Chapter 4 Project

Consider a function $f(x)$ that is at least twice differentiable. In this project, you will show that the second derivative of $f(x)$ at $x = c$ can be found as the limit of so-called **second-order differences**:

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

1. Instead of working with a secant line through the points $(c, f(c))$ and $(c+h, f(c+h))$ like we did when approximating the first derivative, suppose that

   $$y = a_1 x^2 + a_2 x + a_3$$

   is the parabola through the following three points on the graph of f: $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$. Do you expect to always be able to find coefficients $a_1, a_2, a_3 \in \mathbb{R}$ such that the resulting parabola satisfies the desired conditions? Why or why not? Why would you expect $2a_1$ to be "close" to $f''(c)$ if h is "small"? What will happen to $2a_1$ as $h \to 0$? Write a short paragraph answering the above questions.

2. By substituting the points $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$ into $y = a_1 x^2 + a_2 x + a_3$, obtain a system of linear equations in unknowns a_1, a_2, and a_3. Solve the system for the unknown a_1.

3. Use Questions 1 and 2 to argue that $f''(c)$ is the limit of the second-order differences:

   $$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

4. Use l'Hôpital's Rule to verify the result you found in Question 3.