So as $n \to \infty$, a_{n+1}/a_n oscillates between values approaching 0 (for odd n) and values growing unboundedly large (for even n)—in other words, $\lim_{n\to\infty}(a_{n+1}/a_n)$ fails to exist in a fairly extreme fashion. But

$$\sqrt[n]{a_n} = \begin{cases} n^{1/n}/2 & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases}$$

and $n^{1/n} \to 1$ as $n \to \infty$, so

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \tfrac{1}{2} < 1$$

and the series converges by the Root Test.

10.5 Exercises

1–34 Use the Ratio Test to determine whether the series converges or diverges.

1. $\sum_{n=0}^{\infty} \frac{1}{n!}$

2. $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$

3. $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

4. $\sum_{n=1}^{\infty} \frac{\pi^n}{n^\pi}$

5. $\sum_{n=0}^{\infty} \frac{3}{(2n)!}$

6. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

7. $\sum_{n=1}^{\infty} \frac{n^5}{e^n}$

8. $\sum_{n=0}^{\infty} \frac{3}{(2n+1)!}$

9. $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$

10. $\sum_{n=2}^{\infty} \frac{2n}{(n-1)^2}$

11. $\sum_{n=0}^{\infty} \frac{3n+2}{5^{3n+2}}$

12. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

13. $\sum_{n=1}^{\infty} \frac{(3n)!}{4^{n-1}5^n}$

14. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

15. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

16. $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{(n+1)!}$

17. $\sum_{n=1}^{\infty} \frac{4^n}{(n+4)n}$

18. $\sum_{n=1}^{\infty} \frac{n}{\pi^{2n+1}}$

19. $\sum_{n=1}^{\infty} \frac{n^2-4}{2^n}$

20. $\sum_{n=0}^{\infty} \frac{(n!)4^n}{(4n+1)!}$

21. $\sum_{n=2}^{\infty} \frac{2^n}{n(n^2-1)}$

22. $\sum_{n=1}^{\infty} \frac{n^2}{(\ln 2)^n}$

23. $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 3)^n}$

24. $\sum_{n=0}^{\infty} \frac{(3n-2)(3n+2)}{3^n}$

25. $\sum_{n=0}^{\infty} \frac{(n+3)3^n}{n!}$

26. $\sum_{n=1}^{\infty} \frac{n^2}{(\ln \pi)^n}$

27. $\sum_{n=1}^{\infty} \frac{2^{3n-2}}{2n^2+3n}$

28. $\sum_{n=1}^{\infty} \frac{(3n)!}{n^3}$

29. $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$

30. $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

31. $\sum_{n=0}^{\infty} \frac{3^n}{2^n+1}$

32. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

33. $\frac{2}{1\cdot 2\cdot 3} + \frac{4}{2\cdot 3\cdot 4} + \cdots + \frac{2^n}{n(n+1)(n+2)} + \cdots$

34. $\frac{1}{3} + \frac{1\cdot 5}{2\cdot 3\cdot 9} + \cdots + \frac{1\cdot 5\cdot \cdots \cdot (4n-3)}{3^n(n!)(2n-1)} + \cdots$

35–38 Verify that the Ratio Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

35. $\sum_{n=0}^{\infty} \frac{n^2+1}{(n+1)^2}$

36. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$

37. $\sum_{n=1}^{\infty} \frac{\arctan n}{n}$

38. $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n}(n+2)^2}$

39–62 Use the Root Test to determine whether the series converges or diverges.

39. $\sum_{n=1}^{\infty} \left(\frac{5}{n}\right)^n$

40. $\sum_{n=0}^{\infty} \frac{1}{(n+1)^n}$

41. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

42. $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n^n}$

43. $\sum_{n=1}^{\infty} \left(\frac{3n}{n^2+3}\right)^n$

44. $\sum_{n=2}^{\infty} \frac{n^{2n}}{(\ln n)^n}$

45. $\sum_{n=1}^{\infty} \frac{n}{2^{2n}}$

46. $\sum_{n=2}^{\infty} \frac{e^{2n}}{(\ln n)^n}$

47. $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$

48. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

49. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

50. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

51. $\sum_{n=2}^{\infty} \frac{1}{(\ln(\ln n))^n}$

52. $\sum_{n=2}^{\infty} \left(\ln \frac{1}{n}\right)^{2n}$

53. $\sum_{n=1}^{\infty} n\left(\frac{4}{5}\right)^n$

54. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

55. $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$

56. $\sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)^n$

57. $\sum_{n=2}^{\infty} \frac{n^{n/2}}{(\ln n)^n}$

58. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$

59. $\sum_{n=1}^{\infty} \frac{(n+2)^n}{n^{2n}}$

60. $\sum_{n=1}^{\infty} \frac{2}{n^{n+2}}$

61. $\sum_{n=1}^{\infty} \sin^n \frac{1}{n^2}$

62. $\sum_{n=1}^{\infty} \frac{1}{\left(2+\frac{1}{n}\right)^{2n}}$

63–66 Verify that the Root Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

63. $\sum_{n=1}^{\infty} \left(\frac{n}{n+5}\right)^n$

64. $\sum_{n=1}^{\infty} \frac{(2\sqrt{n})^{2n}}{(4n+1)^n}$

65. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

66. $\sum_{n=1}^{\infty} \frac{2n}{(n+1)^3}$

67–70 Suppose that the series $\sum a_n$ satisfies the condition $\lim_{n\to\infty} (a_{n+1}/a_n) = \frac{1}{2}$. Decide whether the given series is convergent.

67. $\sum_{n=1}^{\infty} 3^n a_n$

68. $\sum_{n=1}^{\infty} n^2 a_n$

69. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n a_n$

70. $\sum_{n=1}^{\infty} a_n^2$

71. Prove that for all exponents p, the series $\sum (n^p/2^n)$ is convergent.

72. Prove that $\sum (p^n/n!)$ converges for all $p > 0$.

73. For what positive p-values does the series $\sum (p^n/n)$ converge?

Concept Check

74–79 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample. ($\sum a_n$ is a positive series in each problem.)

74. If $\sum a_n$ is convergent, then $\lim_{n\to\infty} (a_{n+1}/a_n) < 1$.

75. If $\sum a_n$ is divergent, then $\lim_{n\to\infty} \sqrt[n]{a_n} \geq 1$ or the limit doesn't exist.

76. If there is an $N > 0$ such that $\sqrt[n]{a_n} < 1$ for all $n \geq N$, then $\sum a_n$ is convergent.

77. The series $\sum_{n=1}^{\infty} (3^{n^2}/n!)$ is convergent.

78.* If $\sum a_n$ satisfies condition a. of the Ratio Test, then it satisfies condition a. of the Root Test.

79.* If $\sum a_n$ satisfies the condition a. of the Root Test, then it satisfies condition a. of the Ratio Test.

10.5 Technology Exercises

80. The following series, discovered by the great Indian mathematician Srinivasa Ramanujan, converges to $1/\pi$ with amazing speed.

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(26390n+1103)}{(n!)^4 396^{4n}} = \frac{1}{\pi}$$

a. Prove that the series is convergent.

b. Use Ramanujan's series with a graphing utility to approximate π, correct to 31 digits after the decimal. How many terms did you have to use to achieve this accuracy?