1.1 Exercises

1–22 Find the domain and range of the given relation.

1. $R = \{(-3,1), (-3,5), (-3,-1), (0,0), (1,2)\}$

2. $S = \{(3,-1), (2.6,6), (\pi,0.5), (e,100)\}$

3. $T = \{(4,5.98), (-2,-8), (-2,0), (3,\cos 3)\}$

4. $U = \{(4,4), (4,\pi), (\pi,4), (4,0)\}$

5. $F = \{(\text{Tanisha}, \text{swimming}), (\text{Don}, \text{biking}), (\text{Peter}, \text{skating}), (\text{David}, \text{skateboarding})\}$

6. $L = \{(\text{Lin}, \text{Chinese}), (\text{Chuck}, \text{English}), (\text{Sarah}, \text{German}), (\text{Daniel}, \text{Hungarian})\}$

7. $A = \{(x,y) \mid x \in \mathbb{Z},\ y = 2x+3\}$

8. $B = \left\{(x,y) \mid x \in \mathbb{R},\ y = \dfrac{x}{2}\right\}$

9. $C = \{(x,-2x+7) \mid x \in \mathbb{Z}\}$

10. $D = \{(2x,5y) \mid x \in \mathbb{N},\ y = x+1\}$

11. $3x = y + 5$

12. $\sqrt{2}x - 1.2y = 3$

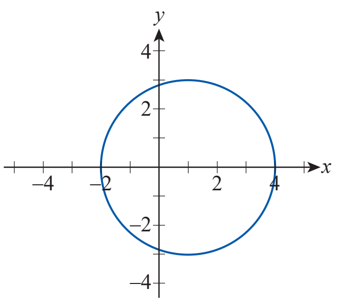
13. $x = 5$

14. $y = \pi$

15. $x = 3y^2 - 1$

16. $y = |x| - 2$

17.



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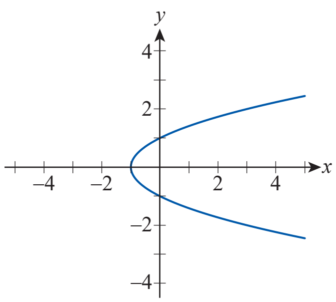

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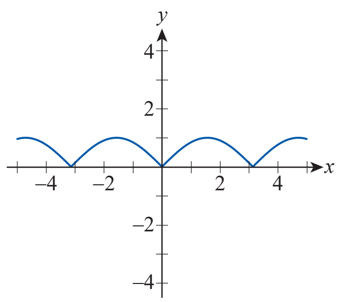

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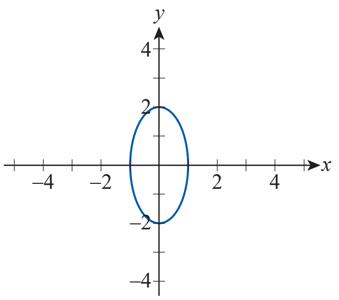

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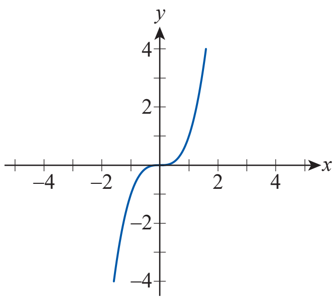

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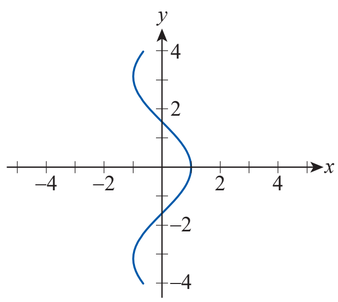


23–26 Find the domain and range of the given relation. Choose an appropriate domain on which the given relation makes sense. (Answers will vary.)

23. $\{(x,y) \mid \text{student } x \text{ is registered for course } y\}$

24. $\{(x,n) \mid x \text{ wears size } n \text{ shoes}\}$

25. $\{(x,y) \mid y \text{ is the father of } x\}$

26. $\{(P,n) \mid \text{person } P \text{ weighs } n \text{ pounds}\}$

27–30 List the ordered pairs in the given relation R on the set $A = \{1, 2, 3, 4, 5\}$.

27. $(a,b) \in R$ if and only if $a = b$

28. $(a,b) \in R$ if and only if $a < b$

29. $(a,b) \in R$ if and only if $a \mid b$

30. $(a,b) \in R$ if and only if $a + b = 5$

31–34 Determine whether the given relation is a function. If the relation is not a function, explain why.

31. $A = \{(1,3),(-2,4),(0,4)\}$

32. $B = \{(0,0),(0,1),(2,3),(4,5),(6,7)\}$

33. $C = \{(-1,2),(\pi,3),(-1,0),(1,2)\}$

34. $D = \{(1,2),(2,1),(3,4),(4,3)\}$

35–40. Determine whether the relations given in Exercises 17–22 are functions. For those that are not, explain why.

41–52 Determine whether the given equation is a function. If the equation is not a function, explain why.

41. $y = 3x - 4$ **42.** $x = 3y - 4$

43. $x^2 + y^2 = 9$ **44.** $x + y^2 = 9$

45. $x^2 + y = 9$ **46.** $y = \sqrt[3]{x}$

47. $x = x^3 - y$ **48.** $xy = 4$

49. $x = \pi$ **50.** $y = \dfrac{3x}{x^2+1}$

51. $F = 5r^2\pi$ **52.** $V = \dfrac{4}{3}r^3\pi$

53–58 Express y explicitly as a function of x from the given relation.

53. $\dfrac{x+3y}{2} = 5$ **54.** $\dfrac{x-3y}{5} = \dfrac{2y+7x}{3}$

55. $3x^2 - y = 5 - x + 2y$

56. $x + 7 - 3y = (x-2)^2 + y$

57. $yx^2 - y = 3x + 1$ **58.** $x + 1 = yx^2$

59–66 Find the value of the given function for **a.** $f(-2)$, **b.** $f(x+1)$, **c.** $f(x+h)$, and **d.** $\dfrac{f(x+h)-f(x)}{h}$.

59. $f(x) = \dfrac{1}{3}x + 2$ **60.** $f(x) = \dfrac{5x-3}{2}$

61. $f(x) = x^2 - 3$ **62.** $f(x) = 3x^2 - 5x + \dfrac{1}{2}$

63. $f(x) = \sqrt{x}$ **64.** $f(x) = \dfrac{1}{\sqrt{x+2}}$

65. $f(x) = \dfrac{1}{x+1}$ **66.** $f(x) = (x-1)^3 + 5$

67–72 Identify the domain, codomain, and range of the given function.

67. $f: \mathbb{N} \to \mathbb{N}, \quad f(x) = x + 1$

68. $g: \mathbb{N} \to \mathbb{Z}, \quad g(x) = 3x - 2$

69. $h: \mathbb{Z} \to \mathbb{Z}, \quad h(x) = x^2$

70. $F: \mathbb{R} \to \mathbb{R}, \quad F(x) = 2x^4 + 1$

71. $G: [0,\infty) \to \mathbb{R}, \quad G(x) = \sqrt{x}$

72. $H: \mathbb{Q}^+ \to \mathbb{Q}, \quad H(x) = \dfrac{1}{x}$ (Note that $\mathbb{Q}^+$ stands for the set of positive rational numbers.)

73–82 Find the implied domain of the given function.

73. $f(x) = \dfrac{x+1}{x^2 - x - 6}$ **74.** $g(x) = \sqrt{3x+2}$

75. $h(x) = \dfrac{2}{\sqrt{x^2 - 4x + 3}}$ **76.** $F(t) = \dfrac{1}{\sqrt{4 - t^2}}$

77. $G(s) = \sqrt{2-s} + \sqrt{s}$ **78.** $D(h) = \dfrac{\dfrac{1}{\sqrt{1+h}} - 1}{h}$

79. $R(x) = \dfrac{1}{|2x+3|}$ **80.** $H(z) = z^{3/2} - 2$

81. $F(\theta) = \dfrac{2}{1 - \cos\theta}$ **82.** $\varphi(x) = \dfrac{5}{\sin x - \dfrac{\sqrt{2}}{2}}$

83–88 Turn the formula into a function by finding the argument(s) of the function. Identify any functions of two variables.

83. $C = 2\pi r$

84. $V = \frac{4}{3}r^3\pi$

85. $C = \frac{5}{9}(F - 32)$

86. $A = 6a^2$

87. $V = \frac{1}{3}b^2h$

88. $E = \frac{1}{2}mv^2$

89–94 Use the vertical line test to decide whether y is a function of x.

89. $y^3 + 1 = x$

90. $2x^2 + 2y^2 = 18$

91. $y^2 + 1 = x$

92. $x = (y - 2)^2$

93. $x = y^3 - 2y$

94. $yx^2 = 1$

95–101 Find all open intervals of monotonicity (intervals where the function is increasing or decreasing) for the given function.

95. $f(x) = (x-1)^2$

96. $g(x) = 4x - x^2$

97. $h(x) = x^3 - 12x$

98. $k(x) = \frac{x^2}{x^2+1}$

99. $F(x) = |x-1|$

100. $G(x) = 2x + |3x-1|$

101. $H(x) = |x+1| + |x-2|$

102–110 Discuss the symmetry of the given equation. Give reasons. (**Hint:** See Example 9.)

102. $y = x^2 - 1$

103. $x = y^2 - 1$

104. $x^4 + y^4 = 5$

105. $|x| + |y| = 2$

106. $x - |y| = 2$

107. $xy = 2$

108. $y = \frac{2x^3 - x}{x^4 + x^2}$

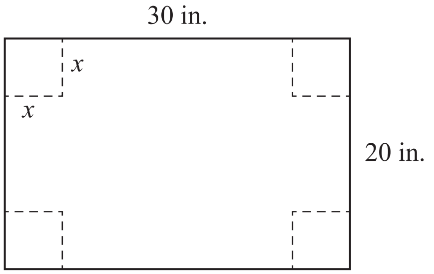
109. $y^2 + 6x = x^3$

110. $y = (x-1)^2$

111. Express the perimeter of a square as a function of its area.

112. Express the area of an equilateral triangle as a function of its perimeter.

113. An open-top box is constructed from a 20 in. by 30 in. piece of cardboard by cutting out a square of side length x from each of the four corners and folding up the sides, as shown in the figure below. Express the volume of the box as a function of x.



114. Express the surface area A of a cube as a function of its volume V.

115. The height of a circular cone is equal to the diameter of its base. Express its volume V as a function of the radius r of the base.

116. Express the volume of a sphere as a function of its surface area.

117. Knowing that water boils at 212 °F, which corresponds to 100 °C, and the fact that freezing occurs at 32 °F, which is 0 °C, obtain the linear function $C(F)$ that expresses the Celsius temperature C as a function of the Fahrenheit reading F.

118.* The organizers of an educational leadership seminar series have found that the seminar attracts 100 participants when the registration fee is set to $150. They estimate that for each increase of $10 in the registration fee, they will end up with 5 fewer registered participants. Express the revenue R as a function of the registration fee F.

```
In[1]:= f[x_] := Piecewise[{{x^2, x < 0}, {3 x + 1, 0 ≤ x < 2},
          {7 - Sqrt[x - 2], x ≥ 2}}]

In[2]:= f[-2]

Out[2]= 4

In[3]:= f[0]

Out[3]= 1

In[4]:= Plot[f[x], {x, -3, 3}]

Out[4]=
```

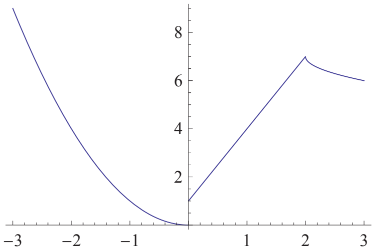


Figure 18

1.2 Exercises

1–8 Identify the degree, leading coefficient, intercepts, and range of the given polynomial function, and then graph the function.

1. $f(x)=\frac{1}{2}x-\frac{3}{2}$
2. $g(x)=-1.2x+4.8$
3. $h(x)=2x^2-3x-2$
4. $u(x)=\frac{1}{2}x^2+x-\frac{3}{2}$
5. $v(x)=x^3-7x+6$
6. $F(x)=10-8x+\frac{x^2}{2}+\frac{x^3}{2}$
7. $G(x)=\frac{x^4}{4}-2x^2$
8. $H(x)=2x^4+12x^3+2x^2-48x-40$

9–16 Find all asymptotes and intercepts of the given rational function and then sketch the graph of the function.

9. $f(x)=\frac{5}{x-1}$
10. $g(x)=\frac{x^2-4}{2x-x^2}$
11. $h(x)=\frac{x^2+3}{x+3}$
12. $u(x)=\frac{x+2}{x^2-9}$
13. $v(x)=\frac{x^2-2x-3}{2x^2-5x-3}$
14. $F(x)=\frac{3x^2+1}{x-2}$
15. $G(x)=\frac{x^2+2x}{x+1}$
16. $H(x)=\frac{x^3-27}{x^2+5}$

17–24 Construct the algebraic function in a finite number of steps. (Answers will vary.)

17. $f(x)=\frac{\sqrt{x^2-1}}{x+1}$
18. $g(x)=\sqrt[3]{\frac{x-1}{-2+x+x^2}}$
19. $h(x)=\sqrt{2x^2+x+1}+3x(2x+1)$
20. $u(x)=13x^3(2-x)+3\sqrt{x}-5x^2(2x-x^2)$
21. $v(x)=\sqrt{2}+\frac{x+3}{\sqrt[5]{2x^2+2x-12}}$
22. $F(x)=\frac{(x^3-4x^2-7x+10)^{2/3}}{\sqrt[5]{x-5}}$
23. $G(x)=\left(x+\left(x+\left(x+(x+1)^3\right)^3\right)^3\right)^3$
24. $H(x)=\sqrt{2x+\sqrt{2x+\sqrt{2x+\sqrt{2x}}}}$

25–34 Simplify the given trigonometric expression.

25. $\dfrac{1-\cos^2\left(\dfrac{\pi}{2}-x\right)}{\cos x}$

26. $\dfrac{1}{\sec^2 x}+\sin x\cos\left(\dfrac{\pi}{2}-x\right)$

27. $\sin\alpha(\csc\alpha-\sin\alpha)$

28. $\dfrac{1}{1+\cos\alpha}+\dfrac{1}{1-\cos\alpha}$

29. $\cot^2\theta-\cos^2\theta\cot^2\theta$

30. $\cos x(1+\tan^2 x)$

31. $\dfrac{\sin\beta}{1+\cos\beta}+\cot\beta$

32. $\dfrac{1}{\cos(-t)\csc(-t)}$

33. $\dfrac{1-\tan^2 x}{\cot^2 x-1}$

34. $\dfrac{\sin x\tan\left(\dfrac{\pi}{2}-x\right)}{\cos x}$

35–38 Graph the given piecewise-defined function. Use open or closed circles as appropriate at the endpoints of the intervals of definition.

35. $F(x)=\begin{cases}-x^2 & \text{if } x\le 0\\ \dfrac{1}{2}x+1 & \text{if } x>0\end{cases}$

36. $G(x)=\begin{cases}-2x-4 & \text{if } x\le -2\\ \dfrac{1}{2}x+\dfrac{3}{2} & \text{if } -2<x\le 1\\ \dfrac{1}{x-1} & \text{if } x>1\end{cases}$

37. $H(x)=\begin{cases}-x & \text{if } x\le 0\\ \sin x & \text{if } 0<x\le\dfrac{\pi}{2}\\ \sqrt[3]{x-\dfrac{\pi}{2}} & \text{if } x>\dfrac{\pi}{2}\end{cases}$

38. $u(x)=\begin{cases}-\sqrt{-x-1} & \text{if } x\le -1\\ \sqrt{1-x^2} & \text{if } -1<x\le 0\\ x^2 & \text{if } x>0\end{cases}$

39–42 Rewrite the given function as a piecewise-defined function, and then graph the function. Use open or closed circles as appropriate at the endpoints of the intervals of definition.

39. $f(x)=|x-1|$

40. $g(x)=\dfrac{x}{|x|}$

41. $h(x)=|\sin x|$

42. $v(x)=|x+2|+|x-3|$

43–45 The greatest integer function is defined as follows: For $x\in\mathbb{R}$, $[\![x]\!]$ is the greatest integer less than or equal to x. For example, $[\![\pi]\!]=3$, $[\![1]\!]=1$, $[\![-1.5]\!]=-2$, and so on.

Use the greatest integer function to sketch the graph of the given function.

43. $f(x)=x-[\![x]\!]$

44. $g(x)=[\![x]\!]-x$

45. $h(x)=[\![\sin x]\!]$

46–48 Simple polynomial functions are used to model real-life phenomena. (**Hint:** See Example 2 for guidance as you work through these problems.)

46. Suppose that while vacationing in Europe, one day you feel a bit dizzy and your host hands you a metric thermometer. Upon checking your temperature, the reading is 39.5 °C. Would you call the doctor? (**Hint:** Recall that the conversion formula between the Fahrenheit and Celsius scales is the linear function $C=\frac{5}{9}(F-32)$. Express F from this formula to answer the question.)

47. Two trains are 630 miles apart, heading directly toward each other. The first train is traveling at 95 mph, and the second train is traveling at 85 mph. How long will it be before the trains pass each other?

48. Jessica started a candle business a few weeks ago and noticed that the relationship between her total cost in producing the candles and the number of candles produced can be modeled by a linear function. She was able to make 3 candles for a total cost of $29, while 7 candles cost her a total of $41 to produce.

a. Find a formula for the total investment as a function of the number of candles produced.

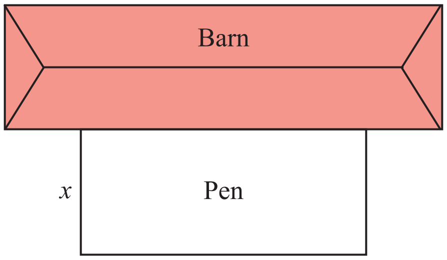
b. Graph the function found in part a. What are the practical meanings of the slope and y-intercept in this particular situation?

c. How much will be Jessica's total cost in producing 50 candles?

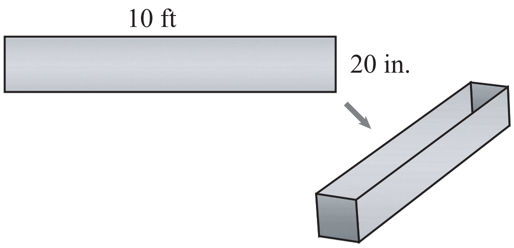
d. If Jessica plans to invest a total of $320 in the next 3 months, how many candles will she be able to produce?

49–60 Find a formula for the quantity to be optimized, and use the location of the vertex of its graph to solve the problem.

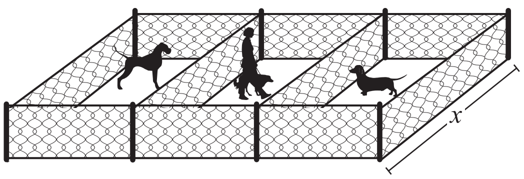
49. A farmer has a total of 200 yards of fencing to enclose a rectangular pen, so that one of the four sides will be the existing wall of a barn. What should the length and width be in order to maximize the enclosed area? (**Hint:** Let x represent the width, and find an expression for the length in terms of x. Then write an expression for the area and analyze the resulting function.)



50. A rancher has a rectangular piece of sheet metal that is 20 inches wide by 10 feet long. He plans to fold the metal into a three-sided channel and weld two rectangular pieces of metal to the ends to form a watering trough 10 feet long. How should he fold the metal in order to maximize the volume of the resulting trough?



51. Cindy wants to construct three rectangular dog-training arenas side by side using a total of 400 feet of fencing. What should the overall length and width be in order to maximize the area of the three combined arenas? (**Hint:** Let x represent the width, as shown, and find an expression for the overall length in terms of x.)



52. Among all the pairs of numbers with a sum of 10, find the pair whose product is maximum.

53. Find the point on the line $2x + y = 5$ that is closest to the origin. (**Hint:** Instead of trying to minimize the distance between the origin and points on the line, minimize the square of the distance.)

54. Among all the pairs of numbers (x, y) such that $2x + y = 20$, find the pair for which the sum of the squares is minimum.

55. Find a pair of numbers whose product is maximum when two times the first number plus the second number is 48.

56. The total revenue for Morris' Studio Apartments is given as the function $R(x) = 100x - 0.1x^2$, where x is the number of apartments rented. What is the number of apartments rented that produces the maximum revenue?

57. The total cost of manufacturing golf clubs is given as the function $C(x) = 800 - 10x + 0.20x^2$, where x is the number of sets of golf clubs produced. How many sets of golf clubs should be manufactured to incur minimum cost?

58. A rock is thrown upward with a velocity of 48 feet per second from the top of a 64-foot-high cliff. What is the maximum height attained by the rock? (**Hint:** Use $h(t) = -16t^2 + 48t + 64$ to describe the height of the rock as a function of time t.)

59. Jason is driving his Mustang GT down a two-lane highway one night, carefully observing the posted speed limit sign of 55 mph. His headlights suddenly illuminate a white-tailed deer, about 120 ft in front of his car, and he immediately hits the brakes. Suppose that the coefficient of friction between his car's tires and the pavement is $\mu = 0.9$. Using the quadratic model from Example 5b in Section 1.1, do you think he will hit the deer? What if he had traveled at 60 mph?

60. A student is throwing a small rubber ball during physical education class at an upward angle so that the horizontal component of the ball's initial velocity is 40 feet per second. If the vertical position function of the ball is given by $h(t) = -16t^2 + 24t + 7$, how far from the student will the ball hit the ground? (**Hint:** First determine how long it will take for the ball to hit the ground. The vertical position h is measured in feet, t in seconds. Ignore air resistance.)

61–72 Trigonometric and exponential functions are used to model real-life situations. (**Hint:** See Examples 6 and 7 for guidance as you work through these problems.)

61. Suppose several potatoes are dumped into the basket of a grocer's scale, which then proceeds to bounce up and down with an amplitude of 4 cm. As discussed in Example 6a, a first approximation to this motion can be given by a trigonometric model. Supposing that the constant ω for the above motion is 6π and that $t = 0$ when the potatoes land in the basket, find the position function for this motion. How long does it take for the basket to complete a full period?

62. The size of a local coyote population in a certain California national forest is estimated to cycle annually according to the function $P(t) = 250 + 20\sin(\pi t/6)$, where t is measured in months, starting on March 1st of each year.

a. What is the approximate size of the population on July 1st?

b. When is the population expected to be the smallest, and what is its size then?

63. A certain species of fish is to be introduced into a new man-made lake, and wildlife experts estimate that the population will grow according to $P(t) = (1000)2^{t/3}$, where t represents the number of years from the time of introduction.

a. What is the doubling time for this population of fish?

b. How long will it take for the population to reach 8000 fish, according to this model?

64. Assuming a current world population of 8 billion people, and exponential growth at an annual rate of 0.9%, what will the world population be in **a.** 10 years and **b.** 50 years?

65. Suppose that a new virus has broken out in an isolated region, and it is spreading exponentially through the villages. The growth of this new virus can be mapped using the following formula where P stands for the number of people in a village, V for the number of infected individuals, and d for the number of days since the virus first appeared.

$$V = P\left(1 - e^{-0.18d}\right)$$

According to this equation, how many people in a village of 300 will be infected after 5 days?

66. The radioactive element polonium-210 decays according to the function $A(t) = A_0 e^{-0.004951t}$, where A_0 is the mass at time $t = 0$, and t is measured in days. The fact that $A(140) = A_0/2$ means that the half-life of polonium-210 is 140 days. What percentage of the original mass of a sample of polonium-210 remains after one year?

67. The half-life of a radioactive material is the time required for an initial quantity to decrease to half its original value. In the case of radium, this is approximately 1600 years.

a. Determine a so that $A(t) = A_0 a^t$ describes the amount of radium after t years, where A_0 is the initial amount at $t = 0$.

b. How much of a 1-gram sample of radium would remain after 100 years?

c. How much of a 1-gram sample of radium would remain after 1000 years?

68. When continuous compounding is used in banking, the balance after t years is described by the formula $A(t) = Pe^{rt}$, where P is the initial amount (or principal) at $t = 0$, and r is the annual interest rate. Suppose Mario made a deposit two years ago, which is compounded continuously at an annual rate of 4.5%. If his current balance is \$1094.17, how much was his initial deposit? How much longer would he have to wait until his initial deposit doubles?

69. The function $f(t) = C(1+r)^t$ models the rise in the cost of a product that has a cost of C today, subject to an average yearly inflation rate of r for t years. If the average annual rate of inflation over the next decade is assumed to be 3%, what will the inflation-adjusted cost of a \$150,000 house be in 10 years?

70. The concentration of a certain drug in the bloodstream after t minutes is given by the formula $C(t) = 0.05\left(1 - e^{-0.2t}\right)$. What is the concentration after 10 minutes?

71. Carbon-11 has a radioactive half-life of approximately 20 minutes, and is useful as a diagnostic tool in certain medical applications. Because of the relatively short half-life, time is a crucial factor when conducting experiments with this element.

a. Determine a so that the formula $A(t) = A_0 a^t$ describes the amount of carbon-11 left after t minutes, where A_0 is the amount at time $t = 0$.

b. How much of a 2-kilogram sample of carbon-11 would be left after 30 minutes?

c. How much of a 2-kilogram sample of carbon-11 would be left after 6 hours?

72. Charles has recently inherited $8000, which he wants to deposit in a savings account. He has determined that his two best bets are an account that compounds annually at a rate of 3.20% and an account that compounds continuously at an annual rate of 3.15%. Which account would pay Charles more interest?

Concept Check

73–82 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

73. The slope of the graph of $y = Ax + B$ is A.

74. The slope of the graph of $y = Ax^2 + Bx + C$ is B.

75. The lines with equations $y = Ax + B$ and $y = -Bx + A$ are perpendicular to each other.

76. A quadratic function can have up to two y-intercepts.

77. If line L_1 has positive slope and L_2 is perpendicular to L_1, then the slope of L_2 is negative.

78. If a polynomial has even degree, then its graph always rises to both the right and the left.

79. All rational functions of the form $p(x)/q(x)$, where $q(x)$ is nonconstant, have at least one asymptote of some kind.

80. Trigonometric functions are transcendental.

81. Logarithmic functions are transcendental.

82. If a population of bacteria grows without restriction from 1000 to 2000 in one hour, then it will grow to 3000 during the next hour.

0.9 and 1.1 in distance from the origin. The rules for coloring other complex numbers in the plane are as follows. Given an initial complex number z not on the gold ring, $f(z)$ is calculated. If the complex number $f(z)$ lies somewhere on the gold ring, the original number z is colored the deepest shade of green. If not, the iterate $f^2(z)$ is calculated. If this result lies in the gold ring, the original z is colored a bluish shade of green. If not, the process continues up to the 12th iterate $f^{12}(z)$, using a different color each time. If $f^{12}(z)$ lies in the gold ring, z is colored red, and if not the process halts and z is colored black.

The idea of recursion can be used to generate any number of similar images, with the end result usually striking and often surprising even to the creator.

1.3 Exercises

1–23 Sketch the graph of the given function by first identifying the more basic function that has been shifted, reflected, stretched, or compressed. Then determine the domain and range of the function.

1. $f(x)=(x+2)^3$ **2.** $G(x)=|x-4|$ **3.** $p(x)=-(x+1)^2+2$ **4.** $g(x)=\sqrt{x+3}-1$

5. $q(x)=(1-x)^2$ **6.** $r(x)=-\sqrt[3]{x}$ **7.** $s(x)=\sqrt{2-x}$ **8.** $F(x)=\frac{|x+2|}{3}+3$

9. $w(x)=\frac{1}{(x-3)^2}$ **10.** $v(x)=\frac{1}{3x}-2$ **11.** $f(x)=\frac{1}{2-x}$ **12.** $k(x)=\sqrt{-x}+2$

13. $b(x)=[\![x-4]\!]+4$ **14.** $R(x)=4-|2x|$ **15.** $S(x)=(3-x)^3$ **16.** $g(x)=-\frac{1}{x+1}$

17. $h(x)=\frac{x^2}{2}-3$ **18.** $W(x)=1-|4-x|$

19. $g(x)=x^2-6x+9$ (**Hint:** Find a better way to write the function.)

20. $h(x)=\frac{|x|}{x}$ (**Hint:** Evaluate h at a few points to understand its behavior.)

21. $W(x)=\frac{x-1}{|x-1|}$ **22.** $S(x)=[\![x-2]\!]$ **23.** $V(x)=-3\sqrt{x-1}+2$

24–29 Write an equation for the function described.

24. Use the function $f(x)=x^2$. Move the function 4 units to the right and 2 units up.

25. Use the function $f(x)=x^2$. Reflect the function across the x-axis and move it 6 units up.

26. Use the function $f(x)=x^3$. Move the function 1 unit to the left and reflect across the y-axis.

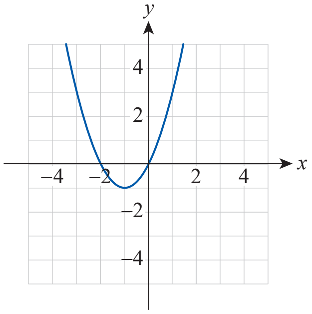
27. Use the function $f(x)=\sqrt{x}$. Move the function 5 units to the left and reflect across the x-axis.

28. Use the function $f(x)=\sqrt{x}$. Reflect the function across the y-axis and move it 3 units down.

29. Use the function $f(x)=|x|$. Move the function 7 units to the left, reflect across the x-axis, and reflect across the y-axis.

30–33 Use your knowledge about transformations to find a possible formula for the function $f(x)$ given by its graph.

30.



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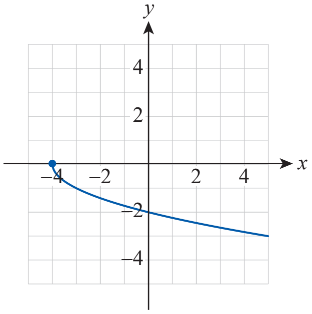

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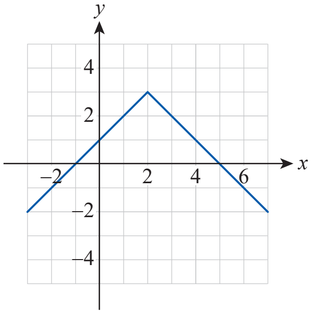

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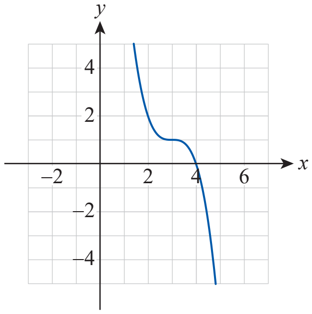


34–45 Use the information given to determine **a.** $(f+g)(-1)$, **b.** $(f-g)(-1)$, **c.** $(fg)(-1)$, and **d.** $(f/g)(-1)$.

34. $f(-1)=-3;\quad g(-1)=5$

35. $f(-1)=0;\quad g(-1)=-1$

36. $f(x)=x^2-3;\quad g(x)=x$

37. $f(x)=\sqrt[3]{x};\quad g(x)=x-1$

38. $f(-1)=15;\quad g(-1)=-3$

39. $f(x)=\dfrac{x+5}{2};\quad g(x)=6x$

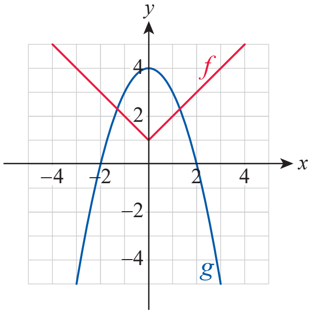
40. $f(x)=x^4+1;\quad g(x)=x^{11}+2$

41. $f(x)=\dfrac{6-x}{2};\quad g(x)=\sqrt{\dfrac{x}{-4}}$

42. $f=\{(5,2),(0,-1),(-1,3),(-2,4)\};$
$g=\{(-1,3),(0,5)\}$

43. $f=\{(3,15),(2,-1),(-1,1)\};\quad g(x)=-2$

44.



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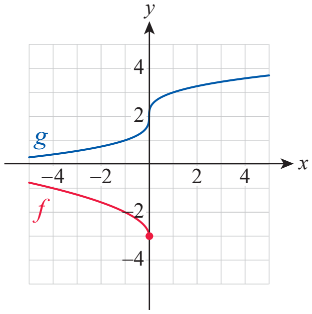


46–53 Find the formula and domain for **a.** $f+g$ and **b.** f/g.

46. $f(x)=|x|;\quad g(x)=\sqrt{x}$

47. $f(x)=x^2-1;\quad g(x)=\sqrt[3]{x}$

48. $f(x)=x-1;\quad g(x)=x^2-1$

49. $f(x)=x^{3/2};\quad g(x)=x-3$

50. $f(x)=3x;\quad g(x)=x^3-8$

51. $f(x)=x^3+4;\quad g(x)=\sqrt{x-2}$

52. $f(x)=-2x^2;\quad g(x)=[\![x+4]\!]$

53. $f(x)=6x-1;\quad g(x)=x^{2/3}$

54–63 Evaluate the expression, if possible, given $f(x)=1/x^2$ and $g(x)=2x+3$.

54. $(f+g)(-7)$

55. $(f+g)(-10)$

56. $(f-g)(-5)$

57. $(f-g)(0)$

58. $(fg)(4)$

59. $(fg)(-3)$

60. $\left(\dfrac{f}{g}\right)(-2)$

61. $\left(\dfrac{f}{g}\right)(0)$

62. $\left(\dfrac{g}{f}\right)(1)$

63. $\left(\dfrac{g}{f}\right)(-6)$

64–73 Use the information given to determine $(f \circ g)(3)$.

64. $f(-5) = 2; \quad g(3) = -5$

65. $f(\pi) = \pi^2; \quad g(3) = \pi$

66. $f(x) = x^2 - 3; \quad g(x) = \sqrt{x}$

67. $f(x) = \sqrt{x^2 - 9}; \quad g(x) = 1 - 2x$

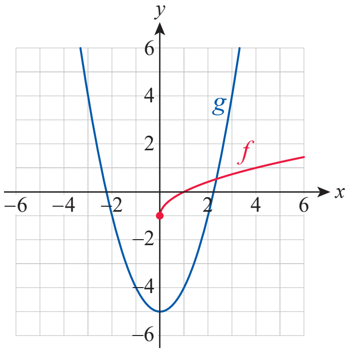
68. $f(x) = 2 + \sqrt{x}; \quad g(x) = x^3 + x^2$

69. $f(x) = x^{3/2} - 3; \quad g(x) = \left[\!\left[\frac{3x}{2} \right]\!\right]$

70. $f(x) = \sqrt{x+6}; \quad g(x) = \sqrt{4x-3}$

71. $f(x) = \sqrt{\frac{3x}{14}}; \quad g(x) = x^4 - x^3 - x^2 - x$

72.



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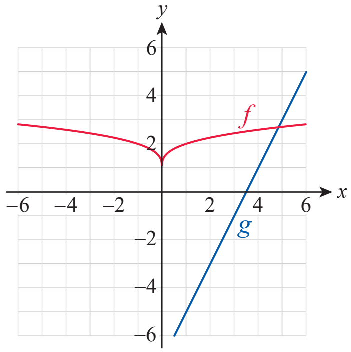


74–87 Find the formula and domain for **a.** $f \circ g$ and **b.** $g \circ f$.

74. $f(x) = \sqrt{x-1}; \quad g(x) = x^2$

75. $f(x) = \frac{1}{x}; \quad g(x) = x - 1$

76. $f(x) = \frac{4x-2}{3}; \quad g(x) = \frac{1}{x}$

77. $f(x) = 1 - x; \quad g(x) = \sqrt{x}$

78. $f(x) = [\![x-3]\!]; \quad g(x) = x^3 + 1$

79. $f(x) = x^2 + 2x; \quad g(x) = 3x^2 + 5$

80. $f(x) = x^2 + 1; \quad g(x) = 3x^2 + 5$

81. $f(x) = \sqrt{x}; \quad g(x) = 2x$

82. $f(x) = \frac{1}{x+7}; \quad g(x) = \frac{2}{x}$

83. $f(x) = \frac{1}{x}; \quad g(x) = \frac{1}{x}$

84. $f(x) = x^2; \quad g(x) = 3x + 1$

85. $f(x) = \sqrt[3]{x}; \quad g(x) = x^3$

86. $f(x) = \sqrt{x-4}; \quad g(x) = x^2 + 2$

87. $f(x) = \frac{3}{1-x}; \quad g(x) = 3x^2$

88–93 Write the given function as a composition of two functions. (Answers will vary.)

88. $f(x) = \sqrt[3]{3x^2 - 1}$

89. $f(x) = \frac{2}{5x-1}$

90. $f(x) = |x-2| + 3$

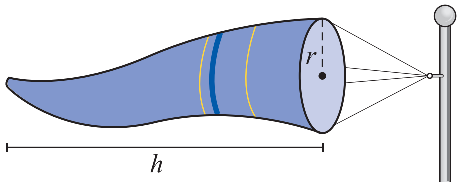
91. $f(x) = x + \sqrt{x+2} - 5$

92. $f(x) = |x^3 - 5x| + 7$

93. $f(x) = \frac{\sqrt{x-3}}{x^2 - 6x + 9}$

94. The volume of a right circular cylinder is given by the formula $V = \pi r^2 h$. If the height h is three times the radius r, show the volume V as a function of r.

95. The surface area of a wind sock is defined by the formula $S = \pi r\sqrt{r^2 + h^2}$ where r is the radius of the base of the wind sock and h is the height of the wind sock. As the wind sock is being knitted by an automated knitter, the height h is increasing with time t as defined by the formula $h(t) = \frac{1}{4}t^2, t \geq 0$. Find the surface area S of the wind sock as a function of time t.



96. The volume of the wind sock described in the previous exercise is given by the formula $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the wind sock and h is the height of the wind sock. If the height h is increasing with time t as defined by the formula $h(t) = \frac{1}{4}t^2,\ t \geq 0$, find the volume V of the wind sock as a function of time t.

97. A widget factory produces n widgets in t hours of a single day. The number of widgets the factory produces is given by the formula $n(t) = 10{,}000t - 25t^2, 0 \leq t \leq 9$. The cost c in dollars of producing n widgets is given by the formula $c(n) = 2040 + 1.74n$. Find the cost c as a function of time t that the factory is producing the widgets.

98. Suppose that $H(x)$ represents the percentage of income spent on a home loan in the year x and $C(x)$ represents the percentage of income spent on a car loan in the year x. If $I(x)$ represents the income in year x, determine the function L that represents the total loan expenses in year x.

99. Given two odd functions f and g, show that $f \circ g$ is also odd. Then verify this fact with the particular functions $f(x) = \sqrt[3]{x}$ and $g(x) = -x^3/(3x^2 - 9)$. (**Hint:** Recall that a function is odd if $f(-x) = -f(x)$ for all x in the domain of f.)

100. Given two even functions f and g, show that the product is also even. Then verify this fact with the particular functions $f(x) = 2x^4 - x^2$ and $g(x) = 1/x^2$. (**Hint:** Recall that a function is even if $f(-x) = f(x)$ for all x in the domain of f.)

101–108 As mentioned in the Interlude, a given complex number c is said to be in the Mandelbrot set if, for the function $f(z) = z^2 + c$, the sequence of iterates $f(0), f^2(0), f^3(0), \ldots$ stays close to the origin (which is the complex number $0 + 0i$). It can be shown that if any single iterate falls more than 2 units in distance (magnitude) from the origin, then the remaining iterates will grow larger and larger in magnitude. In practice, computer programs that generate the Mandelbrot set calculate the iterates up to a predecided point in the sequence, such as $f^{50}(0)$, and if no iterate to this point exceeds 2 in magnitude the number c is admitted to the set. The magnitude of a complex number $a + bi$ is the distance between the point (a,b) and the origin, so the formula for the magnitude of $a + bi$ is $\sqrt{a^2 + b^2}$.

Use the above criterion to determine, without a calculator or computer, if the given complex number is in the Mandelbrot set.

101. $c = 0$ **102.** $c = 1$ **103.** $c = i$ **104.** $c = -1$

105. $c = 1 + i$ **106.** $c = -2$ **107.** $c = 1 - i$ **108.** $c = -1 - i$

Concept Check

109–112 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

109. The graph of any quadratic polynomial is a transformation of the prototypical parabola.

110. The graphs of $y = f(x)$ and $y = f(-x)$ are reflection images of each other.

111. A cubic function can have up to three x-intercepts.

112. If $f(x)$ is an algebraic function and c is a nonzero constant, then $f(cx) = cf(x)$.

1.3 Technology Exercises

113–118 Mentally sketch the graph of the given function by identifying the basic shape that has been shifted, reflected, stretched, or compressed. Then use a graphing utility to graph the function and check your reasoning.

113. $f(x) = -2(3-x)^3 + 5$

114. $f(x) = \dfrac{3}{x+5} - 1$

115. $f(x) = \dfrac{-1}{(x-2)^2} - 3$

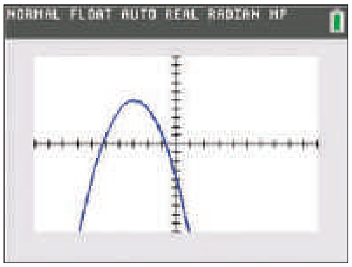
116. $f(x) = -3|x+2| - 4$

117. $f(x) = -\sqrt{1-x} + 2$

118. $f(x) = \sqrt[3]{2+x} - 1$

119–124 Write a possible equation for the function depicted on the graphing calculator. The function is shown in a $[-10,10]$ by $[-10,10]$ window.

119.



