# Chapter 6
## Review Exercises

**1–2** The base of a solid *S* is described in the *xy*-plane along with its cross-sections in a certain direction. Find the volume of *S*.

1. The base of $S$ is the first-quadrant region of the unit disk centered at the origin; the cross-sections perpendicular to the $y$-axis are squares.

2. The base of $S$ is the region bounded by the graph of $y=\sqrt{2-x}$ and the coordinate axes; the cross-sections perpendicular to the $x$-axis are equilateral triangles.

**3–6** Use the disk/washer method to find the volume of the solid generated by rotating the region *R* about the indicated axis.

3. $R$ is bounded by the graph of $y=\sqrt{1-x}$ and the coordinate axes, rotated about the $x$-axis.

4. $R$ is the region of Exercise 3, rotated about the $y$-axis.

5. $R$ is bounded by the graphs of $y = \operatorname{arcsec} x$, $x=\sqrt{2}$, and $y = 0$, rotated about the $y$-axis.

6. $R$ is the first-quadrant portion of the region bounded by the graphs of $y=4/x^2$ and $y = 5 - x^2$, rotated about $x = -2$.

**7–10** Use the shell method to find the volume of the solid generated by rotating the region *R* about the indicated axis.

7. $R$ is bounded by the graph of $y = 2x - x^2$ and the $x$-axis, rotated about the $y$-axis.

8. $R$ is bounded by the graphs of $y = x$, $y = 3 - 2x$, and the $y$-axis, rotated about $x = -1$.

9. $R$ is the first-quadrant region bounded by the graphs of $y = 4x$ and $y = x^3$, rotated about the $x$-axis.

10. $R$ is bounded by the graphs of $y=\sqrt{x}$ and $y = x^2$, rotated about $y = -2$.

**11–14** The given integral represents the volume of a solid of revolution. Describe the solid. (Do not evaluate the integral.)

11. $\int_0^3 \pi x^2\,dx$

12. $\pi\int_0^2\left[1-(y-1)^2\right]dy$

13. $\int_0^2 2\pi x\left(4-x^2\right)dx$

14. $2\pi\int_1^3 y\sqrt{1-(y-2)^2}\,dy$

**15–18** Use the shell method or the disk/washer method to find the volume of the solid obtained by revolving the region bounded by the graphs of the equations about the given axis. Choose the method that seems to work best.

15. $x=\sqrt{\sin y},\quad x=0,\quad 0\le y\le \pi;$ about the $y$-axis

16. $y=x^3-2x^2+x,\quad y=0,\quad x=0,\quad x=1;$ about the $x$-axis

17. $y=\sqrt{x},\quad y=2-x,\quad x=0;$ about the $y$-axis

18. $y=\dfrac{e^x}{x},\quad y=0,\quad x=1,\quad x=2;$ about the $y$-axis

19. Suppose the region bounded by the graph of $y=\sqrt{c^2-x}$ $(c\ne 0)$ and the coordinate axes is revolved about the $y$-axis. Find the resulting solid's volume, and then consider the volume of the inscribed circular cone that results from rotating the line segment $y=c-\dfrac{x}{c}$, $0\le x\le c^2$, about the $y$-axis. Show that the ratio of these two volumes is $\frac{8}{5}$. (This result is due to Pierre de Fermat, the great $17^{\text{th}}$-century mathematician.)

**20–23** Set up, but do not evaluate, an integral defining the arc length of the graph of the equation over the given interval.

20. $y=x^2-x+1;\quad 0\le x\le 3$

21. $x+2y^2=3-y;\quad 1\le y\le 5$

22. $y=\tan x;\quad -\pi/4\le x\le \pi/4$

23. $e^x=\cos y;\quad 0\le y\le \pi/3$

**24–29** Determine the arc length *L* of the curve defined by the equation over the given interval.

24. $y=\sqrt{3}x+1;\quad 0\le x\le 4$

25. $y=\dfrac{4x^{3/2}+1}{6};\quad 0\le x\le 3$

26. $y=\dfrac{x^4+3}{6x};\quad 1\le x\le 3$

27. $y=\dfrac{x^2}{4}-\ln\sqrt{x};\quad 1\le x\le e^2$

28. $y=\dfrac{\sqrt{x}}{6}-2x^{3/2};\quad 0\le x\le 1$

**29.** $y = e^{x/2} + e^{-x/2}; \quad 0 \le x \le 2$

**30–33** Set up, but do not evaluate, an integral defining the surface area of the solid obtained by revolving the given curve about the indicated axis.

