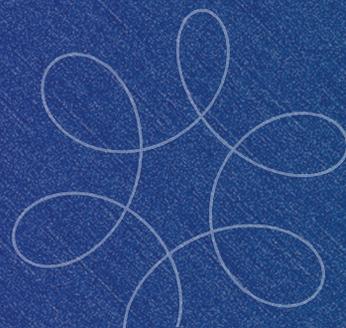# Chapter 1 Project 📝

As time goes on, there is increasing awareness, controversy, and legislation regarding the ozone layer and other environmental issues. The hole in the ozone layer over the South Pole disappears and reappears in a cyclical manner annually. Suppose that over a particular stretch of time the hole is assumed to be circular with a radius growing at a constant rate of 2.6 kilometers per hour.
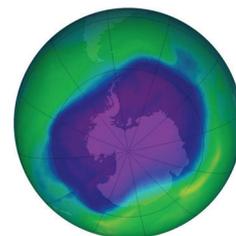
Photo Courtesy of NASA

1. Assuming that t is measured in hours, that $t = 0$ corresponds to the start of the annual growth of the hole, and that the radius of the hole is initially 0, write the radius as a function of time, t. Denote this function by $r(t)$.

2. Use function composition to write the area of the hole as a function of time, t. Denote this function by $A(t)$. Sketch the graph of $A(t)$ and label the axes appropriately.

3. After finding $A(1)$, the area of the ozone hole at the end of the first hour, determine the time necessary for this area to double. How much additional time does it take to reach three times the initial area?

4. Are the two time intervals you found in Question 3 equal? If not, which one is greater? Explain your finding. (Use a comparison of some basic functions discussed in Section 1.2 in your explanation.)

5. What are the radius and area after 3 hours? After 5.5 hours?

6. What is the average rate of change of the area from 3 hours to 5.5 hours?

7. What is the average rate of change of the area from 5.5 hours to 8 hours?

8. Is the average rate of change of the area increasing or decreasing as time passes?

9. What flaws do you see with this model? Can you think of a better approach to modeling the growth of the ozone hole?

Chapter 2 Project 📝
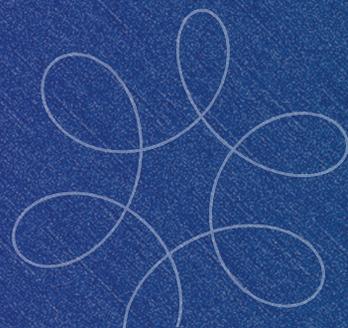
Some years ago, it was common for long-distance phone companies to charge their customers in one-minute increments. In other words, the company charges a flat fee for the first minute of a call and another fee for each additional minute or any fraction thereof (see Exercise 82 in Section 2.5). In this project, we will explore in detail a function that gives the cost of a telephone call under the above conditions.

1. Suppose a long-distance call costs 75 cents for the first minute plus 50 cents for each additional minute or any fraction thereof. In a coordinate system where the horizontal axis represents time t and the vertical axis price p, draw the graph of the function $p = C(t)$ that gives the cost (in dollars) of a telephone call lasting t minutes, $0 < t \le 5$.

2. Does $\lim_{t \to 1.5} C(t)$ exist? If so, find its value.

3. Does $\lim_{t \to 3} C(t)$ exist? Explain.

4. Write a short paragraph on the continuity of this function. Classify all discontinuities; mention one-sided limits and left or right continuity where applicable.

5. In layman's terms, interpret $\lim_{t \to 2.5} C(t)$.

6. In layman's terms, interpret $\lim_{t \to 3^-} C(t)$.

7. In layman's terms, interpret $\lim_{t \to 3^+} C(t)$.

8. If possible, find $C'(3.5)$.

9. If possible, find $C'(4)$.

10. Find and graph another real-life function whose behavior is similar to that of $C(t)$. Label the axes appropriately and provide a brief description of your function.

Chapter 3 Project 📝
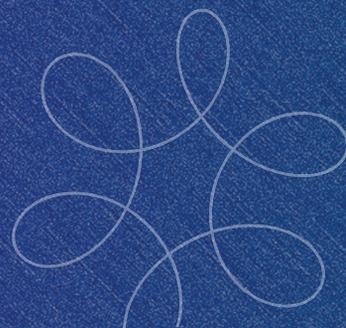
The following table shows the atmospheric pressure p at the altitude of k feet above sea level (pressure is measured in mm Hg; note that this unit of pressure is approximately the pressure generated by a column of mercury 1 millimeter high).

k (ft)	0	1000	2000	3000	4000	5000	6000	7000	8000	9000	10,000
p (mm Hg)	760	733	707	681	656	632	609	586	564	543	523

1. Find the average rate of change of air pressure from sea level to 2000 feet of altitude.

2. Find the average rate of change of air pressure between the altitudes of 4000 and 10,000 feet.

3. Use a *symmetric difference quotient*

$$\frac{p(c+h)-p(c-h)}{2h}$$

   to estimate the instantaneous rate of change of air pressure at 7000 ft by choosing $h = 1000$ ft.

4. Tell whether you expect the answer to Question 2 or 3 to better approximate the instantaneous rate of change of air pressure at altitude 7000 ft. Explain. (**Hint:** Plotting the data on paper may help.)

5.* Explain why you expect the symmetric difference quotient $\dfrac{f(c+h)-f(c-h)}{2h}$ in general to be a better approximation of the instantaneous rate of change of f at $x = c$ than the "regular" difference quotient $\dfrac{f(c+h)-f(c)}{h}$.

6. Use a graphing calculator or computer algebra system to find an exponential regression curve to the given data and plot the curve along with the data on the same screen.

7. Use the exponential function you found in Question 6 to estimate the instantaneous rate of change of air pressure at 7000 ft, and compare with your estimate given in Question 3.

8. Is the instantaneous rate of change increasing or decreasing with altitude? Explain.

Chapter 4 Project ✎
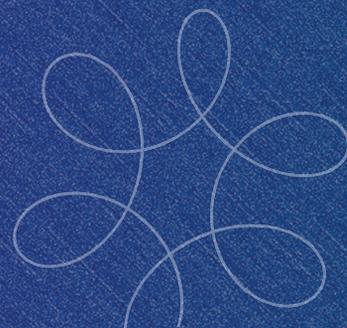
Consider a function $f(x)$ that is at least twice differentiable. In this project, you will show that the second derivative of $f(x)$ at $x = c$ can be found as the limit of so-called **second-order differences**:

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

1. Instead of working with a secant line through the points $(c, f(c))$ and $(c+h, f(c+h))$ like we did when approximating the first derivative, suppose that

$$y = a_1 x^2 + a_2 x + a_3$$

is the parabola through the following three points on the graph of f: $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$. Do you expect to always be able to find coefficients $a_1, a_2, a_3 \in \mathbb{R}$ such that the resulting parabola satisfies the desired conditions? Why or why not? Why would you expect $2a_1$ to be "close" to $f''(c)$ if h is "small"? What will happen to $2a_1$ as $h \to 0$? Write a short paragraph answering the above questions.
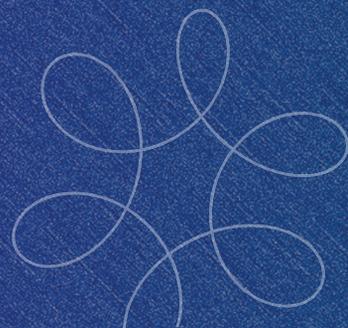
2. By substituting the points $(c-h, f(c-h))$, $(c, f(c))$, and $(c+h, f(c+h))$ into $y = a_1 x^2 + a_2 x + a_3$, obtain a system of linear equations in unknowns a_1, a_2, and a_3. Solve the system for the unknown a_1.

3. Use Questions 1 and 2 to argue that $f''(c)$ is the limit of the second-order differences:

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$$

4. Use l'Hôpital's Rule to verify the result you found in Question 3.

Chapter 5 Project ✎

The topic of this project is the so-called *sine integral function*, which is important for its applications, most notably in electrical engineering and signal processing.

1. Consider the following piecewise defined function:

$$f(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases}$$

    Prove that for any $x \geq 0$, $f(t)$ is integrable on $[0, x]$.

2. The **sine integral function** is defined as follows:

$$\mathrm{Si}(x) = \int_0^x f(t)\,dt, \text{ for } x \geq 0$$

    Prove that $\mathrm{Si}(x)$ is continuous.

3. Find the derivative $\dfrac{d}{dx}\mathrm{Si}(x)$.

4. Without graphing first, write a short paragraph on why you would expect the graph of $\mathrm{Si}(x)$ to be oscillating. Explain why its amplitude is expected to decrease as $x \to \infty$.

5. Find the x-values where the relative maxima and minima of $\mathrm{Si}(x)$ occur.

6. Extend the definition of $\mathrm{Si}(x)$ to negative x-values and prove that for any $a > 0$, $\int_{-a}^{a} \mathrm{Si}(x)\,dx = 0$.

7. Use a computer algebra system to plot the graph of $\mathrm{Si}(x)$ on the interval $[-8\pi, 8\pi]$.

8. Use a computer algebra system to approximate the range of $y = \mathrm{Si}(x)$ to four decimal places.

Chapter 6 Project ✎
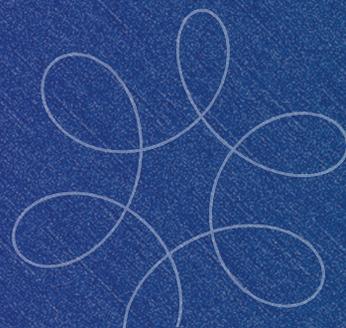
In this project, we will expand upon our explorations from Exercises 48 through 55 of Section 6.5 (also see the discussion preceding those exercises). In particular, we will determine an equation satisfied by the velocity of a projectile launched with initial velocity v_0, taking into consideration that acceleration caused by gravity decreases with altitude. (This is important when objects are launched to great altitudes.) We will then use our equation to find the maximum height attained by the projectile. This will lead us to the value of the so-called *escape velocity*, the velocity needed for an object to be able to overcome the Earth's gravitational field without further propulsion. (In turn, since gravity is conservative, this is the same velocity an object would achieve if pulled in by gravity from an "infinite distance.") We are ignoring all retarding forces (such as air resistance or friction) in this discussion.

1. Recall from Exercise 48 of Section 6.5 that g, the acceleration caused by gravity on a free-falling mass near the Earth's surface, is approximately

$$g = \frac{MG}{R^2},$$

where M and R are the mass and radius of the Earth, respectively, and G is the universal gravitational constant. However, a launched projectile's acceleration caused by gravity is negative (if we are assuming the positive direction is upwards) and actually depends on its height h above the Earth's surface. In particular, use Newton's Law of Gravitation to show that this dependence is given by the equation

$$a(h) = \frac{-gR^2}{(R+h)^2},$$

where $a(0) = -g$, as we would expect. (Actually, $a(h) \approx -g$ when h is negligible compared to the Earth's radius.)

2. Show that if $v = v(h)$ denotes the velocity of the projectile, then

$$\frac{d}{dh}(v^2) = 2\frac{dv}{dt}.$$

(**Hint:** Use the Chain Rule.)

3. Use the above results to show that

$$\frac{d(v^2)}{dh} = \frac{-2gR^2}{(R+h)^2}.$$

4. Integrating both sides of the previous equation with respect to h, show that $v = v(h)$ satisfies the equation

$$v^2 = v_0^2 - 2gR\left(1 - \frac{R}{R+h}\right).$$

(**Hint:** After integrating, use the fact that $v(0) = v_0$.)

5. Use the equation found in Question 4 to find the maximum height attained by the projectile. (**Hint:** Use the fact that $v = 0$ when the projectile reaches its maximum height.)

6. Find a formula for the escape velocity v_e of the projectile; then use the data found in the exercises of Section 6.5 (Exercises 48–55 and the preceding discussion) to express your answer in kilometers per second. (**Hint:** Use the fact that if $v_0 = v_e$, the projectile will "travel to infinity.")

7. Find the escape velocity of the projectile if it is launched on the Moon. (**Hint:** For Moon data, see Exercise 53 of Section 6.5.)

Chapter 7 Project ✎
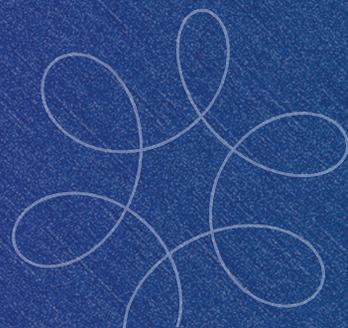
In this project, we will derive a famous infinite product named after its discoverer, the English mathematician John Wallis (1616–1703). Wallis introduced the symbol ∞ for infinity, and in turn he used $1/\infty$ to denote an *infinitesimal* quantity. He contributed to the development of *infinitesimal calculus* (it wasn't until the 19th century that infinitesimals were replaced by limits in the works of Bolzano, Cauchy, and Weierstrass).

1. For a nonnegative integer n, let

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx.$$

Find I_0, I_1, I_2, and I_3.

2. Show that if $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(**Hint:** See Exercise 81 of Section 7.1.)

3. Use Questions 1 and 2 to find I_4, I_5, I_6, and I_7.

4. Show that in general,

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

while

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \cdot \frac{2}{3}.$$

(**Hint:** Observe a pattern or use induction.)

5. Use Question 4 to show that

$$\frac{I_{2n}}{I_{2n+1}} = \frac{3^2 5^2 \cdots \cdot (2n-1)^2 (2n+1)}{2^2 4^2 \cdots \cdot (2n)^2} \cdot \frac{\pi}{2}$$

holds for all n.

6. Show that

$$\frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n}.$$

7. Prove the inequalities

$$I_{2n-1} \geq I_{2n} \geq I_{2n+1}.$$

(**Hint:** Use the definition of I_n from Question 1 and compare the integrands.)

8. Use Questions 6 and 7 to show that

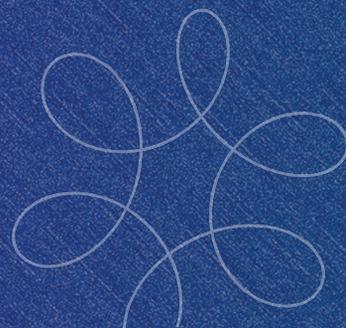$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n},$$

and use this observation to prove that

$$\lim_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

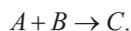9. Use your answers to the previous questions to derive **Wallis' product**:

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2^2 4^2 \cdots \cdot (2n)^2}{3^2 5^2 \cdots \cdot (2n-1)^2 (2n+1)}.$$

Chapter 8 Project ✎

Recall from Section 3.7 our discussion of a chemical reaction where reactants A and B produce a new product substance C, a process represented by

$$A + B \to C.$$

In this project, we will derive and use a differential equation that describes such a process.

1. Suppose that in the above reaction for each gram of reactant A, b grams of B are used to form C. If we start with initial amounts A_0 and B_0, respectively, and $X(t)$ denotes in grams the amount of substance C already formed at time t, find the amount remaining of reactants A and B at any time during the process.

2. Given that the rate of formation of substance C at any time is proportional to the product of the remaining amounts of reactants A and B, respectively, find a differential equation in terms of $X(t)$ that describes the process.

   (As in Question 1, let A_0 and B_0 stand for the initial amounts.)

3. Suppose a product substance C is being formed from reactant substances A and B and that for each gram of substance A, 3 grams of B are used to form C. As in Question 1, let $X(t)$ denote the amount of C formed at time t, and assume that the initial amounts of reactants A and B are $A_0 = 60$ grams and $B_0 = 40$ grams, respectively. Find the initial value problem describing this reaction. (**Hint:** Use your answer to Question 2.)

4. If 20 grams of the product compound forms during the first 5 minutes, use the model you obtained in Question 3 to predict how much of the product compound C is present 10 minutes into the process.

5. Use your model from Question 3 to predict what happens as $t \to \infty$. Interpret your answer.

Chapter 9 Project 📝
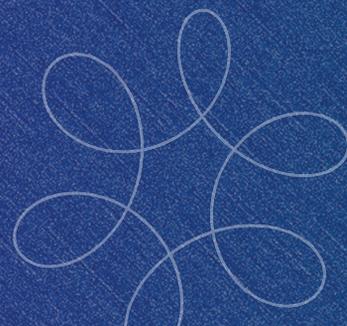
In this project, you will be introduced to a class of parametric curves called *Bézier curves*. They are important for their applications in engineering, computer graphics, and animation. This class of curves is named after Pierre Bézier (1910–1999), a design engineer for the French automaker Renault, who first demonstrated these curves' use in designing automobile bodies in the 1960s. The design advantage of Bézier curves lies in the fact that they can easily be manipulated by moving around their so-called *control points*. In addition, it is easy to smoothly join together several Bézier curves for more complicated shapes.

1. The linear Bézier curve $B_{0,1}(t)$ from $P_0(a_0, b_0)$ to $P_1(a_1, b_1)$ is simply the line segment connecting the two points (note that P_0 and P_1 are the only control points in this case). Verify that this curve can be parametrized as

$$B_{0,1}(t) = (1-t)P_0 + tP_1, \, t \in [0,1],$$

and find $x(t)$ and $y(t)$ corresponding to this parametrization. (In this and subsequent questions, control points will be labeled $P_i(a_i, b_i)$, $0 \le i \le 3$.)

2. The Bézier curve $B_{0,1,2}(t)$ with control points P_0, P_1, and P_2 is a quadratic curve joining the points P_0 and P_2 in such a way that both line segments $\overline{P_0 P_1}$ and $\overline{P_1 P_2}$ are tangent to $B_{0,1,2}(t)$. Intuitively speaking, this means that the curve "starts out at P_0 in the direction of P_1," and "arrives at P_2 from the direction of P_1" (see figure).