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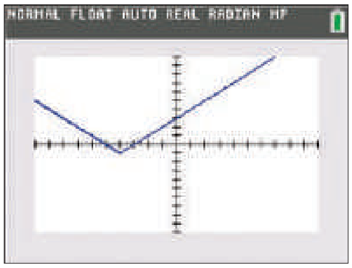

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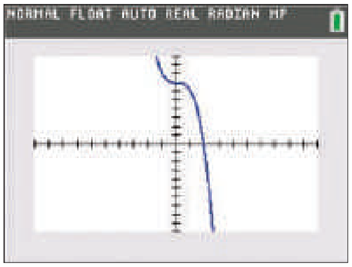

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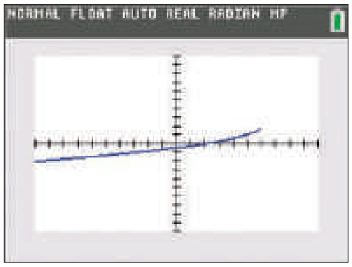

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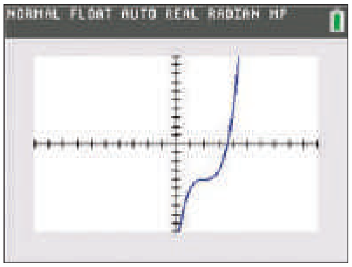

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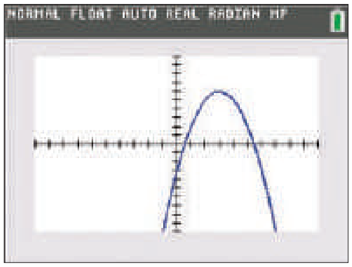


125–127 Use a graphing utility to determine $(f+g)(x)$, $(fg)(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$ for the given pair of functions.

125. $f(x) = (3x+2)^2$; $g(x) = \sqrt{x^2+5}$

126. $f(x) = \dfrac{1}{3x-5}$; $g(x) = (x+2)^3$

127. $f(x) = \dfrac{x+1}{x-1}$; $g(x) = \dfrac{x-1}{x}$

1.4 Exercises

1–12 Graph the inverse of the given relation, and state its domain and range.

1. $R = \{(-4,2),(3,2),(0,-1),(3,-2)\}$
2. $S = \{(-3,-3),(-1,-1),(0,1),(4,4)\}$
3. $y = x^3$
4. $y = |x| + 2$
5. $x = |y|$
6. $x = -\sqrt{y}$
7. $y = \frac{1}{2}x - 3$
8. $y = -x + 1$
9. $y = [\![x]\!]$
10. $T = \{(4,2),(3,-1),(-2,-1),(2,4)\}$
11. $x = y^2 - 2$
12. $y = 2\sqrt{x}$

13–22 Determine if the given function has an inverse function. If not, suggest a domain to restrict the function so that it would have an inverse function. (Answers will vary.)

13. $f(x) = x^2 + 1$
14. $g(x) = (x-2)^3 - 1$
15. $h(x) = \sqrt{x+3}$
16. $s(x) = \frac{1}{x^2}$
17. $G(x) = 3x - 5$
18. $F(x) = -x^2 + 5$
19. $r(x) = -\sqrt{x^3}$
20. $b(x) = [\![x]\!]$
21. $m(x) = \frac{13x-2}{4}$
22. $H(x) = |x - 12|$

23–37 Find the inverse of the given function.

23. $f(x) = x^{1/3} - 2$
24. $g(x) = 4x - 3$
25. $r(x) = \frac{x-1}{3x+2}$
26. $s(x) = \frac{1-x}{1+x}$
27. $F(x) = (x-5)^3 + 2$
28. $G(x) = \sqrt[3]{3x-1}$
29. $V(x) = \frac{x+5}{2}$
30. $W(x) = \frac{1}{x}$
31. $h(x) = x^{3/5} - 2$
32. $A(x) = (x^3+1)^{1/5}$
33. $J(x) = \frac{2}{1-3x}$
34. $k(x) = \frac{x+4}{3-x}$
35. $h(x) = x^7 + 6$
36. $F(x) = \frac{3-x^5}{-9}$
37. $r(x) = \sqrt[5]{2x}$

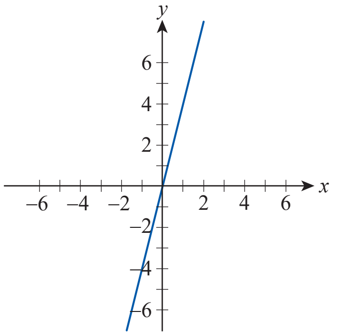
38–45 Show that $f^{-1}(f(x)) = x$ and that $f(f^{-1}(x)) = x$.

38. $f(x) = 2x - 3; \quad f^{-1}(x) = \frac{x+3}{2}$
39. $f(x) = x^2,\ x \ge 0; \quad f^{-1}(x) = \sqrt{x}$
40. $f(x) = \frac{3x-1}{5}; \quad f^{-1}(x) = \frac{5x+1}{3}$
41. $f(x) = \frac{x-5}{2x+3}; \quad f^{-1}(x) = \frac{3x+5}{1-2x}$
42. $f(x) = (x-2)^2,\ x \ge 2; \quad f^{-1}(x) = \sqrt{x} + 2,\ x \ge 0$
43. $f(x) = \sqrt[3]{x+2} - 1; \quad f^{-1}(x) = (x+1)^3 - 2$
44. $f(x) = \frac{1}{x}; \quad f^{-1}(x) = \frac{1}{x}$
45. $f(x) = \frac{1}{1+x},\ x \ge 0; \quad f^{-1}(x) = \frac{1-x}{x},\ 0 < x \le 1$

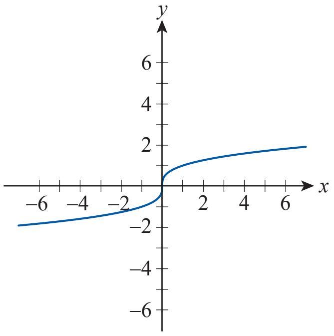
46–51 Match the function with the graph of the inverse of the function (labeled A–F).

46. $f(x) = x^3$
47. $f(x) = x - 5$
48. $f(x) = \sqrt{x-4}$
49. $f(x) = x^2$
50. $f(x) = \frac{x}{4}$
51. $f(x) = \sqrt[3]{x+1}$

A.



B.



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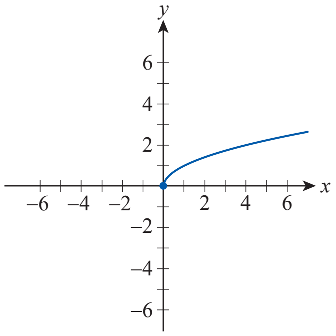
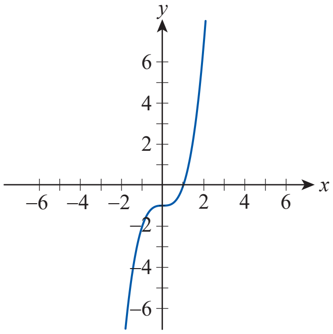
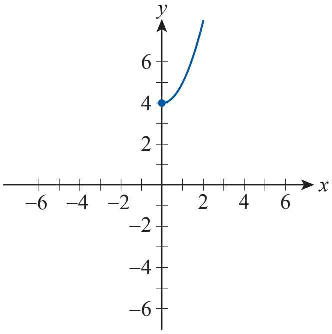


D.



E.



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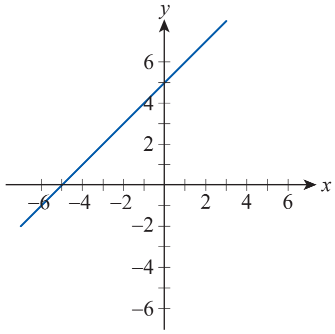


52–60 Match the logarithmic function with its graph (labeled A–I).

52. $f(x)=\log_2 x-1$

53. $f(x)=\log_2(2-x)$

54. $f(x)=\log_2(-x)$

55. $f(x)=\log_2(x-3)$

56. $f(x)=1-\log_2 x$

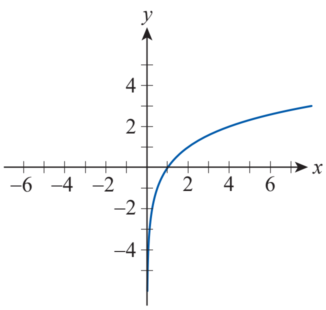
57. $f(x)=-\log_2 x$

58. $f(x)=-\log_2(-x)$

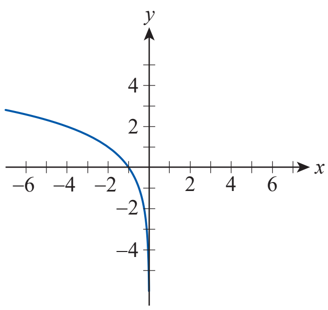
59. $f(x)=\log_2 x$

60. $f(x)=\log_2 x+3$

A.



B.



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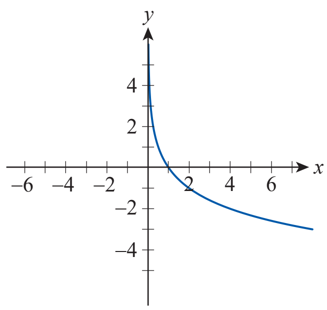

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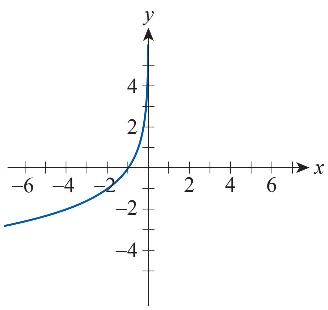

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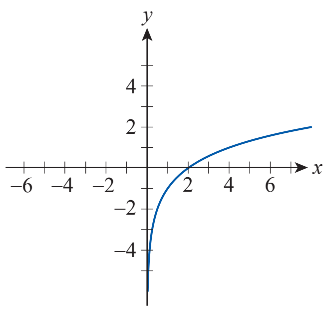
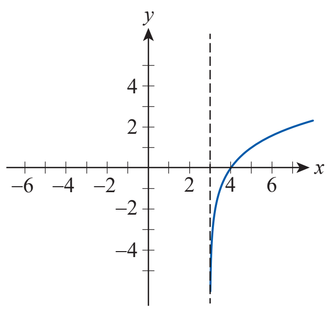


F.



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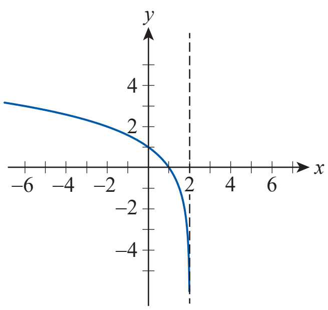

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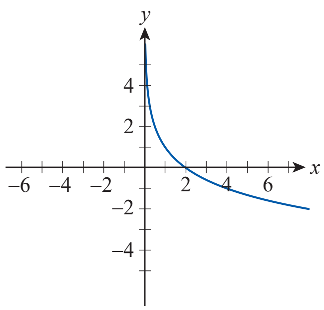

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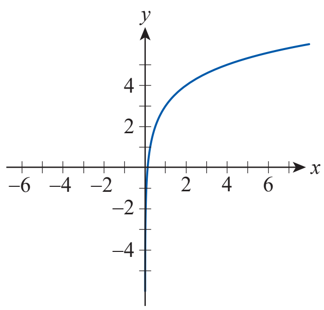


61–72 Sketch the graph of the given function.

61. $f(x)=\log_3(x-1)$

62. $g(x)=\log_5(x+2)-1$

63. $r(x)=\log_{1/2}(x-3)$

64. $p(x)=3-\log_2(x+1)$

65. $q(x)=\log_3(2-x)$

66. $s(x)=\log_{1/3}(5-x)$

67. $h(x)=\log_7(x-3)+3$

68. $m(x)=\log_{1/2}(1-x)$

69. $f(x)=\log_3(6-x)$

70. $p(x)=4-\log_{10}(x+3)$

71. $s(x)=-\log_{1/3}(-x)$

72. $g(x)=\log_5(2x)-1$

73–78 Evaluate the given expression without using a calculator.

73. $\log_4 16$

74. $\log_5\left(25^3\right)$

75. $\ln\left(e^4\right)+\ln\left(e^3\right)$

76. $\log_4\frac{1}{64}$

77. $\ln\left(e^{1.5}\right)-\log_4 2$

78. $\log_2\left(8^{2\log_2 4-\log_2 4}\right)$

79–84 Evaluate the given logarithmic expression to two decimal places. (**Hint:** Use the change of base formula.)

79. $\log_6\left(3^4\right)$

80. $\log_7 14.3$

81. $\log_{1/2}\left(\pi^{-2}\right)$

82. $\log_{1/5} 626$

83. $\ln(\log 123)$

84. $\log_{17} 0.041$

85–90 Use the properties of logarithms to rewrite the given expression as a single term that does not contain a logarithm.

85. $5^{2\log_5 x}$

86. $\log_4 16\cdot\log_x\left(x^2\right)$

87. $e^{2-\ln x+\ln p}$

88. $e^{5\left(\ln\sqrt[5]{3}+\ln x\right)}$

89. $10^{\log\left(x^3\right)-4\log y}$

90. $a^{\log_a b+4\log_a\sqrt{a}}$

91–99 Use the properties of logarithms to expand the given expression as much as possible; that is, decompose the expression into sums or differences of the simplest possible terms. Simplify any numerical expressions that can be evaluated without a calculator.

91. $\ln\frac{\sqrt{x^3}pq^5}{e^7}$

92. $\log_a\sqrt[5]{\frac{a^4b}{c^2}}$

93. $\log\left(\log\left(100x^3\right)\right)$

94. $\log_3(9x+27y)$

95. $\log\frac{10}{\sqrt{x+y}}$

96. $\ln\left(\ln\left(e^{ex}\right)\right)$

97. $\log_2\frac{y^2+z}{16x^4}$

98. $\log\left(\log\left(100{,}000^{2x}\right)\right)$

99. $\log_b\sqrt{\frac{x^4y}{z^2}}$

100–105 Use the properties of logarithms to condense the given expression as much as possible, writing the answer as a single term with a coefficient of 1.

100. $\frac{1}{5}\left(\log_7\left(x^2\right)-\log_7(pq)\right)$

101. $\ln 3+\ln p-2\ln q$

102. $2\left(\log_5\sqrt{x}-\log_5 y\right)$

103. $\log(x-10)-\log x$

104. $2\log\left(a^2b\right)-\log\frac{1}{b}+\log\frac{1}{a}$

105. $3\left(\ln\sqrt[3]{e^2}-\ln(xy)\right)$

106–111 Evaluate the given expression, if possible.

106. $\cos^{-1}\left(\cos\frac{2\pi}{4}\right)$

107. $\sin^{-1}\left(\sin\frac{3\pi}{2}\right)$

108. $\tan\left(\tan^{-1}(0.5)\right)$

109. $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

110. $\cos\left(\cos^{-1}(-0.8)\right)$

111. $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

112–117 Most calculators are not equipped with arccosecant, arcsecant, and arccotangent buttons, but expressions involving these functions can still be evaluated. For example, to evaluate $\csc^{-1}x$, let $\theta=\csc^{-1}x$.

$$\csc\theta=x$$
$$\frac{1}{\sin\theta}=x$$
$$\sin\theta=\frac{1}{x}$$
$$\theta=\sin^{-1}\frac{1}{x}$$

Use the method described above to evaluate the given expression. (Round your answer to four decimal places.)

112. $\csc^{-1} 5$

113. $\sec^{-1}(-0.5)$

114. $\cot^{-1} 150$

115. $\cot^{-1}(-0.2)$

116. $\csc^{-1}(-8.9)$

117. $\sec^{-1} 2$

118–123 Find the value of the given expression without using a calculator.

118. $\sin\left(\arctan\sqrt{3}\right)$

119. $\cos\left(\sec^{-1}(-2)\right)$

120. $\tan(\operatorname{arccot} 1)$

121. $\csc\left(\arccos\left(-\frac{\sqrt{3}}{2}\right)\right)$

122. $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

123. $\sec\left(\csc^{-1}\frac{2\sqrt{3}}{3}\right)$

124–129 Rewrite the given function as a purely algebraic function.

124. $\tan\left(\cos^{-1} x\right)$

125. $\cot\left(\sin^{-1}\frac{2}{x}\right)$

126. $\sec\left(\tan^{-1} 3x\right)$

127. $\tan\left(\sin^{-1}\frac{x}{\sqrt{x^2+3}}\right)$

128. $\sin\left(\sec^{-1} x\right)$

129. $\cos\left(\tan^{-1}\frac{x}{4}\right)$

130–133 Sketch the graph of the given function. Then graph the function using a graphing utility to check your answer.

130. $f(x) = \sin^{-1}(x-3)$

131. $f(x) = \sec^{-1} 2x$

132. $f(x) = \arctan\frac{x}{2}$

133. $f(x) = 2\arccos x$

134–137 An inverse function can be used to encode and decode words and sentences by assigning each letter of the alphabet a numerical value (A = 1, B = 2, C = 3, …, Z = 26). Example: Use the function $f(x) = x^2$ to encode the word CALCULUS. The encoded message would be 9 1 144 9 441 144 441 361. The word can then be decoded by using the inverse function $f^{-1}(x) = \sqrt{x}$. The inverse values are 3 1 12 3 21 12 21 19, which translates back to the word CALCULUS.

Encode or decode the given message using the numerical values A = 1, B = 2, C = 3, …, Z = 26.

134. Encode the message SANDY SHOES using the function $f(x) = 4x - 3$.

135. Encode the message WILL IT RAIN TODAY using the function $f(x) = x^2 - 8$.

136. The following message was encoded using the function $f(x) = 8x - 7$. Decode the message.

41 137 65 145 9 33 33 169 113 89 89 33 193 9 1 89 89

1 105 25 57 113 137 145

33 145 57 113 33 145

137. The following message was encoded using the function $f(x) = 5x + 1$. Decode the message.

91 26 66 26 66 11 26 91 126 76 106 91

96 106 71 11 61 76 16 56

138–139 The energy released during earthquakes can vary greatly, but logarithms provide a convenient way to analyze and compare the intensity of earthquakes. Earthquake intensity is measured on the Richter scale (named for the American seismologist Charles F. Richter, 1900–1985). In the formula that follows, I_0 is the intensity of a just-discernible earthquake, I is the intensity of an earthquake being analyzed, and R is its ranking on the Richter scale.

$$R = \log\frac{I}{I_0}$$

By this measure, earthquakes range from a classification of minor ($R < 4$), to light ($4 \le R < 5$), to moderate ($5 \le R < 6$), to strong ($6 \le R < 7$), to major ($7 \le R < 8$), and finally to great ($R \ge 8$).

Use this information to solve the problem.

138. The 1994 Northridge, California earthquake measured 6.7 on the Richter scale. What was the intensity, relative to a 0-level earthquake, of this event?

139. The April, 2009 Abruzzo earthquake in Italy was 2,000,000 times as intense as a 0-level earthquake. What was the Richter ranking of this tragic event?

140–141 Sound intensity is another quantity that varies greatly, and the measure of how the human ear perceives intensity, in units called decibels, is very similar to the measure of earthquake intensity. If I_0 is the intensity of a just-discernible sound, I is the intensity of the sound being analyzed, and D is its decibel level, we have the formula $D = 10\log(I/I_0)$. Decibel levels range from 0 for a barely discernible sound, to 40 for the level of a quiet conversation, to 80 for heavy traffic, to 120 for a loud rock concert, and finally (as far as humans are concerned) to around 160, at which point the eardrum is likely to rupture.

Use the decibel formula given above to answer the question.

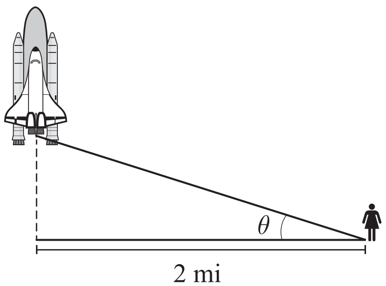
140. A construction worker operating a jackhammer would experience noise with an intensity of 20 watts/meter^2 if it weren't for ear protection. Given that $I_0 = 10^{-12} \text{ watts/meter}^2$, what is the decibel level for such noise?

141. The intensity of a cat's soft purring is measured to be 2.19×10^{-11} watts/meter^2. Given that $I_0=10^{-12}$ watts/meter^2, what is the decibel level of this noise?

142–143 Use inverse trigonometric functions to solve the problem. (Round your answer to four decimal places.)

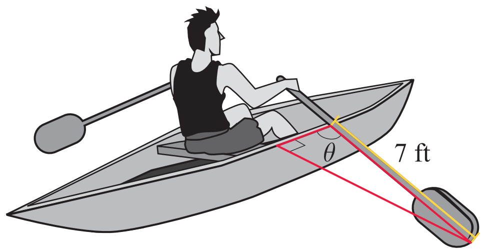
142. Kim is watching a space shuttle launch from an observation spot 2 miles away from the launchpad. Find the angle of elevation to the shuttle for each of the following heights.

a. 0.5 miles **b.** 2 miles **c.** 2.8 miles



143. Jesse is rowing in the men's singles race. The length of the oar from the side of the shell to the water is 7 feet. At what angle is the oar from the side of the boat when the blade is at the following distances from the boat?

a. 2 feet **b.** 3 feet **c.** 5 feet



Concept Check

144–151 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

144. All exponential functions are one-to-one.

145. $\sin(\arcsin x)=x$ for all $x\in[-1,1]$

146. $\arcsin(\sin x)=x$ for all $x\in\mathbb{R}$

147. $\tan(\arctan x)=x$ for all $x\in\mathbb{R}$

148. $\arccos(\cos(3\pi/2))=3\pi/2$

149. The domain of $\arcsin x$ is $[-\pi/2,\pi/2]$.

150. The domain of $f(x)=\cot^{-1}x$ is $\mathbb{R}$.

151. The function $f(x)=\sin(\tan^{-1}x)$ can be represented as an algebraic function.

1.5 Exercises

1–12 Express the given function (using ( ), ^, ×, ÷, etc.) in a format suitable for entering into a graphing utility.

1. $f(x)=1+3x+\sqrt{x}$

2. $g(x)=3x-2+\sqrt[3]{x}$

3. $h(x)=\dfrac{3x}{\sqrt{x-1}}$

4. $k(x)=\dfrac{1+\sqrt{x}}{2-3x}$

5. $u(x)=\dfrac{(-3x+16)^4}{3x+6}$

6. $v(x)=\sqrt{2+5x+\sqrt{x}}$

7. $F(x)=\left(2x^{2/3}+3x^{5/3}\right)^5$

8. $G(x)=\left(9x^{1/5}+2x^{3/5}\right)^{10}$

9. $H(x)=\ln\left(x^2+1\right)+2^{\sqrt{x}}$

10. $K(x)=\dfrac{e^{\cos x}}{\sqrt{\log\left(x^4+2\right)}}$

11. $Q(x)=\dfrac{\arctan x+1}{\left(\cos(\arcsin x)\right)^3}$

12. $R(x)=\sqrt{(\arccos x)^2+\log_2(\tan x)}$

13. During the last 5 years, the advertising manager for a corporation has gathered the following data that show the relationship between the advertising budget (in millions of dollars) and the total sales (in thousands of units).

Advertising and Sales

Advertising budget (x) (in millions)	\$4.50	\$6.50	\$3.50	\$4.20	\$2.60
Units sold (y) (in thousands)	37	46	42	32	29

a. Find the least-squares line of best fit for the data.

b. Estimate the sales if \$4 million is budgeted for advertising.

14. Records at a company for the last 5 years show the following relationship between the units sold (in thousands) and the price of a product.

Sales

Price (p)	\$8.80	\$8.00	\$7.50	\$6.90	\$6.20
Units sold (x) (in thousands)	3.8	5.2	7.3	8.0	9.6

a. Find the least-squares line of best fit for the price in terms of units sold.

b. Estimate the price that should be charged in order to sell 10,000 units.

15. The following data show the amount spent on office-building construction (in thousands) for a particular county during a 6-month period.

Office Construction

Month	Apr	May	Jun	Jul	Aug	Sep
Amount (in thousands)	\$24	\$24	\$30	\$49	\$68	\$69

a. Find the least-squares line of best fit for the data. (Let $x = 1$ correspond to January, $x = 2$ to February, etc.)

b. Estimate the amount spent on construction in October.

16. The annual revenue (in millions of dollars) for a corporation is given in the following table.

Annual Revenue

Year	2017	2018	2019	2020	2021	2022
Revenue (in millions)	\$66	\$82	\$127	\$201	\$310	\$315

a. Find the least-squares line of best fit for the data. (Let $x = 0$ correspond to the year 2017.)

b. Estimate the revenue for 2023.

17. The price of livestock futures is the estimated market price of livestock on the delivery date (end of the indicated month). The cattle futures (in cents per pound) for the months February through July are as follows.

Livestock Futures

Month	Feb	Mar	Apr	May	Jun	Jul
Price (cents per pound)	79.10	76.02	71.80	71.45	71.45	72.50

a. Find the least-squares line of best fit for the data. (Let $x = 1$ correspond to January, $x = 2$ to February, etc.)

b. Estimate the price for August.

18. The total number of foreign tourists visiting the United States between 2000 and 2004, as reported by the US Travel and Tourism Administration, is shown in the following table.

Foreign Tourists Visiting the United States

Year	2000	2001	2002	2003	2004
Tourists (in millions)	25.7	26.3	29.7	34.2	38.3

a. Find the least-squares line of best fit for the data. (Let x represent the number of years passed since 2000.)

b. Estimate the number of foreign tourists who visited the United States during 2006.

1.5 Technology Exercises

19–28 Determine whether there are points where we need to be careful in interpreting the result when using graphing technology to graph the given function. Find all those points and explain. Then use a graphing utility to sketch the graph, using various viewing windows.

19. $f(x) = x^4 \cos\frac{1}{x}$

20. $G(x) = \cos\frac{1}{x-2}$

21. $p(x) = \frac{1}{2}\tan(3x-2)$

22. $g(x) = \sec(2x+1)$

23. $q(x) = \frac{x^2-2x-1}{x+1}$

24. $r(x) = \frac{x^2+1}{x^2-9}$

25. $h(x) = \frac{2x^4+1}{2x^4-1}$

26. $F(x) = \ln(\cos x)$

27. $s(x) = \cos(\ln x)$

28. $t(x) = \sin(\csc x)$

29–40 Use a graphing utility to graph the given function in the window $[-10,10]$ by $[-10,10]$. Explain what appears to be wrong with the picture. Then find a more appropriate window, which reveals the significant parts of the graph, and draw the "improved" graph.

29. $f(x) = \frac{3x-25}{\sqrt{x^2+5}}$

30. $g(x) = (40+3x)\sqrt{16-x}$

31. $h(x) = (3x+4)^2(5x-25)^2$

32. $F(x) = (6x+30)^2(3x-15)^2$

33. $G(x) = 35+17x-x^2-x^3$

34. $H(x) = 210-80x+x^3$

35. $r(x) = \sqrt[3]{x^3-x^2-x-50}$

36. $u(x) = \sqrt[3]{x^4-3x^2-3x-30}$

37. $v(x) = (12-6x-x^2)^{4/3}$

38. $f(x)=(x^3-x-100)^{1/3}$ **39.** $g(x)=x^2\sin\frac{\pi}{x-12}$ **40.** $h(x)=\sec^2\frac{x}{10}$

41–46 Use a graphing utility to graph the given function in a suitable window and find the smallest *y*-value possible. (Use only the given interval, if specified. Round your answer to four decimal places.)

41. $f(x)=x^2-104x+2724$ **42.** $g(x)=\frac{-1-x^2-3x^3}{5^x}$

43. $h(x)=x^3-17x+5;\quad -3\le x\le 5$ **44.** $F(x)=\frac{\sqrt[3]{x}-150}{5+x^2}$

45. $G(x)=x^{1.5}-8x-15$ **46.** $H(x)=x^{1.8}-x-100$

47–52 Use a graphing utility to graph the given function in a suitable window and find the greatest *y*-value possible. (Use only the given interval, if specified. Round your answer to four decimal places.)

47. $f(x)=50-2^x;\quad -10\le x\le 10$ **48.** $g(x)=(x+1)^5-1.5^{x+1}$ **49.** $h(x)=x^{17}-17^x;\quad -2\le x\le 2$

50. $k(x)=x(3^{-x})$ **51.** $F(x)=\frac{-2x}{x^2+1}$ **52.** $G(x)=\frac{3-5x}{\sqrt{3x^2+2}}$

53–58 Use a graphing utility to graph the given function, and describe the characteristics of the graph as *c* varies. Use different viewing windows.

53. $f(x)=x^2-cx$ **54.** $g(x)=\frac{1}{2}x^3-c(x^2+x+1)$ **55.** $h(x)=e^{cx}$

56. $k(x)=\ln(x^2+cx+1)$ **57.** $F(x)=\frac{x}{c}+\cos\frac{c^2x}{c}$ **58.** $G(x)=\frac{cx^2}{x+cx^3}$

59–64 Use a graphing utility to approximate the solution(s) of the given equation, rounded to four decimal places. (**Hint:** Zoom in on the *x*-intercepts or points of intersection as appropriate for each equation.)

59. $x^3-20x-2=0$ **60.** $2x^3=31x+2$ **61.** $3\cos x=\sqrt{x}$

62. $\arctan x=\frac{1}{100}x^5$ **63.** $\ln x=x-2$ **64.** $x+5=e^x$

65–70 Use appropriately large viewing windows on a graphing utility to decide which of the given functions eventually "rises faster" toward infinity.

65. $f(x)=\frac{1}{2}x^3;\quad g(x)=x^2$ **66.** $f(x)=\sqrt{x};\quad g(x)=x$ **67.** $f(x)=5\sqrt{x};\quad g(x)=\frac{1}{5}x$

68. $f(x)=\frac{1}{2}e^x;\quad g(x)=x^2$ **69.** $f(x)=5\log x+5;\quad g(x)=\frac{1}{2}x^5$ **70.** $f(x)=10\arctan x;\quad g(x)=2\sqrt[3]{x}$

71–73 Most graphing utilities have regression capabilities to fit curves other than lines to a given data set. Frequently, depending on the tendency of the data, a quadratic, an exponential, or some other type of curve provides for much better approximation. Most often the choice is the modeler's.

Use the regression capabilities of your technology to build a graphical model and then answer the questions.

71. The following table shows daytime temperatures in El Cajon, CA on a particular spring day from 6:00 a.m. to 12:00 p.m. Find the best-fitting curve and use it to predict the temperatures at 1:00 p.m. and 2:00 p.m.

Daytime Temperatures in El Cajon, CA

Time	6:00 a.m.	7:00 a.m.	8:00 a.m.	9:00 a.m.	10:00 a.m.	11:00 a.m.	12:00 p.m.
Temperature (°F)	47	50	55	61	68	73	75

72. The following table shows the winning times of the Olympic men's 100 m dash champions. Find the best-fitting curve and use it to predict the winning times at the next three Olympics.

Olympic Men's 100 m Dash Winning Times

Year	Time (s)	Year	Time (s)	Year	Time (s)
1896	12.00	1948	10.30	1988	9.92
1900	11.00	1952	10.40	1992	9.96
1904	11.00	1956	10.50	1996	9.84
1908	10.80	1960	10.20	2000	9.87
1912	10.80	1964	10.00	2004	9.85
1920	10.80	1968	9.95	2008	9.69
1924	10.60	1972	10.14	2012	9.63
1928	10.80	1976	10.06	2016	9.81
1932	10.30	1980	10.25	2020	9.80
1936	10.30	1984	9.99		

73. The following table shows acceleration times for the Ferrari Enzo up to 130 mph. Find the best-fitting curve and use it to predict the acceleration times for the Enzo from **a.** 0 to 150 mph and **b.** 0 to 170 mph.

Acceleration Times for Ferrari Enzo

Speed (mph)	0–30	0–40	0–50	0–60	0–70	0–80	0–90	0–100	0–110	0–120	0–130
Time (s)	1.5	2.0	2.7	3.3	3.8	5.0	5.8	6.6	8.0	9.2	10.3

Source: *Car and Driver*

b. As far as the arc length of the curve connecting $(0,0)$ with $(1,1)$, we will illustrate the calculations by first dividing $[0,1]$ into four subintervals, as we did in part a.

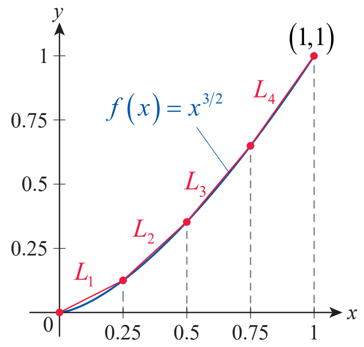


Figure 12

If we label the endpoints as $x_1 = 0$, $x_2 = 0.25$, …, $x_5 = 1$, and connect the points $(x_i, f(x_i))$ on the graph with $(x_{i+1}, f(x_{i+1}))$ for $i = 1, \ldots, 4$, a "crude" first approximation for the arc length will simply be the sum of the lengths of the four resulting line segments (see Figure 12). This can be calculated using the Pythagorean Theorem as follows.

$$\begin{aligned} s_4 &= L_1 + L_2 + L_3 + L_4 \\ &= \sqrt{(0.25)^2 + \left[(0.25)^{3/2}\right]^2} + \sqrt{(0.25)^2 + \left[(0.5)^{3/2} - (0.25)^{3/2}\right]^2} \\ &\quad + \sqrt{(0.25)^2 + \left[(0.75)^{3/2} - (0.5)^{3/2}\right]^2} + \sqrt{(0.25)^2 + \left[1 - (0.75)^{3/2}\right]^2} \\ &\approx 1.4362 \end{aligned}$$

Upon dividing $[0,1]$ into 10 equal parts, a similar, but a bit longer, calculation yields the much better approximation of

$$s_{10} \approx 1.4389.$$

As before, with the help of a computer or programmable calculator we can generate a table of values such as the following.

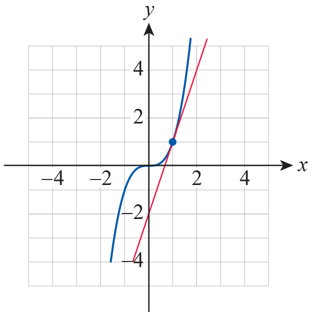
n	50	100	1000	10,000
s_n	1.43966	1.43970	1.43971	1.43971

From the table above, we conclude that the true value of the arc length is approximately $s \approx 1.43971$.

2.1 Exercises

1–6 Estimate the slope of the tangent line shown in the given graph.

1.



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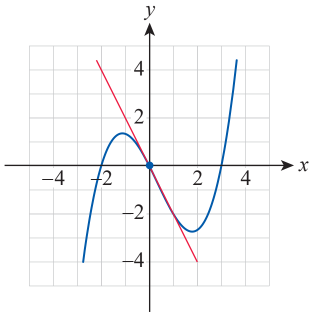

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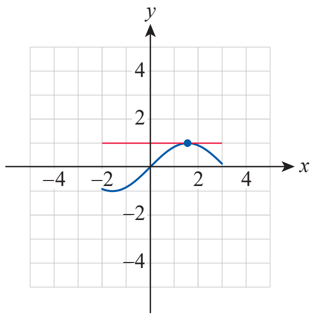

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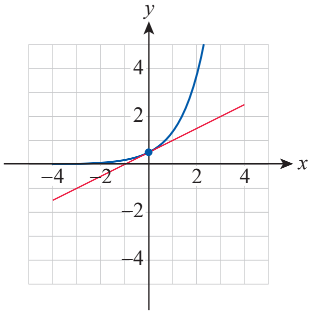
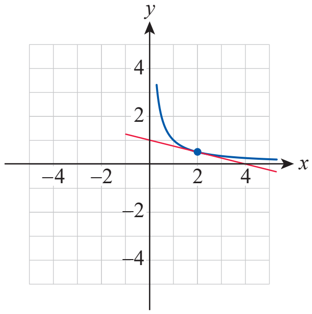


5.



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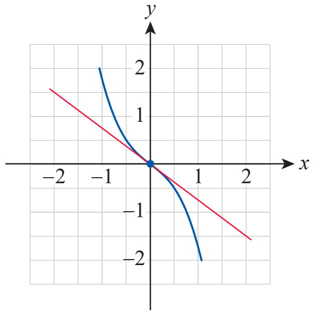


7–18 Use difference quotients to approximate the slope of the tangent to the graph of the function at the given point. Use at least five different h-values that are decreasing in magnitude. (Answers will vary.)

7. $f(x)=1-2x;\quad (1,-1)$

8. $g(x)=\frac{5}{4}x-8;\quad (8,2)$

9. $h(x)=\frac{1}{3}x^2-1;\quad (3,2)$

10. $F(x)=3+x-\frac{x^2}{2};\quad (4,-1)$

11. $G(x)=\frac{1}{4}x^3-x+1;\quad (-2,1)$

12. $k(x)=10-x^{3/2};\quad (4,2)$

13. $H(x)=\ln x+1;\quad (e,2)$

14. $u(x)=\cos x;\quad \left(\frac{\pi}{2},0\right)$

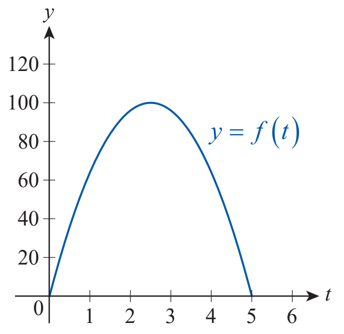
15. $v(x)=\log 2x-1;\quad (5,0)$

16. $w(x)=\tan x;\quad (0,0)$

17. $p(x)=-x^4+1;\quad (1,0)$

18. $q(x)=x^5-x+3;\quad (0,3)$

19. An arrow is shot into the air and its height in feet after t seconds is given by the function $f(t)=-16t^2+80t$. The graph of the curve $y=f(t)$ is shown.



a. Find the height of the arrow when $t=2$ seconds.

b. Find the instantaneous velocity of the arrow when $t=2$ seconds.

c. Find the slope of the line tangent to the curve at $t=2$ seconds.

d. Find the time it takes the arrow to reach its peak.

20. Suppose that a sailboat is observed, over a period of 5 minutes, to travel a distance from a starting point according to the function $s(t)=t^3+60t$, where t is time in minutes and s is the distance traveled in meters.

a. How far will it travel during the first 6 seconds?

b. What is the average velocity during the first 6 seconds?

c. Estimate how fast the boat is moving at the starting point.

d. Estimate how fast the boat is moving at the end of 3 minutes.

21. A model rocket is fired vertically upward. The height after t seconds is $h(t)=192t-16t^2$ feet.

a. What will be its height at the end of the first second?

b. What is the average velocity of the rocket during the first second?

c. Estimate the instantaneous velocity at $t=0$ seconds.

d. Estimate the instantaneous velocity at $t=4$ seconds.

e. When will the velocity be 0? (**Hint:** Start with the initial velocity you found in part c. and use the fact that under the influence of gravity, when air resistance is ignored, vertical upward velocity decreases by 32 ft/s every second. Once you have a guess, test it using a table of difference quotients.)

22. A particle moving in a straight line is at a distance of $s(t)=2.5t^2+18t$ feet from its starting point after t seconds, where $0\le t\le 12$. Estimate the instantaneous velocity at **a.** $t=6$ seconds and **b.** $t=9$ seconds.

23. The distance, in meters, traveled by a moving particle in t seconds is given by $d(t)=3t(t+1)$. Estimate the instantaneous velocity at **a.** $t=0$ seconds, **b.** $t=2$ seconds, and **c.** at time t_0. (**Hint:** Write the difference quotient corresponding to $t=t_0$, simplify, and try to find the value being approached by the expression as h decreases.)

24. The distance, in meters, traveled by a moving particle in t seconds is given by $d(t)=t^2-3t$. Estimate the instantaneous velocity at **a.** $t=0$ seconds, **b.** $t=4$ seconds, and **c.** at time t_0. (See the hint given in Exercise 23c.)

25. After start, on a straight stretch of the track, a race car's velocity changes according to the function $v(t) = -1.8t^2 + 18t$, when $0 \le t \le 10$, t is measured in seconds, and $v(t)$ is measured in meters per second.

a. When does peak velocity occur and what is it? (**Hint:** The graph of $v(t)$ may be helpful.)

b. When does peak deceleration occur?

c. Use difference quotients to estimate peak deceleration. Approximately what multiple of $g \approx 9.81 \text{ m/s}^2$ have you obtained?

26. If we ignore air resistance, a falling body will fall $16t^2$ feet in t seconds.

a. How far will it fall between $t = 2$ and $t = 2.1$?

b. What is its average velocity between $t = 2$ and $t = 2.1$?

c. Estimate its instantaneous velocity at $t = 2$.

27. A student dropped a textbook from the top floor of his dorm and it fell according to the formula $S(t) = -16t^2 + 8\sqrt{t}$, where t is the time in seconds and $S(t)$ is the distance in feet from the top of the building.

a. If the textbook hit the ground in exactly 2.5 seconds, how high is the building?

b. What was the average speed for the trip?

c. What was the instantaneous velocity at $t = 1$ second?

d. What was the velocity of impact?

28–31 Approximate the area of the region between the graph of the function and the x-axis on the given interval. Use **a.** $n = 4$ and **b.** $n = 5$. (Round your answers to four decimal places.)

28. $f(x) = x^2$ on $[0,1]$

29. $g(x) = 16x - x^3$ on $[0,4]$

30. $h(x) = \sin x$ on $[0,\pi]$

31. $F(x) = e^x + 1$ on $[-10,0]$

32–35 Approximate the arc length of the graph of the function on the given interval. Use **a.** $n = 4$ and **b.** $n = 5$. (Round your answers to four decimal places.)