**30.** $y = \sin x; \quad 0 \le x \le \pi; \quad$ about the $x$-axis

**31.** $y = \sqrt{\ln x}; \quad 1 \le x \le e; \quad$ about the $x$-axis

**32.** $y = \sqrt{\ln x}; \quad 1 \le x \le e; \quad$ about the $y$-axis

**33.** $y = x^2; \quad 0 \le x \le 2; \quad$ about $x = -1$

**34–39** Find the surface area of the solid generated by revolving the given curve about the indicated axis.

**34.** $y = \frac{1}{3}x + 2; \quad 0 \le x \le 3; \quad$ about the $x$-axis

**35.** $y = \frac{x^3}{4}; \quad 0 \le x \le 1; \quad$ about the $x$-axis

**36.** $3y = 3x^{3/2} - \sqrt{x}; \quad 1 \le x \le 2; \quad$ about the $x$-axis

**37.** $x = \frac{y^3}{12} + \frac{1}{y}; \quad 1 \le y \le 2; \quad$ about the $y$-axis

**38.** $x = 1.5y^{5/3} - 0.3\sqrt[3]{y}; \quad 0 \le y \le 1; \quad$ about the $y$-axis

**39.** $12xy = 3y^4 + 4; \quad 1 \le y \le 2; \quad$ about $x = -\frac{1}{2}$

**40–43** Find the centroid of the plane region bounded by the given curves. If possible, use symmetry to simplify your calculations.

**40.** $y = 5x - x^2, \quad y = 0$

**41.** $y = 2\sqrt{x}, \quad y = 2x$

**42.** $y = x^{3/5}, \quad x = 0, \quad y = 8$

**43.** $y = x, \quad y(x+1)^2 = 4, \quad x = 0$

**44–47** Find the center of mass of the plane region of varying density that is bounded by the given curves.

**44.** $2y = 4 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = x^2$

**45.** $2y = 4 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = \sqrt{y}$

**46.** $y = x, \quad y = \sqrt{x}; \quad \rho(x,y) = 1 + x$

**47.** $x = 9 - y^2, \quad y = x - 3; \quad \rho(x,y) = y + 3$

**48.** Suppose the density of a baseball bat lying along the positive $x$-axis is given by the function $\rho(x) = (0.033x + 0.5)^2$ ounces per inch $(0 \le x \le 30)$. Find the center of mass of the bat.

**49.** Find the centroid of the upper semielliptical region bounded by the $x$-axis and the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (See Exercise 58 of Section 6.4.)

**50.** Use Pappus' Theorem for volumes to find the volume of the solid generated by revolving the region of Exercise 43 about the $y$-axis.

**51.*** Use Pappus' Theorem for volumes to find a second solution to Exercise 49 using the fact that the area of the semiellipse is $A = \pi ab/2$.

**52.** A 10 cm long unstressed spring requires a force of 2 N to stretch to 14 cm. How much work will be done in stretching the spring an additional 4 cm?

**53.** A 100 ft cable is lifting a 30 lb weight by 20 ft, so that the cable is wound on a cylindrical drum. Find the total work done if the cable weighs 2 lb/ft.

**54.** A 20 m cable is being wound up on a cylindrical drum at a rate of 25 cm/s. If the cable weighs 10 N/m, find the work done from $t = 4$ s to $t = 32$ s.

**55.** A leaky container is being lifted by a crane at 0.5 m/s for 20 s. If the full container originally weighed 300 kg, but is leaking at a rate of 0.1 kg/s, find the work done by the crane. Use $g \approx 9.81\ \text{m/s}^2$.

**56.** The shape of an underground gasoline tank is an inverted circular cone with a top radius of 6 ft and a depth of 18 ft. Find the work done in pumping the full content of the tank to 1 ft above the top of the tank. The weight density of gasoline is $44\ \text{lb/ft}^3$.