Find $x(t)$ and $y(t)$ corresponding to the parametrization

$$B_{0,1,2}(t) = (1-t)B_{0,1}(t) + tB_{1,2}(t), \, t \in [0,1]$$

and verify that $B_{0,1,2}(t)$ satisfies the conditions stated above.
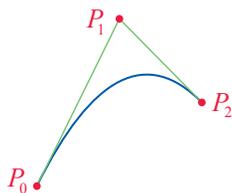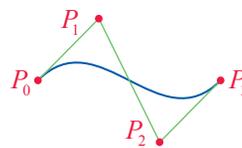
A Quadratic Bézier Curve

3. The cubic Bézier curve $B_{0,1,2,3}(t)$ with control points P_0, P_1, P_2, and P_3 joins P_0 and P_3 so that the line segments $\overline{P_0 P_1}$ and $\overline{P_2 P_3}$ are tangent to $B_{0,1,2,3}(t)$ at P_0 and P_3, respectively (see figure). Verify that the following curve satisfies these conditions:

$$x(t) = a_0(1-t)^3 + 3a_1(1-t)^2 t + 3a_2(1-t)t^2 + a_3 t^3$$
$$y(t) = b_0(1-t)^3 + 3b_1(1-t)^2 t + 3b_2(1-t)t^2 + b_3 t^3, \, t \in [0,1]$$

A Cubic Bézier Curve

4. Show that the parametrization in Question 3 corresponds to

$$B_{0,1,2,3}(t) = (1-t)B_{0,1,2}(t) + tB_{1,2,3}(t).$$

5. Use Question 3 to verify that the Bézier curve with control points $P_0(1,3)$, $P_1(3,7)$, $P_2(6,9)$, and $P_3(8,6)$ has the following parametrization:

$$x(t) = -2t^3 + 3t^2 + 6t + 1$$
$$y(t) = -3t^3 - 6t^2 + 12t + 3$$

6. Find the slope of the curve in Question 5 at **a.** $t = 0$, **b.** $t = \frac{1}{2}$, and **c.** $t = 1$.

7. Use a computer algebra system to graph the Bézier curve of Question 5 along with its control points. If your CAS has animation capabilities, explore what happens if you move around the control points in the plane.

Chapter 10 Project ✎
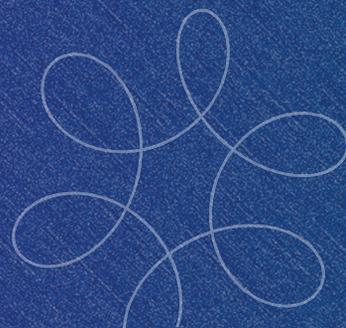
In this project, we are going to expand on our earlier work with the harmonic series. In the process, we will meet a famous constant called *Euler's constant*, also known as the *Euler-Mascheroni constant*. (This number is not to be confused with $e \approx 2.71828$, the natural base, which is also known as Euler's number.)

1. As in Example 6 of Section 10.2, we let s_n stand for the nth partial sum of the harmonic series; that is,

   $$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

   (The partial sum s_n is also called the nth *harmonic number*.) For each $n \geq 1$, we define

   $$d_n = s_n - \ln n.$$

   Prove that $d_n > 0$ for any positive integer n. (**Hint:** Refer to the illustration provided for Exercise 65 of Section 10.2, and start by comparing s_n with $\int_1^{n+1} (1/x)\,dx$.)

2. Prove that $\{d_n\}$ is a decreasing sequence. (**Hint:** Referring again to the figure from Exercise 65 of Section 10.2, fix an n and identify a region whose area is $d_n - d_{n+1}$.)

3. Use an appropriate theorem from the text to show that the sequence $\{d_n\}$ is convergent. Letting $\gamma = \lim\limits_{n \to \infty} d_n$, this limit is called **Euler's constant**. It is important in many applications throughout various areas of mathematics, and like other famous constants (including π and e) can be approximated with great precision using modern computing power. Surprisingly, however, it is not yet known whether γ is rational or irrational!

4. Use the convergence of $\{d_n\}$ to prove that the sequence $a_n = \sum\limits_{i=n}^{2n} \frac{1}{i}$ converges and find its limit.

5. Use a computer algebra system to approximate γ, accurate to the first 10 decimal places.

6. Use the approximate value of γ found in Question 5 to estimate s_n, rounded to 5 decimal places, for **a.** $n = 10{,}000$ and **b.** $n = 2{,}000{,}000$. Compare the latter estimate with the answer for Exercise 125b of the Chapter Review.