32. $f(x) = \sqrt{x}$ on $[0,1]$

33. $g(x) = x^3 + x^2$ on $[-1,0]$

34. $F(x) = \cos x$ on $\left[0,\frac{\pi}{2}\right]$

35. $G(x) = \ln x + 1$ on $[1,2]$

2.1 Technology Exercises

36–39 Use a graphing utility to graph $f(x)$ along with three secant lines at the indicated x-value, corresponding to the difference quotients with h-values of 0.2, 0.1, and 0.01, respectively. Can you come up with a possible equation for the tangent line? Use technology to test your conjecture.

36. $f(x) = x^2; \quad x = 2$

37. $f(x) = -x^3 + x + 1; \quad x = \frac{\sqrt{3}}{3}$

38. $f(x) = \sin x + \cos x; \quad x = 0$

39. $f(x) = 3\sqrt{x}; \quad x = 4$

40–43 Use a graphing utility to graph the given function $f(x)$ along with $D(x) = \frac{f(x+0.001) - f(x)}{0.001}$ in the same coordinate system. Explain how the function values of $D(x)$ are reflected on the graph of $f(x)$.

40. $f(x) = x^4$

41. $f(x) = x(3-x)$

42. $f(x) = \sin x$

43. $f(x) = \ln x$

44–47 Use a graphing utility to find the x-values at which the graph of $f(x)$ does not have a tangent line. Explain.

44. $f(x) = -|x-1| + 1$

45. $f(x) = |x^2 - 4|$

46. $f(x) = |\ln x|$

47. $f(x) = (x-1)^{2/3}$

48–51. Use a computer algebra system to find approximations for the areas in Exercises 28–31 by using **a.** $n = 100$ and **b.** $n = 1000$. (Round your answers to four decimal places.)

52–55. Use a computer algebra system to find approximations for the arc lengths in Exercises 32–35 by using **a.** $n = 100$ and **b.** $n = 1000$. (Round your answers to four decimal places.)

Again we see that, because of rounding errors and other reasons, our technology can sometimes mislead us by giving seemingly conflicting or inaccurate feedback, and we must be aware of this when using it.

We will stick with our guess that

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1-\cos x}{x} = 0,$$

but realize that we haven't actually proved this at all. We will learn how to do that after discussing limit theorems in upcoming sections.

Technology Note **Finding a Limit**

Computer algebra systems such as *Mathematica* provide additional tools for determining limits, but it should always be remembered that software has limitations and can be fooled. *Mathematica* contains the built-in command **`Limit`** that uses the same mathematical facts we will learn in the next two sections to correctly evaluate many types of limits. Its use is illustrated below. (For additional information on *Mathematica* and the use of the **`Limit`** command, see Appendix A.)

```
In[1]:= Limit[(Sqrt[x^2 + 4] - 2) / x^2, x → 0]

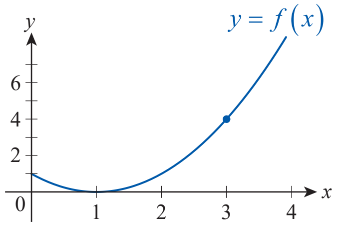
         1
Out[1]=  —
         4
```

Figure 17

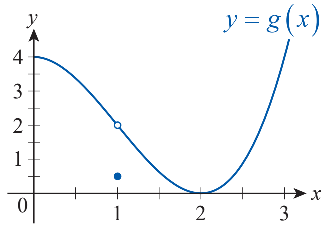
2.2 Exercises

1–4 Use the graph of the function to find the indicated limit (if it exists).

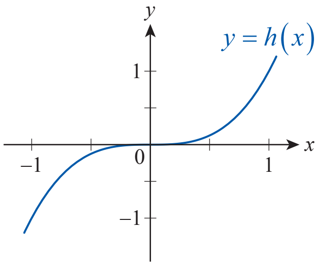
1. $\lim_{x\to 3} f(x)$



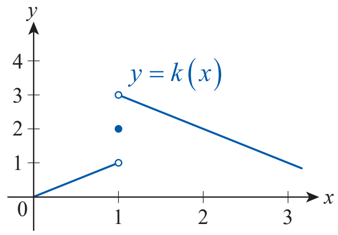
2. $\lim_{x\to 1} g(x)$



3. $\lim_{x\to 0} h(x)$



4. $\lim_{x\to 1} k(x)$



5–12 Create a table of values to estimate the value of the indicated limit without graphing the function. Choose the last x-value so that it is no more than 0.001 units from the given c-value.

5. $\lim_{x\to\sqrt{2}} \frac{x^2-2}{x-\sqrt{2}}$

6. $\lim_{x\to 3} \frac{x^3-9x^2+27x-27}{x-3}$

7. $\lim_{x\to 1} \frac{x^{10}-1}{x-1}$

8. $\lim_{x\to 0} \frac{4\sin x}{3x}$

9. $\lim_{x\to \pi} \frac{2\cos x-1}{1-\sin x}$

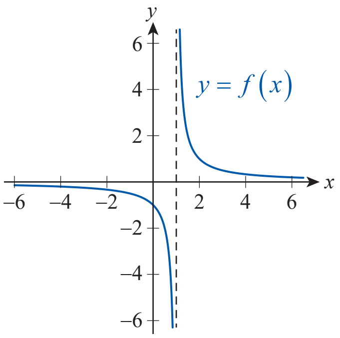
10. $\lim_{x\to 7^-} \frac{x^2-49}{x-7}$

11. $\lim_{x\to 7^+} \frac{x^2+49}{x-7}$

12. $\lim_{x\to 0^+} \frac{\sqrt{4+x}}{x}$

13–24 Use one-sided limit notation to describe the behavior of the function near its vertical asymptote(s).

13.



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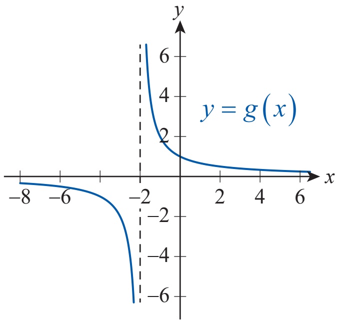

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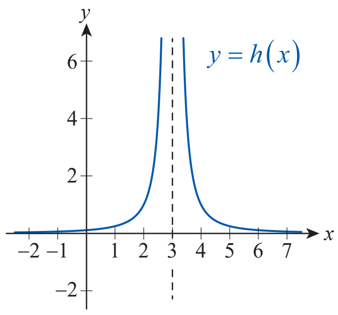

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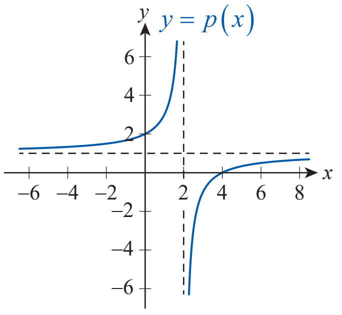

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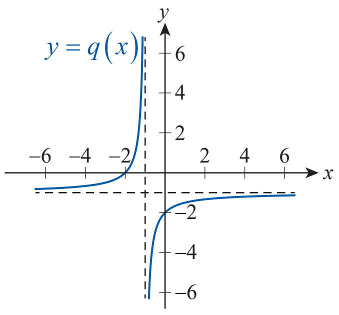

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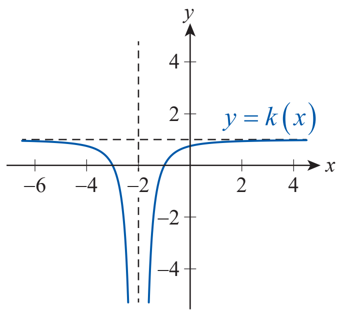

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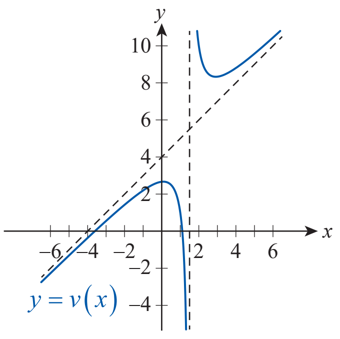

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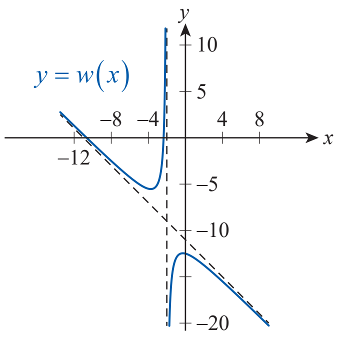

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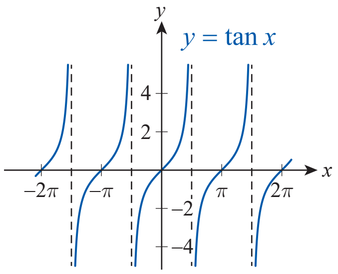

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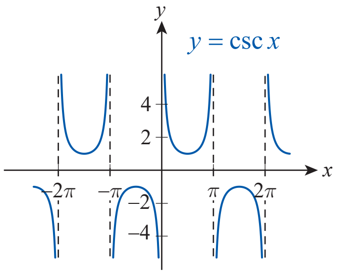

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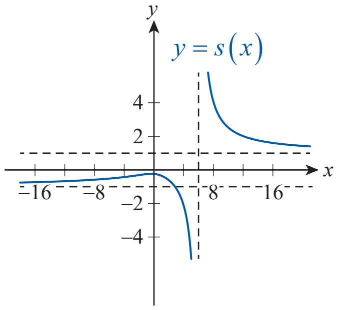

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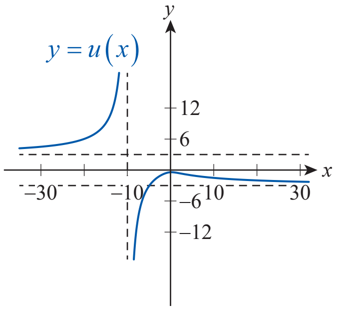
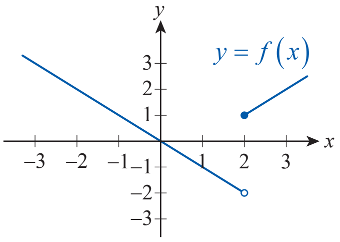


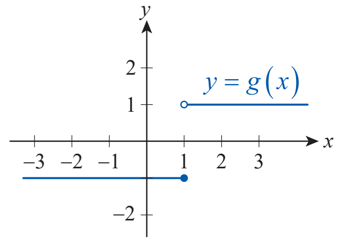
25–36. Consider the functions given in Exercises 13–24. Find their limits at ∞ and $-\infty$ (if they exist). When applicable, use the horizontal asymptote(s) as a guide.

37–46 Use the graph to find the indicated one-sided limits, if they exist.

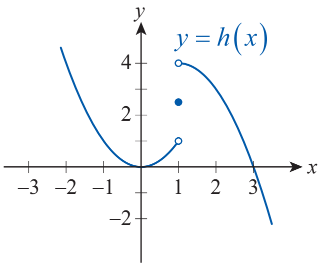
37. **a.** $\lim_{x \to 2^-} f(x)$ **b.** $\lim_{x \to 2^+} f(x)$



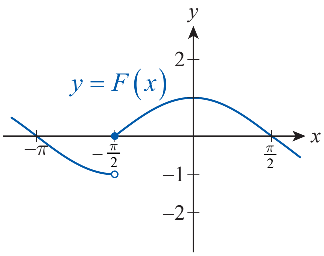
38. **a.** $\lim_{x \to 1^-} g(x)$ **b.** $\lim_{x \to 1^+} g(x)$



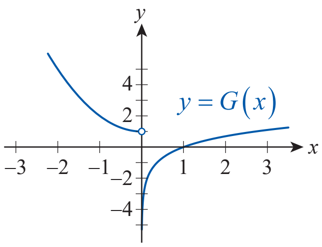
39. **a.** $\lim_{x \to 1^-} h(x)$ **b.** $\lim_{x \to 1^+} h(x)$



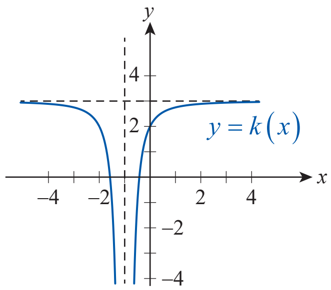
40. **a.** $\lim_{x \to (-\pi/2)^-} F(x)$ **b.** $\lim_{x \to (-\pi/2)^+} F(x)$



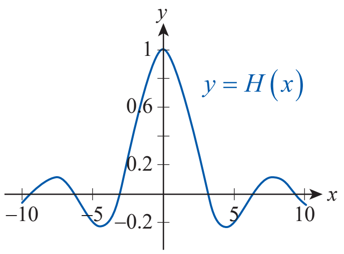
41. **a.** $\lim_{x \to 0^-} G(x)$ **b.** $\lim_{x \to 0^+} G(x)$



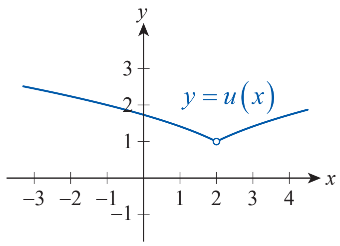
42. **a.** $\lim_{x \to -1^-} k(x)$ **b.** $\lim_{x \to -1^+} k(x)$



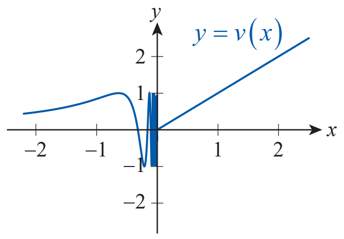
43. **a.** $\lim_{x \to 0^-} H(x)$ **b.** $\lim_{x \to 0^+} H(x)$



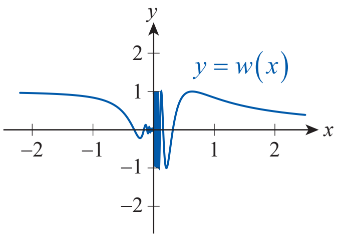
44. **a.** $\lim_{x \to 2^-} u(x)$ **b.** $\lim_{x \to 2^+} u(x)$



45. **a.** $\lim_{x\to 0^-} v(x)$ **b.** $\lim_{x\to 0^+} v(x)$



46. **a.** $\lim_{x\to 0^-} w(x)$ **b.** $\lim_{x\to 0^+} w(x)$



47–58 Use limit notation to describe the unbounded behavior of the given function as *x* approaches ∞ and/or −∞.

47. $f(x) = x^3$

48. $g(x) = x^2 + 2.1x - 1$

49. $h(x) = -x^4 + 0.2x^3$

50. $k(x) = -0.35x^5 + x + 1.35$

51. $F(x) = \sqrt{x+2}$

52. $G(x) = \sqrt[3]{x+1} - 2.3$

53. $H(x) = |x+2|$

54. $K(x) = -|x+2| - 1$

55. $u(x) = |x-1| + |x+2|$

56. $v(x) = e^{x+2}$

57. $s(x) = -10^{-x} + 1$

58. $t(x) = \ln x - 1$

Concept Check

59–63 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

59. If $\lim_{x\to c} f(x)$ does not exist, then $f(x)$ is undefined at $x = c$.

60. If $f(x)$ is undefined at $x = c$, then $\lim_{x\to c} f(x)$ does not exist.

61. If $f(x)$ is defined on $(0, \infty)$ and $y = 0$ is a horizontal asymptote for $f(x)$, then there exists a number $M > 0$ such that if $x > M$ then $f(x) < 1/10^6$.

62. If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c} f(x) = \infty$ or $\lim_{x\to c} f(x) = -\infty$.

63. If $\lim_{x\to c} f(x)$ does not exist, then $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$ or at least one of $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ does not exist.

2.2 Technology Exercises

64–71 Use a graphing utility to decide whether the given limit exists by evaluating the function at several *x*-values approaching the indicated *c*-value. Then graph the function to confirm your findings. Do you obtain misleading graphs when choosing small viewing windows?

64. $\lim_{x\to 2} \dfrac{x^2 - 5x + 6}{x - 2}$

65. $\lim_{x\to 3} \dfrac{x - 2}{x^2 - 5x + 6}$

66. $\lim_{x\to -1.5} \dfrac{x^2 - 2.25}{x + 1.5}$

67. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1}$

68. $\lim_{x\to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

69. $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$

70. $\lim_{x\to 0^+} \cos \dfrac{1}{x}$

71. $\lim_{x\to 0^+} x \cos \dfrac{1}{x}$

72. Evaluate the function $f(x) = \left(1 + \dfrac{1}{x}\right)^x$ for several consecutive positive integers, and try to observe a tendency. Then use a graphing utility to graph $f(x)$ in a large viewing window and try to guess $\lim_{x\to\infty} f(x)$. Have you seen that number before?

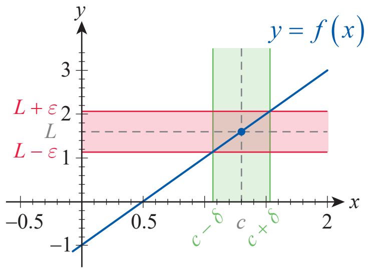
73. Write a program in a graphing calculator or computer algebra system to estimate the limit of an input function as *x* approaches *c*. (Calculate $f(x)$ successively at *x*-values increasingly close to *c* and display the results.) Try your program on Exercises 64–71.

74–81. Use the `Limit` command specific to your computer algebra system to evaluate the limits in Exercises 64–71. Are your previous results confirmed?

2.3 Exercises

1–4 Use the graph to estimate δ corresponding to the given ε satisfying the ε-δ definition of $\lim_{x\to c} f(x) = L$.

1.



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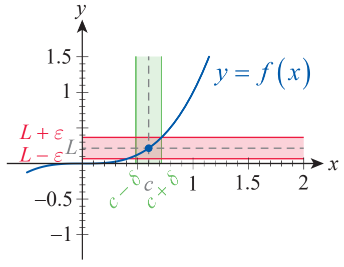

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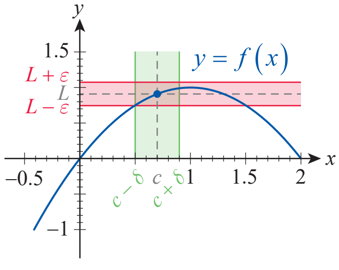

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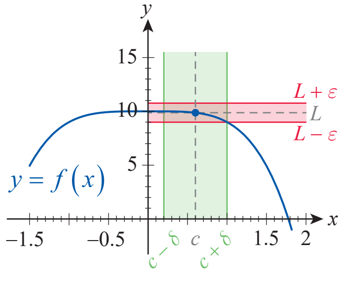


5–10 Calculus students gave the following definitions for the existence of a limit of $f(x)$ at c. Find and correct any errors.

5. "$\lim_{x\to c} f(x)$ exists if for any $\varepsilon > 0$ and real number L there is a $\delta > 0$ such that $0 < |x - L| < \delta$ implies $|f(x) - c| < \varepsilon$."

6. "$\lim_{x\to c} f(x)$ exists and equals L if for any $\varepsilon > 0$ and $\delta > 0$ whenever $0 < |x - c| < \varepsilon$, we have $|f(x) - L| < \delta$."

7. "If there is a real number L such that for an $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \neq c$, we have $|f(x) - L| < \varepsilon$, we say that the limit of the function at c is L."

8. "We say that $\lim_{x\to c} f(x) = L$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$."

9. "We say that $\lim_{x\to c} f(x) = L$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 \le |x - c| \le \delta \Rightarrow |f(x) - L| \le \varepsilon$."

10. "If the real number L is such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$, we say that $\lim_{x\to c} f(x) = L$."

11–20 Find a $\delta > 0$ that satisfies the limit claim corresponding to $\varepsilon = 0.1$, that is, such that $0 < |x - c| < \delta$ would imply $|f(x) - L| < 0.1$.

11. $\lim_{x\to 2}(5x - 1) = 9$

12. $\lim_{x\to 1}(3x + 1) = 4$

13. $\lim_{x\to -1}(-x + 2) = 3$

14. $\lim_{x\to 6}\left(4 - \frac{x}{2}\right) = 1$

15. $\lim_{x\to 0} x^2 = 0$

16. $\lim_{x\to 8} \sqrt[3]{x} = 2$

17. $\lim_{x\to 1} \frac{1}{x} = 1$

18. $\lim_{x\to 0} e^x = 1$

19. $\lim_{x\to 1} \ln x = 0$

20. $\lim_{x\to 0} \cos x = 1$

21–26 Find a number N that satisfies the limit claim corresponding to $\varepsilon = 0.1$, that is, such that $x > N$ (or $x < N$, as appropriate) would imply $|f(x) - L| < 0.1$.

21. $\lim_{x\to\infty} \frac{1}{x} = 0$

22. $\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 2x} = 1$

23. $\lim_{x\to -\infty} \frac{x + 1}{x} = 1$

24. $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2 + x}} = 2$

25. $\lim_{x\to -\infty} e^x = 0$

26. $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

27–32 For the given function $f(x)$, find a $\delta > 0$ corresponding to $M = 100$, that is, such that $0 < |x - c| < \delta$ would imply $f(x) > 100$ (let $N = -100$ if the limit is $-\infty$, in which case $0 < |x - c| < \delta$ should imply $f(x) < -100$).

27. $\lim_{x \to 0} \frac{1}{x^2} = \infty$

28. $\lim_{x \to 0^+} \frac{1}{x} = \infty$

29. $\lim_{x \to -1} \frac{-1}{(x+1)^2} = -\infty$

30. $\lim_{x \to 0^+} \ln x = -\infty$

31. $\lim_{x \to (\pi/2)^-} \tan x = \infty$

32. $\lim_{x \to 0^+} \csc x = \infty$

33–46 Use the ε-δ definition to prove the limit claim. (**Hint:** See Examples 3 and 4 for guidance as you work through these exercises.)

33. $\lim_{x \to 1} (2x+3) = 5$

34. $\lim_{x \to 7} x = 7$

35. $\lim_{x \to c} a = a$

36. $\lim_{x \to 4} \left(\frac{1}{4}x + 1\right) = 2$

37. $\lim_{x \to 0} \left(\frac{1}{2} - 4x\right) = \frac{1}{2}$

38. $\lim_{x \to 9} \left(5 - \frac{x}{3}\right) = 2$

39. $\lim_{x \to 1} x^3 = 1$

40. $\lim_{x \to 0} x^2 = 0$

41. $\lim_{x \to 0} \frac{1}{2}|x| = 0$

42. $\lim_{x \to -2} |x+2| = 0$

43. $\lim_{x \to 1^+} \sqrt{x-1} = 0$

44. $\lim_{x \to 0^-} \left(\sqrt[3]{x} + 1\right) = 1$

45. $\lim_{x \to 1} (x^2 + x) = 2$

46. $\lim_{x \to 3} (3x^2 - 9x + 5) = 5$

47–62 Give the formal definition of the limit claim. Then use the definition to prove the claim. (**Hint:** See Examples 5 and 6 for guidance as you work through these exercises.)

47. $\lim_{x \to \infty} \frac{1+x}{x} = 1$

48. $\lim_{x \to -\infty} \frac{2}{x^2} = 0$

49. $\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$

50. $\lim_{x \to \infty} \frac{1+3x^3}{x^3} = 3$

51. $\lim_{x \to -\infty} 2^x = 0$

52. $\lim_{x \to \infty} (e^{-x} - 1) = -1$

53. $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

54. $\lim_{x \to \infty} (2 \arctan x) = \pi$

55. $\lim_{x \to 0^+} \frac{1}{x} = \infty$

56. $\lim_{x \to 0} \frac{1}{x^4} = \infty$

57. $\lim_{x \to -1} \frac{-1}{(x+1)^2} = -\infty$

58. $\lim_{x \to 0^+} \log x = -\infty$

59. $\lim_{x \to (\pi/2)^+} \tan x = -\infty$

60. $\lim_{x \to 0^+} \csc x = \infty$

61. $\lim_{x \to 2} \frac{-3}{(x-2)^2} = -\infty$

62. $\lim_{x \to -2^-} \frac{x+3}{x+2} = -\infty$

63–67 Decide whether the given limit exists. Prove your conclusion. (**Hint:** See Example 7 for guidance as you work through these exercises.)

63. $\lim_{x \to 0^+} \sin \frac{\pi}{x}$

64. $\lim_{x \to 0^+} x^2 \cos \frac{1}{x}$

65. $\lim_{x \to 0} \frac{|x|}{x}$

66.* $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

67.* $\lim_{x \to 0} g(x)$, where $g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

68. Use ε and δ to state what $\lim_{x \to c} f(x) \neq L$ means.

69. A piston is manufactured to fit into the cylinder of a certain automobile engine. Suppose that the diameter of the cylinder is 82 mm and that the cross-sectional area of the piston is not allowed to be less than 99.89% of that of the cylinder. If both are perfectly round, what does this mean in terms of maximum tolerance for the clearance between the piston and the cylinder wall? (Be sure to identify which function and data take the roles of $f(x)$, c, ε, and δ in this problem.)

70. The tension in a stretched steel wire (in newtons, N) is calculated by the formula $F = E\frac{\Delta L}{L_0}A$, where $E = 2 \times 10^{11}\ \text{N/m}^2$ is the elastic modulus (or Young's modulus) of steel, ΔL is the elongation, L_0 the original length, and A the cross-sectional area (in m^2). Suppose a 1-meter-long steel string of radius 1 millimeter is stretched by 2 millimeters when tuning a musical instrument.

a. Calculate the tension in the string caused by the above tightening.

b. If we are not allowed to overload the string by more than 100 N, what is the tolerance in the amount of stretching? (Be sure to identify the function and data taking the roles of c, ε, and δ in this problem.)

Concept Check

71–73 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

71. If $f(c) = L$, then as x approaches c, $\lim_{x \to c} f(x) = L$.

72. If $\lim_{x \to c} f(x)$ exists and equals L, then $f(c) = L$.

73. If $f(x) < g(x)$ for all $x \neq c$, and both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then $\lim_{x \to c} f(x) < \lim_{x \to c} g(x)$.

2.3 Technology Exercises

74–83 Use a graphing utility to estimate the given limit. By zooming in appropriately, find δ-values that correspond to $\varepsilon = 0.1$. (Answers will vary.)

74. $\lim_{x\to 5}\frac{x^2-5x+6}{x-2}$

75. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

76. $\lim_{x\to 3.5}\frac{x^2-6.25}{x+2.5}$

77. $\lim_{x\to 0}\frac{x-1}{\sqrt{x}-1}$

78. $\lim_{x\to 0}\frac{\sin 3x}{2x}$

79. $\lim_{x\to -\infty}\frac{2x+3}{\sqrt{x^2+1}}$

80. $\lim_{x\to \infty}\frac{\sqrt{9x^2+1}}{x-2}$

81. $\lim_{x\to -\infty}\frac{2x^2+1.5x-7}{\sqrt{x^4+1}}$

82. $\lim_{x\to \infty}\left(\sqrt{x^2+3x+5}-\sqrt{x^2+2x+1}\right)$

83. $\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x$

84–89 Use a graphing utility to locate a vertical asymptote of the given function. Then for such an asymptote $x = c$ find an appropriate value $\delta > 0$ such that $|x-c|<\delta \Rightarrow |f(x)|>10$. (Answers will vary.)

84. $f(x)=\frac{x^2-7}{x^3+x+1}$

85. $f(x)=\frac{3x+1}{2x^4+x-5}$

86. $f(x)=\ln\frac{x^2}{x^2+1}$

87. $f(x)=\tan\left(\frac{1}{2}x+3\right)$

88. $f(x)=\csc(2x+1)$

89. $f(x)=\cot\left(\frac{1}{2}\cos x\right)$

We will end with one last limit theorem that will prove useful in some derivations to follow. (See Appendix E for a proof.)

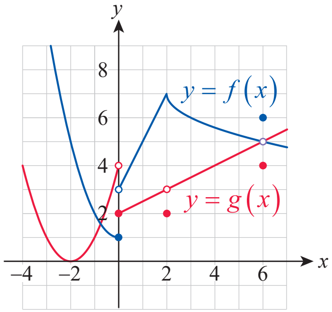
Theorem **Upper Bound Theorem**

If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at c itself, and if the limits of f and g both exist at c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

2.4 Exercises

1–2 Use the graph to find the given limit.



1. a. $\lim_{x\to 0^+}\left[g(x)-2f(x)\right]$ **b.** $\lim_{x\to 2^+}\left[g(x)f(x)\right]$

2. a. $\lim_{x\to 6}\left[g(x)+f(x)\right]$ **b.** $\lim_{x\to 0^-}\dfrac{f(x)}{2g(x)}$

3–20 Use appropriate limit laws to evaluate the given limit.

3. $\lim_{x\to 4} 5$

4. $\lim_{x\to 4} 5x$

5. $\lim_{x\to 3}(2x+1)$

6. $\lim_{x\to 1/2}(3-4x)$

7. $\lim_{x\to -3} x^2$

8. $\lim_{x\to -2}\left(-x^5\right)$

9. $\lim_{x\to 3}\left(2x^2-x+7\right)$

10. $\lim_{x\to -1}\left(3+x-\dfrac{5}{2}x^2\right)$

11. $\lim_{x\to 1/2}\left(2x^3-3x^2+x-4\right)$

12. $\lim_{x\to -2}\left(3x^3-x^5\right)$

13. $\lim_{x\to 1}\dfrac{3x-7}{x+1}$

14. $\lim_{x\to -1}\dfrac{5x+3}{x^2-x}$

15. $\lim_{x\to 3}\left(\dfrac{4x}{11x-x^3}\right)^{1/3}$

16. $\lim_{t\to 1}\left(\dfrac{2t+t^3}{3t^2+1}\right)^{3/2}$

17. $\lim_{x\to -2}\sqrt[3]{5x^4-x^3+3x^2+2x+4}$

18. $\lim_{x\to 4}\sqrt{x^4+2x^2+1}$

19. $\lim_{x\to -3}\left(\dfrac{x^4-5x}{x^3+2x^2-4x}\right)^{4/5}$

20. $\lim_{x\to -5}\sqrt[3]{\left(x^4+2x^3+x^2\right)^2}$

21–44 Use algebra to evaluate the given limit.

21. $\lim_{x\to 6}\dfrac{x^2-36}{x-6}$

22. $\lim_{x\to -7}\dfrac{x+7}{x^2-49}$

23. $\lim_{x\to 3}\dfrac{3-13x+4x^2}{x-3}$

24. $\lim_{x\to 4}\dfrac{x^4-256}{x^2-16}$

25. $\lim_{x\to 5}\dfrac{2x^3-7x^2-14x-5}{x^2-25}$

26. $\lim_{x\to 2}\dfrac{x^3-8}{x^3-2x^2+2x-4}$

27. $\lim_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}$

28. $\lim_{x\to 9}\dfrac{3-\sqrt{x}}{x-9}$

29. $\lim_{x\to 0}\dfrac{\sqrt{x+5}-\sqrt{5}}{x}$

30. $\lim_{x\to 0}\dfrac{\frac{1}{4+x}-\frac{1}{4}}{x}$

31. $\lim_{x\to 2}\dfrac{\frac{1}{3}-\frac{1}{1+x}}{x-2}$

32. If $f(x)=x^2$, find $\lim_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$.

33. If $g(x)=x^2-2$, find $\lim_{h\to 0}\dfrac{g(3+h)-g(3)}{h}$.

34. If $k(x)=1-x+x^2$, find $\lim_{h\to 0}\dfrac{k(2-h)-k(2)}{h}$.

35. If $p(x)=x^3+x$, find $\lim_{x\to 1}\dfrac{p(x)-p(1)}{x-1}$.

36. If $F(x)=\dfrac{1}{x}$, find $\lim_{x\to 1/2}\dfrac{F(x)-F\left(\frac{1}{2}\right)}{x-\frac{1}{2}}$.

37. $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

38. $\lim_{x\to 3}\dfrac{2x^2-3x-9}{x^2-9}$

39. $\lim_{x\to 2}\dfrac{x^4-16}{3x^2-5x-2}$

40. $\lim_{x\to -3}\dfrac{\frac{1}{3}+\frac{1}{x}}{x^3+27}$

41. $\lim_{x\to 2}\dfrac{3-\sqrt{x+7}}{x-2}$

42. $\lim_{x\to 8}\dfrac{8-x}{\sqrt[3]{x}-2}$

43. $\lim_{y\to 0}\left(\frac{1}{y}+\frac{1}{y^2-y}\right)$ **44.** $\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

45–50 Use $\lim_{x\to c} f(x)=3$ and $\lim_{x\to c} g(x)=-2$ to find the limit.

45. $\lim_{x\to c}\left[2f(x)-g(x)\right]$

46. $\lim_{x\to c}\frac{4f(x)+3g(x)}{f(x)-\frac{1}{2}g(x)}$

47. $\lim_{x\to c}\sqrt{\left[f(x)\right]^4+10\left[g(x)\right]^2}$

48. $\lim_{x\to c}\left(\left[f(x)-1\right]^2\sqrt[3]{g(x)}\right)$

49. $\lim_{x\to c}\left(\left[f(x)\right]^2+(x-2)g(x)\right)$

50. $\lim_{x\to c}\left(\frac{f(x)+g(x)}{\left[g(x)\right]^2}\right)^{3/2}$

51–58 Use the limit laws to find the one-sided limit.

51. $\lim_{x\to 0^+}\frac{x}{|x|}$

52. $\lim_{x\to 0^-}\operatorname{sgn} x\cos x$, where $\operatorname{sgn} x=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x=0\\ -1 & \text{if } x<0\end{cases}$

53. $\lim_{x\to 2^+}\frac{\sqrt{x-2}}{3x+1}$ **54.** $\lim_{x\to 1^-}\sqrt{1-x^2}$

55. $\lim_{x\to (1/3)^-}\frac{\sqrt{1-3x}}{6x+5}$

56. $\lim_{x\to 1^+}\left([\![x]\!]-x\right)$ (See the definition of $f(x)=[\![x]\!]$ in Section 1.2, before Exercises 43–45.)

57. $\lim_{x\to 2^+}[\![x]\!]e^x$ **58.** $\lim_{x\to -1^-}\frac{[\![x]\!]\left(2x^2+1\right)}{x+3}$

59–64 Use the Squeeze Theorem to prove the limit claim.

59. $\lim_{x\to 0}x^2\sin\frac{1}{x}=0$ **60.** $\lim_{x\to 0}|x|\cos x=0$

61. $\lim_{x\to\infty}\frac{\cos x}{x}=0$ **62.** $\lim_{x\to -\infty}e^x\sin x=0$

63. $\lim_{x\to 0^+}x^{3/2}e^{\cos(1/x)}=0$ **64.** $\lim_{x\to\infty}\frac{\sin^2 x+1}{2+x}=0$

65. Provide a rigorous proof of the limit claim $\lim_{x\to c}1=1$. (**Hint:** Use the fact that for the constant 1 function, $f(x)=1$ for all x, so in particular, if an $\varepsilon>0$ is given, $|f(x)-1|=|1-1|=0$, which makes the choice of δ "easy.")

66. Provide a rigorous proof of the limit claim $\lim_{x\to c}x=c$. (**Hint:** Since $f(x)=x$ in this problem, for a given $\varepsilon>0$ we need to ensure that $|f(x)-c|=|x-c|<\varepsilon$ as long as $0<|x-c|<\delta$. This observation makes the choice of δ obvious.)

67. Use Exercise 66 and the basic limit laws to prove the Polynomial Substitution Law. (**Hint:** From Exercise 66 and a repeated application of the Product Law, it follows that $\lim_{x\to c}x^k=c^k$. As a next step, from the Constant Multiple Law we can conclude that if $a\in\mathbb{R}$, $\lim_{x\to c}ax^k=ac^k$. From the above claim, a repeated application of the Sum Law will yield the result for a general polynomial.)

68. Use Exercise 67 and the basic limit laws to prove the Rational Function Substitution Law.

69. Combine the Positive Integer Power Law and the Positive Integer Root Law to prove the Rational Power Law. (**Hint:** Assuming first that both m and n are positive, we can write $\lim_{x\to c}\left[f(x)\right]^{m/n}=\lim_{x\to c}\left[\left[f(x)\right]^{1/n}\right]^m=\lim_{x\to c}\left[\sqrt[n]{f(x)}\right]^m$. Now use the Positive Integer Power Law followed by the Positive Integer Root Law to obtain that the above limit is equal to $\left[\lim_{x\to c}\sqrt[n]{f(x)}\right]^m=\left[\sqrt[n]{\lim_{x\to c}f(x)}\right]^m$, from which the result follows. If m is negative, note that $\left[f(x)\right]^{m/n}=1\big/\left[f(x)\right]^{-m/n}$, where $-m$ is positive. Thus if we use the Quotient Law along with the previous argument, we obtain

$$\begin{aligned}\lim_{x\to c}\left[f(x)\right]^{m/n}&=\lim_{x\to c}\frac{1}{\left[f(x)\right]^{-m/n}}\\&=\frac{1}{\lim_{x\to c}\left[f(x)\right]^{-m/n}}\\&=\frac{1}{\left[\lim_{x\to c}f(x)\right]^{-m/n}},\end{aligned}$$

from which the result follows.)

70. Let $D(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$.

Does $\lim_{x\to 0}D(x)$ exist? Prove your answer.

71. Let $F(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x^2 & \text{if } x \text{ is irrational}\end{cases}$.

Does $\lim_{x\to 0}F(x)$ exist? Prove your answer.

72.* Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = K$, then $L = K$. In words, prove that if the limit of f exists at c, then the limit is unique.

73.* Prove that if n and m are positive integers, then $\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$.

74.* Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

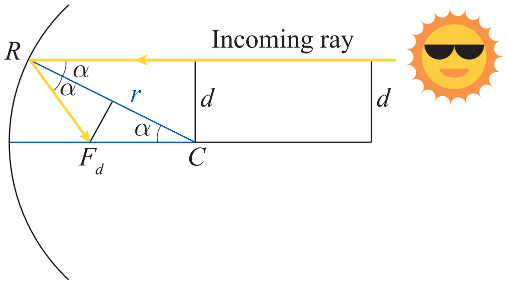
75.* Prove that if $\lim_{x\to c} f(x) = 0$ and $g(x)$ is such that $|g(x)| \le M$ for some number M (such functions are called bounded), then $\lim_{x\to c} [f(x)g(x)] = 0$.

76.* Prove that in Exercise 75, it is sufficient to require the boundedness of g only on an interval around c (except at c itself).

77.* By finding functions f and g such that $\lim_{x\to c} f(x) = 0$ but $\lim_{x\to c} [f(x)g(x)] \neq 0$, show that it is necessary to impose a boundedness condition on g in Exercise 75.

78.* Give examples of f and g to show that **a.** the existence of $\lim_{x\to c} [f(x) + g(x)]$ does not imply the existence of $\lim_{x\to c} f(x)$ and **b.** the existence of $\lim_{x\to c} [f(x)g(x)]$ does not imply the existence of $\lim_{x\to c} f(x)$.

79. A *concave spherical mirror* is a part of the inside of a sphere, silvered to form a reflective surface. The radius r of the sphere is called the mirror's *radius of curvature*. If the size of such a mirror is small relative to its radius of curvature, light rays parallel to its principal axis are reflected through approximately a single point, called *focus*. In the following illustration, C denotes the center, F_d is the focus, while d is the distance between the incoming ray and the principal axis. Note that according to the Law of Reflection, the incoming and reflected rays make the same size angle α with the radius $\overline{CR}$ (this radius is called *normal* to the mirror surface). One way to determine the *focal length* (the distance between the mirror and the focus along the principal axis) is to find the limiting position of F_d as $d \to 0$. Noting that the triangle $\triangle CRF_d$ is isosceles, by similarity we obtain $\frac{CF_d}{(r/2)} = \frac{r}{\sqrt{r^2 - d^2}}$. Use this observation to express CF_d and then determine the focal length of the spherical mirror by taking the limit as $d \to 0$.



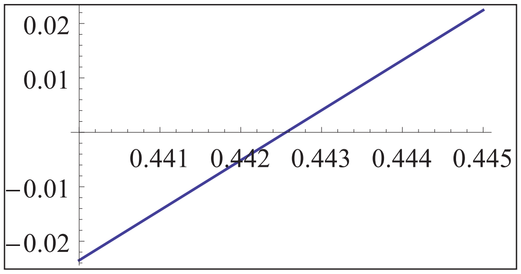
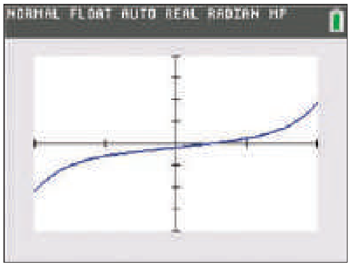


Figure 14

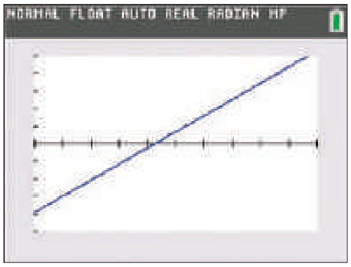
It is clear from Figure 14 that the root is located between 0.442 and 0.443, so we have achieved the desired accuracy; the first two digits after the decimal point are correct. We conclude that the root is $c \approx 0.44$. (Note that the above accuracy could actually have been achieved from carefully eyeballing the graph in Figure 13, while the graph in Figure 14 says even more: we might guess that the root is in fact very close to 0.4426.)

Note that it is also possible to estimate the root by appropriately zooming in on the graph using a graphing calculator (see Figure 15).

Finally, we use the **NSolve** command of *Mathematica* to approximate the solution of $f(x)=0$. The screenshot in Figure 16 shows the feedback we receive from the software. The appearance of complex roots shows the power of the software, but we can ignore them for now and focus on the sole real root. *Mathematica* approximates it as $c \approx 0.442558$.



