10.2 Exercises

1. Suppose that $a_n = 2n^2/(n^3-1)$ and consider the series $\sum_{n=2}^{\infty} a_n$. One of your classmates argues that the series converges, because the numerator of a_n has a lesser degree than the denominator, therefore $\lim_{n\to\infty} a_n = 0$, which makes $\{s_n\}$ convergent, and thus, $\sum_{n=2}^{\infty} a_n$ has a finite sum. Is this argument correct? Why or why not?

2–5 Find the first five terms of the sequence of partial sums $\{s_n\}$ for the given series.

2. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

3. $\sum_{n=0}^{\infty} (-1)^n$

4. $\sum_{n=0}^{\infty} \frac{3^n-1}{3^n}$

5. $\sum_{n=1}^{\infty} \sin\frac{n\pi}{3}$

6. Determine the index n so that the difference between the sum of the series $\sum_{n=1}^{\infty}(1/2^n)$ and the partial sum $\{s_n\}$ (the error) is less than 0.0001.

7. Write a short paragraph on the difference between the sequences $\{a_n\}$ and $\{s_n\}$ for the series in Example 5. (Mention convergence and limits.)

8–17 Determine whether the given geometric series converges. If so, find its sum.

8. $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots$

9. $5-1+\frac{1}{5}-\frac{1}{25}+\frac{1}{125}-\frac{1}{625}+\cdots$

10. $1-1.1+(1.1)^2-(1.1)^3+(1.1)^4-\cdots$

11. $\frac{4}{5}-1+\frac{5}{4}-\frac{25}{16}+\frac{125}{64}-\cdots$

12. $\sum_{n=0}^{\infty} 7^{-n}$

13. $\sum_{n=0}^{\infty} (-0.7)^n$

14. $\sum_{n=0}^{\infty} 3\cdot\left(\frac{11}{12}\right)^n$

15. $\sum_{n=0}^{\infty} 2\cdot(-0.35)^n$

16. $\sum_{n=0}^{\infty} \frac{4}{5}\cdot\left(-\frac{5}{4}\right)^n$

17. $\sum_{n=0}^{\infty} \frac{5}{4}\cdot\left(-\frac{4}{5}\right)^n$

18–19 Find all values of *x* for which the geometric series converges.

18. $\sum_{n=1}^{\infty} 2(1-3x)^{n-1}$

19. $\sum_{n=1}^{\infty} \frac{4}{(2x-5)^{n-1}}$

20–23 Recognize the repeating decimal as a geometric series and write the decimal as a ratio of two integers.

20. $0.\overline{5}$

21. $0.\overline{123}$

22. $0.5\overline{384}$

23. $3.37\overline{9}$

24–29 Find the sum of the series. (**Hint:** Use partial fraction decomposition wherever appropriate to express it as a telescoping series.)

24. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

25. $\sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)}$

26. $\sum_{n=1}^{\infty} \frac{2}{20n^2-5}$

27. $\sum_{n=1}^{\infty} \frac{3}{n^2+5n+6}$

28. $\sum_{n=1}^{\infty} \frac{1}{(4n+3)(4n-1)}$

29. $\sum_{n=1}^{\infty} \frac{2}{12n^2-3}$

30–49 Decide whether the given series converges. If so, find its sum.

30. $\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^n}$

31. $\sum_{n=3}^{\infty} \frac{(-1)^{n-1}n}{n-2}$

32. $\sum_{n=0}^{\infty} (-1)^n\left(\frac{\pi}{4}\right)^n$

33. $\sum_{n=1}^{\infty} \left(\frac{1}{3^n}-\frac{3}{n}\right)$

34. $\sum_{n=1}^{\infty} \frac{1}{3^{1/n}}$

35. $\sum_{n=1}^{\infty} \frac{2^n+2^{2n}}{5^n}$

36. $\sum_{n=0}^{\infty} \left(2\sqrt{3}\right)^{2-n}$

37. $\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{\ln n}$

38. $\sum_{n=0}^{\infty} \left(\frac{\pi-1}{e}\right)^n$

39. $\sum_{n=1}^{\infty} \left(3^{-n}-4^{-n}\right)$

40. $\sum_{n=1}^{\infty} \frac{n}{3n+2}$

41. $\sum_{n=1}^{\infty} \frac{3^n-1}{3^n}$

42. $\sum_{n=1}^{\infty} (-1)^n\left(\frac{e}{\pi}\right)^n$

43. $\sum_{n=0}^{\infty} \frac{2\cdot 3^n+5\cdot 7^n}{11^n}$

44. $\sum_{n=0}^{\infty} \frac{2+3^n-5^n}{7^n}$

45. $\sum_{n=0}^{\infty} (-1)^n\frac{n}{n+2}$

46. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

47. $\sum_{n=1}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$

48. $\sum_{n=2}^{\infty} \left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$

49. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n-1}$

50. The series $\sum_{n=1}^{\infty}\left(1/2^n\right)$ converges and its sum is 1, as we have seen in Example 1. Examine what happens when we **a.** drop the first three terms of the series, and **b.** adjoin the three terms 3 + 2 + 1 to the series. Conclude that deleting or adjoining finitely many terms to a series may change its sum, but not the fact of convergence. Can you provide a rigorous proof of this more general statement?

