Chapter 9 Project

In this project, you will be introduced to a class of parametric curves called *Bézier curves*. They are important for their applications in engineering, computer graphics, and animation. This class of curves is named after Pierre Bézier (1910–1999), a design engineer for the French automaker Renault, who first demonstrated these curves' use in designing automobile bodies in the 1960s. The design advantage of Bézier curves lies in the fact that they can easily be manipulated by moving around their so-called *control points*. In addition, it is easy to smoothly join together several Bézier curves for more complicated shapes.

1. The linear Bézier curve $B_{0,1}(t)$ from $P_0(a_0, b_0)$ to $P_1(a_1, b_1)$ is simply the line segment connecting the two points (note that P_0 and P_1 are the only control points in this case). Verify that this curve can be parametrized as

$$B_{0,1}(t) = (1-t)P_0 + tP_1,\ t \in [0,1],$$

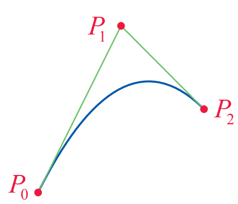
and find $x(t)$ and $y(t)$ corresponding to this parametrization. (In this and subsequent questions, control points will be labeled $P_i(a_i, b_i)$, $0 \le i \le 3$.)

2. The Bézier curve $B_{0,1,2}(t)$ with control points P_0, P_1, and P_2 is a quadratic curve joining the points P_0 and P_2 in such a way that both line segments $\overline{P_0P_1}$ and $\overline{P_1P_2}$ are tangent to $B_{0,1,2}(t)$. Intuitively speaking, this means that the curve "starts out at P_0 in the direction of P_1," and "arrives at P_2 from the direction of P_1" (see figure).

Find $x(t)$ and $y(t)$ corresponding to the parametrization

$$B_{0,1,2}(t) = (1-t)B_{0,1}(t) + tB_{1,2}(t),\ \ t \in [0,1]$$

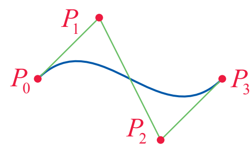
and verify that $B_{0,1,2}(t)$ satisfies the conditions stated above.



A Quadratic Bézier Curve

3. The cubic Bézier curve $B_{0,1,2,3}(t)$ with control points P_0, P_1, P_2, and P_3 joins P_0 and P_3 so that the line segments $\overline{P_0P_1}$ and $\overline{P_2P_3}$ are tangent to $B_{0,1,2,3}(t)$ at P_0 and P_3, respectively (see figure). Verify that the following curve satisfies these conditions:

$$x(t) = a_0(1-t)^3 + 3a_1(1-t)^2 t + 3a_2(1-t)t^2 + a_3t^3$$
$$y(t) = b_0(1-t)^3 + 3b_1(1-t)^2 t + 3b_2(1-t)t^2 + b_3t^3,\ t \in [0,1]$$



A Cubic Bézier Curve

4. Show that the parametrization in Question 3 corresponds to

$$B_{0,1,2,3}(t) = (1-t)B_{0,1,2}(t) + tB_{1,2,3}(t).$$

5. Use Question 3 to verify that the Bézier curve with control points $P_0(1,3)$, $P_1(3,7)$, $P_2(6,9)$, and $P_3(8,6)$ has the following parametrization:

$$x(t) = -2t^3 + 3t^2 + 6t + 1$$
$$y(t) = -3t^3 - 6t^2 + 12t + 3$$

6. Find the slope of the curve in Question 5 at **a.** $t = 0$, **b.** $t = \frac{1}{2}$, and **c.** $t = 1$.

7. Use a computer algebra system to graph the Bézier curve of Question 5 along with its control points. If your CAS has animation capabilities, explore what happens if you move around the control points in the plane.