$f(x)=x^5+9x-4$
on $[-2,2]$ by $[-100,100]$



$f(x)=x^5+9x-4$
on $[0.4,0.5]$ by $[-0.5,0.5]$

Figure 15 Zooming In on a Root with a Graphing Calculator

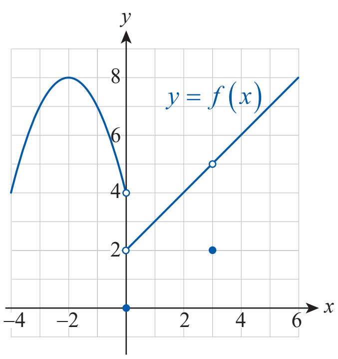
```
In[1]:= NSolve[x^5 + 9 x - 4 == 0, x]
Out[1]= {{x → -1.32389 - 1.2337 i},
         {x → -1.32389 + 1.2337 i}, {x → 0.442558},
         {x → 1.10261 - 1.24271 i}, {x → 1.10261 + 1.24271 i}}
```

Figure 16

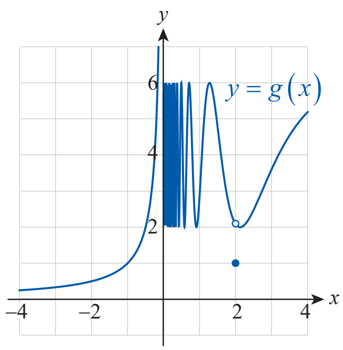
2.5 Exercises

1–2 Find all points of continuity as well as all points of discontinuity for the given function. For any discontinuities, identify those of the three continuity criteria that fail to hold.

1.



2.



3. Sketch a graph of a function (a formula is not necessary) that has a removable discontinuity at $x=-1$, a jump discontinuity at $x=2$, but is right-continuous at 2. (Answers will vary.)

4. Sketch a graph of a function that has an infinite discontinuity at $x=0$ and an oscillating discontinuity at $x=5$ so that it is still left-continuous at 5. (Answers will vary.)

5–29 Find and classify the discontinuities (if any) of the function as removable or nonremovable.

5. $f(x)=\dfrac{1}{x}$

6. $g(x)=\dfrac{-2}{x-3}$

7. $h(x)=\dfrac{x^2-9}{x-3}$

8. $k(x)=\dfrac{x^2-2x}{x^2+5x-14}$

9. $u(x)=\dfrac{x^2-9}{x-2}$

10. $v(x)=\dfrac{x-1}{x^2+2x-3}$

11. $w(x)=\begin{cases} x+1 & \text{if } x\le 0 \\ \dfrac{1}{2}x^2+1 & \text{if } x>0 \end{cases}$

12. $f(x)=\begin{cases} \dfrac{1}{2}x-2 & \text{if } x\le 4 \\ x^3+1 & \text{if } x>4 \end{cases}$

13. $g(x)=\begin{cases} \tan x & \text{if } x<\dfrac{\pi}{2} \\ \cos x & \text{if } x\ge\dfrac{\pi}{2} \end{cases}$

14. $h(x)=\begin{cases} \cos x & \text{if } x\le 0 \\ \tan x+1 & \text{if } x>0 \end{cases}$

15. $F(u)=\dfrac{u-4}{\sqrt{u}-2},\quad u\ge 0$

16. $G(s)=\dfrac{s}{\sqrt{s+4}-2},\quad s\ge -4$

17. $H(t)=\dfrac{t}{\sqrt{t^2+2}}$

18. $u(x)=\cos\dfrac{1-x^2}{1-x}$

19. $v(t)=|\sin t|$

20. $K(x)=|x+2|+|x-1|$

21. $F(t)=\dfrac{t}{t^2-1}$

22. $G(t)=\dfrac{t}{t^2+1}$

23. $H(x)=|x+2|$

24. $k(x)=\dfrac{|x-4|}{x-4}$

25. $s(x)=[\![x+2]\!]$

26. $t(x)=4-[\![x]\!]$

27. $u(z)=[\![z^2]\!]$

28. $v(x)=x[\![x]\!]$

29. $w(x)=x\left[\!\left[\dfrac{1}{x}\right]\!\right]$

30–33 Use the ε-δ definition to prove that the function is continuous.

30. $f(x)=\dfrac{1}{x}$

31. $g(x)=3x-2$

32. $F(x)=x^3$

33. $G(x)=\sqrt{x}$

34–39 Use the theorems of this section to discuss the continuity of the function.

34. $F(x)=\sqrt{\dfrac{x}{x^2+7x+12}}$

35. $G(x)=\sqrt{\dfrac{x^4-x^3-11x^2+9x+18}{2x^3+x}}$

36. $H(x)=\cos\dfrac{2\ln(x-3)+1}{\sqrt[3]{x^2-2x-15}}$

37. $f(x)=\arctan\dfrac{x}{\sqrt{3-x^2}}$

38. $g(x)=\ln(\arcsin(\pi x+1))$

39. $h(x)=\dfrac{\csc(\pi x+1)}{\sin(\pi e^{x+2})}$

40. Prove the alternate formulation of continuity, that is, the statement that a function f is continuous at the point c if and only if $\lim_{h\to 0} f(c+h)=f(c)$.

41–44 Use the alternate formulation of continuity to prove that the function is continuous.

41. $f(x)=3x-5x^2$

42. $g(x)=\cos x$

43. $h(x)=\tan x$

44. $k(x)=e^x$

45–50 Identify the removable discontinuities and define the continuous extension of the function.

45. $f(x)=\dfrac{x^2+x-12}{x-3}$

46. $g(x)=\dfrac{x^3-2x^2-x+2}{x^2-3x+2}$

47. $h(x)=\dfrac{x-1}{\sqrt{x}-1}$

48. $F(x)=\dfrac{\sqrt{x+1}-2}{x-3}$

49. $G(x)=2^{-1/x^2}$

50. $H(x)=x\cos\dfrac{\pi}{x}$

51–54 Discuss the continuity of the function on the given closed interval.

51. $S(x)=\sqrt{16-x^2}$ on $[-4,4]$

52. $T(x)=\left[\!\left[\frac{x}{3}\right]\!\right]$ on $[0,3]$

53. $U(x)=\begin{cases}\dfrac{1}{x^2-9} & \text{if } |x|<3\\ 0 & \text{if } |x|=3\end{cases}$ on $[-3,3]$

54. $V(x)=\begin{cases}x^2\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ on $\left[0,\dfrac{1}{\pi}\right]$

55-58 Find the value of a (or the values of a and b, where applicable) such that f is continuous on the entire real line.

55. $f(x)=\begin{cases}0 & \text{if } x\leq 0\\ ax & \text{if } 0<x<1\\ 2x+3 & \text{if } x\geq 1\end{cases}$

56. $f(x)=\begin{cases}x^3 & \text{if } x\leq 3\\ ax^2 & \text{if } x>3\end{cases}$

57. $f(x)=\begin{cases}-x^2 & \text{if } x<1\\ ax+b & \text{if } 1\leq x\leq 3\\ (x-3)^2+2 & \text{if } x>3\end{cases}$

58. $f(x)=\begin{cases}\cos x & \text{if } x\leq 0\\ -(x-a)^2+b & \text{if } 0<x<2\\ \dfrac{1}{2}x-2 & \text{if } x\geq 2\end{cases}$

59. Prove that if $f(x)$ is continuous and $f(c)>0$, then there is a $\delta>0$ such that $f(x)>0$ for all $x\neq c$ in the interval $(c-\delta,c+\delta)$.

60. Prove that the Dirichlet function

$$\xi(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous at every real number.

61. Prove that the function

$$f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x^2 & \text{if } x \text{ is irrational}\end{cases}$$

is continuous only at the single point $c=0$.

62. Prove that if the functions f and g are both continuous on $\mathbb{R}$ and they agree on the rationals (i.e., $f(x)=g(x)$ for all $x\in\mathbb{Q}$), then $f=g$.

63–68 Decide whether the Intermediate Value Theorem applies to the given function on the indicated interval. If so, find c as guaranteed by the theorem. If not, find the reason.

63. $f(x)=-x^2+x+3$ on $[0,3]$; $f(c)=1$

64. $g(x)=2x^3-x^2-1$ on $[-1,2]$; $g(c)=0$

65. $h(x)=\dfrac{x}{x+2}$ on $[0,4]$; $h(c)=\dfrac{1}{2}$

66. $F(x)=\dfrac{2x}{x-1}$ on $[0,2]$; $F(c)=2$

67. $G(x)=[\![x-2]\!]$ on $[-2,2]$; $G(c)=-\dfrac{1}{2}$

68. $H(x)=\sin\dfrac{3x+2}{2}$ on $\left[-\dfrac{2}{3},\dfrac{\pi-2}{3}\right]$; $H(c)=\dfrac{1}{2}$

69–74 Use the Intermediate Value Theorem to prove that the given equation has a solution on the indicated interval.

69. $x^3-7.5x^2+1.2x+1=0$ on $[-1,0]$

70. $2x^3+x+10=0$ on $[-2,1]$

71. $\cos x=x^2$ on $[0,\pi]$

72. $\ln x-\sqrt{x-2}=0$ on $[2,5]$

73. $\dfrac{5}{x^2+2}=1$ on $[-3,-1]$

74. $\cot\dfrac{\pi x}{4}-\dfrac{x}{x+2}=-\dfrac{1}{2}$ on $[1,3]$

75. Suppose that the outside temperature in Columbia, SC, on a summer morning at 7:00 a.m. is 74 °F, and it shoots up to 98 °F by 1:00 p.m. Assuming that temperature changes continuously, prove that sometime between 7:00 a.m. and 1:00 p.m. the temperature was exactly 88.35 °F.

76. Prove that if f is continuous and never 0 on the interval $[a,b]$, then either $f(x)>0$ for every x in $[a,b]$, or $f(x)<0$ for every x in $[a,b]$.

77.* (Existence of n^{th} roots) Prove that if b is a positive real number and n is a positive integer, then there is a positive real number c such that $c^n=b$. (**Hint:** Consider the continuous function $f(x)=x^n$ on the interval $[0,b+1]$.)

78.* Prove that a circle of diameter d has a chord of length c for every number c between 0 and d.

79. Use the function

$$f(x)=\begin{cases}\sin\frac{\pi}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

to prove that the converse of the Intermediate Value Theorem is false; in other words, a function may possess the Intermediate Value Property without being continuous.

80.* (Fixed Point Theorem) Prove that if the function $f:[a,b]\to[a,b]$ is continuous, then there is a number c in $[a,b]$ with $f(c)=c$ (i.e., c is "fixed," or "not being moved," by f).

81.* A hermit leaves his hut at the foot of a mountain one day at 6:00 a.m. and sets out to climb all the way to the top. He arrives at 6:00 p.m. and realizes that it is too late to go back, so he sets up camp for the night. At 6:00 a.m. the following day, he starts hiking back to his hut, taking the exact same route as the day before. This time, however, it is mostly downhill, so he makes much better time and arrives home at 2:00 p.m. Prove that there is a point along the hermit's route that he passed at exactly the same time on both days. (**Hint:** Apply the Intermediate Value Theorem or the Fixed Point Theorem.)

82. For certain international calls, a phone company charges 31 cents for the first minute and 10 cents for each additional minute or any fraction thereof. Graph the cost as a function of time, find a formula for it, and discuss the significance of its discontinuities. (**Hint:** Use the greatest integer function to construct your answer.)

83.* If Δt denotes the length of the time interval between two events as measured by an observer on a spaceship moving at speed v, and ΔT is the length of the same time interval as measured from Earth, then the formula relating the two quantities is given by

$$\Delta T=\frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}},$$

where c is the speed of light. This phenomenon is called *time dilation*, and it follows from the theory of relativity. In essence, it says that a clock moving at speed v relative to an observer is perceived by the same observer to run slower.

a. Explain why we don't normally notice the time dilation effect in everyday life.

b. What is the significance of the discontinuity of ΔT (as a function of v)?

Concept Check

84–88 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

84. If f is both left- and right-continuous at c, then f is continuous at c.

85. Any function f has an interval (a,b) on which it is continuous.

86. If c is a discontinuity of f, but f does not have a vertical asymptote at c, then c is a removable or jump discontinuity.

87. If $\lim_{x\to c} f(x)=L$, and $f(c)=L$, then f is continuous at c.

88. If c is a discontinuity of f, but $\lim_{x\to c^+} f(x)$ exists, then c is a removable or jump discontinuity.

2.5 Technology Exercises

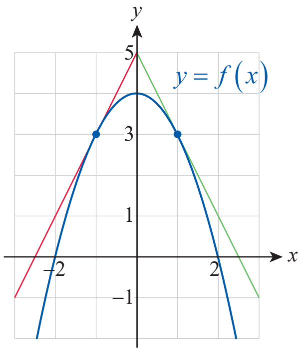
89–94. Use a graphing utility to solve the equations given in Exercises 69–74 to four decimal places.

95–100. Use a graphing utility to graph the functions of Exercises 34–39 and explain how the graphs support your discussions of continuity in the aforementioned exercises.

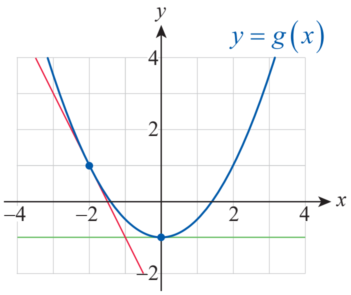
2.6 Exercises

1–2 Use the graph to estimate the derivative at the given points.

1. **a.** $x_1 = -1$ **b.** $x_2 = 1$



2. **a.** $x_1 = -2$ **b.** $x_2 = 0$



3–14 Find the equation of the tangent line to the graph of $f(x)$ at the given point.

3. $f(x) = x^2 - 2;\quad (2,2)$

4. $f(x) = 3x - 2x^2;\quad (-1,-5)$

5. $f(x) = \frac{1}{2}x + 4;\quad (2,5)$

6. $f(x) = 1 - 5x;\quad (0,1)$

7. $f(x) = x^3;\quad (2,8)$

8. $f(x) = 5x - 2x^3;\quad (-1,-3)$

9. $f(x) = \sqrt{x+1};\quad (0,1)$

10. $f(x) = 2\sqrt{1-3x};\quad (-1,4)$

11. $f(x) = \frac{1}{x};\quad \left(\frac{1}{2},2\right)$

12. $f(x) = \frac{5}{1-2x};\quad \left(-1,\frac{5}{3}\right)$

13. $f(x) = \frac{1}{\sqrt{x}};\quad \left(4,\frac{1}{2}\right)$

14. $f(x) = \frac{2}{\sqrt{x+1}};\quad (3,1)$

15–38 Use the definition (also called the *limit process*) to find the derivative function f' of the given function f. Find all x-values (if any) where the tangent line is horizontal.

15. $f(x) = 2$

16. $f(x) = 2x$

17. $f(x) = 4x + 5$

18. $f(x) = 3 - \frac{2}{5}x$

19. $f(x) = 3x^2$

20. $f(x) = 4 - 2x^2$

21. $f(x) = \frac{1}{2}x^2 + 5x - 7$

22. $f(x) = x - \frac{1}{3}x^2$

23. $f(x) = x^3 + x$

24. $f(x) = 7 + x - 3x^2 + x^3$

25. $f(x) = x^4$

26. $f(x) = \frac{1}{2x}$

27. $f(x) = \frac{5}{2x-4}$

28. $f(x) = \frac{x-2}{x+2}$

29. $f(x) = \frac{2x+1}{x-3}$

30. $f(x) = \frac{2}{x^2}$

31. $f(x) = \frac{1}{x^2+1}$

32. $f(x) = \frac{2}{x^2-2x}$

33. $f(x) = \sqrt{5x}$

34. $f(x) = \frac{1}{\sqrt{5x}}$

35. $f(x) = \sqrt{2x+1}$

36. $f(x) = \frac{1}{\sqrt{x-2}}$

37. $f(x) = \sqrt{x^2+1}$

38. $f(x) = \frac{1}{\sqrt{x^2+1}}$

39–44 Find the equation of a tangent line to the graph of the function that is parallel to the given line.

39. $f(x) = x^2 + 3;\quad y - 6x + 1 = 0$

40. $g(x) = 2x - x^2;\quad y - 5 = 4x$

41. $h(x) = \frac{1}{2x};\quad x + 2y = 3$

42. $F(x) = \frac{1}{x-3};\quad y + 4x + 7 = 0$

43. $G(x) = \frac{1}{\sqrt{x}};\quad 54y + x = 1$

44. $H(x) = \frac{1}{\sqrt{x^2-7}};\quad 27y + 4x - 2 = 0$

45–56 Use the alternate form of the definition of the derivative $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ to evaluate the given slope.

45. $f(x)=5-\frac{1}{4}x;\quad f'(3.6)$

46. $g(x)=x^2+1;\quad g'(-1)$

47. $h(x)=(x+2)^2;\quad h'(3)$

48. $F(t)=\frac{1}{t-3};\quad F'(2)$

49. $G(x)=\frac{2}{5-x};\quad G'(7)$

50. $k(t)=\sqrt{t+5};\quad k'(11)$

51. $u(x)=2\sqrt{1-x};\quad u'(-3)$

52. $v(x)=\frac{1}{x^2+1};\quad v'(0)$

53. $w(s)=\frac{1}{\sqrt{s+4}};\quad w'(5)$

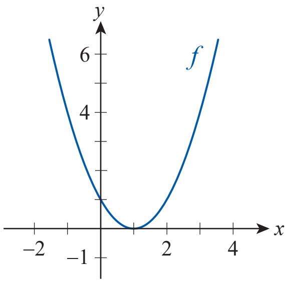
54. $F(t)=t^3-t;\quad F'(1)$

55. $G(s)=s^4;\quad G'(-2)$

56. $H(x)=\frac{2}{\sqrt{x^2+1}};\quad H'(0)$

57–60 Match the graph of f with the graph of its derivative f' (labeled A–D).

57.



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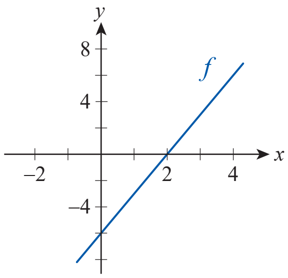

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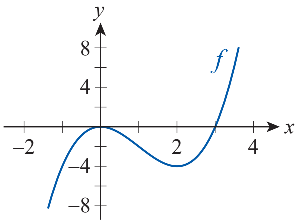

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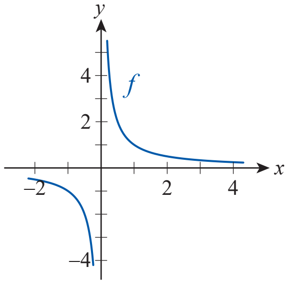
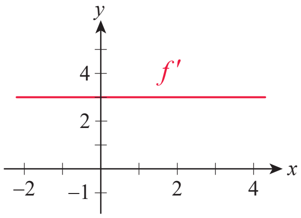


A.



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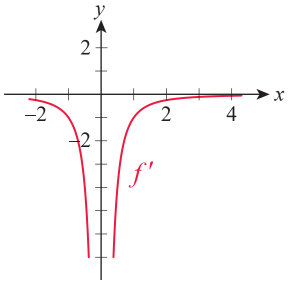

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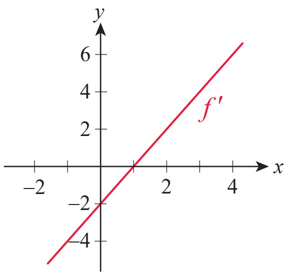
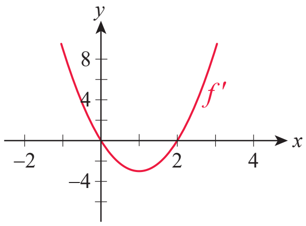


D.



61–65 Sketch the graph of a function f possessing the given characteristics. (A formula is useful, but not necessary.)

61. $f(0)=1,\ f'(0)=0,\ f'(x)<0$ for $x<0$, $f'(x)>0$ for $x>0$

62. $f(1)=0,\ f'(1)=0,\ f'(x)\geq 0$ on the entire real line

63. $f(x)>0$ on the entire real line, $f'(x)<0$ on the entire real line

64. $f(1)=1,\ f'(1)=-1,\ f'$ is nonzero on the entire real line

65. $f(1)=5,\ f'(x)=5$ on the entire real line

66. Prove that if $f(x)=c$ (a constant function), then $f'(x)=0$.

67. Use the definition of the derivative to prove that if $f(x)=x$, then $f'(x)=1$.

68. Generalize Exercise 67 to prove that if $f(x)$ is a linear function, then $f'(x)$ is constant.

69.* Use the definition of the derivative to prove that if $f(x)=x^n$ for a positive integer n, then $f'(x)=nx^{n-1}$.

70.* Recall from Section 1.1 that a function f is even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$ throughout its domain. Prove that the derivative of an even function is odd and, vice versa, an odd function has an even derivative.

71.* Find the equation of the line tangent to the graph of $f(x)=1/x$ at the point $(c, f(c))$. Prove that the area of the triangle bounded by the tangent line and the coordinate axes is the same for all $c\neq 0$.

72. The position function of a moving particle is given by $p(t)=t^2-3t+1$ feet at t seconds. Find all points in time where the particle's speed is $1\ \text{ft/s}$. When does it come to a momentary stop?

73. Repeat Exercise 72 with the position function $p(t)=\frac{1}{9}t^3-t^2+\frac{8}{3}t$.

74. A baseball is hit vertically upward with an initial speed of $80\ \text{ft/s}$. When does it slow down to $32\ \text{ft/s}$? How high does it go and how long is it aloft? (**Hint:** Use the position function $h(t)=-16t^2+80t$. Ignore air resistance.)

75. A rock is thrown upward from the edge of a 150 ft high cliff with an initial velocity of $48\ \text{ft/s}$.

a. Calculate the velocity and speed of the rock when it is exactly 32 ft above the person's hand.

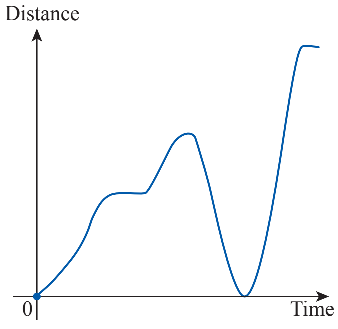
b. How high does it go and when does it reach the bottom of the cliff?

c. What is the velocity of impact?

(**Hint:** Use $h(t)=-16t^2+48t+150$ as the position function, where h is in feet, t in seconds. Ignore air resistance.)

76. A package is dropped from a small airplane 122.5 meters above Earth. If we ignore air resistance, how much time does the package need to reach the ground and what is the speed of impact? (**Hint:** The position function is $h(t)=-4.9t^2+122.5$ meters, where t is measured in seconds.)

77. The following graph is a position function of a student's car relative to her home as she drove to class one morning. From the graph, recreate a possible story of her trip, mentioning distance, velocity, speed, and so forth.



78. A manufacturer has determined that the revenue from the sale of x cell phones is given by $R(x)=94x-0.03x^2$ dollars. The cost of producing x telephones is $C(x)=10{,}800+34x$ dollars.

a. Find the profit function $P(x)$ and any break-even points.

b. Find $P(200)$, $P(400)$, and $P(600)$.

c. Find the marginal profit function $P'(x)$.

d. Find $P'(200)$, $P'(400)$, and $P'(600)$.

79. The owner of a leather retailer has determined that he can sell x attaché cases if the price is $p = D(x) = 46 + 0.25x$ dollars ($D(x)$ is often called the demand function). The total cost for these cases is $C(x) = 0.15x^2 + 6x + 190$ dollars.

a. Find the profit function $P(x)$. (**Hint:** Find the revenue function $R(x)$ first.)

b. Find any break-even points.

c. Find $P(25)$, $P(30)$, and $P(40)$.

d. Find the marginal profit function $P'(x)$.

e. Find $P'(25)$, $P'(30)$, and $P'(40)$.

80. The average cost $\bar{C}(x)$ of manufacturing x units of a certain product is $C(x)/x$, where $C(x)$ is the total cost function.

a. Find the average cost function if

$$C(x) = 30 + 2x + 0.003x^2.$$

b. What is the rate of change of average cost?

c. What value of x results in a minimum average cost? (**Hint:** Use the fact that when average cost is a minimum, its rate of change is 0. Alternatively, use technology to graph $C(x)$ for $x \geq 0$ and zoom in on the lowest point.)

81. The average manufacturing cost function for a product is given by $\bar{C}(x) = 20x^{-1} + 3$. Determine the cost function and the marginal cost function for the product. (**Hint:** See Exercise 80.)

2.6 Technology Exercises

82–105. Referring back to the functions given in Exercises 15–38, use a graphing utility to sketch the graph of f along with that of f' in the same viewing window. Compare the graphs and describe their relationship.

3.1 Exercises

1–12 Find the derivative of the given function at the specified point and express your answer using the differential notation due to Leibniz.

1. $f(x)=7;\quad x=1$

2. $g(x)=\frac{1}{2}x-5;\quad x=-1$

3. $h(x)=\frac{1}{2}x^2-5;\quad x=0$

4. $F(t)=\frac{1}{5}t+2t^2;\quad t=\frac{1}{5}$

5. $G(s)=\frac{1}{3}s^3-s;\quad s=-3$

6. $H(t)=\frac{1}{2}t^4+t^2;\quad t=-2$

7. $K(z)=\frac{5}{3z+1};\quad z=0$

8. $T(t)=\frac{2t-3}{t+1};\quad t=\sqrt{5}-1$

9. $w(z)=\frac{1}{z^2+2};\quad z=\sqrt{2}$

10. $A(t)=\sqrt{2t};\quad t=0$

11. $Q(y)=\frac{1}{\sqrt{3y}};\quad y=1$

12. $B(u)=\sqrt{u^2+1};\quad u=-2\sqrt{2}$

13–24 Find the derivative of the function and use the differentiation operator D_x to express your answer.

13. $f(x)=\pi^2$

14. $g(x)=1-\frac{2}{3}x$

15. $h(t)=\frac{3}{2}t^2+10t-1$

16. $F(s)=4-s+\frac{1}{2}s^2+s^3$

17. $G(y)=\frac{1}{3y}$

18. $H(t)=\frac{t-3}{t+3}$

19. $S(z)=\frac{-3}{z^2}$

20. $T(u)=\frac{4}{u^2-3u}$

21. $R(v)=\sqrt{2v-3}$

22. $f(y)=\frac{2}{\sqrt{y+2}}$

23. $X(y)=2y^4$

24. $u(x)=\frac{2}{\sqrt{2x^2+1}}$

25–33 Find the first, second, and third derivatives of the function. Then graph the function along with its derivatives in the same coordinate system and compare the graphs. (**Hint:** See Example 4.)

25. $f(x)=\frac{5}{2}x-1$

26. $g(x)=x^2+5$

27. $h(x)=-\frac{1}{2}x^2+x-\frac{3}{2}$

28. $U(x)=-x^3$

29. $V(x)=\frac{1}{3}(x-2)^3$

30. $F(t)=t^4-1$

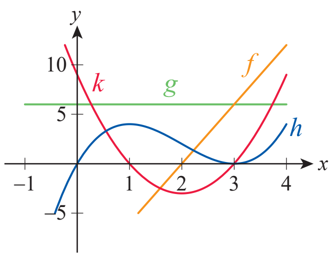
31. $G(x)=2(x-1)^4$

32. $H(x)=\frac{1}{x}$

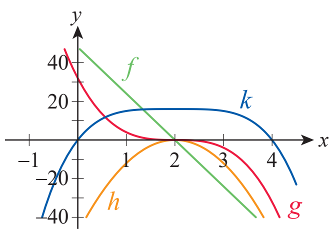
33. $K(s)=\frac{-2}{s-1}$

34–37 The graphs of the position, velocity, acceleration, and jerk of a moving particle are given. Decide which one is which, label them accordingly, and explain.

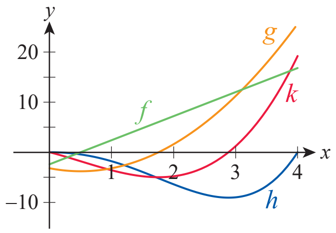
34.



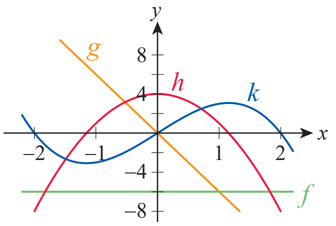
35.



36.

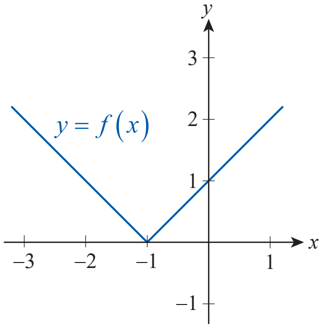


37.

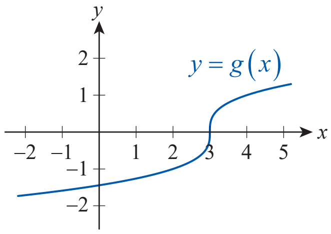


38–43 Use the given graph of the function to find all x-values where the function is differentiable.

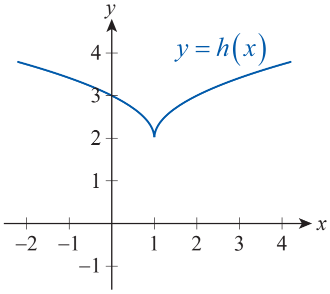
38.



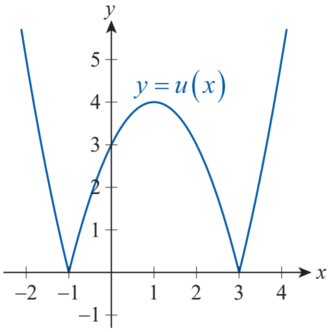
39.



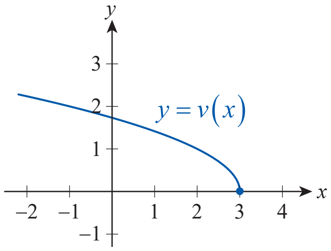
40.



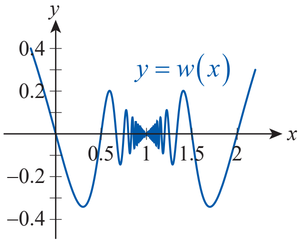
41.



42.



43.



44–58 Find all points where the function is not differentiable. For each of those points, find the one-sided derivatives (if they exist).

44. $f(x)=|x+5|$

45. $g(x)=|x+2|-|x-4|$

46. $h(x)=(x-1)^{2/3}$

47. $F(x)=\sqrt[3]{x-1.5}+2$

48. $H(x)=\sqrt{1.8-x}$

49. $k(x)=\sqrt{3-x^2}$

50. $G(x)=\dfrac{x^2}{x^2-9}$

51. $m(x)=|x^2-6x+5|$

52. $A(t)=[\![t-4]\!]$

53. $B(x)=x-[\![x]\!]$

54. $F(t)=\begin{cases}\dfrac{1}{2}t\cos\dfrac{1}{t} & \text{if } t\neq 0\\ 0 & \text{if } t=0\end{cases}$

55. $H(z)=\begin{cases}\sqrt{z}\sin\dfrac{\pi}{z} & \text{if } z>0\\ 0 & \text{if } z=0\end{cases}$

56. $P(x)=\begin{cases}\sqrt[3]{x-1} & \text{if } x<1\\ (x-1)^2 & \text{if } x\geq 1\end{cases}$

57. $G(x)=\begin{cases}2x+2 & \text{if } x\leq -2\\ -\dfrac{1}{2}x^2 & \text{if } x>-2\end{cases}$

58. $S(t)=\begin{cases}\dfrac{1}{t} & \text{if } t\leq 1\\ t & \text{if } t>1\end{cases}$

59. Prove that the function

$$f(x)=\begin{cases}x^2\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

is differentiable at 0. Contrast this result with Example 5.

60. The position function of a car crashing head-on at 62 mph during a crash test is $x(t)=-196t^2+27.78t$, where x is measured in meters and t in seconds. Find the deceleration of the dummy inside the car. What multiple of g is this (where g is the gravity constant)?

61. The position from its starting point of a small plane preparing for takeoff is given by $x(t)=1.1t^2$ meters (t is measured in seconds).

a. What is the acceleration of the plane?

b. How long does it take for the plane to reach the minimum takeoff speed of 33 m/s?

c. What is the minimum required runway length for this type of plane?

62. The position function of a theme park thrill ride moving along a straight line is $x(t)=\frac{14}{3}t^3+10t$ ft ($0\leq t\leq 3$, t is measured in seconds). Find the velocity, acceleration, and jerk. How far from starting position are the cars at the end of the 3-second time interval?

63. The **symmetric derivative** of a function f at a point c is defined as

$$f'_{sym}(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

a. Prove that if f is differentiable at c, then its symmetric derivative exists and $f'_{sym}(c) = f'(c)$.

b. Give an example of a function g and a point c such that $g'_{sym}(c)$ exists, but g is not differentiable at c.

64. Sketch the graph of $f(x) = -x^2 + 2x$ and its derivative on the interval $[0,2]$ in the same coordinate system. Where (on which interval) is f' positive? Where is f' negative? Identify those intervals where f is increasing versus decreasing. Do you see a connection? Can you give an intuitive reason for your findings?

65. Repeat Exercise 64 for the function $g(x) = 1/x^2$. Sketch both g and g' on their entire domains and summarize your observations. Can you formulate a general conjecture?

Concept Check

66–69 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

66. If f is continuous at c, then f is differentiable at c.

67. If f is differentiable at c, then f is continuous at c.

68. If f is differentiable at c, then

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c - \Delta x) - f(c)}{-\Delta x}.$$

69. If both one-sided derivatives of f at c exist, then f is differentiable at c.

3.1 Technology Exercises

70–75 Use a graphing utility to graph the given function along with its derivative in the same viewing window and answer the questions of Exercise 64. (**Hint:** Use the differentiation capabilities of your technology to find f' first.)

70. $f(x) = \frac{1}{3}x^3 - 5x^2 - 1$

71. $f(x) = \frac{x^2}{x^2 - 4}$

72. $f(x) = \frac{x}{x^2 + 1}$

73. $f(x) = -\cos x$

74. $f(x) = \sin^2 x$

75. $f(x) = e^{1/(x^2+1)}$

76–79 Use a graphing utility to find the first four derivatives of f; then graph them along with f in the same viewing window and compare the graphs.

76. $f(x) = \frac{1}{2}x^5 - 2x^4 - 5x^3 + 7$

77. $f(x) = \arctan x$

78. $f(x) = \frac{2x}{x - 3}$

79. $f(x) = x \sin\left(\frac{1}{10}x - 1\right)$

80–83 Use a graphing utility to graph the function and identify all points where the function is not differentiable. Explain.

80. $G(x) = (x^2 - 4)^{2/5}$

81. $H(t) = |2t - 1|^{2/3}$

82. $P(x) = \begin{cases} \arctan x & \text{if } x < 0 \\ x^{3/2} & \text{if } x \ge 0 \end{cases}$

83. $L(t) = \sqrt{|t|} \sin \frac{1}{|t|}$

3.2 Exercises

1–12 Use the appropriate rules from this section to find the derivative of the given function.

1. $f(x)=5-2x$
2. $g(x)=\frac{4}{5}x+2$
3. $h(x)=\frac{1}{2}+2x-3x^2$
4. $F(x)=x^3-2x^2+\frac{1}{2}x-77$
5. $G(x)=\frac{1}{2}x^4+2x^3-x^2+3.2x+\sqrt{2}$
6. $k(x)=x^{11}-0.2x^{10}+\frac{\pi}{3}x^3+\pi$
7. $H(x)=x^8+\sqrt{2}x^5-2x^4$
8. $R(t)=t^{100}-2t^{59}+\pi t^{38}+et$
9. $S(z)=4z^3-3\sqrt{z}+11.2$
10. $Q(s)=\frac{1}{3s}-\sqrt{2}s+\sqrt{2s}$
11. $T(r)=\pi r^2+2e^r+\pi^2$
12. $N(t)=e^{2+t}+\frac{t+1}{t}+pt$

13–20 Use the Product Rule to find the indicated derivative. Then find the answer without the use of the Product Rule, by multiplying first, and compare your answers.

13. $\frac{d}{dx}\left[(x+2)(3x+5)\right]$
14. $\frac{d}{dx}\left[(3x+7)(x^2+2x)\right]$
15. $\frac{d}{dx}\left[(x^2-6)(2x^2+5x)\right]$
16. $\frac{d}{dx}\left[(2x^3+3x^2)(4x^2-2x+5)\right]$
17. $\frac{d}{dx}\left[\left(\frac{1}{3}x^3+\frac{7}{5}x^5\right)\left(\frac{2}{x}-4x^2\right)\right]$
18. $\frac{d}{dx}\left[(e^x+3)(e^2-5)\right]$
19. $\frac{d}{dt}\left[(3+2\sqrt{t})(4\sqrt{t}-5)\right]$
20. $\frac{d}{ds}\left[s\left(-3-\frac{1}{3}s^3\right)(s^4+2s)\right]$

21–32 Use the Reciprocal Rule or Quotient Rule to determine the derivative of the function.

21. $f(x)=\frac{1}{1-2x}$
22. $g(x)=\frac{1}{4x-2x^2}$
23. $h(x)=\frac{2}{2x^3-5x^2+3x+1}$
24. $F(x)=\frac{e}{e^x-\sqrt{x}}$
25. $G(x)=\frac{2x+1}{x-5}$
26. $k(x)=\frac{3x-4}{2x^2+5}$
27. $H(x)=\frac{x^3-3x^2}{2x^3+5x^2+1}$
28. $A(x)=\frac{6\sqrt{x}}{3x-4}$
29. $B(u)=\frac{u^2}{\sqrt{u}+1}$
30. $f(t)=\frac{4-\sqrt{t}}{t^2+3}$
31. $g(t)=\frac{3-t}{4-5\sqrt{t}}$
32. $w(s)=\frac{1+2e^s}{3e^s+5}$

33–38 Differentiate the quotient by simplifying it algebraically first.

33. $f(x)=\frac{30x^2-10x^6}{5x}$
34. $g(x)=\frac{1+5x+x^2}{x}$
35. $h(x)=\frac{3x^{1/2}-5x^{3/2}+7x^{5/2}-9x^{7/2}}{x^{1/2}}$
36. $F(x)=\frac{2(\sqrt{x})^3+3\sqrt{x}}{\sqrt{x}+(\sqrt{x})^3}$
37. $G(x)=\frac{\frac{6}{x^2}-\frac{5}{x}+1}{\frac{1}{x}-\frac{3}{x^2}}$
38. $H(x)=\frac{2-\frac{1}{e^x}}{2e^{-x}}$

39–61 Using the rules of this section, differentiate the given function.

39. $f(t)=t^{1/2}(4t+3)$
40. $g(x)=x^2\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$
41. $h(s)=\left(5+\frac{1}{s}\right)\left(s^2+\frac{1}{5}\right)$
42. $F(x)=\frac{1}{x}+\frac{2}{x^2}$
43. $G(x)=3x^{-5}+2x^{-3}$
44. $k(s)=s^2\left(\frac{3}{s}+\frac{1}{s-1}\right)$
45. $H(t)=\sqrt{t}(9-t^2)$
46. $K(x)=\frac{\frac{1}{x^2}-3}{x+2}$
47. $w(z)=z\left(2+\frac{4}{4-\sqrt{z}}\right)$

48. $L(T)=T^{-3}\left(2-4T^{-2}\right)$

49. $r(x)=\dfrac{x-a^2}{x+a^2}$

50. $Q(t)=\dfrac{at+b}{ct+d}$

51. $F(x)=e^x\left(2+\sqrt{x}\right)$

52. $E(s)=\dfrac{2+se^s}{e^s-s}$

53. $C(x)=\dfrac{a}{a+\dfrac{a}{x}}$

54. $D(x)=\dfrac{x}{x+\dfrac{x}{a}}$

55. $G(s)=\dfrac{3s^2}{2e^s+s}$

56. $S(t)=\left(4-\sqrt{t}\right)\left(2-e^t\right)$

57. $L(y)=\left(y^4-3y^3\right)\left(y^2-2y^5\right)$

58. $h(z)=\dfrac{1}{ae^z+z}$

59. $H(s)=\dfrac{a}{b+ce^s}$

60. $T(x)=(x+2)\left(2x^2-x\right)\left(x^3+5\right)$

61. $L(t)=t\left(2e^t+\sqrt{t}\right)\left(\dfrac{1}{t}-1\right)$

62–67 Find the first, second, and third derivatives of the function.

62. $f(x)=2x+5$

63. $g(x)=\dfrac{x}{x+1}$

64. $h(x)=3\sqrt{x}$

65. $F(x)=2-x+5x^2-\pi x^3$

66. $V(z)=2z^2+\dfrac{2}{z^2}$

67. $W(t)=3t^2+3e^t$

68–71 Find a function f that satisfies the given conditions. (**Hint:** A polynomial is the most natural choice. Answers will vary.)

68. $f(0)=2,\ f'(0)=1,$ and $f''(0)=-1.$

69. $f(0)=0$ and f has horizontal tangent lines at $x=2$ and $x=-2$.

70. $f(0)=1,\ y=x+1.5$ is tangent to the graph at $x=1$, and $y=5.5-x$ is tangent to the graph at $x=3$.

71. $f(1)=5,\ f'(1)=8,\ f$ has a horizontal tangent line at $x=-1$, and $f'''(x)=6.$

72–75 Find a formula for the k^{th} derivative of the function. (**Hint:** Calculate the first few derivatives and try to recognize an emerging pattern.)

72. $f(x)=x^n$

73. $g(x)=\dfrac{1}{x}$

74. $h(x)=xe^x$

75.* $q(x)=x^n e^x$

76. If $f(1)=2,\ f'(1)=1,\ g(1)=-1,$ and $g'(1)=3,$ find the following function values.

a. $(f-g)'(1)$

b. $(fg)'(1)$

c. $\left(\dfrac{f}{g}\right)'(1)$

77. If $f(3)=-1,\ f'(3)=5,\ g(3)=\tfrac{1}{2},$ and $g'(3)=-2,$ find the following function values.

a. $(2f+5g)'(3)$

b. $(4fg)'(3)$

c. $\left(\dfrac{f}{2g}\right)'(3)$

78–82 Find the equation of the line tangent to the graph of the function at the given point.