51–52 Use the definition of series convergence to prove that the series is convergent.

51. $\sum_{n=1}^{\infty}\frac{1}{2^n}$

52. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

53. Use Example 5 to show that it is possible for $\sum_{n=1}^{\infty}\left(a_n-b_n\right)$ to converge if both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are divergent.

54. Give an example of two divergent series such that $\sum_{n=1}^{\infty}\left(a_n+b_n\right)$ is convergent.

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57. Prove that the series $\sum_{n=0}^{\infty}\left(a+nd\right)$ $(a, d\in\mathbb{R})$ converges if and only if $a=d=0$. (Such series are called *arithmetic series*.)

58. In one of Zeno's famous motion paradoxes, Achilles races a tortoise, giving the tortoise a 100-meter head start. Even though we assume that Achilles is ten times faster, the statement is that Achilles will never actually catch the tortoise. The reasoning goes as follows. Soon after the start, Achilles will reach the starting point of the tortoise, but by that time, the tortoise will have advanced 10 meters. Achilles' job is to quickly cover that 10-meter distance, but during that time, the tortoise will have advanced, namely, a meter, and so on. Give a calculus-based solution to the paradox by proving that Achilles will actually catch the tortoise and find the total distance Achilles will have run when it happens.

59. By examining partial sums and using properties of logarithms, prove that $\sum_{n=3}^{\infty}\ln\frac{n}{n-2}$ diverges.

60. Prove that $\sum_{n=0}^{\infty}\left(1-x\right)^n$ is convergent if $0<x<2$, and find its sum.

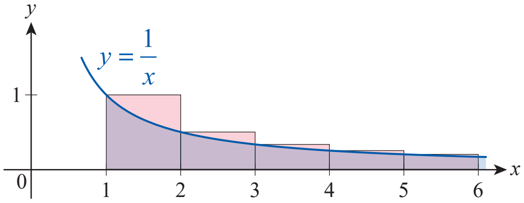
61. Prove the Sum Law for convergent series: $\sum\left(a_n+b_n\right)=\sum a_n+\sum b_n$. (**Hint:** Fix n, and write the statement for the n^{th} partial sums first; then take the limits and use the appropriate limit laws.)

62. Prove the Difference Law for convergent series: $\sum\left(a_n-b_n\right)=\sum a_n-\sum b_n$. (See the hint given in Exercise 61.)

63. Prove the Constant Multiple Law for convergent series: $\sum ka_n=k\sum a_n$. (See the hint given in Exercise 61.)

64. Suppose a large state injects a 2-billion-dollar stimulus package into its economy. Consumers and businesses in the state save approximately 30 percent of that money and respend 70 percent. Of that latter amount, approximately 70 percent is again spent, and so on. What is the total spending generated by the stimulus package? (**Hint:** Find the sum of the geometric series that models the process. In economics, this is called the *multiplier effect*.)

65. Use the figure below and the fact that $\int_1^{\infty}\left(1/x\right)dx=\infty$ to argue that the harmonic series diverges. (We will refine this idea in Section 10.3 and use it to "test" the convergence of various series.)



66. Prove that if the series $\sum a_n$ is convergent, then $\sum\left(1/a_n\right)$ is a divergent series.

67. Suppose that for the series $\sum a_n$ and $\sum b_n$ there exists a natural number N such that for all $n>N$, $a_n=b_n$. Prove that the series either both converge or both diverge.

68.* Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then it satisfies the so-called *Cauchy criterion for convergence*: For any $\varepsilon > 0$, there is a corresponding natural number N such that $\left|\sum_{n=n_1+1}^{n_2} a_n\right| < \varepsilon$ for all $n_1, n_2 > N$. (**Hint:** Choose an appropriately small ε-neighborhood around the sum of the series, noticing that $\left|\sum_{n=n_1+1}^{n_2} a_n\right| = \left|s_{n_2} - s_{n_1}\right|$.)

69.* Use Exercise 68 to prove that $\sum_{n=1}^{\infty}(1/n!)$ is convergent.

70. Prove that if $\sum_{n=1}^{\infty} a_n$ is a *positive series*, that is, $a_n > 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence $\{s_n\}_{n=1}^{\infty}$ of its partial sums is bounded.

71. Prove that $\sum_{n=1}^{\infty}\left(1/\sqrt{n}\right)$ diverges. (**Hint:** Prove that the n^{th} partial sum $s_n \geq \sqrt{n}$.)

72.* Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series such that $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \geq b_n$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} b_n$ is convergent. (**Hint:** Examine partial sums.)

73.* Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series such that $\sum_{n=1}^{\infty} a_n$ is divergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} b_n$ is divergent. (See the hint given in Exercise 72.)

74. Use Exercise 72 to prove that $\sum_{n=1}^{\infty}\left(1/n^2\right)$ is convergent. (**Hint:** Use the inequality below.)