**57.** Find the total work done in Exercise 56 if the tank initially is filled to only half of its capacity.

**58.** The shape of a water tank can be approximated by rotating the graph of $y = x^2$, $-2 \le x \le 2$, about the $y$-axis (units in meters). Find the work required to fill up the empty tank through an opening at its lowest point. Use $9810\ \text{N/m}^3$ for the weight density of water.

**59.*** A cup of soda of height 5 inches can be approximated by a frustum of a circular cone, standing on its smaller base of radius 1.25 inches, with a top opening of radius 2 inches. Find the work done in drinking a full cup of soda through a straw, supposing that the end of the straw is 2 inches above the rim of the cup. Use the weight density of water, $62.4\ \text{lb/ft}^3$, for the soda.

**60.** A cylindrical underground tank of radius 3 ft is positioned horizontally, and is half full of gasoline. Find the force exerted by the gasoline on one end of the tank. (See Exercise 56 for the weight density of gasoline.)

**61.** Use the centroid approach (Example 8 in Section 6.5) to provide a second solution to Exercise 60. (**Hint:** See Exercise 52 of Section 6.4.)

**62.** Use the centroid approach to provide a second solution to Exercise 56. (**Hint:** See the discussion preceding Exercises 58–64 of Section 6.5 and use the fact that the centroid of a right circular cone is on the axis of the cone, one-fourth of the way from the base toward the vertex.)

**63.** One end of a dam is an isosceles trapezoid, standing vertically on its shorter base of 15 meters. The top width is 30 meters, while the depth of the dam is 20 meters. Find the force exerted by the water on one end of the dam. Use $9810\ \text{N/m}^3$ for the weight density of water.

**64.*** The water depth in an 18 ft by 35 ft pool increases uniformly from the shallow end of 3 ft to the deep end of 8 ft. Find the fluid force exerted on the bottom of the pool. Use $62.4\ \text{lb/ft}^3$ for the weight density of water.

**65.** Find the energy expended in moving a 1.5-ton satellite to an altitude of 220 miles above the surface of Earth. Assume the radius of Earth is approximately 4000 miles. Express your answer in the units foot-pounds. (**Hint:** Ignore the work done in accelerating the satellite, air resistance, as well as the weight of the launching vehicle and fuel.)

**66–67** Find the indicated function value. If it doesn't exist, say so.

**66.** **a.** $\sinh 0$ **b.** $\operatorname{csch} 0$

**67.** **a.** $\coth^{-1} 0$ **b.** $\operatorname{sech} 0$

**68.** Use the definition to show that $\tanh x > 0$ if $x > 0$ and $\tanh x < 0$ if $x < 0$.

**69.** Prove that if $f(x) = \tanh x$ or $f(x) = \coth x$, then $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$.

**70.** Suppose that $e^x = f(x) + g(x)$, where $f(x)$ is even and $g(x)$ is odd. Prove that in this case, we must necessarily have $f(x) = \cosh x$ and $g(x) = \sinh x$. (**Hint:** Start by expressing $e^{-x}$ in terms of $f(x)$ and $g(x)$.)

