10.4 Exercises

1–32 Use the Direct Comparison Test to determine whether the series converges or diverges. (Wherever applicable, combine the Direct Comparison Test with previous techniques, such as the Integral Test, *p*-series test, etc.)

1. $\sum_{n=1}^{\infty} \frac{2}{n^3+1}$

2. $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$

3. $\sum_{n=2}^{\infty} \frac{4n}{\sqrt{n^3-1}}$

4. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)}$

5. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+\sqrt{n}}$

6. $\sum_{n=1}^{\infty} \frac{2}{n^n}$

7. $\sum_{n=0}^{\infty} \frac{n^3+1}{n^4+1}$

8. $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

9. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+\sqrt{n}}$

10. $\sum_{n=0}^{\infty} \frac{1}{1+2^n}$

11. $\sum_{n=0}^{\infty} \frac{5n^2}{n^4+1}$

12. $\sum_{n=2}^{\infty} \frac{5n^2}{n^3-1}$

13. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{5n^3+2}}$

14. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

15. $\sum_{n=2}^{\infty} \frac{2}{\ln n}$

16. $\sum_{n=1}^{\infty} \frac{2}{n-\ln n}$

17. $\sum_{n=0}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+2}}$

18. $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$

19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1+\sqrt{n}}}$

20. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$

21. $\sum_{n=2}^{\infty} \frac{3n}{e^n \ln n}$

22. $\sum_{n=0}^{\infty} \sqrt{ne^{-n^2}}$

23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{ne^{\sqrt{n}}}}$

24. $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2}$

25. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3+1}$

26. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^4+2n+3}}$

27. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n^2-1}}$

28. $\sum_{n=0}^{\infty} \frac{2}{e^{2n}+n^{3/2}}$

29. $\sum_{n=2}^{\infty} \frac{\ln n}{n^{5/2}}$

30. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}-\frac{1}{2}}$

31. $\sum_{n=2}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}\ln n}$

32. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

33–50 Use the Limit Comparison Test to determine whether the series converges or diverges.

33. $\sum_{n=1}^{\infty} \frac{2n^2}{n^3+1}$

34. $\sum_{n=2}^{\infty} \frac{2n^2}{n^4-1}$

35. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}}$

36. $\sum_{n=1}^{\infty} \frac{n^4}{\sqrt{2n^9+n^5+3n^3+2}}$

37. $\sum_{n=5}^{\infty} \frac{2n+4}{n(n-2)(n-4)}$

38. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{2n}+\ln n}$

39. $\sum_{n=1}^{\infty} \frac{\sqrt{n}\ln(n+1)}{n^3}$

40. $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

41. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (**Hint:** Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.)

42. $\sum_{n=1}^{\infty} \frac{3^{1/n}-1}{3^{1/n}}$ (**Hint:** Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$.)

43. $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{\sqrt{n}\ln n}$

44. $\sum_{n=1}^{\infty} \frac{\sqrt{2+\frac{1}{n}}}{2n^2}$

45. $\sum_{n=1}^{\infty} \frac{ne^{-n^2}}{1+e^{-n}}$

46. $\sum_{n=1}^{\infty} \frac{n^2}{3n^{5/2}+1}$

47. $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n^2+3}}$

48. $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n^3+3}}$

49. $\sum_{n=0}^{\infty} \frac{n+3}{\sqrt{n^4+3}}$

50. $\sum_{n=0}^{\infty} \frac{n+3}{\sqrt{n^5+3}}$

51–70 Use any test covered so far in the text to determine whether the series converges or diverges.

51. $\sum_{n=0}^{\infty} \frac{1}{2^n+3}$

52. $\sum_{n=0}^{\infty} \frac{2^n}{2^n+3}$

53. $\sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{n+1}}-\frac{2}{\sqrt{n+2}}\right)$

54. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt[3]{n}}$

55. $\sum_{n=0}^{\infty} \frac{1}{2^{n^2}-3}$

56. $\sum_{n=1}^{\infty} \frac{n}{2^{-n}+3^{-n}}$

57. $\sum_{n=1}^{\infty} \frac{\ln n+1}{\sqrt{n}}$

58. $\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}+\cos n}$

59. $\sum_{n=1}^{\infty} \frac{1-\sin n}{n^{3/2}}$

60. $\sum_{n=1}^{\infty} \frac{n-3}{n^2\sqrt{n}}$

61. $\sum_{n=1}^{\infty} \frac{1}{\tan^{-1} n}$

62. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln(\ln n))}$

63. $\sum_{n=3}^{\infty} \frac{1}{\ln n - 1}$

64. $\sum_{n=3}^{\infty} \frac{\sqrt{n-2}}{n\sqrt{n^2-1}}$

65. $\sum_{n=1}^{\infty} \frac{n+3}{n3^n}$

66. $\sum_{n=1}^{\infty} \frac{2^{n-1}+2}{2^n}$

67. $\sum_{n=1}^{\infty} \frac{3^n+4^n}{4^n+5^n}$

68. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

69. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

70. $\sum_{n=0}^{\infty} \left(\frac{n}{n+2}\right)^n$

71. Prove part b. of the Limit Comparison Test. (**Hint:** Choose $K > 0$ and a natural number $N_0 \geq N$ for which $0 \leq a_n/b_n \leq K$ for all $n \geq N_0$ and apply the Direct Comparison Test.)