78. $f(x)=\dfrac{x^2+1}{x};\quad (1,2)$

79. $w(x)=\dfrac{8}{x^2+4};\quad (2,1)$

(This curve is called the **witch of Agnesi**.)

80. $g(x)=\dfrac{2}{\sqrt{x}+1};\quad (1,1)$

81. $h(x)=\dfrac{2e^x}{x^2};\quad (1,2e)$

82. $k(x)=\dfrac{2x}{2+x^2};\quad \left(1,\dfrac{2}{3}\right)$

(This curve is called a **serpentine**.)

83. Find the equation of the **normal line** to the graph of the function $s(x)=\dfrac{9x}{x^2+9}$ at the point $(0,0)$. (We call a line **normal** to the graph at a point if it is perpendicular to the line tangent to the graph at the same point.)

84. Repeat Exercise 83 for the graph of the function $h(x)=\dfrac{e^x}{x^4+2}$ at $\left(0,\tfrac{1}{2}\right)$.

85–96 Find all x-values where the graph of the function has a horizontal tangent line, or prove that the graph has no horizontal tangent line.

85. $f(x)=x^2-2x$

86. $g(x)=2x^3-3x^2-12x+1$

87. $h(x)=\dfrac{2}{x^2}$

88. $F(x)=\dfrac{2}{x^2+1}$

89. $G(x)=\dfrac{1}{2}e^x-2$

90. $k(x)=x^2-a$

91. $H(x)=\dfrac{1}{x^2-a}$

92. $f(x)=\sqrt{x+5}$

93. $g(x)=\dfrac{x^2}{x^2+5}$

94. $F(x)=\dfrac{2x-1}{x^2}$

95. $P(x)=2x^3+3x^2+3x-5$

96. $Q(x)=e^x-x$

97. The line $y=8x+b$ is tangent to the graph of $f(x)=ax^2$ at $x=2$. Find the values of a and b.

98. Repeat Exercise 97 with the line $y=-2x+b$ that is tangent to the graph of $g(x)=-x^2+ax$ at $x=2$.

99. Show that the graphs of $y=e^{x/2}$ and $y=\dfrac{1}{(x+1)^2}$ intersect at $x=0$ in such a way that their respective tangent lines are perpendicular at their point of intersection.

100–103 Find the equation(s) of the line(s) tangent to the graph of f and passing through the indicated point, which does not lie on the graph of f. (**Hint:** If the point of tangency is $(x,f(x))$, then the slope of the tangent line going through (a,b) is $f'(x)=\dfrac{f(x)-b}{x-a}$.)

100. $f(x)=x^2;\ (0,-1)$

101. $f(x)=e^x;\ (0,0)$

102. $f(x)=\sqrt{x};\ (-2,0)$

103. $f(x)=\dfrac{1}{x};\ (-1,0)$

104–107 Assuming that f and g are differentiable functions, differentiate the given expression.

104. $\dfrac{f(x)}{x}$

105. $\dfrac{xf(x)}{g(x)}$

106. $e^xg(x)$

107. $\dfrac{f(x)e^x}{g(x)+2}$

108. Use the definition of the derivative to prove the Constant Multiple Rule. (**Hint:** For a given constant k, start out by using the definition $\dfrac{d}{dx}[kf(x)]=\lim\limits_{h\to 0}\dfrac{kf(x+h)-kf(x)}{h}$, and let k "pass through" the limit sign.)

109. Use the Product Rule and mathematical induction to provide a third proof of the Positive Integer Power Rule. (**Hint:** The base case of $n=1$ should be obvious. After setting up the induction hypothesis of $(x^n)'=nx^{n-1}$, find the derivative of x^{n+1} by treating it as $x^{n+1}=x\cdot x^n$ and use the Product Rule along with the induction hypothesis.)

110. Use the Product Rule to arrive at a rule for $\dfrac{d}{dx}[f(x)g(x)h(x)]$.

111. Use the Product Rule to arrive at a formula for $\dfrac{d}{dx}[f(x)]^2$, and then use Exercise 110 to find the formula for $\dfrac{d}{dx}[f(x)]^3$. Do you recognize a pattern?

112. Use mathematical induction to prove the **Generalized Positive Integer Power Rule**:

$$\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}f'(x).$$

(The hint provided in Exercise 109 might prove helpful, but you will need to appropriately modify your induction hypothesis.)

113. Use the Sum Rule and mathematical induction to prove that any finite sum of functions $f_1(x)+f_2(x)+\cdots+f_n(x)$ can be differentiated termwise, that is,

$$[f_1(x)+f_2(x)+\cdots+f_n(x)]'=f_1'(x)+f_2'(x)+\cdots+f_n'(x).$$

In particular, polynomials can be differentiated termwise. (Look at the hint provided in Exercise 109.)

114. Use Exercise 113 and the results of this section to prove that the derivative of a polynomial is always another polynomial.

115. A Formula One race car was moving in a parabolic curve with equation $y=\sqrt{x}$ when it hit an oil patch and the driver lost control at the point $(4,2)$. The car left the track along the tangent line at the same point. Where did he hit the tire wall if the equation of the tire wall was $y=4$? (Fortunately, there were no injuries.)

116. The position function of a golf ball rolling on an incline is given by $d(t)=2t^2+3t$, where d is measured in meters, t in seconds. Find the ball's velocity and acceleration at $t=4$ seconds.

117. The velocity function of a moving particle is given by $v(t)=\frac{50t}{t+10}$ ft/s. Find its acceleration at **a.** $t=2$ seconds and **b.** $t=10$ seconds.

118. The position function of a moving object is given by $p(t)=2t^3-6t$ ft. Find its position and acceleration at the instant when its velocity changes directions.

119. The position function of an object dropped by an astronaut on the moon is $h(t)=-0.81t^2+1.5$, where h is measured in meters, t in seconds. What is the acceleration due to gravity on the moon? How long does it take for the above object to reach the ground and what is the speed of impact?

120. The radius of a spherical balloon being inflated increases according to the function $r(t)=2+5\sqrt[3]{t}$, where r is measured in centimeters and t in seconds. Find the rate of change of the balloon's volume and surface area with respect to time at $t=8$ seconds.

Concept Check

121–129 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

121. If $y=\pi x^n$, then $y'=n\pi x^{n-1}$.

122. If $y=\pi^n$, then $y'=n\pi^{n-1}$.

123. If $y=\pi/x^n$, then $y'=\pi/\left(nx^{n-1}\right)$.

124. If $y=e^x$, then $y'=xe^{x-1}$.

125. If $y=\pi e^x$, then $y'=\pi e^x$.

126. If p is a fifth-degree polynomial, then its sixth derivative is 0.

127. If $F(x)=\frac{f(x)}{g(x)}$, then $\frac{d}{dx}F(x)=\frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)}$.

128. The jerk of a free-falling object is 0.

129. If a polynomial $p(x)$ has degree n, then its derivative has degree $n-1$.

3.2 Technology Exercises

130–135. Use the differentiation capabilities of a graphing utility to check your answers for Exercises 62–67, and then graph each function along with its derivatives in the same viewing window.

136–140. Referring back to Exercises 78–82, verify your answers by using a graphing utility to graph each function and its indicated tangent line in the same viewing window.

3.3 Exercises

1–12 Use the results of this section to find the indicated limit.

1. $\lim_{x\to 0}\frac{\sin 3x}{2x}$
2. $\lim_{x\to 0}\frac{-\sin\frac{x}{2}}{5x}$
3. $\lim_{x\to 0}\frac{\sin \pi x}{x}$
4. $\lim_{x\to 0}\frac{\tan 4x}{5x}$
5. $\lim_{x\to 0}\frac{\cos 5x-1}{2x}$
6. $\lim_{x\to 0}\frac{\cos x-1}{\sin x}$
7. $\lim_{x\to 0}\frac{x+\tan x}{\sin x}$
8. $\lim_{\beta\to 0}\frac{\csc\beta-\cot\beta}{\beta\csc\beta}$
9. $\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}$
10. $\lim_{\alpha\to 0}\frac{\tan(\alpha^2)}{\alpha}$
11. $\lim_{t\to 0}\frac{2t+3\tan t}{\sin t}$
12. $\lim_{\theta\to 0}\frac{\cos\theta\sin\theta-\sin\theta}{\theta^2}$

13–30 Differentiate the given function.

13. $f(x)=2\sin x-5\cos x$
14. $g(x)=3x^2+2\tan x$
15. $h(x)=x\cos x$
16. $F(x)=2.5x(1-\cot x)$
17. $G(x)=2\sqrt{x}\sec x$
18. $k(x)=\pi x\sin x+\pi x$
19. $L(x)=-3e^x(\csc x+\cot x)$
20. $f(x)=2\cos 2x-2\cos x$
21. $g(x)=\cot^2 x$
22. $h(x)=\frac{\tan x}{x}$
23. $F(t)=\frac{1-\cos t}{t^2}$
24. $W(x)=\frac{1+\cos x}{1+\sin x}$
25. $R(z)=\frac{e^z+\sin z}{z}$
26. $N(w)=\frac{2\sqrt{w}-\sec w}{\sqrt{w}}$
27. $B(x)=\frac{\frac{1}{\sin x}-\sin x}{\cos x}$
28. $G(y)=y\cot y\csc y$
29. $T(s)=s^2e^s\cot s$
30. $r(t)=\frac{1}{t\sin t\cos t}$

31–36 Find all points where the function has a horizontal tangent line.

31. $f(x)=\frac{1}{2}x+\sin x$
32. $g(x)=x+\sin 2x$
33. $h(x)=\sec^2 x$
34. $T(s)=\tan s-s$
35. $K(u)=\tan u+\cot u$
36. $F(t)=\frac{1-\sin t}{1-\cos t}$

37–40 Find all x-values where the tangent line to the graph of the function is parallel to the given line.

37. $f(x)=\sin x+\frac{3}{2};\quad y-x=\frac{3}{2}$
38. $g(x)=\cot x;\quad y+2x=\pi$
39. $G(x)=\frac{x}{3}-\tan x;\quad x+y=5$
40. $F(x)=\sin x\cos x;\quad 2x+2y=7$

41–44 Find the equation of the tangent line to the graph of the given function at the indicated point.

41. $f(x)=2x\cos x;\quad (0,0)$
42. $g(x)=\tan x-\sec x;\quad (0,-1)$
43. $h(x)=2\csc x-\sin x;\quad \left(\frac{\pi}{2},1\right)$
44. $k(x)=\frac{\cot x}{x};\quad \left(\frac{\pi}{4},\frac{4}{\pi}\right)$
45. Let us assume that for some function f, we have $f(0)=1$ and $f'(0)=2$. Let $F(x)=f(x)\tan x$, $G(x)=f(x)/\cos x$, and $H(x)=f(x)\sin x\cos x$. Find $F'(0)$, $G'(0)$, and $H'(0)$.

46–48 Verify the trigonometric identity by differentiating both sides of the equation. (**Hint:** If $f'(x)=g'(x)$, it doesn't necessarily follow that $f(x)=g(x)$. In general, we can only conclude that $f(x)=g(x)+c$ for some constant c.)

46. $\tan x\cot x=1$
47. $(1-\cos\theta)(1+\cos\theta)=\sin^2\theta$
48. $\frac{1}{1-\cos x}+\frac{1}{1+\cos x}=2\csc^2 x$

49–52 Find $f'(x)$, $f''(x)$, and $f'''(x)$. Observing a pattern, find a formula for $f^{(n)}(x)$.

49. $f(x)=\sin x$
50. $f(x)=\cos x$
51. $f(x)=e^x\sin x$
52. $f(x)=e^x\cos x$

53. Provide a second proof of the limit statement $\lim\limits_{\theta \to 0} (\cos\theta - 1)/\theta = 0$ by multiplying both the numerator and denominator by $\cos\theta + 1$ to obtain

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}.$$

Then by the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, you obtain

$$\lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)}$$
$$= \lim_{\theta \to 0} \frac{\sin\theta(-\sin\theta)}{\theta(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \lim_{\theta \to 0} \frac{-\sin\theta}{\cos\theta + 1}.$$

Conclude the argument by using the first limit statement in the lemma at the beginning of this section.

54. Prove that $\frac{d}{dx}(\cos x) = -\sin x$ by mimicking the proof of the theorem "Derivative of Sine." (**Hint:** You will need the angle sum identity $\cos(u + v) = \cos u \cos v - \sin u \sin v$.)

55. Provide an alternative proof of the fact that $\frac{d}{dx}(\sin x) = \cos x$ by using the identity

$$\sin x - \sin c = 2\sin\frac{x - c}{2}\cos\frac{x + c}{2}.$$

(**Hint:** Rewrite the difference quotient $\frac{\sin x - \sin c}{x - c}$ as $\frac{2\sin\frac{x-c}{2}\cos\frac{x+c}{2}}{x - c}$. Let $c \to x$, and use the lemma from the beginning of this section.)

56. Use the definition of the derivative and the lemma from the beginning of this section to show that $(\sin 3x)' = 3\cos 3x$. Generalize to obtain that if $k \in \mathbb{R}$, $(\sin(kx))' = k\cos(kx)$.

57. Repeat Exercise 56 with $f(x) = \cos(kx)$.

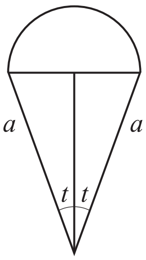
58. Find a constant a such that the graphs of $f(x) = a\sin x$ and $g(x) = a\cos x$ intersect at right angles, that is, their respective tangent lines are perpendicular at their point(s) of intersection.

59. Prove the remaining two cases of the theorem "Derivatives of Tangent, Cotangent, Secant, and Cosecant," namely, the statements

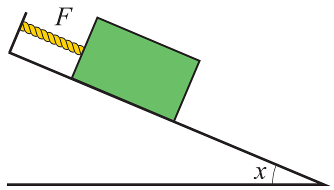
$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \text{and} \quad \frac{d}{dx}(\sec x) = \sec x \tan x.$$

(**Hint:** Mimic the proof presented in the text, using the derivatives of sine and cosine along with appropriate differentiation rules.)

60. The cross-section of an ice cream cone is an isosceles triangle, with the angular opening at the bottom being $2t$ (radians). Assuming that the ice cream sits on top of the cone in the shape of a perfect hemisphere, let V_i = volume of the ice cream, V_c = volume of the cone. Express both of these volumes in terms of t, and then compute $\lim\limits_{t \to 0^+} \frac{V_i}{V_c}$.



61. An object is tied to the top of an inclined surface of variable angle of elevation so that the rope is parallel to the surface. The tension in the rope is given by $F = mg(\sin x - \mu\cos x)$, where m is the mass of the object, g is the gravity constant, and μ is the coefficient of friction (assume all units are metric units).



a. What is the rate of change of F with respect to x?

b. For what x-value (if any) is this rate of change equal to 0?

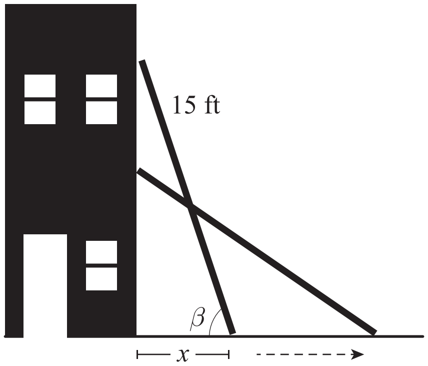
62. Suppose an object oscillating in fluid obeys the position function $y = 10e^{-0.2t}\cos(2\pi t)$, where y is the distance from equilibrium, measured in centimeters with upward displacement considered positive, and t is measured in seconds. Such motion is called *damped harmonic motion*. Can you see why?

a. Find the position, velocity, and acceleration at $t = 3.5$ seconds.

b. What is the maximum displacement of the object and when does it occur?

(**Hint:** Use the definition of the derivative to find the derivatives of $e^{-0.2t}$ and $\cos(2\pi t)$. You may also want to review Exercise 57 for the latter.)

63. A 15 ft ladder is leaning against a wall, making an angle of β with the horizontal, when it starts sliding. If x denotes the distance of the bottom of the ladder from the wall, find the rate of change of x with respect to β when $\beta = \pi/6$ (or 30°). Interpret the result.



64. A man is pulling his child on a sled at a constant rate, via a rope that makes an angle of α with the horizontal. Since there is no acceleration, the pulling force satisfies the equation $F\cos\alpha = \mu(mg - F\sin\alpha)$, where μ is the coefficient of friction, m is the total mass of the sled and child, and g is the gravity constant.

a. Express F as a function of α.

b. Find the rate of change of F with respect to α.

c. What is the above rate when $\alpha = 60°$?

d. When (if ever) is this rate of change 0?

3.3 Technology Exercises

65–70 Use a graphing utility to find the derivative of $f(x)$. Then graph f along with its derivative on the same screen. By zooming in, if necessary, find at least two x-values where the graph of f has a horizontal tangent line. What can you say about f' at such points? (Answers will vary.)

65. $f(x) = \dfrac{x}{1+\cos x}$

66. $f(x) = \dfrac{1-\sec x}{1+\sec x}$

67. $f(x) = \dfrac{\csc x}{x}$

68. $f(x) = \dfrac{\sin x}{\cos x + \tan x}$

69. $f(x) = \cos x(\cot x + \tan x)$

70. $f(x) = \dfrac{\cot x}{\sec x + x\cos x}$

71. Find the maximum velocity and acceleration values in Exercise 62 by using a graphing utility to graph the velocity and acceleration functions of the oscillating object.

3.4 Exercises

1–9 Identify $f(x)$ and $u=g(x)$ such that $F(x)=f(u)=f(g(x))$. Also find $h(x)$ wherever $F(x)=f(g(h(x)))$. (Answers will vary.)

1. $F(x)=(3x-2.5)^6$
2. $F(x)=2(x^3-5x^2+\pi)^{-4}$
3. $F(x)=2\sqrt[3]{x^2-9}$
4. $F(x)=\dfrac{-3}{5+\sqrt{x^3+x}}$
5. $F(x)=\sin\dfrac{1}{x^2+1}$
6. $F(x)=3\cos\left(\dfrac{\tan x}{2}\right)$
7. $F(x)=\csc(3e^x)$
8. $F(x)=\sec\left(e^{2+\sqrt{x}}\right)$
9. $F(x)=\dfrac{3}{\sqrt{\ln(x^2+1)}}$

10–60 Find the derivative of the given function.

10. $f(x)=(2x^2+x)^7$
11. $g(x)=3(x^5-\pi x^2+7.5)^{11}$
12. $h(x)=\dfrac{1}{2}(x^8+5x^3-ex)^{100}$
13. $F(x)=-3(5+2\sqrt{x})^{-5}$
14. $G(x)=(2x^2-3x+1)^{2/3}$
15. $k(x)=-5(x^5-2x^3+10.5x)^{-2/5}$
16. $f(x)=\sqrt{2-4x}$
17. $g(x)=\sqrt{x^2-5x+2}$
18. $h(x)=(4x+5)^{21}(3x-7)^{13}$
19. $q(x)=2(x^3-5x)^{2/3}(x+3)^{5/4}$
20. $r(t)=\dfrac{1}{3t+1}$
21. $k(z)=\dfrac{1}{1+5z-2z^2}$
22. $F(x)=\left(\dfrac{2x-3}{1-7x}\right)^{10}$
23. $S(v)=\left(\dfrac{2v+1}{v^2-5}\right)^{-3}$
24. $G(y)=\left(\dfrac{3y^2-1}{2+4y}\right)^{7/5}$
25. $T(s)=\left(\dfrac{s^2-1}{s^2+1}\right)^{-2/3}$
26. $G(x)=\dfrac{(5-\pi x^2)^2}{(1+2x)^3}$
27. $H(x)=\dfrac{\sqrt{x^2-2}}{(x^2+2)^2}$
28. $R(x)=\sqrt{\dfrac{1}{x^2-1}}$
29. $B(t)=\sqrt[3]{\dfrac{t}{2t^2+1}}$
30. $K(s)=\sqrt{\dfrac{2s-5}{3s+1}}$
31. $t(x)=\sin(\cos x)$
32. $Q(x)=2\tan(\sin x)$
33. $P(x)=x\tan^2 x$
34. $w(x)=\cot(x^2)$
35. $U(z)=5\sec^2 z$
36. $R(x)=x\sqrt{\sin x}$
37. $C(x)=\sin^2(\tan x)$
38. $U(v)=\csc\left(\dfrac{v}{\cos v}\right)$
39. $V(x)=e^{\cos x}$
40. $R(\theta)=e^{\theta\tan\theta}$
41. $w(x)=\sin\sqrt{2x+1}+e^{\tan\sqrt{2x+1}}$
42. $t(x)=10^{\sqrt{x}}$
43. $f(x)=\pi 2^{\sin(\pi x)}$
44. $u(x)=2^{x^2}-4^{\sqrt{x}}$
45. $t(s)=\tan(2^s)$
46. $u(x)=\cot^2(2^{\sin x})$
47. $E(x)=5^{5^x}$
48. $K(x)=\sqrt[3]{3^x}+3^{\sqrt[3]{x}}$
49. $N(x)=\cos^2\left(e^{\cos(x^2)}\right)$
50. $u(t)=\tan^3(t^3+3^t)$
51. $C(x)=\cos^2(x^2)$
52. $F(x)=5^{x^5}$
53. $t(s)=\sqrt{\cos(10^s)}$
54. $G(t)=\sec^{-3}(5^t)$
55. $H(s)=\sin(2^s)\tan(2^s)$
56. $w(s)=\sin(\tan(2^s))$
57. $T(z)=\sin(e^z)+e^{\sin z}$

58.* $q(x)=\sin(\cos(\tan(\cot x)))$

59.* $U(\theta)=\theta+\tan(\theta+\tan(\theta+\tan\theta))$

60.* $v(x)=\left(1+\left(2+(3+4x)^5\right)^6\right)^7$

61–68 Find an equation for the tangent line to the graph of the given function at the specified point.

61. $f(x)=\sqrt{2x^2+1};\quad (2,3)$
62. $g(x)=(x^2+3x+4)^{2/3};\quad (1,4)$
63. $q(x)=\cos(\tan x);\quad (0,1)$
64. $S(x)=\sin(x^2)+\sin^2 x;\quad (0,0)$
65. $M(x)=\dfrac{e^{\cos x}}{x};\quad \left(\pi,\dfrac{1}{e\pi}\right)$
66. $a(x)=10^{\sqrt{x}};\quad (1,10)$
67. $h(x)=\dfrac{3x+1}{\sqrt{x^2+3}};\quad (1,2)$
68. $u(x)=\pi^{\pi^{\sin x}};\quad (0,\pi)$

69–76 Find all x-values where the line tangent to the given curve is horizontal.

69. $f(x)=(x^2-8x+15)^{100}$

70. $g(x)=\dfrac{2x+3}{x^2-2}$

71. $h(x)=\sqrt{x^2+1}$

72. $T(x)=\tan^{10}x$

73. $w(x)=\sec(x^2+2)$

74. $t(x)=\cos(\cos x)$

75. $k(x)=e^{x/(x^2+1)}$

76. $q(x)=\pi^{\cos^2 x}$

77–84 Determine the second derivative of the function.

77. $p(x)=(x^2+5)^{20}$

78. $r(t)=\sqrt{t^2+5}$

79. $g(x)=5\cos^2 x$

80. $c(x)=e^{\tan x}$

81. $F(t)=t\sin(t^2)$

82. $d(x)=5^{5^x}$

83. $G(x)=\sin^2 x+\cos^2 x$

84. $U(s)=\sec\sqrt{s}$

85. Suppose that $f(1)=1,\ f'(1)=-2,\ g(1)=1,$ and $g'(1)=5.$ If $F(x)=(f\circ g)(x)$ and $G(x)=(g\circ f)(x),$ find $F'(1)+G'(1).$

86. Let $P(x)=x(x+1)(x+2)\cdots(x+10).$ If $F(x)=(P\circ P)(x),$ find the value of $F'(0).$

87. Find a formula for the n^{th} derivative of $f(x)=\cos(kx),\ k\in\mathbb{R}.$ (**Hint:** Use the Chain Rule and recognize a pattern.)

88. Repeat Exercise 87 for the function $g(x)=2^{kx}.$

89. Use the Chain Rule to prove that the function $f(x)=\sin(1/x^2)$ is differentiable for $x\neq 0.$

90. Use the Chain Rule to construct a second proof of the Quotient Rule. (**Hint:** Rewrite $f(x)/g(x)$ as $f(x)\cdot[g(x)]^{-1}$.)

91. Use the Chain Rule to prove that the derivative of an even function is odd and vice versa.

92. Find all points where the line tangent to the graph of $y=\sqrt[3]{\cos x}$ is horizontal, as well as those where it is vertical.

93.* A spherical balloon is being inflated so that its radius is increasing at a rate of $dr/dt=0.1$ in./s. Find the rate at which the volume of the balloon is increasing when its radius is $r=4$ in. (**Hint:** Notice that $V(t)=V(r(t))$ and use the Chain Rule.)

94.* Pouring sand is forming a conical shape so that the radius of the bottom of the cone is always twice its height throughout the process. If the height of the cone is increasing at a rate of $dh/dt=0.5$ mm/s, find the rate at which the volume of the cone is increasing when its height is $h=50$ mm. (See the hint given in Exercise 93.)

95. The position function of a vibrating loudspeaker cone is given by $x(t)=10^{-3}\cos 1500t,$ where distance is measured in meters, time in seconds. As indicated by the position function, the cone is at one of its extreme positions at $t=0$. Use the above information to find **a.** the maximum velocity of the cone and **b.** the maximum acceleration of the cone.

96. The position function for damped harmonic motion of an object of mass m is

$$x(t)=Ae^{-\frac{k}{2m}t}\cos(\omega t),$$

where A is the amplitude and k and ω are constants specific to the motion. Find the velocity and acceleration functions for this motion.

97. Unless conditions are extreme, most gases obey the so-called *Ideal Gas Law*, which says $PV=nRT$, where P stands for pressure measured in pascals (Pa), V for volume, n for the number of moles (mol) of gas in the container, T denotes temperature measured in kelvins (K), and R is the *universal gas constant*, which is the same for all gases. Suppose 5 moles of gas are being slowly compressed by a piston in a container so that $dV/dt=-2\cdot 10^{-8}\ \text{m}^3/\text{s}.$ Assuming that temperature is being kept constant at $T=293$ K throughout the process, find the rate of change of pressure with respect to time when $V=10^{-3}\ \text{m}^3$. (Use $R\approx 8.315\ \text{J}/(\text{mol}\cdot\text{K})$.)

3.4 Technology Exercises

98–99 The Maclaurin polynomial of order 2 of the function $f(x)$ is used to approximate $f(x)$ near $x=0$. It is defined as

$$P_2(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2.$$

Find the Maclaurin polynomial of order 2 for $f(x)$. Then use a graphing utility to graph f along with its Maclaurin polynomial. (We will learn more about Maclaurin polynomials in Section 10.8.)

98. $f(x)=\cos(\sin x)$

99. $f(x)=\dfrac{1}{x^2+1}$

3.5 Exercises

1–12 Use implicit differentiation to determine dy/dx for the given equation. Then check your answer by expressing y explicitly and using differentiation rules.

1. $x + y^2 = 2$

2. $xy = 3$

3. $x^2 - y^2 = 1$

4. $4x^2 + 25y^2 = 100$

5. $3xy^2 = x - 5$

6. $y^2\sqrt{x} = 2x^2 + 1$

7. $y\sqrt{x+2} = xy - 2$

8. $2x^2y - 3y - x - 1 = 0$

9. $2y\cos x - xy = x + 3$

10. $ye^x + 2y - 1 = 0$

11. $\frac{2}{x} - \frac{3}{y} = 4$

12. $x^2\sqrt{y} - x^2 - 1 = e^2$

13–20 Find dx/dy by implicit differentiation. Then check your answer by expressing x explicitly in terms of y and differentiating with respect to y using differentiation rules.

13. $x - y^2 = 0$

14. $xy - y^3 = 3y$

15. $x^3 + y^3 = 1$

16. $-5xy^2 + 4xy - 3y^2 - y - 2 = 0$

17. $xy + 3\sin y = e^y$

18. $xy = \sqrt{y^2+1} - 5x$

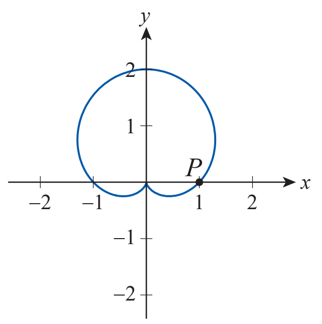
19. $\sqrt[3]{8x^3 - 5y^4} = 3$

20. $(y+2)\sqrt{x+3} = \sqrt{y}$

21–28 Use implicit differentiation to find the equations of the tangent and normal lines at point P for the well-known curve.

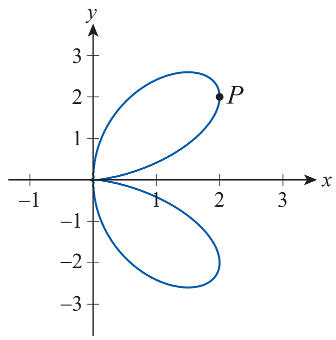
21. $x^2 + y^2 - (x^2 + y^2 - y)^2 = 0; \quad P(1,0)$

Cardioid



22. $(x^2 + y^2)^2 - 8y^2x = 0; \quad P(2,2)$

Bifolium



23. $(x^2 + 4)y = 8; \quad P(2,1)$

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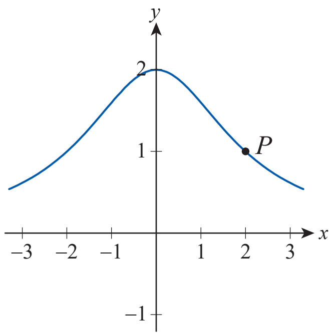
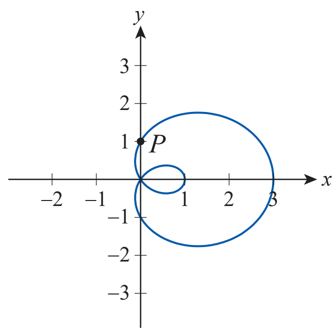


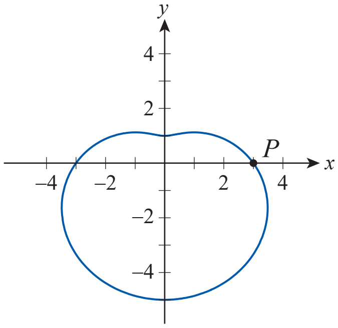
24. $x^2 + y^2 = (x^2 + y^2 - 2x)^2; \quad P(0,1)$

Limaçon



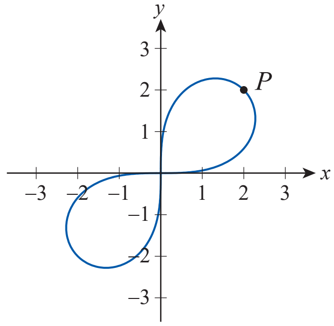
25. $9\left(x^2+y^2\right)=\left(x^2+y^2+2y\right)^2$; $P(3,0)$

Dimpled Limaçon



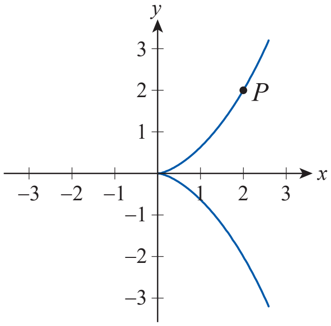
26. $\left(x^2+y^2\right)^2=16xy$; $P(2,2)$

Lemniscate



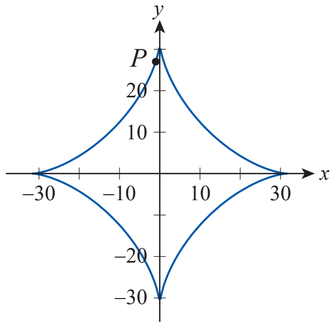
27. $(6-x)y^2=2x^3$; $P(2,2)$

Cissoid



28. $x^{2/3}+y^{2/3}=10$; $P(-1,27)$

Astroid



29–44 Find dy/dx by implicit differentiation.

29. $x^4+y^4=1$

30. $\sqrt{x}+\sqrt{y}=4$

31. $x^3y^4-x^4y^3=1$

32. $y=\cos(x-2y)$

33. $(x+y)^3+3=x+y$

34. $e^{xy}=e^x+e^y$

35. $\sin^2 x+\cos^2 y=\tan\left(x^2+y^2\right)$

36. $\sqrt{x^2+y^2}=2x$

37. $\dfrac{x+3y^2}{y-x^2}=2x+1$

38. $\dfrac{y}{x^3}-\dfrac{x}{y^3}=x^3y^3$

39. $\sqrt{2xy}=3y-5x$

40. $y-x=x^4y^4$

41. $\tan x=\sin y-2xy$

42. $e^x\tan x=y+\cos y$

43. $\sqrt{\sin x+\cos x}=\sec(x+y)$

44. $(\tan x+\cot y)^2=1+x$

45. Find ds/dt by implicit differentiation: $s^2t^3-2t=\sqrt{s}$.

46. Find dt/ds by implicit differentiation: $s\sin t=t\cos s$.

47–52 Find d^2y/dx^2 by implicit differentiation.

47. $4y^2-x^2=4$

48. $y-x=xy-2$

49. $xy^2+5=x$

50. $y^3=xy+1$

51. $x^3+y^3=3$

52. $\sqrt{x}+\sqrt{y}=2$

53. Notice that for a circle centered at the origin, any line tangent to the curve in the first quadrant has negative slope; this is consistent with our observation that $dy/dx<0$ when $x>0$ and $y>0$ (see Example 5). Verify that the sign of dy/dx in each of the quadrants is what we would expect.

54. Verify that the sign of the second derivative d^2y/dx^2 of the circle in Example 5 is what we would expect in each quadrant. (**Hint:** Traverse the circle from left to right and examine whether the first derivative is increasing or decreasing; then draw a conclusion regarding the sign of the second derivative.)

55–58 Find all points on the given curve where it has horizontal or vertical tangent lines.

55. $xy^2 - x^2y = \frac{1}{4}$

56. $x^2 - xy + y^2 = \frac{1}{4}$ (Rotated ellipse)

57. $(x^2 - 2x + 5)y = 5$

58. $xy + y^2x^2 = 1$

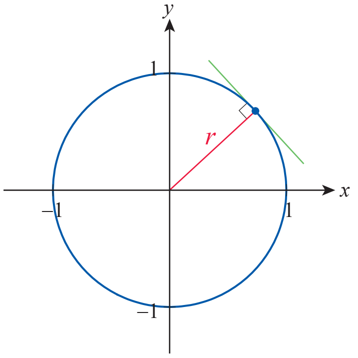
59. Two graphs are called *orthogonal* if their respective tangent lines are perpendicular at their point(s) of intersection. Show that the graphs of $x^2 - y^2 = 5$ and $xy = 6$ are orthogonal.

60. Generalizing Exercise 59, show that the families of curves $x^2 - y^2 = a$ and $xy = b$ are orthogonal for a, $b \in \mathbb{R}$. (Such families of curves are called *orthogonal trajectories*.)

61. Repeat Exercise 60 for the families $x^2 + y^2 = a$ and $y - bx = 0$.

62. Repeat Exercise 60 for the families $x^2 + y^2 = ax$ and $x^2 + y^2 = by$.

63. Use implicit differentiation to prove that a tangent line to a circle is always perpendicular to the radius connecting the center and the point of tangency. (**Hint:** We can assume without loss of generality that the circle is a unit circle in the xy-coordinate system, centered at the origin.)



64. Use implicit differentiation to prove that the equation of the line tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\frac{x_0}{a^2}x + \frac{y_0}{b^2}y = 1$.

65. Use implicit differentiation to find the equations of the two lines tangent to the ellipse $2x^2 + y^2 = 2$ that pass through the point $(0, 2)$.

66. An object of mass m is attached to a spring and is moving along the x-axis so that its position and velocity satisfy the equation $m(v - v_0)^2 = -kx^2$, where v_0 represents the initial velocity. Use implicit differentiation to verify Hooke's Law; that is, prove that the restoring force exerted by the spring satisfies $F = -kx$. (**Hint:** Differentiate and use Newton's Second Law of Motion, which states that $F = ma$.)

3.5 Technology Exercises

67–70. Use the implicit graphing capabilities of a graphing utility to graph the curves along with the tangent lines you found in Exercises 55–58 and visually verify that your answers are correct.

71–73. Use a graphing utility to graph the families of curves in Exercises 60–62 for several different values of the parameters a and b. Visually verify that they are orthogonal.

74. Beautiful, "irregular" curves can be created by using a graphing utility to plot graphs of equations such as the following.

$$(x^2 - 1)(x - 2)(x - 3) = (y - 1)(y - 2)(y - 3)$$

Graph the above equation and explain why this graph cannot be that of a function. Then try experimenting by slightly modifying the above equation and thus creating your own curves. (Answers will vary.)

75. Repeat Exercise 74 starting with the equation $x^5 - 3x^3 - x^2 = -y^5 + 3y^3 - y^2$.

3.6 Exercises

1–15 Use the Derivative Rule for Inverse Functions to determine $\left(f^{-1}\right)'(a)$ for the indicated value of *a*. (In these and subsequent exercises, the domain of *f* is assumed to have been restricted so that the inverse exists and is differentiable, whenever appropriate.)

1. $f(x)=x^3;\quad a=8$

2. $f(x)=2x-1;\quad a=5$

3. $f(x)=\sqrt{x};\quad a=3$

4. $f(x)=\sqrt[3]{x+2};\quad a=-1$

5. $f(x)=x^2+5;\quad a=9$

6. $f(x)=x^{3/2};\quad a=27$

7. $f(x)=\dfrac{2x}{x-1};\quad a=4$

8. $f(x)=\dfrac{5}{(x-1)^3};\quad a=5$

9. $f(x)=\dfrac{3}{x^2+2};\quad a=1$

10. $f(x)=e^{2x};\quad a=5$

11. $f(x)=10^x;\quad a=10$

12. $f(x)=2^{\sqrt{x}};\quad a=8$

13. $f(x)=\sin x;\quad a=\dfrac{\sqrt{3}}{2}$

14. $f(x)=2\tan^{-1}x;\quad a=\dfrac{\pi}{2}$

15. $f(x)=\sin\left(x^2\right);\quad a=\sin 0.01$

16–30 Determine the value of $\left(g^{-1}\right)'(b)$ at the given point (assume that the domain of *g* is appropriately restricted so that g^{-1} exists). (**Note:** Do *not* attempt to find a formula for g^{-1}.)

16. $g(x)=x^5+2x+1;\quad b=g(1)$

17. $g(x)=x^6-11x^4+x;\quad b=g(-1)$

18. $g(x)=x^{100}+x^{50}+1;\quad b=g(-1)$

19. $g(x)=\sqrt{x^4+x^2};\quad b=g(-2)$

20. $g(x)=\left(3x^8+x^3+1\right)^{3/2};\quad b=g(1)$

21. $g(x)=\left(2x^9-3\sqrt{x}\right)^{2/5};\quad b=g(1)$

22. $g(x)=x^5+x+2;\quad b=2$

23. $g(x)=x^{17}+2x^{11}-2x+3;\quad b=4$

24. $g(x)=\dfrac{x^3+8}{\sqrt{x+1}};\quad b=g(2)$

25. $g(x)=\dfrac{x+1}{x^3};\quad b=\dfrac{3}{8}$

26. $g(x)=e^{x^4-x+2};\quad b=g(-2)$

27. $g(x)=x\sin x;\quad b=\dfrac{\pi}{2}$

28. $g(x)=10^{\cos\left(x^3+x\right)};\quad b=g(1)$

29. $g(x)=\tan\sqrt{x};\quad b=g(1)$

30. $g(x)=x^3e^{x^2+1};\quad b=e^2$

31–48 Determine the derivative of the given function.

31. $f(x)=\ln\left(x^3\right)$

32. $g(x)=(\ln x)^3$

33. $h(x)=\ln\left(x^2+3\right)$

34. $F(x)=\ln\left(x\sqrt{x^2+4}\right)$

35. $G(x)=x\ln\sqrt{x^2+4}$

36. $k(x)=\ln\dfrac{2x}{x^2+1}$

37. $L(x)=\dfrac{\ln 2x}{x^2+1}$

38. $f(x)=\ln\sqrt{\dfrac{x+3}{2x+5}}$

39. $g(x)=\ln\sqrt[3]{\dfrac{x+3}{x-3}}$

40. $H(x)=\ln(\ln x)$

41. $F(t)=\ln\left(\sqrt{t^2+4}+2t\right)$

42. $L(s)=\ln\dfrac{\sqrt{s^2+2}}{s^4+s^2+1}$

43. $T(x)=\ln|\cos x|$

44. $C(t)=\ln\left(\sin^2 t+1\right)$

45. $v(x)=\cos 2x(\ln(\cos 2x))$

46. $F(t)=\dfrac{\log_5 t}{t^2}$

47. $w(x)=x\log x$

48. $t(x)=\log_{3/2}\left(\left(5x^2+4\right)^{3/2}\right)$

49–66 Use logarithmic differentiation to find y'.

49. $y=(x+1)(x+2)(x+3)(x+4)$

50. $y=\dfrac{(x+1)(x+2)}{(x+3)(x+4)}$

51. $y=\sqrt[3]{(2x-1)(x-5)(3x+1)}$

52. $y=\dfrac{\left(x^2-1\right)^{2/3}\left(5x^3+3\right)}{\left(x^2+x+2\right)\left(x^4-10\right)^{3/4}}$

53. $y=\dfrac{\sqrt[3]{x^3-5x^2+7}\,(x+2)}{x^{2/3}\sqrt{3x^2+4}}$

54. $y=\sqrt[3]{\dfrac{x^3-2x^2+1}{(x^2-1)(x^3+5)}}$

55. $y=\dfrac{x^2\sqrt[5]{x^3+3}}{\sqrt[4]{x^4+4}}$ **56.** $y=x^{x^2}$

57. $y=(\sin x)^{1/x}$ **58.** $y=(2x^2+1)^{\tan x}$

59. $y=(\cos x)^{\sqrt{x}}$ **60.** $y=\left(\sqrt[3]{x}\right)^{\sqrt[3]{x}}$

61. $y=(\ln x)^x$ **62.** $y=\dfrac{(\ln x)^x(x^3-1)}{e^x+2}$

63. $y=x^{x^x}$ **64.** $y=(\sin x)^{\cos x}$

65. $y=(\ln x)^{\sin x}$ **66.** $y=(e^x)^x$

67. Mimic the procedure seen in the text to find a formula for the derivative of $y=\cos^{-1}x$.

68. Find a formula for the derivative of $y=\sec^{-1}x$ (see Exercise 67).

69. Find a formula for the derivative of $y=\cot^{-1}x$ (see Exercise 67).

70–93 Determine dy/dx. (Recall that $\arcsin x$ is just a different notation for $\sin^{-1}x$, and the same holds for the other inverse trigonometric functions.)

70. $y=\cos^{-1}(x^2)$ **71.** $y=\tan^{-1}(2x+1)$

72. $y=x\arcsin x$ **73.** $y=\ln(\arctan x)$

74. $y=(\operatorname{arccot} x)^2$ **75.** $y=\arccos\sqrt{x}$

76. $y=\tan^{-1}x+\dfrac{x}{1+x^2}$

77. $y=\arccos x-x\sqrt{1-x^2}$

78. $y=\dfrac{\operatorname{arccot} x}{x}$ **79.** $y=\arctan(e^x)$

80. $y=\operatorname{arccot}(\ln 3x)$ **81.** $y=\dfrac{1-\arctan x}{1+\arctan x}$

82. $y=\arccos x\cdot\operatorname{arccot} x$ **83.** $y=\left(\arcsin(x^3)\right)^2$

84. $y=\sec^{-1}\left(e^{x^2}\right)$ **85.** $y=\sec^{-1}(x^2+1)$

86. $y=\csc^{-1}(e^{-x})$ **87.** $y=\sec^{-1}\sqrt{x^2+1}$

88. $y=\sin(\arccos 3x)$ **89.** $y=(\arctan x)^x$

90. $y=(\arcsin x)^{\ln x}$

91. $y=\cos\left(\operatorname{arccsc}(x^2+1)\right)$

92. $y=\tan\left(\operatorname{arcsec}\sqrt{1+e^{2x}}\right)$

93. $y=\cos\left(\operatorname{arccot}\dfrac{x-1}{\sqrt{2x-1}}\right)$

94–99 Find the equation of the line tangent to the graph of $y=f(x)$ at the indicated x-value. (If needed, round your answer to three decimal places.)

94. $f(x)=\log_2(x^2+1);\quad x=1$

95. $f(x)=\dfrac{(2+x)2^{\ln x}}{x^2e^x};\quad x=1$

96. $f(x)=\arcsin(\ln 3x);\quad x=\dfrac{1}{3}$

97. $f(x)=x\arccos\dfrac{x}{4}-\ln\dfrac{1}{x^2+1};\quad x=2$

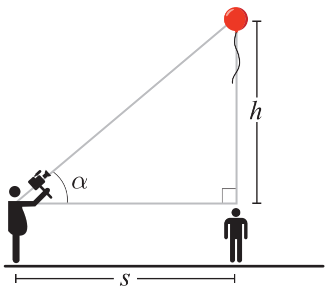
98. $f(x)=(\sin x)^x;\quad x=\dfrac{\pi}{2}$

99. $f(x)=x^{\ln(\arctan x)};\quad x=1$

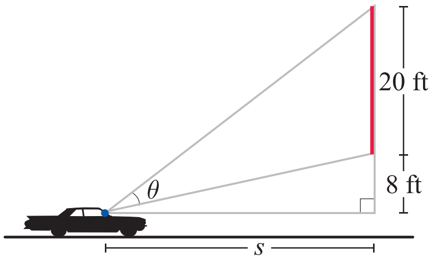
100. Differentiate $f(x)=\arcsin x+\arccos x$. What information about f can you glean from your answer?

101. Repeat Exercise 100 for the function $f(x)=\arcsin(1/x)-\operatorname{arccsc} x$.

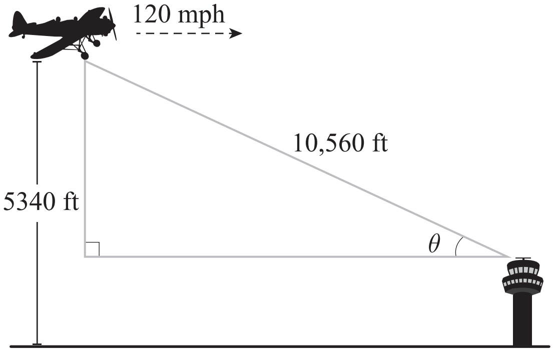
102. A father is filming his child releasing a helium-filled balloon. Assuming that the balloon rises vertically, let the distance between father and child be denoted by s and the height of the balloon, measured from the child, be denoted by h. Find a formula for the angle of elevation α of the camera as it is following the rise of the balloon. Then differentiate with respect to time to find $d\alpha/dt$.



103. The height of the screen of a drive-in movie theater is 20 ft and it is mounted 8 ft above the eye level of a driver who is parked s feet from the screen. Find a formula for the angle θ at which the screen is viewed by this driver. Then differentiate to find the rate of change of the viewing angle as a function of the distance s.



104. An air traffic controller observes a small plane flying horizontally toward the tower and determines from the instrument readings that the distance between the tower and the plane is 10,560 ft, the flying altitude is 5340 ft, and the speed of the plane is 120 mph.



a. Find the angle of elevation θ at which the controller first sees the plane, if the tower is 60 ft high.

b.* Find the angular rate of change $d\theta/dt$ when the plane is 1.25 miles from the controller.

105. Give an alternative definition to $\cot^{-1} x$ so as to make the function continuous and satisfy the identity $\cot^{-1} x = (\pi/2) - \tan^{-1} x$. Graph the function. (**Hint:** Appropriately restrict the domain of $\cot x$. You might also think about the relationship between the graph of the function to be defined and that of $\tan^{-1} x$.)

Concept Check

106–109 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

106. The tangent lines to the graphs of $\ln x$ and $\ln 3x$ have the same slope for all x.

107. If $y = \log \pi$, then $y' = \frac{1}{\ln 10} \cdot \frac{1}{\pi}$.

108. The derivative of $\csc^{-1} x$ is negative everywhere.

109. The functions $f(x) = \ln x$ and $g(x) = \log_c x$ are constant multiples, hence so are their derivatives.

3.7 Exercises

1. Work through Example 1 with the following version of the growth model: $P(t) = P_0 a^t$, where a is treated as the (initially unknown) "growth constant." (**Hint:** Since P doubles every hour, $P(1) = 2P(0)$ gives $P_0 \cdot a^1 = 2P_0$.)

2. In an effort to control vegetation overgrowth, 100 rabbits are released in an isolated area that is free of predators. After one year, it is estimated that the rabbit population has increased to 500. Assuming exponential population growth, what will the population be after another six months?

3. The population of a certain inner-city area is estimated to be declining according to the model $P(t) = 237{,}000e^{-0.018t}$, where t is the number of years from the present. What does this model predict the population will be in 10 years?

4. A population of squirrels is growing in a Louisiana forest with a monthly growth constant of 6 percent. If the initial count is 100 squirrels, how many are there in a year? (**Hint:** Let $N(t)$ stand for the number of squirrels after t months, and note that $N'(t) = 0.06N(t)$. Mimic the steps of Example 1 or, alternatively, make use of the fact that $\frac{d}{dt}(a^t) = (\ln a)a^t$.)

5. The process of radioactive decay is akin to population growth in the sense that the rate of decay is proportional to the amount of material present at any given time. Therefore, it shouldn't come as a surprise that this process can be modeled with the same type of function. Suppose that $A(t)$ stands for the amount of a certain radioactive material at time t, and that it is decaying in a way that the rate of decay satisfies $\frac{d}{dt}A(t) = -0.1A(t)$ (note the negative sign), where t is measured in days. If we start with 1000 g of material, how much is left after 10 days?

6. In Exercise 67 of Section 1.2, we defined the *half-life* of a radioactive substance to be the amount of time required for half of the substance to decay. Find the half-life of the material in Exercise 5.

7. Carbon-11 has a radioactive half-life of approximately 20 minutes, and is useful as a diagnostic tool in certain medical applications. Because of the relatively short half-life, time is a crucial factor when conducting experiments with this element.

   a. Determine a so that $A(t) = A_0 a^t$ describes the amount of carbon-11 left after t minutes (as usual, A_0 is the amount at time $t = 0$).

   b. How much of a 2 kg sample of carbon-11 would be left after 30 minutes?

   c. How much of a 2 kg sample of carbon-11 would be left after 6 hours?

8. The half-life of radium-226 is approximately 4 days. Determine what percentage of the initial amount is left after two weeks.

9. According to Newton's Law of Cooling, the rate of change of temperature of a cooling object is proportional to the temperature difference between the object and the surrounding medium, that is,

$$\frac{dT(t)}{dt} = k\left[T(t) - T_s\right],$$

   where $T(t)$ is the temperature of the object at time t, and T_s is the temperature of its surroundings. Suppose that a cup of 180 °F coffee is left in a 72 °F room and cools to 122 °F in five minutes. How long does it take for the coffee to cool down to 85 °F? (**Hint:** Introduce a new variable for the temperature difference, $D(t) = T(t) - 72$, and then observe that Newton's law translates into the equation $D'(t) = k \cdot D(t)$. Now mimic the procedure seen in Example 1.)

10.* According to the Stefan-Boltzmann Law, the radiation energy emitted by a hot object of temperature T is $R(T) = kT^4$, where T is measured in kelvins. Use this to find a formula for the rate of change of energy emitted by the coffee of Exercise 9. (**Hint:** Use the following formula to convert degrees Fahrenheit to kelvins: $K = \frac{5}{9}(F - 32) + 273$.)

11. A snowplow is moving at a constant speed of $3\ \text{m/s}$, and the snow it is pushing is accumulating at a rate of $100\ \text{kg/s}$. What extra force is necessary for the engine of the snowplow to maintain constant speed despite the increasing mass?

12. When washing his car, Brad is aiming a water hose at the side of the car, with water leaving the hose at a rate of 1 liter per second, with a speed of 15 m/s. If we ignore any "splash backs," what force does the water exert on the side of the car? (**Hint:** Use the equation $F = dP/dt$. See Example 2.)

13. Use Example 3 to find a formula for the force (in newtons) exerted on a scale by a rope dropped on it if the rope is 2 meters long and each centimeter of it has a mass of 20 grams. (As in the text, let x stand for the length of the segment of the rope that has already landed on the scale.)

14. Referring back to Exercise 12, suppose that after washing the car, with almost half of the contents of his 20-liter bucket still left, Brad pours it with a quick move into a much smaller 5-liter container that is sitting on the ground. Find the force exerted on the bottom of the smaller container by the incoming water at the instant when it starts overflowing. (**Hint:** Modify and use the result of Example 3.)

15. Suppose that the snowplow of Exercise 11 is 3500 kg and starts accelerating from 3 m/s at a rate of 0.1 m/s^2. Assuming there is no snow accumulation this time, find the force exerted by the engine.

16.* Consider the accelerating snowplow of Exercise 15, this time assuming the same accumulation rate for the snow as in Exercise 11. Find the force exerted by the engine at $t = 2$ seconds.

17. Show that the velocity v of an object that has fallen a distance x from rest satisfies the equation $v^2 = 2xg$. (**Hint:** Velocity increases at a constant rate from 0 to v, so the distance x can be calculated as the product of the average velocity and time: $x = v_{ave} \cdot t = \left[(0+gt)/2\right]t = \frac{1}{2}gt^2$.)

18. Derive the result of Exercise 17 in an alternative way, using the fact that if air resistance is ignored, the potential energy of an object with mass m at altitude x, which is calculated as $E_p = mgx$, is turned into kinetic energy upon impact, which is calculated as $E_{kin} = \frac{1}{2}mv^2$.

19. In an attempt to escape from a predator, a small fish is swimming vertically downward at a rate of 75 cm/s. Find the rate of change of water pressure around the fish. Express your answer in atm/s. (**Hint:** Use the fact that underwater pressure at a depth of d meters is approximately $P(d) = 1 + 0.097d$ standard atmospheres (atm), where 1 atm = 101.325 kPa.)

20. Hydrogen may be obtained from water by a process called electrolysis, according to the process $2H_2O \to 2H_2 + O_2$. If we measure the production of hydrogen at $2.5\text{ mol}/(\text{L}\cdot\text{h})$ (i.e., 2.5 moles of hydrogen per liter per hour), what will be the concentration of the newly obtained oxygen in 3 hours?

21. The combustion of ammonia gas (NH_3) produces nitrogen and water according to the process $4NH_3 + 3O_2 \to 2N_2 + 6H_2O$. Supposing that the rate of combustion is $0.5\text{ mol}/(\text{L}\cdot\text{s})$, what is the rate of the production of water? How many milliliters of water are produced in two seconds? (**Hint:** Use the fact that the approximate molar mass of hydrogen is 1 g, while that of oxygen is 16 g. Also, the mass of 1 mL of water is 1 g.)

22. Magnesium is a flammable metal, and because of its bright light it has traditionally been used in camera flashes, illumination of mine shafts, fireworks, and flares. The reaction itself is described by $2Mg + O_2 \to 2MgO$. If magnesium burns in a chamber at an initial rate of 1.5 g/s, find the rate (in $\text{mol}/(\text{L}\cdot\text{s})$) at which the concentration of O_2 is decreasing in the chamber. (**Hint:** The approximate molar mass of magnesium is 24 g.)

23–29 Use the technique of linearization to determine the answer.

23. A manufacturer of small remote-controlled cars found its weekly revenue to be $R(x) = 160x - 0.3x^2$ dollars when x units are produced and sold.

a. Use marginal revenue to estimate the extra revenue when production is increased from 12 to 13 units.

b. Use the revenue function to calculate the actual revenue increase. Compare your answers.

24. Suppose the monthly cost of producing x units of a particular commodity is $C(x) = 75x^2 + 200x + 5100$ dollars, while the revenue function is $R(x) = 32x(120 - x)$.

a. Use marginal cost to find the added expense in increasing production from 5 to 6 units.

b. Use marginal revenue to estimate the revenue generated by raising production from 5 to 6 units.

c. Find the actual increases in cost and revenue by producing and selling the sixth unit, and compare these numbers to your estimations from parts a. and b.

25. Repeat Exercise 24 for $C(x) = \frac{486}{7}x^2 + 98x + 120$ and $R(x) = -19.5x^2 + 2526x + 442$.

26. A manufacturer models the total cost of producing n hundreds of a particular pocket calculator by the function $C(n) = \frac{1}{900}n^2 + 2500n + 3100$ dollars. Market research shows that all products will be sold at the price of $p(n) = \frac{2}{5}\left(120 - \frac{1}{4}n\right)$ dollars per calculator.

a. Use the marginal cost function to estimate the cost of raising the level of production from 1000 to 1100 calculators.

b. Use the marginal profit function to estimate the additional profit if the level of production is raised from 1000 to 1100 calculators.

c. Find the actual increases in cost and profit when production is raised as in parts a. and b., and compare these values with your estimates obtained in the previous parts.

27. A lumber company estimates the cost of producing x units of a product to be $C(x) = 0.1x^2 + 2250x + 1450$ dollars, while the price of each unit has to be $p(x) = 50(128 - 0.2x)$ in order to sell all x units. However, seasonal supply of raw materials makes x dependent on time so that $x(t) = -0.35(t-6)^2 + 192$, where t is measured in months. Use the value of the marginal profit at $t = 2$ to estimate the change in profit during the third month.

28. Repeat Exercise 27 if $x(t) = 185 + 10\sin^2\left(\frac{\pi}{6}t\right)$.

29. A child playing on the beach is pouring sand from a bucket, forming a sand cone that is growing in such a way that its height is always half of the radius of its circular base. Estimate the change in volume of the sand cone as its height grows from 3 to 4 inches.

30–39 Slightly generalize the technique of linearization to find the answer. For cases where Δx is not 1, estimate the change in a function $f(x)$ by $\Delta f = f'(x)\Delta x$.

30. Suppose the cost of manufacturing x units of a certain commodity was found to be $C(x) = 35x^2 + 20x + 780$ dollars, and suppose the current production level is 50 units. Use linearization to estimate how the cost changes if production is raised to 50.25 units.

31. Repeat Exercise 30 with an increase in production from 13 to 13.5 units.

32. Repeat Exercise 24 with an increase in production from 6 to 6.75 units.

33. Use marginal analysis to estimate the changes in cost and profit of Exercise 26 when production is decreased from 1000 to 950 calculators. Then find the actual changes and compare them with your estimates.

34. An ice cube with a side length of 1.5 inches starts melting in such a way that its sides are decreasing by 0.1 inches per minute. Use linearization to estimate the change in the cube's volume during the second minute.

35. According to MRI scans, a benign tumor in a patient had a radius of 1.6 cm when it was first discovered, and it is growing by 1 mm each month. Assuming the tumor is spherical, estimate the change in its volume during the first week.

36. The daily output of a small factory is $n(I) = 50\sqrt{I}$ units, where I is the owner's investment measured in dollars. If the current investment is \$100,000, use linearization to estimate how much additional capital is needed to increase the daily output by 5%.

37. Suppose that $w(t)$, the number of workers at a certain factory at time t, has been decreasing at a rate of 2.5 percent, relative to the size of the workforce, due to a recent recession. At the same time, however, the workers' average productivity $p(t)$ has been increasing by 4 percent due to extra training and the inherent fear of a job loss. Find the change in gross productivity. (**Hint:** See Example 5.)

38. A factory outputs $n(I) = 200\sqrt{I(t)}\sqrt[3]{g(t)}$ units daily, where I is the total investment measured in dollars and g stands for gross productivity. If the total investment is decreasing by 1 percent while the gross productivity is increasing by 2.1 percent (both in the relative sense), find the relative change in the daily output of the factory.

39. Repeat Exercise 38 if $I(t)$ is decreasing by 2 percent, the number of workers is decreasing by 3 percent, but worker productivity is increasing by 5.1 percent. (**Hint:** Recall the equation from Example 5: $g(t) = w(t)p(t)$.)

If $r = 20$ cm and $dV/dt = 200\text{ cm}^3/\text{s}$, then we have

$$\frac{dS}{dt} = \left(\frac{2}{20\text{ cm}}\right)\left(200\text{ cm}^3/\text{s}\right) = 20\text{ cm}^2/\text{s},$$

the answer we obtained before.

But we might try to attack the problem in a different manner. If we *could* express S in terms of V, we could differentiate both sides with respect to t and again arrive at a relation between dS/dt and dV/dt. In pursuit of this, note that

$$\frac{r}{3}S = \frac{r}{3}\left(4\pi r^2\right) = \frac{4}{3}\pi r^3 = V,$$

so

$$S = \frac{3}{r}V.$$

But then this seems to imply that

$$\frac{dS}{dt} = \frac{dS}{dV}\cdot\frac{dV}{dt} = \frac{3}{r}\cdot\frac{dV}{dt},$$

which is not the result of $\frac{2}{r}\cdot\frac{dV}{dt}$ that we found above! What has gone wrong?

The answer is that just as V is a function of r, r can also be said to be a function of V; that is, as V changes, it certainly implies a change in r. In fact,

$$r = cV^{1/3},$$

where $c = \left[3/(4\pi)\right]^{1/3}$ (this comes from solving $V = \frac{4}{3}\pi r^3$ for r). So,

$$S = \frac{3}{r}V = \frac{3}{cV^{1/3}}V = \frac{3}{c}V^{2/3}.$$

We leave it as an exercise (Exercise 15) to show that if this last equation is differentiated with respect to t and simplified, the result is again $\frac{dS}{dt} = \frac{2}{r}\cdot\frac{dV}{dt}$.

Moral: Watch out for hidden functional relationships between the variables in your equations!

3.8 Exercises

1. A theme park ride is descending on a parabolic path that can be approximated by the equation $y = -\frac{1}{90}x^2 + 90$ (distance is measured in feet). If the horizontal component of its velocity is a constant 6 ft/s, find the rate of change of its elevation when $x = 22.5$.

2. Adapt your solution from Exercise 1 to find dx/dt at $x = 30$ feet if the equation of the ride's path is $y = 0.01(x-95)^2 - 2.25$ and $dy/dt = -20\text{ ft/s}$.

3–10 Find the rate of change using the given information.

3. $\dfrac{dy}{dt}$ at $y=3$, if $y=\sqrt{x+2}$ and $\dfrac{dx}{dt}=1$

4. $\dfrac{dy}{dt}$ at $x=2$, if $x^2+y^2=5$, $y>0$, and $\dfrac{dx}{dt}=3$

5. $\dfrac{dx}{dt}$ at $y=0.5$, if $y=\dfrac{1}{x}$ and $\dfrac{dy}{dt}=-2$

6. $\dfrac{dx}{dt}$ at $x=1$, if $xy^2=\dfrac{1}{4}$ and $\dfrac{dy}{dt}=-0.25$

7. $\dfrac{dy}{dt}$ at $x=0$, if $y=\dfrac{x+2}{x^2+1}$ and $\dfrac{dx}{dt}=-5.2$

8. $\dfrac{dy}{dt}$ at $y=\dfrac{1}{2}$, if $y=\dfrac{1}{2}e^{-x}$ and $\dfrac{dx}{dt}=25$

9. $\dfrac{dy}{dt}$ at $x=-\dfrac{3\pi}{4}$, if $y=2\sin\left(x+\dfrac{\pi}{4}\right)$ and $\dfrac{dx}{dt}=7.4$

10. $\dfrac{dx}{dt}$ at $y=\dfrac{\pi}{4}$, if $x=\cot y$ and $\dfrac{dy}{dt}=-3.35$

11. The length of a rectangle is increasing at a rate of 5 in./s, while its width is decreasing at 2 in./s. Find the rate of change of its area when its length is 45 in. and its width is 25 in.

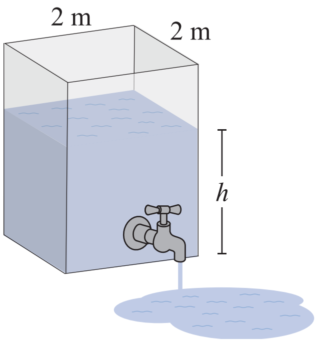
12. Find a formula for the rate of change of the distance from the origin of a point moving on the graph of $f(x)=x^2$ when $x=2$ and $dx/dt=3$ units per second.

13. Find the rate of separation between the moving points (x_1, y_1) and (x_2, y_2) on the graph of $y=\sin x$ when $x_1=\pi/2$ if they start at the origin at the same time, and the horizontal components of their velocities are $dx_1/dt=\frac{1}{2}$ units per second and $dx_2/dt=-\frac{1}{2}$ units per second, respectively.

14. A Ferris wheel of radius 34 feet needs 3 minutes to complete a full revolution. At what rate is a rider descending when she is 51 feet above ground level?

15. Using the notation of Example 5, use the Chain Rule to differentiate the equation $S=(3/c)V^{2/3}$ with respect to time to obtain $\dfrac{dS}{dt}=\dfrac{2}{r}\cdot\dfrac{dV}{dt}$. (**Hint:** After differentiating, make use of the equation $r=cV^{1/3}$ again.)

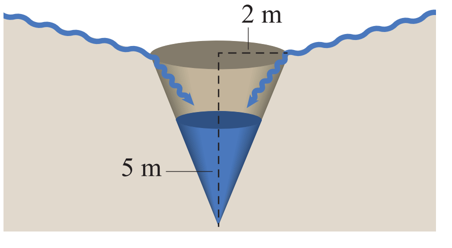
16. Rework Example 1 assuming that the tank is a rectangular prism with a 2 m by 2 m square base.



17. Rework Example 1 again, this time assuming that the tank is an inverted, right square pyramid with height 4 m and a 2 m by 2 m base. How fast is the level of water falling when its depth is 2 m?

18. A spectator is tracking a stunt plane at an air show with his camera. If the plane is on a near-vertical path, rising at a speed of 100 feet per second, and the camera is 400 feet from the point on the ground directly below the plane, how fast is the camera angle (measured with respect to the ground) changing when the plane's altitude is 400 feet? How fast is the distance between the camera and the plane increasing at that instant?

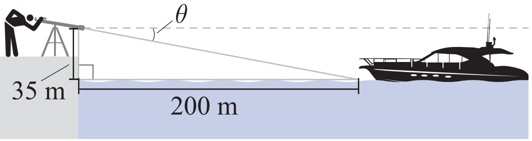
19. A cistern in the form of an inverted circular cone is being filled with water at the rate of 75 liters per minute. If the cistern is 5 meters deep, and the radius of its opening is 2 meters, find the rate at which the water level is rising in the cistern half an hour after the filling process began. (**Hint:** 1 m^3 = 1000 L)



20. Repeat Exercise 19, this time assuming that the cistern is in the form of a pyramid with a 4-by-4-meter square opening.

21. A ship passes a lighthouse at 3:15 p.m., sailing to the east at 10 mph, while another ship sailing due south at 12 mph passes the same point half an hour later. How fast will they be separating at 5:45 p.m.?

22. A tourist at scenic Point Loma, California uses a telescope to track a boat approaching the shore. If the boat moves at a rate of 10 meters per second, and the lens of the telescope is 35 meters above water level, how fast is the angle of depression of the telescope (θ) changing when the boat is 200 meters from shore?

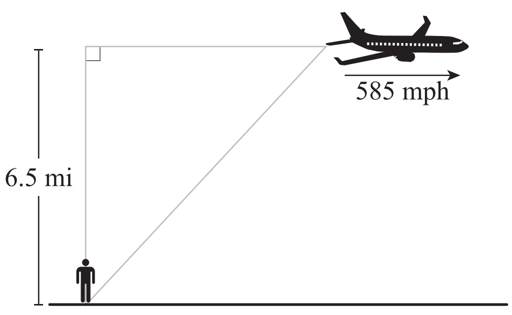


23. When preparing cereal for her child, a mother is pouring milk into a bowl, the shape of which can be approximated by a hemisphere with a radius of 6 in. If milk is being poured at a rate of $4 \text{ in.}^3/\text{s}$, how fast is the level of milk rising in the bowl when it is 1.5 inches deep? (**Hint:** The volume of fluid of height h in a hemispherical bowl of radius r is $V = \pi h^2\left(r - \frac{1}{3}h\right)$.)

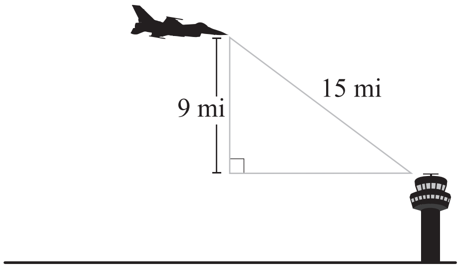
24. Suppose that in Exercise 29 of Section 3.7, the sand is being poured at a rate of 8 cubic inches per second. Find the rate of change of the height of the cone when it is 4 inches tall.

25. When finished playing in the sand, the child of Exercise 24 takes advantage of a nice wind and starts flying his kite on the beach. When the kite reaches an altitude of 60 feet the wind starts blowing it horizontally away from the child at a rate of 15 feet per second while maintaining the altitude of the kite. How fast does the child have to be letting out the string when 100 feet are already out?

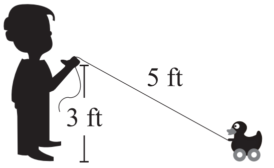
26. A passenger airplane, flying at an altitude of 6.5 miles at a ground speed of 585 miles per hour, passes directly over an observer who is on the ground. How fast is the distance between the observer and the plane increasing 3 minutes later?



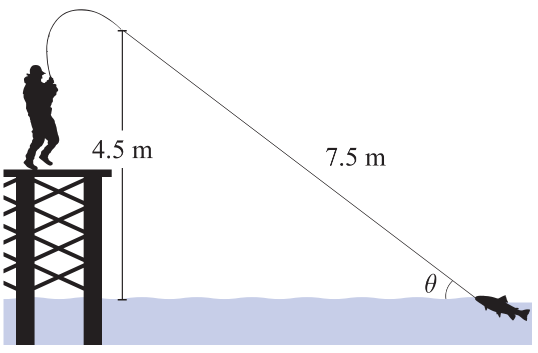
27. A military plane is flying directly toward an air traffic control tower, maintaining an altitude of 9 miles above the tower. The radar detects that the distance between the plane and the tower is 15 miles and that it is decreasing at a rate of 950 miles per hour. What is the ground speed of the plane?



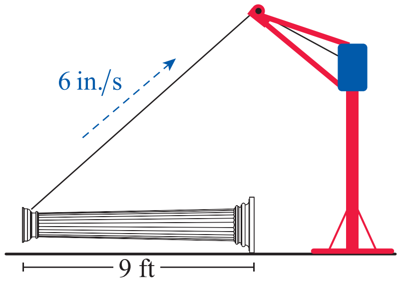
28. A child is retrieving a wheeled toy that is attached to a string by pulling in the string at a rate of 1 foot per second. If the child's hands are 3 feet from the ground, at what rate is the toy approaching when 5 feet of the string are still out?



29. A fisherman is reeling in a fish at a rate of 20 centimeters per second. If the tip of his fishing rod is 4.5 meters above the water, and we are assuming that the fish is near the water surface throughout the process, how fast is it approaching when 7.5 meters of fishing line are still out? How fast is the angle θ between the fishing line and the water increasing at that instant?



30. A construction worker is using a winch to pull a 9-foot column to a vertical position. If the winch is in the exact position where the top of the installed column is supposed to be, and the rope is being pulled at the rate of 6 inches per second, at what rate is the angle between the column and the ground changing when it is $\pi/6$ radians? At what rate is the top of the column rising vertically at that instant? (Assume the base of the column doesn't slip during lifting.)



31. The volume of a cube is decreasing at a rate of $150 \text{ mm}^3/\text{s}$. What is the rate of change of the cube's surface area when its edges are 30 mm long?

32. The acute angles of a rhombus are increasing at a rate of 0.25 radians per second. If the sides of the rhombus are 20 cm, at what rate is the area of the rhombus increasing when the acute angles are $\pi/3$ radians?

33. A 35-foot-by-18-foot rectangular pool, whose depth increases uniformly from 3 feet to 8 feet (along the 35-foot side), is being filled with water at the rate of 4.5 cubic feet per minute. You observe that water appears to be "creeping up" on the angled bottom much faster than it rises along the vertical walls. Find the rate at which the water rises along the angled bottom at the instant when the water level is 2 feet at the deep end of the pool.

34. Considering again the pool of Exercise 33, suppose that it is measured that the water is climbing upward along the angled bottom at a rate of 3.02 in./min when the water level is 1 foot at the deep end. Assuming that the pump is working at the same rate of 4.5 cubic feet per minute, use this information to prove that the pool has a leak, and find the rate at which water is leaking out of the pool.

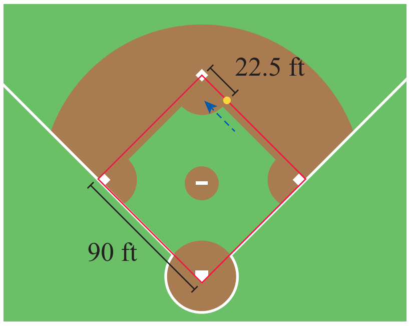
35. A trough that is 5 meters long and 1 meter across at the top has a cross-section in the form of an isosceles trapezoid and both of its endplates are vertical. The altitude of the trapezoid is 40 centimeters, and the shorter base is 20 centimeters long. If the trough is being filled at the rate of 30 liters per minute, how fast is the water level rising at the instant when the water's depth is 20 centimeters?

36. Rework Example 3, this time assuming that, after turning right, Susan drove up a short ramp to reach an elevated highway 20 feet above the other drivers. She is now proceeding levelly at a distance of 40 feet, as measured along the ground, from the intersection.

37. An electrician is working on top of a 15 ft ladder that is leaning against the wall when its bottom starts sliding at a rate of 1 ft/s. Fortunately, a fellow worker catches it when the ladder's bottom is 5 ft from the wall. How fast is the top of the ladder (along with the electrician) sliding down the wall at that instant?

38. Adam is arriving home one evening in his SUV and is slowly approaching his garage door at a rate of 5 ft/s when the sensor lights come on. If the lights are mounted directly above the door at a height of 15 ft from the ground and Adam's SUV is 6 ft tall, at what rate is the length of the car's shadow shrinking when it is 25 ft from the garage door? What is the speed of the tip of the car's shadow?

39. A baseball player is running from first base to second base at 25 feet per second. At what rate is his distance increasing from home plate when he is 22.5 feet from second base? (**Hint:** The baseball diamond is a 90-foot-by-90-foot square.)



40.* A container in the shape of a cone, standing on its circular base, is being filled with water at the rate of 1.5 cubic feet per minute. If the radius of the base is 2 feet and the height of the cone is $2\sqrt{3}$ feet, how fast is the water level rising when it is 2 feet deep? (**Hint:** The volume of liquid in a partially filled conical tank is $V = \frac{1}{3}\pi d\left(R^2 + Rr + r^2\right)$, where R is the radius of the base, r is the radius of the top of the liquid, and d is its depth.)

41.* Italian police are chasing a criminal down a narrow street at a speed of 90 kilometers per hour. If the blue light on the top of the car is rotating counterclockwise at a rate of 1 rotation per second, and the buildings are only 3 meters from the car on the right, how fast is the beam moving on the wall at the instant when it is already 6 meters ahead of its source?

42.* When studying for a calculus test, Roger accidentally pushes his book over the edge of his 2.5 ft high desk. If his 6 ft tall lamp is standing 3 ft from where the textbook fell down, how fast was the book's shadow moving when the text hit the ground? (Ignore air resistance. Use $g \approx 32 \text{ ft/s}^2$.)

43.* A wall clock has a 10 in. minute hand and a 6 in. hour hand. At what rate are the tips of the hands approaching each other at 3 o'clock?

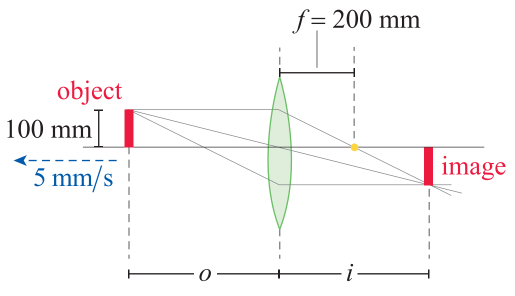
44.* The *lens equation*, easily derivable from geometric similarity for a thin converging lens, is

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f},$$

where o (the *object distance*) and i (the *image distance*) are the respective distances of the object and the image from the lens, and f is the *focal length* of the lens. Suppose a 100 mm high object is being slowly moved away from a lens at a speed of 5 mm/s. The focal length of the lens is 200 mm.

a. Find the rate at which the image changes its location when the object distance is 600 mm.

b. Find the rate at which the image changes its size at the same instant.



45. Suppose that the torque output of an automobile engine, as a function of engine speed, is approximated by

$$T(s)=\left(-0.001/150^4\right)(s-3000)^4+160 \text{ lb-ft},$$

where s is measured in revolutions per minute (rpm), and that the engine revs up from 0 to 5000 rpm (assume no gear shift takes place).

a. Use a graphing utility to graph the torque as a function of s on the interval $[0,5000]$ (this is called the engine's *torque curve*).

b. If the power output of the engine, measured in horsepower (hp), is calculated by $P=\frac{1}{5252}sT(s)$ hp, and the engine is revving up according to the function $s(t)=1000t$ (t is measured in seconds), find the rate of change of the power output at $t=3$ seconds.

3.9 Exercises

1–12 Find the linearization of the function at the given value.

1. $f(x)=x^3-x;\quad x=1$
2. $g(x)=\sqrt{x-3};\quad x=4$
3. $h(x)=\left(x^4-5x^2+1\right)^7;\quad x=0$
4. $k(x)=\left(x^2+1\right)^{-2};\quad x=2$
5. $C(\theta)=\cos\theta;\quad \theta=0$
6. $T(\theta)=\tan\theta;\quad \theta=0$
7. $F(t)=\left(t^2+5t-6\right)^{-1/3};\quad t=2$
8. $r(x)=\dfrac{1}{x+4};\quad x=-3$
9. $t(u)=\dfrac{u+2}{u^2-15};\quad u=-4$
10. $v(x)=\sin\pi x;\quad x=\dfrac{1}{6}$
11. $G(z)=e^z;\quad z=0$
12. $U(s)=\ln\left(s^4+1\right);\quad s=1$

13–24 Find the value of the differential *dy* for the given values of *x* and *dx*.

13. $y=3x^2+x;\quad x=1,\quad dx=0.2$
14. $y=x\sqrt{x-5};\quad x=6,\quad dx=0.01$
15. $y=\dfrac{4x+1}{x-3};\quad x=2,\quad dx=0.1$
16. $y=\sec x;\quad x=\dfrac{\pi}{4},\quad dx=\dfrac{1}{8}$
17. $y=x^{3/2}+x^{-3/2};\quad x=4,\quad dx=\dfrac{1}{16}$
18. $y=\ln x+\dfrac{1}{\ln x};\quad x=e,\quad dx=0.01$
19. $y=x\tan x;\quad x=-\dfrac{\pi}{4},\quad dx=\dfrac{1}{4}$
20. $y=e^{\sqrt{x^2+3}};\quad x=1,\quad dx=0.001$
21. $y=\sqrt{\ln(x+1)};\quad x=e-1,\quad dx=\dfrac{-1}{e^2}$
22. $y=\arctan x;\quad x=-1,\quad dx=\dfrac{-1}{2^5}$
23. $y=\dfrac{\tan x}{x^2+1};\quad x=\dfrac{\pi}{3},\quad dx=-0.1$
24. $y=\cos(\arcsin x);\quad x=0.6,\quad dx=-0.16$

25–28 Calculate the values of *dy* and Δy and then use graph paper to draw the curve near the given point, indicating all three of the line segments *dx*, *dy*, and Δy.

25. $y=\dfrac{1}{2}x^2;\quad x=1,\quad dx=\dfrac{1}{2}$
26. $y=\tan x;\quad x=0,\quad dx=\dfrac{\pi}{6}$
27. $y=2^x;\quad x=1,\quad dx=\dfrac{1}{4}$
28. $y=\dfrac{1}{x^2};\quad x=1,\quad dx=-\dfrac{1}{4}$

29–40. Find the values of Δy and compare them with *dy* at the indicated points for the curves given in Exercises 13–24.

41–48 Use linear approximation to approximate the given number. Compare this approximation to the actual value obtained using a calculator. Round your answer to four decimal places. (**Hint:** First identify $f(x)$ and *c*; then find and appropriately evaluate $L(x)$.)

41. $\sqrt{9.1}$
42. $(1.01)^3$
43. $(7.9)^{2/3}$
44. $\dfrac{1}{10.1}$
45. $\sqrt[5]{31}$
46. $\cos 1$
47. $\ln 2.7$
48. $e^{1.05}$

49. Prove the Power and Quotient Rules for differentials.
   a. $d\left(x^n\right)=nx^{n-1}dx$
   b. $d\left(\dfrac{u}{v}\right)=\dfrac{vdu-udv}{v^2}$

50. Use the equations for V and dV from Example 4 to prove that the propagated error in the calculated volume of a sphere, in percentage terms, is three times larger than the margin of error in the measured radius; that is,

$$\frac{dV}{V}=3\cdot\frac{dr}{r}.$$

51. Prove or disprove that an analogous equation to that obtained in Exercise 50 is true for a cube; that is, if the measured side length of a cube is *a* units with a margin of error of *da*, then

$$\frac{dV}{V}=3\cdot\frac{da}{a}.$$

52–71 Use differentials or linearization to provide the requested approximations.

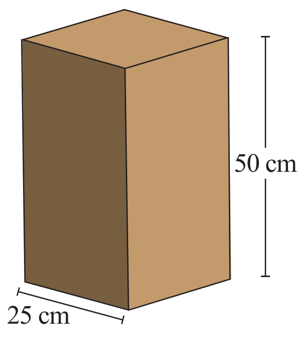
52. The side of a square was measured to be 9.5 cm with a possible error of 0.5 mm. Approximate the propagated error in the calculated area of the square. Express your answer as a percentage error.

53. The radius of a circular disk was measured to be $10\frac{1}{8}$ inches. Estimate the maximum allowable error in the measurement of the radius if the percentage error in the calculated area of the disk cannot exceed 2.5 percent.

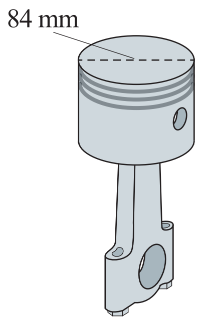
54. The base and altitude of a triangle were measured to be 7 and 9 inches, respectively. If the possible error in both cases is $\frac{1}{16}$ inches, approximate the propagated error when computing the area of the triangle.

55. Two sides of a triangle were measured to be 60 and 80 mm, respectively, while the included angle is 60 degrees. If the margin of error of the linear measurements is 0.1 mm, while that of the angle measurement is 0.1 degrees, find the possible propagated error in the calculated area of the triangle.

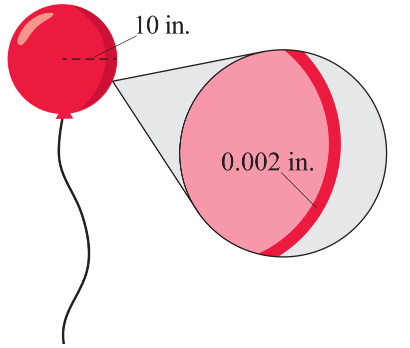
56. A box in the shape of a rectangular prism has a square base. If the edge of the base is 25 cm and the height is 50 cm, both with a possible measurement error of 0.2 mm, estimate the propagated errors in both the computed volume and surface area of the box. Express both answers as percentage errors.



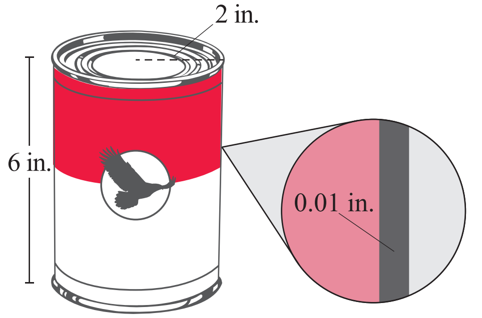
57. A cylindrical piston of diameter 84 mm is being manufactured for an automobile engine. If the maximum percentage error in the measurement of the diameter is 0.05%, estimate the greatest possible value of the propagated error in the computed cross-sectional area of the piston. Express your answer as a percentage error.



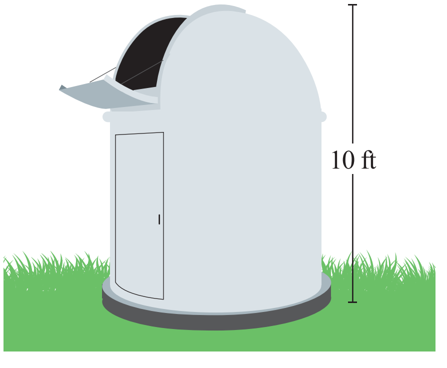
58. If the radius of an inflated balloon is 10 inches and the thickness of its wall is 0.002 inches, estimate the volume of the material it is made of. (Assume the balloon is perfectly spherical.)



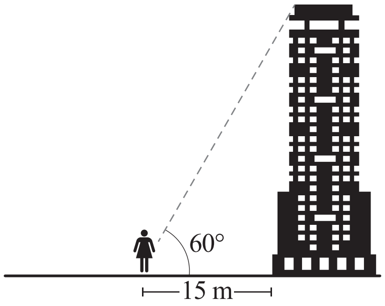
59. A tin can has a circular base of radius 2 inches and a height of 6 inches. If the thickness of its walls is 0.01 inches, estimate the volume of the material it is made of.



60. The exterior of a small private observatory needs to be painted. The building is approximately a circular cylinder with a hemisphere on top. The radius of the base is 3.5 feet and the height of the entire structure is 10 feet. Express the volume as a function of the radius of the base and use linearization to estimate the amount of paint that will provide a coat that is $\frac{1}{32}$ inches thick.



61. A trigonometry student stands 15 meters from a building and measures the angle of elevation to the top of the building as 60°. How accurate does her angle measurement have to be if she wants her propagated percentage error in estimating the height of the building to be no more than 5%?



62. Referring to Exercise 44 of Section 3.8, estimate the change in image distance when the object distance increases from 60 cm to 61 cm.

63. The magnetic force experienced by a wire carrying a current I in an external magnetic field of uniform strength B is found from the equation

$$F = BIL\sin\theta,$$

where L is the length of the wire (measured in meters), and θ is the angle between the directions of B and I.

a. Find the magnetic force on a 50 cm wire if $B = 0.03\ \text{N}/(\text{A}\cdot\text{m})$, $I = 25$ amperes (A), and $\theta = 30°$.

b. Estimate the change in force if θ is increased to 33°.

c. Calculate the true value of the change and compare it with your approximation.

64. Estimate the change in the force in Exercise 63 if θ is increased to 33°, I is increased to 27 A, and B is decreased to $0.025\ \text{N}/(\text{A}\cdot\text{m})$.

65. The kinetic energy (in J) of a moving object is found from the equation $E_{kin} = \frac{1}{2}mv^2$, where m is the mass (in kg) of the object and v is its velocity (in m/s). Estimate the change in kinetic energy of a 1400 kg car that is accelerating from $100\ \text{km/h}$ (approx. 62 mph) to $112\ \text{km/h}$ (approx. 70 mph). What is the estimated percentage change?

66. When air resistance is negligible, the speed of impact of an object falling from height h is $v_i = \sqrt{2hg}$. Suppose that a rock is dropped from a height of 5 meters.

a. Find the speed of impact as the rock hits the ground.

b. Approximate the height from which the rock has to be dropped in order to increase the speed of impact by 10 percent. Express the height difference in both absolute and relative (percentage) terms.

c. Find the true value of the above height and compare it with your approximation.

67.* The volume of a cube of side length a is being determined by immersing the cube into a container of water and measuring the volume of the displaced water, and then the surface area is calculated. Estimate the percentage error we can allow in the measurement of the volume if the calculated surface area cannot differ from the true value by more than 2%. Can you generalize the result?

68. The profit function for a company is found to be $P(x) = -1.2x^2 + 500x - 2600$, where x is the number of units manufactured. If the current production level is 100 units, estimate the percentage change in profit if production is raised to 110 units.

69. For the company in Exercise 68, estimate how much the company has to increase production from 100 units in order to achieve a 10 percent profit increase.

70.* The diameter of the bottom of a 4.5-inch-tall paper cup is 2.5 inches, while the diameter of its opening is 3.5 inches. If the cup is filled with iced soda to a depth of 4.3 inches and an additional 1-cubic-inch ice cube is dropped in, predict whether the cup will overflow. (**Hint:** See Exercise 40 of Section 3.8 for help in finding the volume of soda in the cup.)

71. Suppose the velocity function of a moving object is $v(t) = 1/(1+t^2)$, and that it is moving in the positive direction along the x-axis. If you know that its location at $t = 2$ is $x = 5$, estimate its position half a second later.

72. The actual error in measurement is sometimes called absolute error, while the percentage error is referred to as relative. Write a short paragraph comparing absolute, relative, and propagated errors. Illustrate with a concrete example.

73. Examine the answer you obtained for Exercise 57. Can you state and prove a result, analogous to the one in Exercise 50, for the radius and cross-sectional area of a right circular cylinder? Explain.

Concept Check

74–80 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. Since the differential dx is an increment, its value is always positive.

75. If $f(x) = k$, then $df = 0$.

76. If f is linear, then $\Delta f / \Delta x = df / dx$.

77. If f is differentiable at c, then $\lim_{\Delta x \to 0} (\Delta f / \Delta x) = df / dx$.

78. Propagated error is also called percentage error.

79. The differential dy is always a bit less than Δy.

80. If f is increasing and $dx < 0$, then $dy > \Delta y$.

3.9 Technology Exercises

81–92. Use a graphing utility to graph the functions given in Exercises 13–24 in the same viewing window along with their linear approximations at the specified x-values. Use the `Zoom` and `Trace` features to find the maximum value for dx so that the approximation is accurate to 0.01.

$$30,000+\frac{50,000(x-3)}{\sqrt{1+(3-x)^2}}=0$$

$$\frac{(x-3)}{\sqrt{1+(3-x)^2}}=-\frac{30,000}{50,000}$$

$$\frac{(x-3)^2}{1+(3-x)^2}=\frac{9}{25}$$ Square both sides.

$$25(x-3)^2=9+9(x-3)^2$$ Note that $(3-x)^2=(x-3)^2$.

$$16(x-3)^2=9$$

$$x-3=\pm\frac{3}{4}$$ Divide by 16 and take the square root.

$$x=3\pm\frac{3}{4}$$

The point $3+\frac{3}{4}$ does not actually solve the original equation, so we only have to evaluate C at the two endpoints 0 and 3 and at the critical point $3-\frac{3}{4}=\frac{9}{4}$.

$$C(0)=50,000\sqrt{10}\approx\$158,114$$

$$C\left(\frac{9}{4}\right)=\$130,000$$

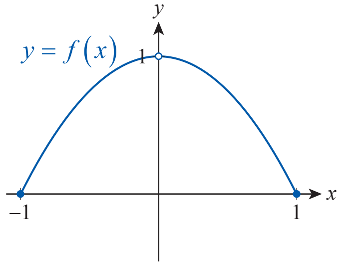
$$C(3)=\$140,000$$

From this comparison, we see that the minimal cost of installation can be achieved by laying the cable with an underground run of $\frac{9}{4}=2\frac{1}{4}$ kilometers and a diagonal run underwater of $\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{\frac{25}{16}}=1\frac{1}{4}$ kilometers.

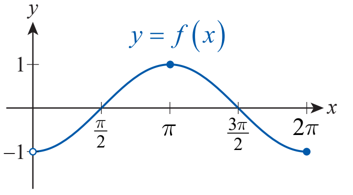
4.1 Exercises

1–4 Use the graph as an aid to identify the absolute extrema, if they exist, for the given function on the specified domain.

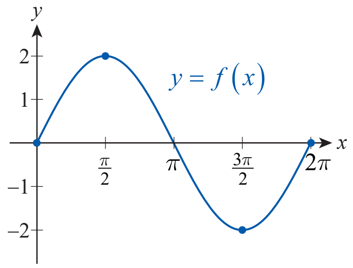
1. $f(x)=-x^2+1;\quad D=[-1,0)\cup(0,1]$



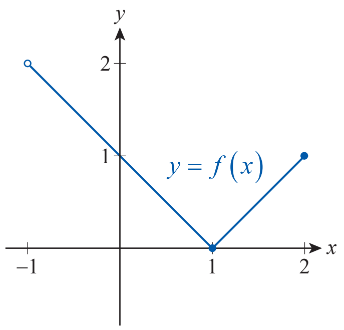
2. $f(x)=-\cos x;\quad D=(0,2\pi]$



3. $f(x)=2\sin x;\quad D=[0,2\pi]$

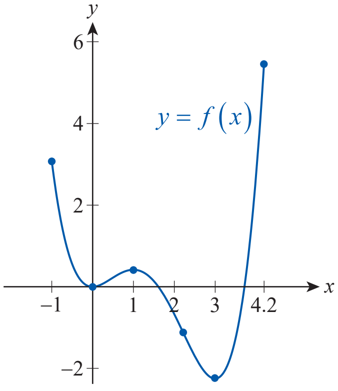


4. $f(x)=|x-1|;\quad D=(-1,2]$



5–8 Use the graph to decide whether each highlighted point is a critical point, and then find and classify all relative and absolute extrema for the function over the given interval.

5.



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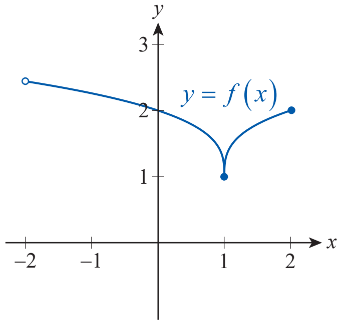

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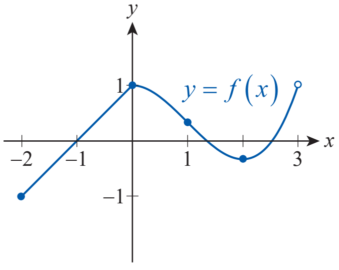

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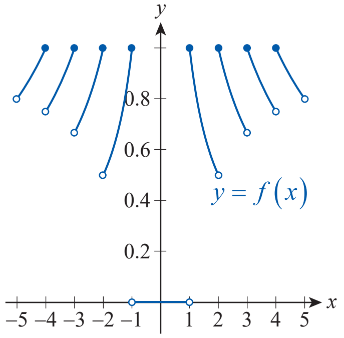


9–20 Use graph paper to sketch the graph of the given function on the specified domain, and then use the graph to visually identify and classify any absolute extrema.

9. $f(x)=2x+1;\quad D=[0,3]$

10. $g(x)=-x-1;\quad D=(-1,2]$

11. $h(x)=\dfrac{1}{2}x-3;\quad D=\mathbb{R}$

12. $u(x)=-x^2;\quad D=(-1,1)$

13. $v(x)=(x+1)(x-3);\quad D=[-2,4]$

14. $k(x)=(x-4)^4;\quad D=\mathbb{R}$

15. $K(x)=x^7;\quad D=\mathbb{R}$

16. $m(x)=e^{-x+2};\quad D=[2,\infty)$

17. $n(x)=\cos\pi x;\quad D=\left(0,\dfrac{3}{2}\right]$

18. $F(x)=\dfrac{1}{(x+1)^2};\quad D=\mathbb{R}$

19. $G(x)=\dfrac{1}{x^2+1};\quad D=\mathbb{R}$

20. $H(t)=\arcsin t;\quad D=[-1,1]$

21–37 Sketch by hand the graph of a function f on the specified domain, with the specified properties. (Answers will vary.)

21. Defined on $[2,4]$, absolute maximum at 2, absolute minimum at 4

22. Defined on $[-1,2]$, absolute maximum at 0, absolute minimum at 1

23. Defined on $[-5,5]$, absolute maximum at 1, absolute minimum at 5

24. Defined on $\mathbb{R}$, absolute minimum at 2, no absolute maximum

25. Defined on $[-3,2]$, absolute maximum at -2, absolute minimum at 0, relative maximum at 1

26. Defined on $[0,6]$, absolute maximum at 2, relative minimum at 4, no absolute minimum

27. Defined on $[-1,1]$, absolute maximum occurs twice, no minimum

28. Defined on $(1.5,7)$, continuous, has relative maximum and minimum, but no absolute maximum or minimum

29. Defined on $(1.5,7)$, continuous, has both absolute maximum and minimum

30. Defined on $[-2,4]$, two relative maxima, but no absolute maximum

31. Defined on $(0,\infty)$, continuous, no relative or absolute extrema

32. Defined on $(-1,3]$, continuous, no absolute minimum, one relative minimum, absolute maximum occurs twice

33. Defined on $\mathbb{R}$, both the absolute maximum and absolute minimum occur infinitely often

34. Defined on $(0,\infty)$, infinitely many relative maxima and minima, no absolute maximum or minimum

35. Differentiable on $\mathbb{R}$, has one critical point, but no extrema

36. Defined on $(0,10)$, not differentiable at 5, but absolute maximum occurs at 5

37. Defined on $(0,10)$, discontinuous at 5, but absolute maximum occurs at 5

38–55 Find all critical points, if they exist, for the given function.

38. $f(x)=x^2-7x+1.5$

39. $g(x)=2x^3+3x^2-12x+1.5$

40. $h(x)=x^3+1.5x^2+3x-2.5$

41. $u(x)=-\frac{3}{2}x+2$

42. $v(x)=x^4-\frac{16}{3}x^3+2x^2+24x-1$

43. $k(x)=|2x-3|$

44. $K(x)=|3x^2+3x-18|$

45. $m(x)=\frac{2-x}{x^2-x+2}$

46. $n(x)=\frac{|x^2-2|}{2x^2+4}$

47. $F(t)=\sqrt{3+3t^2}$

48. $G(x)=x^{3/2}-3\sqrt{x}$

49. $T(s)=2\sqrt[3]{s}(s-2)$

50. $r(v)=\frac{v-1}{\sqrt{v}}$

51. $s(\alpha)=\cos\alpha+\cos^2\alpha$

52. $u(z)=\cot z+2z$

53. $t(x)=\sqrt{x}\ln x$

54. $U(t)=e^t\sin t$

55. $a(t)=\cos(\arctan t)$

56–77 Find all absolute extrema of the function on the given closed interval.

56. $f(x)=4x-x^2$ on $[0,6]$

57. $g(x)=3x^2-30x+7$ on $[0,8]$

58. $h(x)=x^3+1.5x^2-6x+3.5$ on $[-4,3]$

59. $u(x)=3x^4-8x^3+6x^2-24x-9$ on $[0,3]$

60. $v(x)=\frac{x^4}{4}-2x^2+4$ on $[-2,2]$

61. $k(x)=\frac{x^4}{2}+2x^3-x^2-6x+\frac{1}{2}$ on $[-3,1]$

62. $f(x)=|x+3|\cdot|x-3|$ on $[-4,4]$

63. $m(x)=|x+3|+|x-3|$ on $[-4,4]$

64. $n(x)=\frac{3x}{2x^2+2}$ on $[-4,4]$

65. $g(x)=\frac{x^2+5}{x+2}$ on $[-1.5,1.5]$

66. $F(x)=\frac{1}{1+x^2}$ on $[-10,10]$

67. $G(t)=\frac{1}{\sqrt{t}}+\sqrt{t}$ on $\left[\frac{1}{4},4\right]$

68. $k(s)=(s^2-1)\sqrt{s}$ on $[0,2]$

69. $r(z) = \sin(\arccos z)$ on $[-1,1]$

70. $G(x) = \arctan x$ on $[-1,1]$

71. $w(x) = x\sqrt{8-x^2}$ on $\left[-2\sqrt{2}, 2\sqrt{2}\right]$

72. $T(s) = s^2 e^{-s}$ on $[0,10]$

73. $r(x) = (\cos x)e^x$ on $\left[0, \frac{3\pi}{2}\right]$

74. $L(x) = x\ln x$ on $\left[\frac{1}{e^2}, e\right]$

75. $t(x) = \ln((e-1)\sin \pi x + 1)$ on $[0,1]$

76. $U(x) = \sqrt[3]{x}(x-3)$ on $[-1,3]$

77. $V(x) = 5 + (5+x)x^{5/7}$ on $[-5,1]$

78–89 Find and classify the absolute extrema, if they exist, of the function over the given domain.

78. $f(x) = 3x - 2;\quad D = (0,2]$

79. $g(x) = x^2 - 4;\quad D = (-2,2)$

80. $h(x) = 2x^3 - 5x;\quad D = \mathbb{R}$

81. $K(z) = \sqrt{4-z^2};\quad D = (-2,2)$

82. $r(z) = -\frac{2}{z};\quad D = [2,\infty)$

83. $n(x) = \frac{1}{(x+3)^2};\quad D = (-3,\infty)$

84. $t(x) = 10^{x/2};\quad D = \mathbb{R}$

85. $L(x) = \ln(x+1);\quad D = [0,\infty)$

86. $F(x) = \sec x;\quad D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

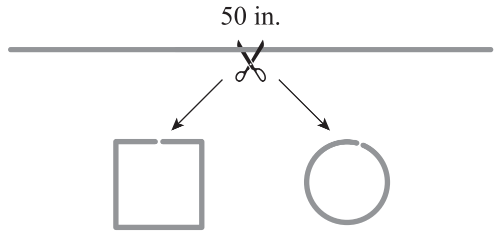
87. $t(z) = 2\cos \pi z + 2;\quad D = \mathbb{R}$

88. $u(x) = x - [\![x]\!];\quad D = [1,3)$

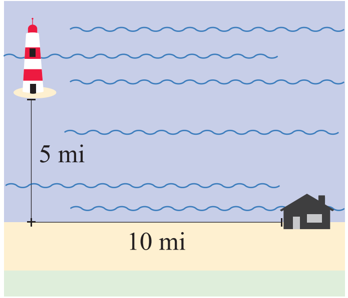
89. $v(x) = \arctan x;\quad D = \mathbb{R}$

90. Find two numbers whose sum is 50 and whose product is as large as possible. (**Hint:** Denote the numbers by x and $50 - x$, and maximize the product.)

91. A 50-inch piece of wire is cut into two pieces, which are then bent into a square and a circle, respectively. Where should the wire be cut in order to minimize the sum of the areas of these two shapes? (**Hint:** Start with the notation of Exercise 90, and use appropriate formulas from geometry.)



92. A lighthouse is 5 miles off a straight shoreline. Ten miles down the coast is a restaurant where the lighthouse keeper is planning to meet his friends. If he can row at 2.5 mph and walk at 4 mph, where should he land in order to make the fastest possible time to the restaurant?



93. Referring to Exercise 26 of Section 3.7, find the number of calculators that have to be produced in order to maximize profit.

94. The power output of a 12-volt car battery when a resistor is connected to it, is given by the formula $P = 12I - (R+r)I^2$, where I is the current (in amperes), and r stands for the (typically very small) so-called internal resistance of the battery. Suppose we are starting a car with a starter motor of resistance $R = 0.16$ ohms, and that the internal resistance of the battery is $r = 0.016$ ohms. Find the current that corresponds to the battery's maximum power output.

Concept Check

95–105 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

95. If f attains both its absolute minimum and absolute maximum values on a closed interval, then f is a continuous function.

96. A continuous function on a closed interval can attain its absolute extrema only at critical points.

97. If $f(x)$ is a differentiable function and k is a constant, then $f(x)$ and $f(x)+k$ have the same critical points.

98. If $f(x)$ is a differentiable function and k is a nonzero constant, then $f(x)$ and $kf(x)$ have the same critical points.

99. If $f(x)$ is a differentiable function and k is a nonzero constant, then $f(x)$ and $f(x+k)$ have the same critical points.

100. If $f(x)$ is a differentiable function and k is a nonzero constant, then $f(x)$ and $f(kx)$ have the same critical points.

101. If $f(x)$ has a maximum at c, then so does $f(-x)$ at $-c$.

102. If $f(x)$ has a maximum at c, then $-f(x)$ has a minimum at c.

103. A function $f(x)$ can have more than one absolute maximum value.

104. If $f(x)$ is continuous on a closed interval I, then it attains its minimum value on I.

105. If $f(x)$ has no maximum on a closed interval I, then $f(x)$ must be discontinuous on I.

4.1 Technology Exercises

106–127. Use a graphing utility to verify the answers you obtained for Exercises 56–77.

$$f(4)+2=4f'(c)$$
$$f(4)=4f'(c)-2\le 4\cdot 3-2$$
$$f(4)\le 10$$

Hence the largest possible value of $f(4)$ is 10.

Example 6 Using Corollary 2 of the Mean Value Theorem

Find the unique function f whose derivative is $3x^2$ and whose graph passes through $(1,5)$.

Solution

By now, we have differentiated enough polynomials to recognize $3x^2$ as the derivative of x^3. If we let $g(x)=x^3$, then g and f have the same derivative; and by Corollary 2, it must be the case that $f(x)=g(x)+C$ for some constant C. We can determine C as follows.

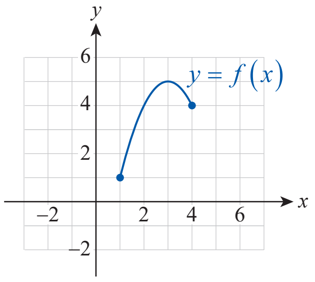
$$f(1)=5 \qquad \text{The graph of } f \text{ passes through } (1,5).$$
$$g(1)+C=5$$
$$1^3+C=5$$
$$C=4$$

Now it can be easily verified that $f(x)=x^3+4$ satisfies the given criteria, and by Corollary 2 it is the unique function to do so.

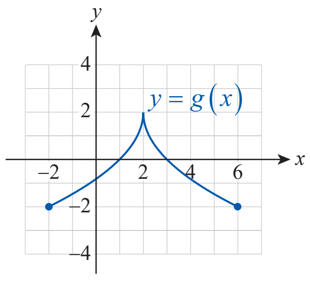
4.2 Exercises

1–4 Use the graph of the function to visually estimate the value of c in the given interval that satisfies the conclusion of the Mean Value (or Rolle's) Theorem; then check your guess by calculation. If such a c doesn't exist, explain why.

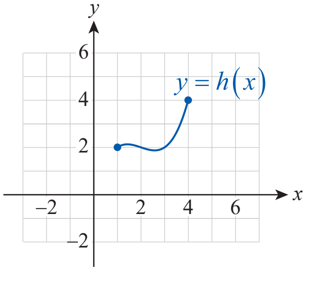
1. $f(x)=-x^2+6x-4$ on $[1,4]$



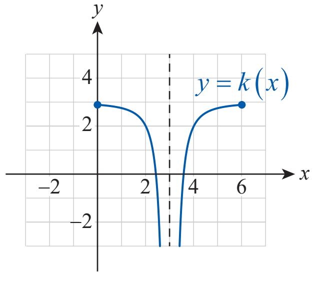
2. $g(x)=-2\sqrt{|x-2|}+2$ on $[-2,6]$



3. $h(x)=\dfrac{x^3}{3}-2x^2+\dfrac{11x}{3}$ on $[1,4]$



4. $k(x)=-\dfrac{1}{(x-3)^2}+3$ on $[0,6]$



5–8 Prove that the equation has exactly one real solution on the given interval.

5. $x^5 - 3x^2 = 25$ on $[2,3]$

6. $5x^3 + 7x = 9$ on R

7. $\arctan x = 3 - x$ on $[0,3]$

8. $\tan x = \cos x$ on $\left(0, \frac{\pi}{2}\right)$

9–20 Determine whether Rolle's Theorem applies to the function on the given interval. If so, find all possible values of *c* as in the conclusion of the theorem. If the theorem does not apply, state the reason.

9. $f(x) = -x^2 + 4x - 3$ on $[1,3]$

10. $g(x) = x^3 - 5x^2 + 2x + 10$ on $[-1,4]$

11. $h(x) = 2x^4 - x^3 + 6x^2 + x - 8$ on $[1,3]$

12. $F(x) = \dfrac{3}{(x-1)^2}$ on $[0,2]$

13. $G(x) = \dfrac{1}{x^2+1}$ on $[-3,3]$

14. $k(x) = \dfrac{x}{x^2-1}$ on $[-3,3]$

15. $H(x) = x^{4/5} - 10$ on $[-10,10]$

16. $m(x) = -\cos x$ on $[0,4\pi]$

17. $T(z) = \cot z$ on $[0,5\pi]$

18. $F(x) = \sec x$ on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

19. $w(t) = \csc t$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

20. $A(x) = |x+1| - 5$ on $[-6,4]$

21–32 Determine whether the Mean Value Theorem applies to the function on the given interval. If so, find all possible values of *c* as in the conclusion of the theorem. If the theorem does not apply, state the reason.

21. $f(x) = |x-1| + 3$ on $[-1,2]$

22. $g(x) = -\frac{1}{2}x + 4$ on $[0,8]$

23. $h(x) = -x^2 + 4x + 4$ on $[-1,4]$

24. $F(x) = \frac{1}{2}x^3 - x^2$ on $[-2,2]$

25. $G(x) = \dfrac{2x}{x+1}$ on $[-2,3]$

26. $k(x) = \dfrac{x+1}{2x-5}$ on $[-1,2]$

27. $H(x) = \dfrac{5}{x^2+5}$ on $[-5,5]$

28. $m(x) = (x-4)^{2/3} + 2$ on $[1,5]$

29. $C(x) = \sqrt{1-x^2}$ on $[-1,0]$

30. $L(t) = \ln|t+1|$ on $[-3,1]$

31. $u(z) = \tan z$ on $[0,\pi]$

32. $v(s) = \arctan s$ on $[0,1]$

33. If $f(-2) = 2$ and $f'(x) \le 2$ for all x, what is the largest possible value of $f(2)$?

34. If $g(1) = 4$ and $g'(x) \ge -3.5$ for all x, what is the smallest possible value of $g(3)$?

35. One of your classmates claims that he found a function f such that $f(-5) = -1$, $f(5) = 1$, and $f'(x) \le 0.1$ for all x. Explain how you know that he made an error in his calculations.

36. If $|F'(x)| \le 2.5$ for all x, prove that $|F(7) - F(3)| \le 10$.

37. Find the function f that passes through $(3,1)$ and whose derivative is $3x^2 - 2x + 1$.

38. Find the function g that passes through $(\pi/2, 0)$ and whose derivative is $2\cos x$.

39. Suppose that the velocity function of a moving object is $v(t) = -2t + 5$ and that its position at $t = 1$ is -8 units from the origin. Find a formula for the position function.

40. Find the velocity and position functions of an object thrown vertically upward from an initial height of 20 ft with initial velocity $v(0) = 42$ ft/s. (The acceleration caused by gravity is $a = -32\,\text{ft/s}^2$. Ignore air resistance.)

41. Generalize Example 1 by proving the following statement: if $f(x)$ is differentiable on $[a,b]$, $f'(x) \ne 0$ on (a,b), and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has exactly one real solution.

42. Two highway patrol cars are stationed 7 miles apart along a straight highway where the speed limit is 65 mph. The first patrol car clocks a red Porsche at 64 mph, and five minutes later, the second police car clocks it at 61.5 mph. Explain why the driver is pulled over and issued a ticket.

43. A plane leaves London Heathrow Airport to arrive at Houston Intercontinental Airport ten and a half hours later. The distance between the two airports is 4830 miles. Explain why the plane must have reached a speed of 450 mph at least twice during the trip.

44. Two MotoGP riders finish a race in a tie. Show that there was at least one moment during the race when the two riders had the exact same speed. (**Hint:** Ignoring the difference in their starting positions, consider the difference of the two position functions and use Rolle's Theorem.)

45. Suppose that $f'(x) = x$ for all $x \in \mathbb{R}$. Prove that there exists a constant C such that $f(x) = \frac{1}{2}x^2 + C$. (**Hint:** Use Corollary 2 from the text.)

46. Use Corollary 2 of this section to prove the well-known trigonometric identity $\cos^2 x + \sin^2 x = 1$. (**Hint:** Use the corollary for the functions $f(x) = \cos^2 x + \sin^2 x$ and $g(x) = 1$, and argue that necessarily $C = 0$.)

47. Follow the hint given in Exercise 46 to prove the identity $\sin^{-1} x + \cos^{-1} x = \pi/2$.

48. Let $f(x) = \cot \pi x$ and $g(x) = \cot \pi x + [\![x]\!]$. Show that $f'(x) = g'(x)$, but $f(x) - g(x)$ is not constant. Why does this not contradict Corollary 2 of this section?

49. Suppose that for a function $f(x)$, the second derivative $f''(x)$ exists for all x on an interval I. Prove that if f has three zeros on I, then its second derivative f'' also has a zero. (**Hint:** Apply Rolle's Theorem on f and then on f'.)

50. Suppose that f is twice differentiable on $\mathbb{R}$ and a and b are two successive zeros of f'. Prove that f can have at most one zero on (a, b). (**Hint:** Start by assuming that f has at least two zeros, and apply Rolle's Theorem.)

51. Suppose that f is continuous on $[a, b]$, differentiable on (a, b), and $c \in (a, b)$ with $f'(c) = 0$. Does it follow that $f(a) = f(b)$? Explain.

52.* Suppose that f is twice differentiable on $\mathbb{R}$ and $f'(c) = 0$ for some $c \in \mathbb{R}$. If $f''(c) \neq 0$, prove that there exist $a, b \in \mathbb{R}$ such that $c \in (a, b)$ and $f(a) = f(b)$.

53. Suppose that the rabbit population at a game preserve is observed monthly and is found to have increased from 550 to 1150 rabbits in a year's time. Explain why there must exist a time during the year when the population is increasing at a rate of 50 rabbits per month.

54. Show that if f is a quadratic function, that is, $f(x) = Ax^2 + Bx + C$, then on any interval $[a, b]$, the point c satisfying the conclusion of the Mean Value Theorem is the midpoint of the interval.

55.* Prove that for all $0 < \alpha < \beta < \pi/2$, $(\beta - \alpha)\cos\beta \leq \sin\beta - \sin\alpha \leq (\beta - \alpha)\cos\alpha$. (**Hint:** Apply the Mean Value Theorem for the function $f(x) = \sin x$ on the interval $[\alpha, \beta]$.)

56. Use the Mean Value Theorem to prove the inequality $\ln(1 + x) \leq x$ for all $x \geq 0$. (**Hint:** For $x > 0$, consider $f(t) = \ln(1 + t)$ on the interval $[0, x]$ and apply the Mean Value Theorem.)

57. Prove the inequality $\sqrt[3]{1 + x} \leq 1 + \frac{1}{3}x$ for $x \geq 0$. (Adapt and follow the hint given in Exercise 56.)

58.* Use the Mean Value Theorem to show that $\lim_{x\to\infty}\left(\sqrt{x} - \sqrt{x - 5}\right) = 0$. (**Hint:** Apply the theorem to the function $f(x) = \sqrt{x}$ on the interval $[x - 5, x]$ for a fixed x-value. Then let x approach infinity.)

59. Use the Mean Value Theorem to prove the following inequality for all $\alpha, \beta \in \mathbb{R}$.

$$|\arctan\alpha - \arctan\beta| \leq |\alpha - \beta|$$

60.* Generalizing Exercise 59, prove that if f is differentiable and $|f'(x)| \leq M$ for some $M > 0$, then for all x, y in the domain of f, $|f(x) - f(y)| \leq M|x - y|$. (Such a function f is said to have the Lipschitz property with constant M.)

61. Use Exercise 60 to prove that $f(x) = \cos 3x$ has the Lipschitz property with constant 3.

62.* We say that the function $f(x)$ has a *fixed point* $c \in \mathbb{R}$ (or that it leaves c fixed) if $f(c) = c$. Prove that if f is differentiable on $\mathbb{R}$ and $f'(x) < 1$ for all x, then f can have no more than one fixed point.

63.* Prove Cauchy's Mean Value Theorem, which is stated as follows: If f and g are both continuous on the closed interval $[a,b]$ and differentiable on (a,b), and if $g'(x) \neq 0$ on (a,b) and $g(a) \neq g(b)$, then there is a point $c \in (a,b)$ for which $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$. (**Hint:** Apply Rolle's Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\left(g(x)-g(a)\right).)$$

Concept Check

64–68 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

64. There are situations when the Mean Value Theorem applies but Rolle's Theorem doesn't.

65. There are situations when Rolle's Theorem applies but the Mean Value Theorem doesn't.

66. If a and b are zeros of the function $f(x)$, then there exists a point c in (a,b) such that $f'(c) = 0$.

67. If a and b are zeros of the polynomial $p(x)$, then there exists a point c in (a,b) such that $p'(c) = 0$.

68. If $f'(x) = 0$ for all x in the domain of f, then $f(x)$ is a constant function.

4.2 Technology Exercises

69–72 Use a graphing utility to graph the function on the interval $[a,b]$ along with its secant line through the points $(a,f(a))$ and $(b,f(b))$ on the same screen. Then find and graph the line(s) tangent to the graph that are parallel to the secant line.

69. $f(x) = \sqrt{x}$ on $[0,9]$

70. $f(x) = \frac{x^3}{2} - x^2 + x + 7.5$ on $[1,3]$

71. $f(x) = \frac{3x}{x+2}$ on $[-1,1]$

72. $f(x) = \sin^2 x + 3x$ on $[0,\pi]$

Solving $f''(x)=0$ gives us the possible inflection points 0 and 1 (note that 0 is both a critical point and a potential inflection point).

We proceed to evaluate the signs of f' and f'' on the intervals of interest.

Interval	$(-\infty,0)$	$(0,\frac{3}{2})$	$(\frac{3}{2},\infty)$
Sign of f'	$f'(-1)=-10<0$	$f'(1)=-2<0$	$f'(2)=8>0$
Monotonicity of f	Decreasing	Decreasing	Increasing

Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
Sign of f''	$f''(-1)=24>0$	$f''(\frac{1}{2})=-3<0$	$f''(2)=24>0$
Concavity of f	Concave up	Concave down	Concave up

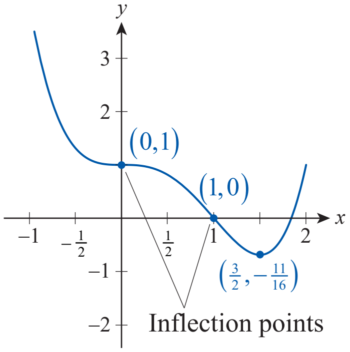


Figure 10
$f(x)=x^4-2x^3+1$

We now have a wealth of information about f at our disposal, and can easily answer all of the questions about its graph.

First, the two tables above identify the intervals of constant monotonicity and concavity and whether f is increasing, decreasing, concave up, or concave down on each. Note how those intervals correspond to the graph of f shown in Figure 10.

Second, we can use the First Derivative Test to realize that f does not have a relative extremum at the critical point 0, but does have a relative minimum at $\frac{3}{2}$. And since the concavity changes from up to down at 0 and from down to up at 1, both 0 and 1 are inflection points.

We could also have used the Second Derivative Test to determine that f has a relative minimum at the critical point $\frac{3}{2}$, even though we never actually evaluated f'' at the point. The reason is that we know f'' exists on the open interval $(1,\infty)$ containing $\frac{3}{2}$ (indeed, f'' exists everywhere), and that it is positive on the interval. In particular, it is positive at $\frac{3}{2}$.

The last step is to evaluate f at the two inflection points and the one relative extremum—those values are shown in Figure 10. Note that f has no absolute maximum value, but the relative minimum at $\frac{3}{2}$ is also its absolute minimum.

Example 4 illustrates that the tools of calculus can answer questions about particular points of interest on the graph of a function. We will explore this idea further in Section 4.5, but we will acquire one additional tool before doing so.

4.3 Exercises

1–22 Determine the intervals of monotonicity of the given function.

1. $f(x)=x^2-4x+1$

2. $g(x)=\frac{3}{2}x+5$

3. $h(x)=\frac{2}{3}x^3+4x^2-10x+\frac{5}{3}$

4. $F(x)=0.75x^4+x^3-15x^2+24x+7$

5. $G(x)=-x^4-2x^3+8x^2-6x+1$

6. $k(x)=(x^2-4)(x^2-3)$

7. $m(x)=-\frac{x^6}{3}-\frac{2x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x$

8. $n(x)=\frac{-2}{x+1}$

9. $H(x)=\frac{x^2+3}{x+3}$

10. $R(t)=\frac{t+2}{t^2-9}$

11. $C(x) = \dfrac{3x^2 + 1}{x - 2}$

12. $A(x) = |x + 4| - 1$

13. $f(t) = 2.5 - |t - 3.125|$

14. $t(x) = x^{2/3} + 2$

15. $w(s) = \sqrt{s}(s - 1)$

16. $F(x) = 6 - (x - 3)^{3/5}$

17. $G(x) = \sin^2 x + 1$

18. $m(x) = 2^{-x} + 2^{2x}$

19. $g(t) = -2\sqrt{t}e^{-2t}$

20. $L(x) = x \ln x$

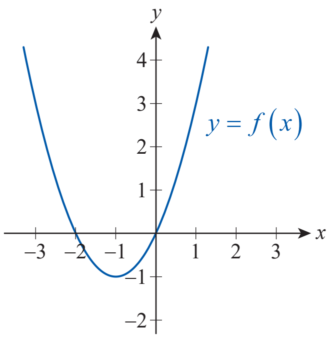
21. $A(x) = 0.5x^{1/5}(x^2 - 4)$

22. $U(s) = s\sqrt{3 - s^2}$

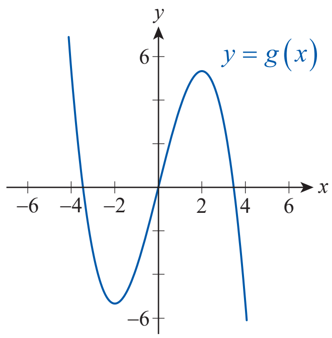
23–42. Use the First Derivative Test to classify the relative extrema, if any, of the functions given in Exercises 3–22.

43–50 Identify all intervals of monotonicity as well as intervals of concavity for the graphed function. Find all local extrema and inflection points, if any.

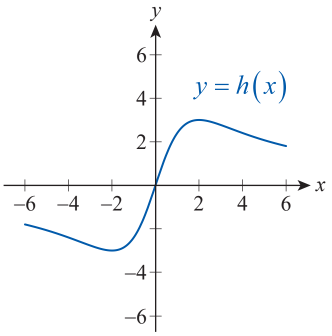
43. $f(x) = x^2 + 2x$



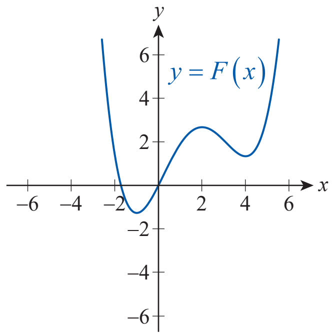
44. $g(x) = -\dfrac{1}{3}x^3 + 4x$



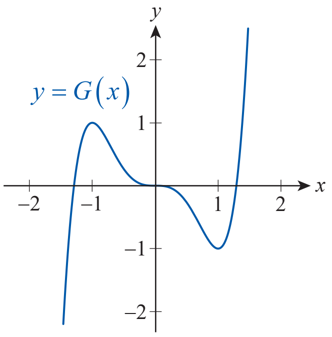
45. $h(x) = \dfrac{12x}{x^2 + 4}$ (Serpentine)



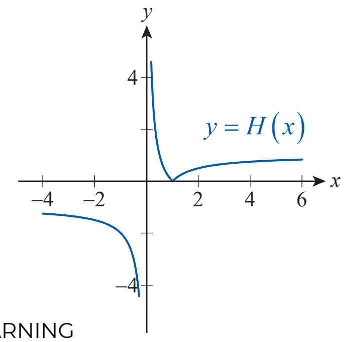
46. $F(x) = \dfrac{1}{16}x^4 - \dfrac{5}{12}x^3 + \dfrac{1}{4}x^2 + 2x$



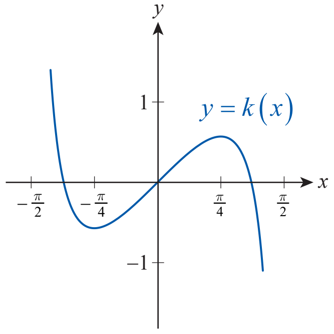
47. $G(x) = 1.5x^5 - 2.5x^3$



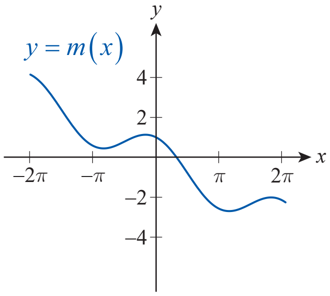
48. $H(x) = \dfrac{|1 - x|}{x}$



49. $k(x) = 2x - \tan x, \quad |x| < \dfrac{\pi}{2}$



50. $m(x) = \cos x - \dfrac{x}{2}, \quad |x| \le 2\pi$



51–62 Determine the intervals of concavity of the given function.

