Chapter 10
Review Exercises

1–2 Starting with $n = 0$, list the first five terms of the given sequence.

1. $a_n = \left(\dfrac{1-n}{n^2+1}\right)^n$

2. $a_n = \dfrac{2^{n+1}-n}{n!}$

3–4 Find the first five terms of the given recursively defined sequence.

3. $a_1 = 0, \quad a_n = 3a_{n-1} + 2$

4. $a_1 = 0, \quad a_2 = 1, \quad a_n = 2a_{n-2} - a_{n-1}$

5–6 Use the definition of the limit of a sequence to prove the limit statement.

5. $\lim\limits_{n\to\infty} \dfrac{2}{n^3+1} = 0$

6. $\lim\limits_{n\to\infty} \dfrac{2n+1}{n} = 2$

7–10 Find the limit of the sequence if it converges or prove that the sequence diverges.

7. $a_n = \dfrac{n^2-2}{3n+1}$

8. $a_n = \sqrt{n^2+1} - n$

9. $a_n = \left(1+\dfrac{2}{n}\right)^{3n}$

10. $a_n = 2\tan^{-1}\left(n^2\right)$

11–12 Use the Squeeze Theorem to prove that the given sequence converges.

11. $\left\{\dfrac{(-1)^n \cos n}{n^2}\right\}_{n=1}^{\infty}$

12. $\left\{\dfrac{\sin 2n}{2^n}\right\}_{n=1}^{\infty}$

13–14 Recognize the repeating decimal as a geometric series and write the decimal as a ratio of two integers.

13. $0.\overline{219}$

14. $0.2\overline{19}$

15. Use the Bounded Monotonic Sequence Theorem to prove that the recursive sequence $a_1 = \sqrt{3}, a_{n+1} = \sqrt{3a_n}$ converges, and find its limit.

16. Suppose that $\lim\limits_{n\to\infty} a_n = -\infty$ and $\{b_n\}$ is a sequence such that there is an N with $b_n \le a_n$ for all $n \ge N$. Prove that $\lim\limits_{n\to\infty} b_n = -\infty$.

17. Prove that if $\lim\limits_{n\to\infty} a_n = \infty$, then the sequence has a smallest term.

18. Let $\{a_n\}$ be a positive null sequence. Prove that $\lim\limits_{n\to\infty}\left(1/a_n\right) = \infty$. (See Exercises 92 and 95 in Section 10.1 for the definitions of a null sequence and a positive sequence, respectively.)

19. Suppose that the dosage of a certain medication is d milligrams once a day, to be taken at the same time of day.

a. Assuming that after 24 hours, half of the initial amount is still present in the bloodstream, find a recursive formula for the sequence $\{a_n\}$, the amount of medication present in the bloodstream right after the n^{th} dose is taken.

b. Find $\lim\limits_{n\to\infty} d_n$. (This is how much medication is in the patient's bloodstream, if it is taken over a long time period.)

20–21 Use partial sums to prove that the given series is convergent.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$

22–25 Decide whether the series converges. If so, find its sum.

22. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^{-n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+1)}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{2+2n}{n^2+2n}$

25. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n-1}{5^{n-2}}$

26. Referring back to Exercise 66 of Section 10.2, supposing that the positive series $\sum a_n$ is divergent, what can you say about $\sum\left(1/a_n\right)$? (See Exercise 70 in Section 10.2 for the definition of positive series.)

27–32 Use the Integral Test to determine whether the series converges or diverges.

27. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+4}$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{4}{n^2+4}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{2^n}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2+e^{1/n}}{n^3}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln\left(\arctan n\right)}{n^2+1}$

33–34 Find the smallest possible n to approximate the sum of the series within the indicated error ε and provide the requested estimate.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}; \quad \varepsilon = 0.04$

34. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}; \quad \varepsilon = 0.03$

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35–38 Use the Direct Comparison Test to determine whether the series converges or diverges.

35. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n-1}$

36. $\displaystyle\sum_{n=2}^{\infty}\frac{n^2+1}{n^3-1}$

37. $\displaystyle\sum_{n=0}^{\infty}\frac{2}{2^n+\sqrt{n}}$

38. $\displaystyle\sum_{n=0}^{\infty}\frac{\sin^4 n}{n^2+1}$

39–42 Use the Limit Comparison Test to determine whether the series converges or diverges.

39. $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{\sqrt{n^5+1}}$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{1+\ln n}}{n^2}$

41. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^3+n}}{n^{5/2}}$

42. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^4+1}}$

43–50 Use the Ratio or Root Test, as appropriate, to determine whether the series converges or diverges.

43. $\displaystyle\sum_{n=1}^{\infty}\frac{n^7}{7^n}$

44. $\displaystyle\sum_{n=5}^{\infty}\frac{5^n}{(n-5)!}$

45. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n\cdot n!}$

46. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n(n!)}{(2n-1)!}$

47. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{4^n+1}$

48. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{2^{n+1}}$

49. $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^{2n}}{n^{n+1}}$

50. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)^n$

51–54 Determine whether the alternating series converges and give a reason for your answer.

51. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n^2}$

52. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{e^n\sqrt{n}}$

53. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n-1}{5n+1}$

54. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n^5}$

55–56 Approximate the sum of the alternating series, accurate to at least the indicated number of decimal places. How many terms did you use? According to the n^{th}-remainder estimate for the error (see Section 10.6), how many terms guarantee the indicated accuracy? (Note that the n^{th}-remainder estimate is not necessarily "sharp".)

55. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$; accurate to 2 decimal places

56. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}$; accurate to 3 decimal places

57–62 Determine whether the given series converges absolutely, converges conditionally, or diverges.

57. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+1}$

58. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$

59. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^{3/2}}$

60. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^6}$

61. $\displaystyle\sum_{n=3}^{\infty}\left(\frac{2-n}{4n}\right)^n$

62. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^{n+1}n}{4n-2}$

63–68 Determine the interval of convergence for the given power series.

63. $\displaystyle\sum_{n=0}^{\infty}2^n x^n$

64. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^3+2}$

65. $\displaystyle\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{n!(x-1)^n}{n^4}$

67. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+1)^n}{(n+1)3^n}$

68. $\displaystyle\sum_{n=2}^{\infty}\frac{(2x-6)^n}{\sqrt{n}\ln n}$

69–70 Determine the interval of convergence for the series, and the limiting function of the series on that interval.

69. $\displaystyle\sum_{n=0}^{\infty}(2x-3)^n$

70. $\displaystyle\sum_{n=0}^{\infty}\frac{(3x-1)^n}{2^n}$

71–72 Find the power series expansion for the given function about 0. What is the radius of convergence? (**Hint:** Use the same approach as in Exercises 41–48 of Section 10.7.)

71. $f(x)=\dfrac{1}{1+3x}$

72. $f(x)=\dfrac{1}{2-4x^2}$

73. Find the power series representation of $f(x)=1/x$, centered at $a=2$. Differentiating twice, find the series for $g(x)=1/x^3$ around $a=2$. What is the radius of convergence? (**Hint:** Start by rewriting $1/x$ as $\dfrac{1}{2-(2-x)}$.)

74–81 Determine the Taylor series (or Maclaurin series if the center is not specified) of the given function about the indicated point. Find the radius of convergence. (**Hint:** Do not use the definition.)

74. $f(x)=e^{x^4/2}$

75. $f(x)=\cos(2x^3)$

76. $f(x)=\dfrac{1}{x+3};\quad a=2$

77. $f(x)=\ln x;\quad a=3$

78. $f(x)=\sin\dfrac{x}{2}$

79. $f(x)=\sin^2 x$
(**Hint:** Use a trigonometric indentity.)

80. $f(x)=x^2e^{3x^2}$

81. $f(x)=\dfrac{x}{x+2};\quad a=1$

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82–83 Use the definition to find the first four nonzero terms of the Taylor series generated by the given function about the indicated point.

82. $f(x) = \cot x; \quad a = \pi/2$

83. $f(x) = \csc x; \quad a = \pi/4$

84–86 Find the first five nonzero terms of the Maclaurin series generated by the given function by using operations on familiar series (try not to use the definition).

84. $f(x) = xe^x - \sin x$ **85.** $f(x) = \dfrac{\cos x}{x+1}$

86. $f(x) = \cos(\sin x)$ (**Hint:** Substitute the series of $\sin x$ into that of $\cos x$.)

87–89 Use Taylor series to approximate the given function value or definite integral to within the indicated accuracy. How many (nonzero) terms did you need?

87. $\cos 1; \quad$ error $\le 10^{-6}$

88. $\displaystyle\int_0^1 e^{-x^3}\,dx; \quad$ error $\le 10^{-4}$

89. $\sqrt[4]{1.2}; \quad$ error $\le 10^{-5}$ (**Hint:** Use a binomial series.)

90. Use series expansion to verify the trigonometric limit $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = 0.$

91–92 Use Taylor series to find the indicated limit.

91. $\displaystyle\lim_{x\to 0}\frac{e^{-x}-1}{x}$ **92.** $\displaystyle\lim_{x\to 0}\frac{x\sin 2x}{1-e^{x^2}}$

93. Using an appropriate power series, find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(n/2^n\right)$. (**Hint:** Start with the familiar series expansion of $1/(1-x)$, and use termwise differentiation in its interval of convergence; then make an appropriate substitution.)

94. Using an appropriate Taylor series, verify that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2.$

95–98 Find the first six nonzero terms of the binomial series expansion of the indicated function.

95. $f(x) = \dfrac{1}{\sqrt{x^2+1}}$ **96.** $f(x) = \dfrac{1}{\sqrt[3]{x+1}}$

97. $f(x) = \sqrt{2x^3+1}$ **98.** $f(x) = \left(1-\dfrac{x}{2}\right)^{4/3}$

99–100 Find a power series solution of the equation. In each case, use the initial conditions $y(0) = a$ and $y'(0) = b$.

99. $y'' + 4y = 0$ **100.** $y'' - \dfrac{2}{(1-x)}y' = 0$

101. Find a power series solution of the differential equation $y' + y = 0$ satisfying the initial condition $y(0) = 2$. Then solve the equation by traditional means and conclude that the solutions are equal.

102. Use the Maclaurin series you obtained in Exercise 86 to find $f^{(6)}(0)$.

103.* Prove that if every bounded monotonic sequence converges, then the Completeness Property of real numbers holds. (**Hint:** Let S be a set that is bounded above by M. Let $a_0 = M$ and for each $n > 0$, let s_n be the least positive integer such that $s_n M/2^n$ is an upper bound for S. Letting $a_n = s_n M/2^n$, proceed to show that a_n converges and its limit is the least upper bound of S. The existence of the greatest lower bound can be shown analogously. Note that this statement is the converse of Exercise 74 in Section 10.1, showing that the Bounded Monotonic Sequence Theorem is equivalent to the Completeness Property of $\mathbb{R}$.)

104. Prove that the alternating series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln(n^p)}{n}$ converges for all values of p.

105. Let $\displaystyle\sum_{n=1}^{\infty}a_n$ be a divergent series and assume that $\displaystyle\sum_{n=1}^{\infty}b_n$ is convergent. Prove that $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n)$ is divergent.

106. Suppose that $\displaystyle\sum_{n=1}^{\infty}a_n$ is divergent and $k \ne 0$. Prove that $\displaystyle\sum_{n=1}^{\infty}ka_n$ is also divergent.

107. Decide whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+2+\cdots+n}$ converges or diverges. Prove your answer.

108. If a_n and b_n are positive sequences such that $\displaystyle\sum_{n=1}^{\infty}a_n$ is convergent and b_n is a null sequence, prove that $\displaystyle\sum_{n=1}^{\infty}a_n b_n$ is convergent.

109. Let $a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$. Show that $a_n \to 0$, yet $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ diverges. Why does this not contradict Leibniz's Test?

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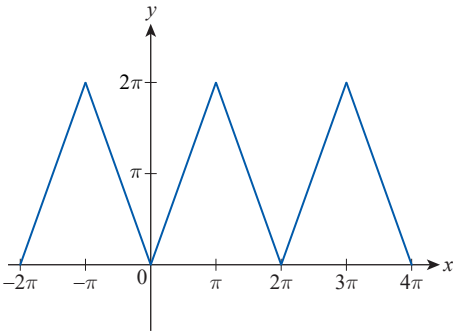
110.* Use Taylor's formula to provide a proof of the Second Derivative Test as follows. Assuming that $f'(c)=0$, use Taylor's formula to conclude that $f(x)=f(c)+\frac{1}{2}f''(a)(x-c)^2$, for some a between x and c. Then examine the signs of $f(x)-f(c)$ and $f''(a)$. Next, assuming $f'(c)=f''(c)=0$ and $f'''(c)\neq 0$, argue that $f(c)$ is neither a relative maximum nor a minimum. (Assume initially that f is continuosly differentiable through at least the third order; then think about whether you can relax this condition.)

111. Find a second solution to Exercise 71 using long division.

112. Find the Fourier series expansion of the 2π-periodic extension of the function graphed below.



Concept Check

113–124 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

113. If a_n is monotonically decreasing, then $\lim_{n\to\infty} a_n = -\infty$.

114. If $\{a_n\}$ is convergent, then $\{a_n/n\}$ is a null sequence.

115. If $\{a_n/n\}$ is a null sequence, then $\{a_n\}$ is convergent.

116. If $\{a_n\}$ is convergent, then $\{a_{n+1}-a_n\}$ is a null sequence.

117. If $\{a_n\}$ is monotonically decreasing to zero, then $\sum_{n=1}^{\infty}(-1)^n a_n$ is absolutely convergent.

118. If $\{a_n\}$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

119. If $\sum_{n=1}^{\infty}|a_n|$ is divergent, then either $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty}(-a_n)$ is divergent.

120. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty}|a_n|$ is divergent.

121. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

122. If the power series $\sum_{n=1}^{\infty} a_n x^n$ diverges at $x=c$, then it diverges at $x=-c$.

123. All power series converge at infinitely many x-values.

124. There is a power series whose convergence set is empty.

Chapter 10
Technology Exercises

125–127 Use a graphing utility to solve the problem.

125. We already know that the harmonic series diverges to infinity, and that it does so at a very slow pace. In this exercise, we will examine this series a bit further.

a. Find out how many terms are needed for the partial sum of the harmonic series to exceed 12.

b. What is the sum of the first 2 million terms? (Compare with Example 4 of Section 10.3. Notice that this calculation takes a bit of time even for today's powerful technology!)

126. The simple series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ was shown by Gregory and Leibniz to converge to $\pi/4$ (hence its name, the *Gregory series*). However, it converges rather slowly. Find out how many terms of this series are necessary to approximate π accurate to two decimal places.

127. **a.** Graph $y=\sin x$ and its 11^{th}-order Maclaurin polynomial on the same screen, over the interval $[-4\pi, 4\pi]$. Visually estimate the subinterval over which you find the approximation acceptable.

b. Repeat part a. with the 21^{st}-order Maclaurin polynomial.

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