72. Prove part c. of the Limit Comparison Test. (**Hint:** One possibility is to notice that $\lim_{n\to\infty}(b_n/a_n) = 0$ and make use of the argument you gave in Exercise 71.)

73. Suppose that $\sum a_n$ is a positive series, $\{a_n\}$ is monotonically decreasing, and $\{na_n\}$ is convergent with $\lim_{n\to\infty} na_n \neq 0$. Prove that $\sum a_n$ diverges. (**Hint:** Use the Limit Comparison Test to compare $\sum a_n$ with an appropriate series. For the definition of a positive series, see Exercise 70 in Section 10.2.)

74–77 Use Exercise 73 to provide a quick proof of the divergence of the given series.

74. $\sum_{n=1}^{\infty} \frac{n^2}{3n^3+1}$

75. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$

76. $\sum_{n=2}^{\infty} \frac{2}{\ln n}$

77. $\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$

78.* (The Cauchy Condensation Test) Suppose that $\sum_{n=1}^{\infty} a_n$ is a positive series and $\{a_n\}$ is monotonically decreasing. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. (**Hint:** Fix n and k, and let s_n be the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$, while letting c_k denote the k^{th} partial sum of the "condensed" series $\sum_{n=0}^{\infty} 2^n a_{2^n}$. Group the terms of $\sum_{n=1}^{\infty} a_n$ as

$$\sum_{n=1}^{\infty} a_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots,$$

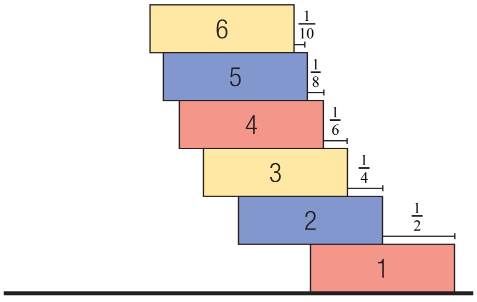
while $\sum_{n=0}^{\infty} 2^n a_{2^n}$ can be written as

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots.$$

Using the monotonicity of $\{a_n\}$, notice that if $n < 2^k$, then $s_n \leq c_k$, while if $n > 2^k$, then $s_n \geq \frac{1}{2}c_k$. Use these observations to show that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded if and only if that of $\sum_{n=0}^{\infty} 2^n a_{2^n}$ is bounded. Now use Exercise 70 from Section 10.2 to finish your argument.)

79. Use Exercise 78 to give a new proof of the p-series test (see Example 2 of Section 10.3).

80.* Suppose we stack idealized building blocks (perfectly rectangular, perfectly homogeneous, perfectly smooth, perfectly level, etc.) with the indicated fraction of each block protruding to the right of the block above it, as shown in the figure below. If we continue building in this way, that is, by making sure exactly $1/(2n)$ of the n^{th} block protrudes to the right, how high can we build the structure before it topples? Assuming vertical sunrays (and an infinitely high sun), how far to the left will the shadow of the structure extend? (**Hint:** Starting from the bottom, consider the substructure consisting of the first n blocks and the location of its centroid relative to the right edge of the $(n+1)^{\text{th}}$ block.)



81.* Prove that a positive series $\sum a_n$ converges if and only if the series $\sum \frac{a_n}{a_n+1}$ converges. (**Hint:** Show that $a_n < 2a_n/(a_n+1)$ if n is large enough.)

Concept Check

82–91 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

82. If $\sum a_n$ and $\sum b_n$ are positive series, $\sum b_n$ diverges, and $a_n < b_n$ for all n, then $\sum a_n$ converges.

83. If $\sum a_n$ and $\sum b_n$ are convergent positive series, then $\sum a_n b_n$ is convergent.

84. If $\lim_{n\to\infty}(a_n/b_n) = 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

85. If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum b_n$ converges, then $\sum a_n$ converges.

86.* If $\sum a_n$ is a convergent positive series, then $\sum \ln(1+a_n)$ is convergent.

87. If $\sum a_n$ is a convergent positive series and $\sum b_n$ and $\sum c_n$ are positive series such that $b_n c_n = a_n$ for all n, then both $\sum b_n$ and $\sum c_n$ are convergent.

88. If $\sum a_n$ is a convergent positive series and $\sum b_n$ and $\sum c_n$ are positive series such that $b_n + c_n = a_n$ for all n, then both $\sum b_n$ and $\sum c_n$ are convergent.

89. If $p > 1$, then $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ diverges.

90.* The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

91.* The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges.