Chapter 10 Project

In this project, we are going to expand on our earlier work with the harmonic series. In the process, we will meet a famous constant called *Euler's constant*, also known as the *Euler-Mascheroni constant*. (This number is not to be confused with $e \approx 2.71828$, the natural base, which is also known as Euler's number.)

1. As in Example 6 of Section 10.2, we let s_n stand for the n^{th} partial sum of the harmonic series; that is,

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(The partial sum s_n is also called the n^{th} *harmonic number*.) For each $n \geq 1$, we define

$$d_n = s_n - \ln n.$$

Prove that $d_n > 0$ for any positive integer n. (**Hint:** Refer to the illustration provided for Exercise 65 of Section 10.2, and start by comparing s_n with $\int_1^{n+1} (1/x)\,dx$.)

2. Prove that $\{d_n\}$ is a decreasing sequence. (**Hint:** Referring again to the figure from Exercise 65 of Section 10.2, fix an n and identify a region whose area is $d_n - d_{n+1}$.)

3. Use an appropriate theorem from the text to show that the sequence $\{d_n\}$ is convergent. Letting $\gamma = \lim_{n\to\infty} d_n$, this limit is called **Euler's constant**. It is important in many applications throughout various areas of mathematics, and like other famous constants (including π and e) can be approximated with great precision using modern computing power. Surprisingly, however, it is not yet known whether γ is rational or irrational!

4. Use the convergence of $\{d_n\}$ to prove that the sequence $a_n = \sum_{i=n}^{2n} \frac{1}{i}$ converges and find its limit.

5. Use a computer algebra system to approximate γ, accurate to the first 10 decimal places.

6. Use the approximate value of γ found in Question 5 to estimate s_n, rounded to 5 decimal places, for **a.** $n = 10{,}000$ and **b.** $n = 2{,}000{,}000$. Compare the latter estimate with the answer for Exercise 125b of the Chapter Review.