Use this series to approximate $\ln 2$ to three decimal places. How many terms did you use? (**Hint:** Start with the series approximating $\ln\frac{1}{2}=-\ln 2$.)

54. Find the power series representation of $f(x)=1/x$ centered at $a=3$. What is the interval of convergence? (**Hint:** Start by rewriting $1/x$ as $\frac{1}{3-(3-x)}$ and proceed along the lines of Exercise 41–48.)

55. Find the second and third derivatives of the power series of Example 4. In both cases, express your answer as a power series and in closed form.

56. Find the closed form of the series $\sum_{n=1}^{\infty}\frac{(2x-3)^n}{2n}$ by first determining the closed form of the series $\sum_{n=0}^{\infty}(2x-3)^n$. What is the interval of convergence?

57. Find the closed form of the series $\sum_{n=2}^{\infty}9(n^2-n)(3x-1)^{n-2}$ by first determining the closed form of the series $\sum_{n=0}^{\infty}(3x-1)^n$. What is the interval of convergence? (**Hint:** Differentiate twice.)

58. Integrate the series you found in Exercise 43 to obtain a series expansion for $F(x)=\arctan x$ about $a=0$. Find the radius of convergence for the series you obtained.

59. Use differentiation and the result of Exercise 43 to find a series expansion of $g(x)=2x/(x^2+1)^2$. Find the radius of convergence for the series you obtained.

60. Even when differentiated, a convergent power series retains its radius of convergence. But as we have noted, the behavior at the interval endpoints can change—in particular, the original series may converge at an endpoint, while its derivative diverges. Verify this by examining the intervals of convergence of f and its first two derivatives for

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n^2+1}.$$

61. Show that for any natural number $k\in\mathbb{N}$, the power series expansion of $\frac{(k-1)!}{(1-x)^k}$ is

$$\sum_{n=0}^{\infty}(n+k-1)(n+k-2)\cdots(n+2)(n+1)x^n.$$

62.* In general, series of functions are not nearly as well behaved with regards to termwise differentiation and integration as power series are. As an illustration, examine the convergence set (the set of x-values for which the series converges) of $f(x)=\sum_{n=0}^{\infty}\frac{\sin(3^n x)}{2^n}$, and that of its first derivative. (**Note:** This is not a power series.)

63. Verify that $y=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ satisfies the differential equation $y'=2xy$. (**Hint:** Differentiate term by term.)

64. Find a power series solution of the differential equation, $y'=2y$ satisfying the initial condition $y(0)=1$. Then solve the equation by traditional means and conclude that the solutions are equal. (**Hint:** Starting with undetermined coefficients, write $y=\sum_{n=0}^{\infty}a_n x^n$, obtain the power series for both y' and $2y$, and finally equate terms.)

65. Find a power series solution of the differential equation, $y''+4y=0$ satisfying the initial conditions $y(0)=1$, $y'(0)=0$. Then solve the equation by traditional means and conclude that the solutions are equal. (See the hint given in Exercise 64.)

Concept Check

66–70 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

66. If the series $\sum a_n(x+1)^n$ converges at $x=-4$, then it must converge at $x=1$.

67. If the series $\sum a_n(x+1)^n$ converges at $x=-4$, then it must converge at $x=2$.

68. If the series $\sum a_n x^n$ converges at $x=1$, then $\sum na_n x^{n-1}$ must also converge at $x=1$.

69. If a series converges on an interval, we may differentiate it term by term to obtain the derivative of its sum.

70. If the interval of convergence of the series $\sum a_n x^n$ is $(-a,a)$ for an $a\in\mathbb{R}^+$, then the interval of convergence of $\sum a_n(x-a)^n$ is $(0,2a)$.