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

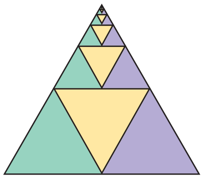
75. Use Exercise 73 and Example 6 to show that $\sum_{n=1}^{\infty}\left(1/n^p\right)$ diverges for all $0 < p < 1$.

76.* Two trains, 200 km apart, are on a collision course toward each other, each traveling at a rate of $50\ \text{km/h}$. A fly is zigzagging between the trains, flying at $75\ \text{km/h}$. Assuming constant rates and that the fly turns around in zero time, how much total distance will the fly be able to cover before being crushed to death by the trains upon their impending collision? (**Hint:** First, find the time required for the first "leg" of the flight, then, taking into consideration how much the original distance of 200 km between the trains has shrunk during this time, find the time required for the second "leg" of the fly's flight. Conclude that the time required for each leg is a constant times that required for the previous leg. Consequently, the fly's total time will be the sum of a geometric sequence; use this to find the total distance covered by the fly.)

77. Solve Exercise 76 "the easy way," without using an infinite series, that is, simply using the fact that the fly's total travel time equals the time necessary for the trains to reach each other. According to a well-authenticated story, when John von Neumann (born János Neumann), the great Hungarian American mathematician of the 20th century, was challenged with a version of this problem, he answered correctly within a few seconds. "Interesting," his challenger remarked, "most people try to solve this problem using infinite series." "Why," came von Neumann's reply, "that is how I did it!"

78–81 The following figures were generated by Rick Mabry as "pictorial proofs" for the convergence of various series. For example, dividing an equilateral triangle into four congruent parts and iterating the process, as below, provides illustration for the fact that $\sum_{n=1}^{\infty}\left(1/4^n\right) = \frac{1}{3}$ (assume that in the figure below, the area of the original triangle is 1; you can visually check what portion of the area is occupied by each color).

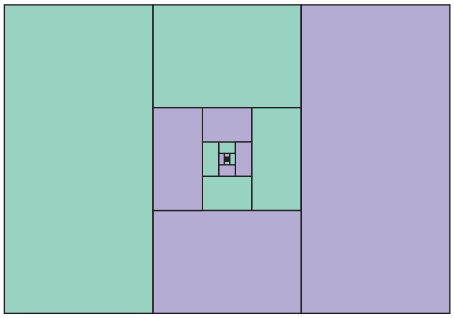
Source: www.lsus.edu/rick-mabry



$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$

In Exercises 78–81, use the visual approach discussed above to identify the convergent series illustrated by the figure.

78.



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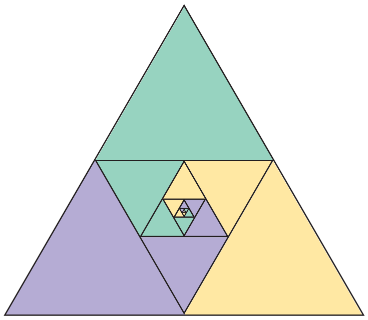

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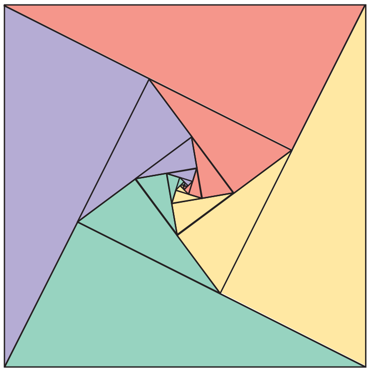

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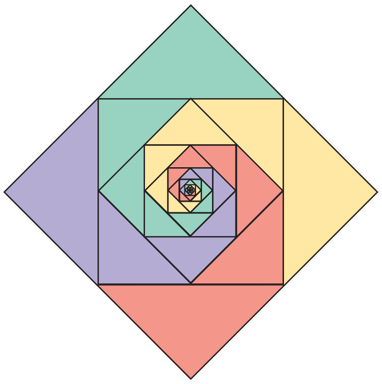

10.2 Technology Exercises

82–85 Often we can readily show that a particular series is convergent, but finding the sum may be extremely challenging, if not impossible. For example, it is not difficult to prove that the series $\sum_{n=1}^{\infty}(1/n^p)$ is convergent if $p > 1$ (see Exercise 74 and, for the full story, Example 2 in Section 10.3), but finding the sum often defies the best efforts of mathematicians. For example, letting $S(p)$ denote the sum of the above series for a particular $p > 1$, we know that $S(2) = \pi^2/6$ (this surprising result was first proven by Euler), but formulas for the sums for odd *p*-values, such as $S(3)$, $S(5)$, etc. are still unknown.

In Exercises 82–85, use a graphing utility to verify the indicated sums.

82. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

83. $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

84. $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$

85. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = \frac{1}{e}$

86. On the same screen, graph the function you obtained in Exercise 60 along with the partial sums $p(x) = \sum_{n=0}^{N}(1-x)^n$ on the interval $(0,2)$, for higher and higher *N*-values. What do you see?