**71.** Verify the following identity.

$$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$$

**72–81** Find the derivative or integral as indicated.

**72.** $\frac{d}{dx}\left[\sinh^2\left(\frac{6x-1}{2}\right)\right]$ **73.** $\frac{d}{dx}\left(e^{2x}\tanh x\right)$

**74.** $\frac{d}{dx}\left[\operatorname{csch}\left(x^2+2\right)\right]$

**75.** $\frac{d}{dx}\left[\ln\left(\sinh(5x-2)\right)\right]$

**76.** $\int\frac{\tanh\sqrt{x}}{2\sqrt{x}}\,dx$ **77.** $\int\frac{\sinh x}{\cosh^2 x}\,dx$

**78.** $\int\coth x\operatorname{csch}^2 x\,dx$ **79.** $\int\frac{dx}{\sqrt{x^2-4x+5}}$

**80.** $\int_0^1\frac{2x}{\sqrt{1+9x^4}}\,dx$ **81.** $\int\frac{e^x}{\sqrt{e^{2x}-1}}\,dx;\quad x>0$

**82.** Prove the following formula for the derivative of the inverse hyperbolic cosecant function.

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$$

**83–85** In the 16th century, Galileo observed the additive property of velocities. For example, if a boat travels at 5 mph in the direction of flow on a river that in turn flows at 2 mph, then the velocity of the boat relative to the shore will be $u = 5 + 2 = 7$ mph. However, according to Albert Einstein's theory of special relativity, the relative velocity between any two objects can never exceed $c$, the speed of light. This contradicts Galileo's observation. (On a theoretical level, just think of a "river" flowing at $\frac{3}{4}c$, with a "boat" on it traveling at, say, $\frac{1}{2}c$. The velocity relative to shore would be $u = \left(\frac{3}{4}+\frac{1}{2}\right)c = \frac{5}{4}c$!) Instead of the Galilean $u = v_1 + v_2$, Einstein discovered the following relativistic addition formula for velocities.

$$u = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

In Exercises 83–85, refer to the above formula.

**83.** Use the addition formula for $\tanh x$ (Exercise 33 of Section 6.6) to verify the following identity.

$$\tanh^{-1}\left(\frac{u}{c}\right) = \tanh^{-1}\left(\frac{v_1}{c}\right) + \tanh^{-1}\left(\frac{v_2}{c}\right)$$

**84.** Suppose a fighter plane fires a missile at 500 mph in the forward direction at a moment when the plane itself is flying at 900 mph. Use Einstein's relativistic formula to find the missile's velocity relative to Earth and compare it with Galileo's prediction of 500 + 900 = 1400 mph. Approximate the speed of light by $3\times10^8$ m/s.

**85.** In this exercise, we are going to up the numbers of Exercise 84 significantly. Suppose a rocket is traveling away from Earth at a speed of $0.7c$ and fires another rocket at $0.5c$. Use Einstein's formula to calculate the velocity of the second rocket relative to Earth.

## Concept Check

**86–92** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

**86.** The disk method is based on the idea of integrating slices.

**87.** When both the disk method and the shell method are applied to calculate the volume of a solid of revolution, the variable of integration is always the same.

**88.** If the area of the region bounded by $y = f(x)$ and $y = g(x)$ is $A$, then the volume of the solid obtained by revolving the same region about the $x$-axis is $V = \pi A^2$.

**89.** If the area of the region bounded by $y = f(x)$ and $y = g(x)$ is $A$, then the volume of the solid obtained by revolving the same region about the $x$-axis never equals $\pi A^2$.

**90.** $\lim_{x\to-\infty} \tanh x = -1$

**91.** $\cosh 2x = 2\sinh^2 x + 1$

**92.** The work needed to pump fluid out of a tank through an opening on its top equals the total weight of the fluid multiplied by the distance traveled by its center of mass.

# Chapter 6 Technology Exercises

**93–96** Use a graphing utility to find (or approximate) the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the indicated axis.

**93.** $y = \sin(x^2), \quad y = 0, \quad x = 0, \quad x = \sqrt{\pi};$ about the $x$-axis

**94.** $y = \arccos x, \quad y = 0, \quad x = 0, \quad x = 1;$ about the $x$-axis

**95.** $y = \sinh^{-1} x, \quad y = 0, \quad x = 4;$ about the $y$-axis

**96.** $y = x^2 \sin^2 x, \quad y = 0, \quad x = 0, \quad x = \pi;$ about the $y$-axis

**97–98** Use a graphing utility to find the arc length of the graph of the equation over the given interval.

**97.** $y = \dfrac{1}{x^2+1}; \quad -1 \le x \le 1$

**98.** $y = \sin x; \quad 0 \le x \le \pi$

**99–100** Use a graphing utility to find the surface area of the solid generated by revolving the given curve about the indicated axis.

**99.** $y = \sin x; \quad 0 \le x \le \pi;$ about the $x$-axis

**100.** $y = \sqrt{\ln x}; \quad 1 \le x \le e;$ about the $y$-axis