51. $f(x) = 4x - x^2$

52. $g(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} + x + 1$

53. $h(x) = 4x^3 - 3x^2 - 36x + 5$

54. $k(x) = -x^4 + 5x + 1$

55. $F(x) = x^4 + 4x^3 + 72x$

56. $G(x) = -x^4 + 6x^3 + 24x^2 - 4x + 2$

57. $k(x) = 0.3x^5 + x^4 - 3x^3 + 12x + 4$

58. $v(x) = \dfrac{5}{x-2}$

59. $m(x) = \dfrac{x^2+5}{x-5}$ **60.** $F(x) = \dfrac{2^x}{x}$

61. $H(x) = 9^x + 3^{-x}$ **62.** $u(x) = (x-2)^{5/7}$

63–82 Use the first and second derivatives to identify the intervals of monotonicity, extrema, intervals of concavity, and inflection points of the given function.

63. $f(x) = \dfrac{1}{2}x + 5$ **64.** $g(x) = x^2 - 8x + 3.5$

65. $h(x) = -\dfrac{1}{2}x^2 + 5x + \dfrac{8}{3}$ **66.** $F(x) = 2x^3 + 3x^2 - 7$

67. $G(x) = -4x^3 - 3x^2 + 18x + 10$

68. $K(x) = 0.5x^4 + 2x^3 - 6x^2 - 16x + 19.5$

69. $L(x) = -x^4 + 12x^2 - 20x + 3$

70. $m(x) = -\dfrac{3}{x^2}$

71. $n(x) = \dfrac{x+1}{x-2}$ **72.** $H(x) = \dfrac{2x}{x^2-4}$

73. $r(x) = \dfrac{2x^2+1}{x-4}$ **74.** $t(x) = \dfrac{5}{4}\left(x - \dfrac{4}{5}\right)^{4/5}$

75. $F(x) = \dfrac{(x-1)^2}{x^2-1}$ **76.** $f(x) = \sqrt[3]{x} - x$

77. $g(x) = x^{2/3}\left(\dfrac{2}{3} - x\right)$ **78.** $h(x) = x\sqrt{9-x^2}$

79. $u(x) = \sqrt{x}e^{-x}$ **80.** $v(x) = \sin^2 x - 2\cos x$

81. $k(x) = \cos x - \sin x$ **82.** $L(x) = -x^2 \ln|x|$

83–90 Sketch a graph of a function satisfying the given conditions. (Answers will vary.)

83. f is differentiable on $\mathbb{R}$, has both a local maximum and minimum, but no global extrema.

84. f is differentiable on $\mathbb{R}$, f' has a zero, and f has no local extrema.

85. f is differentiable on $\mathbb{R}$, f is an odd function, $f'(x) > 0$ on $(-4, 4)$, $f'(x) \le 0$ elsewhere, and $\lim_{|x|\to\infty} f(x) = 0$.

86. $\lim_{|x|\to\infty} f(x) = 1$ and $f(0) = 0$ is a global minimum.

87. $f(x)$ is everywhere positive on $\mathbb{R}$, but its derivative is everywhere negative.

88. $f'(x) < 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$.

89. f has vertical asymptotes at $x = \pm 2$, a horizontal asymptote at $y = 1$, f is an even function, $f'(x) < 0$ on $(0, 2)$ and $(2, \infty)$, f has a local maximum at 0, $f''(x) < 0$ on $(0, 2)$, $f''(x) > 0$ on $(2, \infty)$, and f has no absolute extrema.

90. $f(0) = f(2) = 0$, $f'(1) = f'(3) = f'(4) = 0$, $f''(1) < 0$, $f''(2) = 0$, and $\lim_{x\to\infty} f(x) = 0$.

91–96 The function $p(t)$ gives the position, relative to its starting point, of an object moving along a straight line. Identify the time intervals when the object is moving in the positive versus negative direction, as well as those intervals when it is accelerating or slowing down. Find the times when the object changes direction as well as when its acceleration is zero.

91. $p(t) = 2t^2 - 3t + 2.5$

92. $p(t) = 5t - \frac{1}{2}t^2$

93. $p(t) = 2t^3 - 15t^2 + 24t$

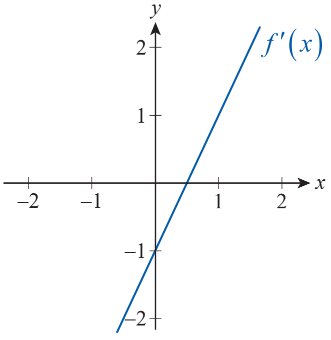
94. $p(t) = -2t^3 + 22.5t^2 - 66t$

95. $p(t) = e^{-t} \sin t$

96. $p(t) = \frac{1-t}{2t+2}$

97–102 The graph of the derivative $f'(x)$ of the function $f(x)$ is given. Use it to sketch an approximate graph of $f''(x)$. Then try to sketch a possible graph of $f(x)$ as well. (**Note:** There are many possible correct answers for the graph of f. Can you see why?)

97.



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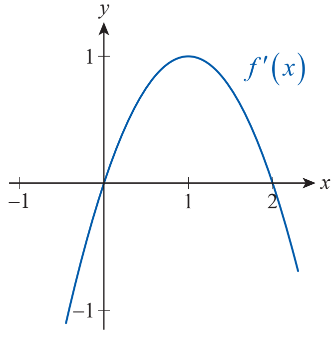

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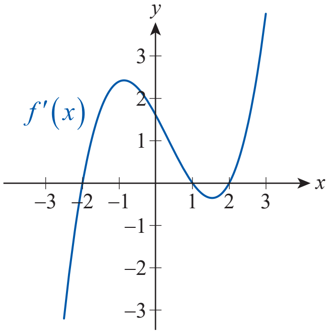

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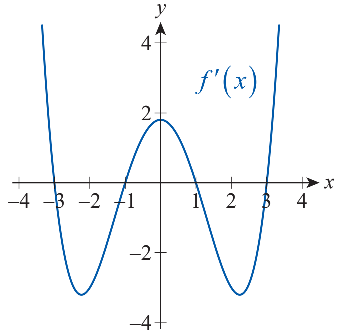

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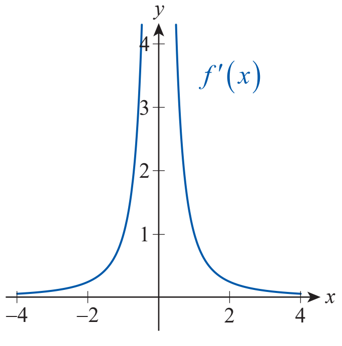
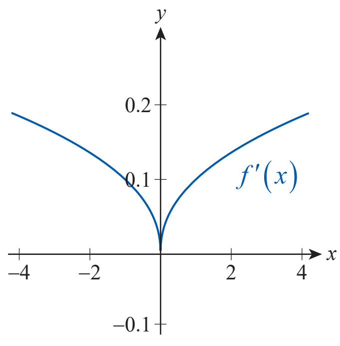


102.



103. Suppose $T(t)$ is the outside temperature, over a 24-hour period on a typical spring day where you live (t is measured in hours, with $t = 0$ corresponding to midnight). Given the following data, what time(s) of day might c_i represent ($i = 1, 2, \ldots, 6$)? Explain your choice(s).

a. $T'(c_1) > 0, \quad T''(c_1) < 0$

b. $T'(c_2) > 0, \quad T''(c_2) > 0$

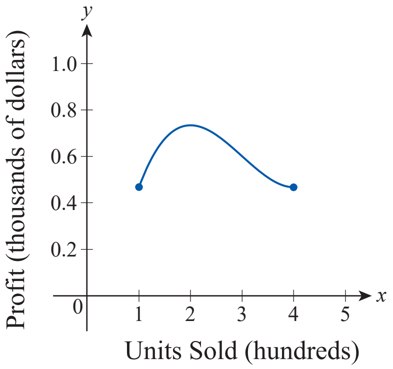
c. $T'(c_3) = 0, \quad T''(c_3) > 0$

d. $T'(c_4) < 0, \quad T''(c_4) < 0$

e. $T'(c_5) < 0, \quad T''(c_5) > 0$

f. $T'(c_6) = 0, \quad T''(c_6) < 0$

104. The graph below shows the profit (in thousands of dollars) from a product per hundreds of units sold. Use the graph to visually estimate the production level at which the marginal profit starts increasing.



105. An aftermarket auto accessories company manufactures StopTheMess trunk liners and organizers. The overhead cost of operating the plant is \$5000 per month and the cost of manufacturing each item is \$20. The company estimates that 200 liners can be sold monthly for \$50 apiece, and that sales will increase by 10 liners per month for each dollar decrease in price.

a. Find a formula for the profit function $P(n)$, where n is the number of trunk liners manufactured (suppose $200 \le n \le 350$).

b. Identify the intervals on which P is increasing or decreasing, and find the production level that maximizes profits.

106. The strength of an electric field due to a charged ring obeys the equation

$$E = \frac{kqx}{\left(x^2 + R^2\right)^{3/2}}$$

where q is the electric charge measured in coulombs (C), $k \approx 8.99 \cdot 10^9 \text{ Nm}^2/\text{C}^2$, R is the radius of the ring and x is the distance to the charge in meters. Find a formula for the rate of change of E as x increases. What happens to E and dE/dx as $x \to \infty$?

107. Suppose that $f'(x) = (x+4)(x+1)^2(x-3)^3$. By examining the zeros of $f'(x)$, identify the x-coordinates of the local maxima and minima of $f(x)$. (**Hint:** Recall what you learned about multiplicities of zeros and sign changes of polynomials.)

108. Repeat Exercise 107 for $g(x)$ if $g'(x) = 3x(x-2)(x+5)^4\left(x-\frac{1}{3}\right)^2(x+1)^3$.

109. Use derivatives to prove that if $x \in (-\infty, 1)$, then $1/(1-x) \ge 1 + x$. (**Hint:** Start by assuming that $x \ge 0$. Rewriting the inequality as $f(x) \ge g(x)$, show that $D(x) = f(x) - g(x)$ is increasing, while $D(0) = 0$. To handle the case of $x < 0$, use the fact that $D'(x) < 0$ along with $D(0) = 0$.)

110. Prove that quadratic functions cannot have any inflection points, while cubic functions have exactly one. What can you say about fourth-degree polynomials?

111. Suppose we know that the derivative of a function f is $f'(x) = a/(x^2+1)$ for some nonzero $a \in \mathbb{R}$. Prove that f is increasing or decreasing everywhere on $\mathbb{R}$.

112. Generalize Exercise 111 by proving the following statement: If f is differentiable on an interval I and $f'(x) \ne 0$ in the interior of I, then f is increasing or decreasing everywhere on I. (**Hint:** Indirectly assume that f' changes signs, and use the Darboux property of derivative functions.)

113. Use Exercise 112 to find conditions under which the general cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ is decreasing everywhere on $\mathbb{R}$.

114. Prove that a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one inflection point. Find its first coordinate, assuming that the three real roots of $p(x)$ are x_1, x_2, and x_3.

115. Determine the conditions on the coefficients of the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ in order for $p(x)$ to have a local maximum at $x = 0$ and a local minimum at $x = 4$.

116.* Consider the following function.

$$f(x) = \begin{cases} 2x^2 & \text{if } x \text{ is irrational} \\ 4x^2 & \text{if } x \text{ is rational} \end{cases}$$

Use f to explain why the changing of signs of the derivative is not necessary for a function to have a local extremum.

117. By considering $f(x) = \begin{cases} x & \text{if } x \le 1 \\ 3-x & \text{if } x > 1 \end{cases}$, explain why the First Derivative Test can't be used for a discontinuous function.

118. Suppose that $f(x)$ and $g(x)$ are at least twice differentiable and that both their first and second derivatives are positive everywhere on an interval I. Which of the following statements can you prove from these conditions? Prove those statements that are true, and provide counterexamples for the rest.

a. $f(x)+g(x)$ is increasing on I.

b. $f(x)+g(x)$ is concave up on I.

c. $f(x)\cdot g(x)$ is increasing on I.

d. $f(x)\cdot g(x)$ is concave up on I.

e. $f(g(x))$ is increasing on I.

f. $f(g(x))$ is concave up on I.

119.* Use mathematical induction to prove the following generalization of the Second Derivative Test:

Suppose that the derivatives of all orders of the function f exist at c, up to $f^{(2k)}(c)$, and that $f'(c)=f''(c)=\cdots=f^{(2k-1)}(c)=0$, but $f^{(2k)}(c)\neq 0$. Then if $f^{(2k)}(c)<0$, f has a relative maximum at c; if $f^{(2k)}(c)>0$, f has a relative minimum at c.

120.* Use mathematical induction to prove that if f is $(2k+1)$-times differentiable at c, $f'(c)=f''(c)=\cdots=f^{(2k)}(c)=0$, but $f^{(2k+1)}(c)\neq 0$, then f has a point of inflection at c.

Concept Check

121–128 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

121. Not all fourth-degree polynomials have inflection points.

122. A function with no inflection points cannot change concavity.

123. If $f'(x)$ is negative on $(-\infty,c)$ and positive on (c,∞), then f has a minimum at c.

124. If $f(x)$ and $g(x)$ are decreasing, then so is $(f+g)(x)$.

125. If $f(x)$ and $g(x)$ are decreasing, then so is $(f\cdot g)(x)$.

126. A polynomial of degree n cannot have more than $n-1$ extrema on $\mathbb{R}$.

127. If $c\in\mathbb{R}$ is a critical point, then the function has a local minimum or a local maximum at c.

128. If $f''(c)=0$, then c is an inflection point.

4.3 Technology Exercises

129–130 Use a graphing utility to graph the given function along with its first and second derivatives on the same screen. Use your graphs to explain the behavior of the function with regard to the signs and values of its derivatives.

129. $f(x)=(x^2+1)\sqrt{9-x^2}$ on $[-3,3]$

130. $g(x)=\sqrt{x}\cos x-\sin 2x$ on $[0,4\pi]$

131. The table below shows the temperature of a pediatric patient over a 24-hour period (measurements were taken every two hours, starting at midnight). Use the regression capabilities of a graphing utility to approximate the data by a fourth-degree polynomial. When do you estimate the patient's temperature to have been the highest? The lowest? When did the highest rates of increase and decrease occur?

Patient's Temperature

Time	12 a.m.	2 a.m.	4 a.m.	6 a.m.	8 a.m.	10 a.m.	12 p.m.
Temp (°F)	99.9	99.5	99.1	98.9	98.7	98.8	99.4
Time	2 p.m.	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 a.m.	
Temp (°F)	100.0	102.1	101.9	101.3	101.0	99.9	

132. In the first few months following the launch of a new product, monthly sales were given by the function $S(t)=300t^2/(t^2+2)$, where t is measured in months.

a. Use a graphing utility to graph the function over the first year, and estimate when the rate of growth in sales was greatest.

b. Use the differentiation capabilities of a computer algebra system to check your estimate in part a.

133–136 Use a graphing utility to graph the given function for different values of the parameter(s). Examine how the values of the parameter(s) affect the number of local extrema. How about inflection points?

133. $f(x)=x^4+cx^3;\quad 1\le c\le 3$

134. $g(x)=0.5x^5+cx^4-dx;\quad 0\le c,d\le 3$

135. $h(x)=\cos x-\sin(cx);\quad 0\le c\le 4$

136. $k(x)=\sin^2(cx)\cos(dx);\quad 0\le c,d\le 5$

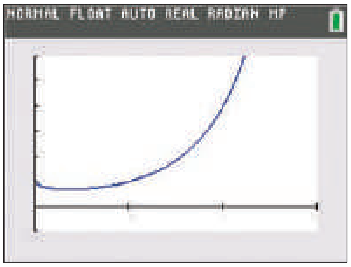


Figure 3

$y = x^x$ on $[0,3]$ by $[-1,6]$

$$\ln y = x \ln x = \frac{\ln x}{\frac{1}{x}},$$

which gives us a limit of indeterminate form ∞/∞. So,

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0,$$

and hence $x^x \to e^0 = 1$ as $x \to 0^+$ (don't forget this last step!).

Example 9 Using L'Hôpital's Rule to Find a Limit of Indeterminate Form ∞^0

Determine $\lim_{x\to\infty} x^{1/x}$.

Solution

The base has a limit of ∞ and the exponent has a limit of 0. We proceed as in the last two examples.

$$y = x^{1/x}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x} \qquad \text{Indeterminate form } \infty/\infty$$

Applying L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

and therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^0 = 1$.

4.4 Exercises

1–12 Evaluate the limit using the theorems of Chapter 2. Then decide whether L'Hôpital's Rule is applicable and, if so, use it to check your answer.

1. $\lim_{x\to 3} \frac{2x^2 - 18}{x-3}$

2. $\lim_{x\to -2} \frac{x^3+8}{x+2}$

3. $\lim_{x\to 0} \frac{\sin x}{2x}$

4. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

5. $\lim_{x\to 0} \frac{\cos x}{x}$

6. $\lim_{x\to\infty} \frac{6x^2 - x + 7}{x - 3x^2}$

7. $\lim_{x\to -\infty} \frac{5x^2 - 2x + 1}{2.5x^3 - 3x^2 + 6}$

8. $\lim_{x\to 0} \frac{2x}{\sqrt{x+3}-\sqrt{3}}$

9. $\lim_{x\to 0} \frac{\sec x}{x}$

10. $\lim_{x\to 0^+} \left(\sqrt{x}\right)^{1/x}$

11. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x\sqrt{x+1}}\right)$

12. $\lim_{x\to 0} \frac{x}{3\tan x}$

13–16 Two functions are in competition to determine the indicated limit. Identify the type of the indeterminate form, and fill out the table to decide which function dominates.

13. $\lim_{x\to\infty} f(x)$, where $f(x)=\dfrac{\sqrt{5x^3+7}}{0.2x^2+1}$

x	1	10	100	1000	10,000	100,000
$f(x)$						

14. $\lim_{x\to\infty} g(x)$, where $g(x)=\dfrac{0.5\sqrt{x}}{\ln(x+1)}$

x	1	10	100	1000	10,000	100,000
$g(x)$						

15. $\lim_{x\to\infty} h(x)$, where $h(x)=x^{100}e^{-x}$

x	1	10	100	1000	10,000	100,000
$h(x)$						

16. $\lim_{x\to 0} k(x)$, where $k(x)=(\sin x)^x$

x	1	0.5	0.1	0.01	0.001	0.0001
$k(x)$						

17–48 Check whether L'Hôpital's Rule applies to the given limit. If it does, use it to determine the value of the limit. If it does not, find the limit some other way. (When necessary, apply L'Hôpital's Rule several times.)

17. $\lim_{x\to\infty}\dfrac{2x+5}{x^2-7}$

18. $\lim_{x\to\infty}\dfrac{4-2.5x}{x+3}$

19. $\lim_{x\to-\infty}\dfrac{1.5x^3-2x^2+x+9}{x^2+2.1x-4}$

20. $\lim_{x\to-\infty}\dfrac{4.5x^4+x^3-2}{3-1.5x^4}$

21. $\lim_{x\to 0^+}\dfrac{\sqrt{x}}{\ln x}$

22. $\lim_{x\to\infty}\dfrac{x\sin x}{e^{-x}}$

23. $\lim_{x\to\infty}\dfrac{\frac{1}{x}+2}{2x+1}$

24. $\lim_{t\to 0}\dfrac{t}{\sqrt{2t+9}-3}$

25. $\lim_{x\to\infty}\dfrac{\sin x+2\ln x}{x^2+5}$

26. $\lim_{x\to 0}\dfrac{\sin x-x}{1-\cos x}$

27. $\lim_{t\to 0}\dfrac{1-\cos t}{3t}$

28. $\lim_{x\to -1^+}\dfrac{\sin\sqrt{x+1}}{x+1}$

29. $\lim_{x\to 0^+}\dfrac{x^{3/2}}{\ln(\cos x)}$

30. $\lim_{x\to 0}\dfrac{\ln(\sec^2 x)}{\sqrt{x}}$

31. $\lim_{x\to\infty}\dfrac{\ln x}{\ln(x^2+3x)}$

32. $\lim_{x\to 0}\dfrac{\log_{10}(x^2+2x+1)}{\log_{10}(x+1)}$

33. $\lim_{x\to 0}\dfrac{x}{3^{x/2}-1}$

34. $\lim_{x\to\infty}\dfrac{2^x}{x^2-3x+4}$

35. $\lim_{x\to 0}\dfrac{\sin x-x}{3x^2}$

36. $\lim_{\phi\to 0^+}\dfrac{1-\cos\phi}{\csc\phi}$

37. $\lim_{\alpha\to 0}\dfrac{\alpha}{e^{\sin\alpha}-1}$

38. $\lim_{\theta\to 0}\dfrac{\theta\tan\theta}{1-\cos\theta}$

39. $\lim_{t\to\infty}\dfrac{\ln(t+1)}{e^{-t}\sin t}$

40. $\lim_{t\to\pi}\dfrac{(\cos(2t)-1)^2}{t-\pi}$

41. $\lim_{\theta\to\pi/2}\dfrac{\left(\theta-\frac{\pi}{2}\right)^2}{\ln(\sin\theta)}$

42. $\lim_{x\to\infty}\dfrac{x+2^x}{5^x-x}$

43. $\lim_{x\to\infty}\dfrac{4^x+x^2}{3^x-x}$

44. $\lim_{x\to\infty}\dfrac{\ln(\ln x)}{x\ln x}$

45. $\lim_{x\to 0^+}\dfrac{\log_2(1+x)}{\log_3(\sin x+1)}$

46. $\lim_{x\to\infty}\dfrac{\log_4(2x+1)}{\log_5(x-4)}$

47. $\lim_{x\to 0^+}\dfrac{\log_4(x+1)}{\log_3 x}$

48. $\lim_{x\to 0}\dfrac{3^x-1}{x3^x}$

49–74 Identify the indeterminate product, quotient, difference, or power, and use L'Hôpital's Rule to find the limit. If the limit is not of indeterminate form, say so and find it by other means.

49. $\lim_{x\to 0^+} x\ln x$

50. $\lim_{x\to\infty}\dfrac{\sqrt{2x^2+1}}{x+3}$

51. $\lim_{x\to 0} x\cos\frac{\pi}{x}$

52. $\lim_{x\to\infty}(\ln x)^{-1/x}$

53. $\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x$

54. $\lim_{x\to 0^+}(-\ln x)^x$

55. $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{2}{x-1}\right)$

56. $\lim_{x\to 0^+}\left(\frac{1}{x}+\ln x\right)$

57. $\lim_{x\to 4^+}\left(\frac{32}{x^2-16}-\frac{x}{x-4}\right)$

58. $\lim_{x\to 0^+} x^{\left(x^2\right)}$

59. $\lim_{x\to 0^+}\left(2^x-x\right)^{1/x}$

60. $\lim_{x\to 0^+}(1-x)^{1/x}$

61. $\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)^{\csc x}$

62. $\lim_{x\to\infty}\left(\sqrt{x^2-3x}-\frac{3}{x^2+1}\right)$

63. $\lim_{x\to\infty}(x-1)^{1/x}$

64. $\lim_{x\to\infty}\frac{\ln x}{x^{7/5}}$

65. $\lim_{x\to 0}(\cos x)^{\cot x}$

66. $\lim_{x\to 0^-}(\cot x)^{\cos x}$

67. $\lim_{x\to 0^+}\tan x\sec x$

68. $\lim_{x\to\infty}\frac{x^{100}}{3^x}$

69. $\lim_{x\to\infty}\frac{\ln\left(100x^2+e^x\right)}{100x}$

70. $\lim_{x\to 0^+} 2\sqrt{x}\csc x$

71. $\lim_{x\to 0}(1+2x)^{1/x}$

72. $\lim_{x\to(\pi/2)^-}\left(\frac{\pi^2}{4}-x^2\right)\sec x$

73. $\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}$

74. $\lim_{x\to 1} x^{1/(1-x)}$

75–85 Find the limit. If applicable, use L'Hôpital's Rule (as many times as appropriate).

75. $\lim_{x\to\infty}\frac{2x^5+x^3-4}{e^x}$

76. $\lim_{x\to\infty}\frac{\cos x}{2^x}$

77. $\lim_{x\to\infty} x\sin\frac{1}{x}$

78. $\lim_{x\to\infty} x^{1/x^3}$

79. $\lim_{x\to 0^+} x^{x^x}$

80. $\lim_{x\to 0^+}\left(x^x\right)^x$

81. $\lim_{x\to\infty} x^{1/x^n},\quad n\in\mathbb{Z}^+$

82. $\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}$

83. $\lim_{x\to 0}\frac{\sin x-x}{2x-e^x+e^{-x}}$

84. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{x^2}$

85. $\lim_{x\to 0}\frac{\sin x-x}{\tan x-x}$

86–91 Find the error(s) in the limit calculation.

86. $\lim_{x\to 0}\frac{1-\sin x}{x}=\lim_{x\to 0}\frac{-\cos x}{1}=-1$ (Incorrect!)

87. $\lim_{x\to 2}\frac{x^2-2}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4$ (Incorrect!)

88. $\lim_{x\to-\infty}\frac{5^x+1}{5^x}=\lim_{x\to-\infty}\frac{(\ln 5)5^x}{(\ln 5)5^x}=1$ (Incorrect!)

89. $\lim_{x\to 0^+} x\cot x=\lim_{x\to 0^+}(1)\left(-\csc^2 x\right)$

$=(1)(-\infty)=-\infty$ (Incorrect!)

90. $\lim_{x\to 0} x\sin\frac{1}{x}=\lim_{x\to 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$=\lim_{x\to 0}\frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}$

$=\lim_{x\to 0}\cos\frac{1}{x}$

$=$ does not exist (Incorrect!)

91. $\lim_{x\to 0^-}\frac{\cos x-x^2-1}{x^4-2x^3}=\lim_{x\to 0^-}\frac{-\sin x-2x}{4x^3-6x^2}$

$=\lim_{x\to 0^-}\frac{-\cos x-2}{12x^2-12x}$

$=\lim_{x\to 0^-}\frac{\sin x}{24x-12}=0$ (Incorrect!)

92–106 Convince yourself that the initial use of L'Hôpital's Rule is not helpful in finding the limit. If possible, try to find a way to make use of the theorem, or evaluate the limit in some other way.

92. $\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x}}$

93. $\lim_{x\to\infty}\frac{\sqrt[3]{x+1}-2}{\sqrt{x^2+2}}$

94. $\lim_{x\to\infty}\frac{2^x+3^x}{5^x}$

95. $\lim_{x\to\infty}\frac{5^x-6^x}{7^x+8^x}$

96. $\lim_{x\to\infty}\frac{2^{-x}}{x^{-1}}$

97. $\lim_{x\to\infty}\left(\frac{1}{x+1}\right)^{-x^3}$

98. $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{e^{-x}}$

99. $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

100. $\lim_{x\to 0}\frac{\csc x}{\cot x}$

101. $\lim_{x\to 0^+}\left(\cot x-\frac{5x+1}{x}\right)$

102. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

103. $\lim_{x\to\pi^+}(\cot x)^{\sin x}$

104. $\lim_{x\to\infty} 2^{-x}x\ln x$

105. $\lim_{x\to 0}(\sin x)^{\tan x}$

106. $\lim_{x\to(\pi/2)^-}\left(\dfrac{1}{\dfrac{\pi}{2}-x}-\tan x\right)$

107–110 Find the limit of the sequence by considering the function you obtain after replacing *n* with the real variable *x*.

107. $\lim_{n\to\infty}\dfrac{n^2+1}{2^n}$

108. $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

109. $\lim_{n\to\infty}\sqrt[n]{n}$

110. $\lim_{n\to\infty}\dfrac{2^n+5^n}{6^n}$

111–114 Use L'Hôpital's Rule to prove the assertion.

111. $\lim_{x\to 0}\dfrac{\sin(kx)}{x^k}=\infty \quad (k>1)$

112. $\lim_{x\to\infty}\dfrac{p(x)}{e^{kx}}=0 \quad (p(x) \text{ is a polynomial, } k>0)$

113. $\lim_{x\to\infty}\dfrac{(\ln x)^n}{x^k}=0 \quad (n\in\mathbb{N},\ k>0)$

114. $\lim_{x\to\infty}\dfrac{a^x}{x^n}=\infty \quad (a>1,\ n\in\mathbb{N})$

115–122 Find the value(s) of *c* satisfying the conclusion of Cauchy's Mean Value Theorem. If the theorem doesn't apply, explain why.

115. $f(x)=x,\quad g(x)=x^2+1;\quad [0,1]$

116. $f(x)=x^3-1,\quad g(x)=x^2+2x;\quad [-1,1]$

117. $f(x)=x^3-x,\quad g(x)=-x^2+2x+3;\quad [-1,3]$

118. $f(x)=x^3,\quad g(x)=-x^2;\quad [-2,3]$

119. $f(x)=x^2+3x,\quad g(x)=3x^2-5x+3;\quad [-1,3]$

120. $f(x)=\dfrac{1}{x},\quad g(x)=\ln x;\quad [1,2]$

121. $f(x)=\cos x,\quad g(x)=\sin x;\quad \left[-\dfrac{\pi}{2},0\right]$

122. $f(x)=x^2-5x-9,\quad g(x)=x^3+x+10;\quad [-3,2]$

123–124 Prove that $f(x)$ has a removable discontinuity at $x=0$. Then find the value of *c* so as to make *f* continuous.

123. $f(x)=\begin{cases}\dfrac{3\tan x-2x}{5x^2+3x} & \text{if } x\neq 0\\ c & \text{if } x=0\end{cases}$

124. $f(x)=\begin{cases}(e^x-\sin 2x)^{2/x} & \text{if } x\neq 0\\ c & \text{if } x=0\end{cases}$

125. Recall the following compound interest formula for the value of an investment of *P* dollars after *t* years, compounded *n* times a year at an annual interest rate of *r*.

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

Use L'Hôpital's Rule to prove that if we let $n\to\infty$, we obtain the following continuous compounding formula.

$$A=Pe^{rt}$$

126. The strength of an electric field due to a disk charge is obtained from the formula

$$E(x)=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

where σ is the electric charge per unit area (in $\mathrm{C/m^2}$), $\varepsilon_0=8.85\cdot 10^{-12}\ \mathrm{C^2/Nm^2}$, R is the radius of the ring, and x is the distance to the charge in meters. Use L'Hôpital's Rule to confirm that $E(x)\to 0$ as $x\to\infty$. How is E affected by σ and R at a given distance? What happens to the rate of change of E as x increases? (**Hint:** Apply L'Hôpital's Rule to dE/dx as $x\to\infty$.)

127. Marquis de l'Hôpital first illustrated the rule named after him in his 1696 textbook, *Analyse des Infiniment Petits*. He used an example where the objective was to find

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

for $a>0$. Determine the above limit.

4.4 Technology Exercises

128–131 Check whether the limit is of indeterminate form, and then use a graphing utility to evaluate the limit.

128. $\lim_{x\to 1^+}(x-1)^{\ln x}$

129. $\lim_{x\to 0^+}\tan x\ln x$

130. $\lim_{x\to 0^+}x^{x^{x^x}}$

131. $\lim_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$

132–133 Use a graphing utility to graph the function for different values of the parameter *c*. Examine how the values of the parameter affect the indicated limit.

132. $\lim_{x\to\infty}\left(1+\dfrac{1}{cx}\right)^x$

What happens to the limit when $|c|\to\infty$?

133. $\lim_{x\to 0^+}\dfrac{1-c^x}{cx}$

What happens to the limit when $c\to\infty$?

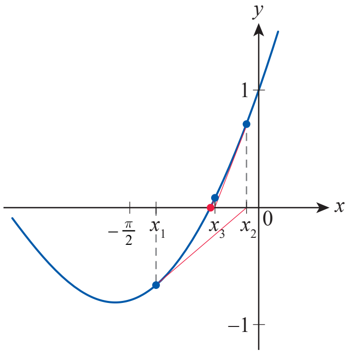


Figure 10
Zoomed-In Graph of
$f(x) = e^x + \sin x$

The largest negative root of f lies somewhere to the right of $-\pi/2$, as the close-up in Figure 10 indicates. An initial guess of 0 would work well to begin Newton's method, but for illustrative purposes we use $x_1 = -1.25$. Applying Newton's method, we obtain the following formula for x_{n+1}.

$$x_{n+1} = x_n - \frac{e^{x_n} + \sin x_n}{e^{x_n} + \cos x_n}$$

The first few approximations are as follows.

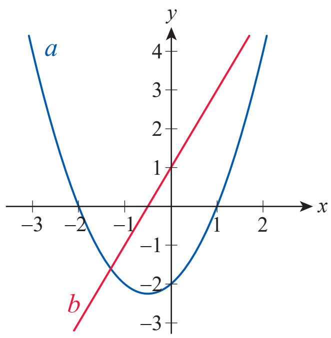
$x_1 = -1.25$	$x_2 \approx -0.149219$
$x_3 \approx -0.534414$	$x_4 \approx -0.587419$
$x_5 \approx -0.588532$	$x_6 \approx -0.588533$

Since x_5 and x_6 agree to five decimal places, the largest negative root of f is approximately -0.58853.

4.5 Exercises

1–4 The graphs of the first and second derivatives of a function f are given. Identify which one is which, and then sketch a possible graph of f. (Answers for the graph of f will vary.)

1.



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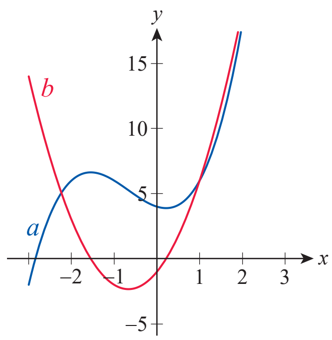

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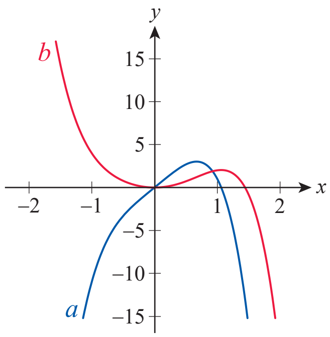
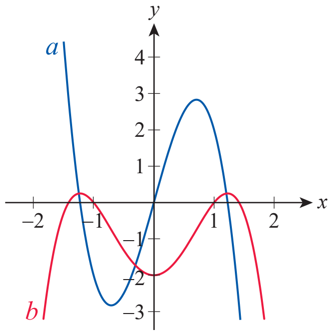


4.



5–48 Use the curve-sketching strategy to construct a graph of the function.

5. $f(x) = x^3 + 3x^2 - 9x$

6. $g(x) = -x^3 + 2x^2 - x + 4$

7. $h(x) = \frac{1}{4}x^4 + \frac{5}{3}x^3 + x^2 - 8x$

8. $F(x) = -\frac{3}{4}x^4 + x^3 + 9x^2 + 2$

9. $G(x) = (x^2 - 1)(x^2 - 2)$

10. $k(x) = x^5 - 2x^3 - 8x + 1$

11. $L(x) = x^5 - 3x^2$

12. $m(x) = 4x^3 - 5x^4$

13. $n(x) = \frac{-3}{x - 2}$

14. $H(x) = \frac{x^2 + 2}{x + 2}$

15. $R(x)=\dfrac{x}{x^2-4}$

16. $r(x)=\dfrac{2x^2+1}{x-3}$

17. $A(x)=|x-3|-2$

18. $f(x)=1.5-|x-2.2|$

19. $w(x)=x^{2/5}+\dfrac{2}{5}$

20. $u(x)=(x-2)\sqrt{x}$

21. $F(x)=2-(x-1)^{3/5}$

22. $G(x)=\sin^2 x-1$

23. $h(x)=e^{-x}+e^{2x}$

24.* $H(x)=-2\sqrt{x}\cdot 2^{-2x}$

25. $P(x)=x\ln x$

26. $H(x)=0.3\sqrt[3]{x}\left(x^2-1\right)$

27. $G(x)=x\sqrt{4-x^2}$

28. $L(x)=\dfrac{3}{x-2}$

29. $m(x)=\dfrac{x^2+7}{x-7}$

30. $K(x)=\dfrac{e^x}{x}$

31. $F(x)=e^x-e^{-2x}$

32. $v(x)=(x-1)^{3/5}$

33. $m(x)=-\dfrac{5}{(x-2)^2}$

34. $R(x)=\dfrac{x+2}{x-4}$

35. $G(x)=\dfrac{-x}{x^2-1}$

36. $t(x)=\dfrac{2x^2+2}{x-4}$

37. $H(x)=\dfrac{3}{4}\left(x-\dfrac{4}{3}\right)^{4/3}$

38. $w(x)=\dfrac{(x-1)^2}{2x^2-2}$

39. $k(x)=x-\sqrt[3]{x}$

40. $F(x)=x^{4/5}\left(x-\dfrac{4}{5}\right)$

41. $c(x)=x\sqrt[3]{1-x^2}$

42. $G(x)=-\sqrt{x}e^{-x}$

43. $Z(x)=2\sin x-\cos^2 x$

44. $K(x)=\sin x-\cos x$

45. $L(x)=x^{5/3}\ln|x|$

46. $G(x)=\sqrt{4x^2+3}$

47. $u(x)=7-\sqrt{9x^2+2x+1}$

48. $z(x)=e^{\cos x}$

49–54 First prove that $\lim\limits_{x\to\pm\infty}\left(f(x)-g(x)\right)=0$. This means that when $x\to\pm\infty$, the graph of $f(x)$ approaches that of $g(x)$. Use this observation as an aid in graphing $f(x)$. (In this case, we say that $f(x)$ is asymptotic to $g(x)$.)

49. $f(x)=\dfrac{x^3+5}{x+2},\quad g(x)=x^2-2x+4$

50. $f(x)=\dfrac{(x+1)^4+2}{3x+3},\quad g(x)=\dfrac{1}{3}(x+1)^3$

51. $f(x)=\sqrt{4x^2+5},\quad g(x)=|2x|$

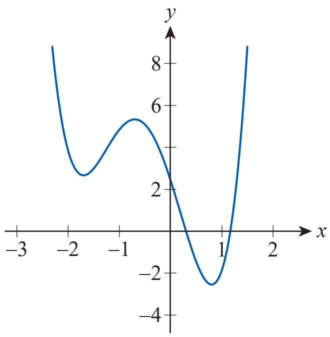
52. $f(x)=\sqrt{x^2-4x+5},\quad g(x)=|x-2|$

53. $f(x)=\sqrt[3]{x}+\dfrac{1}{x^2},\quad g(x)=\sqrt[3]{x}$

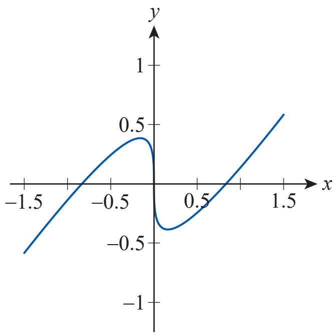
54. $f(x)=\sin x+\dfrac{1}{x},\quad g(x)=\sin x$

55–56 Sketch on paper a few of the tangent lines that are used to approximate the largest root of the indicated function by Newton's method, using the starting values of –1, 0, and 1, respectively. Does the method always work? Explain.

55.



56.



57–60 Use Newton's method to approximate the given number to five decimal places.

57. $\sqrt[4]{50}$

58. $\sqrt[10]{10}$

59. $\ln 5$

60. $\ln 100$

61–70 Use Newton's method to approximate the zero(s) of the given function to five decimal places. Restrict the domain to the given interval where indicated.

61. $f(x)=x^3-x+2$

62. $f(x)=2x^3+x^2-5x+1$

63. $f(x)=x^4-6.1x^3+4.7x^2-12.2x+5.4$

64. $f(x)=0.25x^4-2x^2+x+0.69$

65. $f(x)=x^5+x+1$

66. $f(x)=2x^5-5x^4+2x^3-4x^2+1$

67. $f(x)=4.2x-\sqrt{x+3}$

68. $f(x)=\sqrt{2+x^2}-1.1x$

69. $f(x)=2x^2-\cos(x-1);\quad \left(0,\dfrac{\pi}{2}\right)$

70. $f(x)=\sin(2x+1)-\dfrac{x}{2};\quad (0,1)$

71–76 Use Newton's method to solve the equation on the given interval. Approximate the root to six decimal places.

71. $\sin x = x^2$ on $\left(0, \frac{\pi}{2}\right)$

72. $2 - x^3 = e^x$ on $\mathbb{R}$

73. $x^4 = \arctan x$ on $(0, \infty)$

74. $\ln x = 2 - \sqrt{x}$ on $(0, \infty)$

75. $\cos x = \tan x$ on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

76. $\log_{1/2} x = \sin x$ on $(0, \infty)$

77–80 Recall from Exercise 62 of Section 4.2 that $c \in \mathbb{R}$ is said to be a fixed point of $f(x)$ if $f(c) = c$. Use Newton's method to approximate to four decimal places the fixed point(s) of the function on the given interval.

77. $f(x) = e^{-x}$ on $(0, \infty)$

78. $f(x) = \cos x$ on $\mathbb{R}$

79. $f(x) = 2\cot x$ on $(0, 2\pi)$

80. $f(x) = \log_{1/2} x$ on $(0, \infty)$

81–82 Use Newton's method to find the critical point(s) of the function correct to five decimal places.

81. $f(x) = x^5 - x^3 - 5x$

82. $f(x) = x^2 \sin x, \quad 0 < x < \pi$

83–87 Perform the first few iterations of Newton's method for the given function with the indicated first guess, and explain why the method doesn't work.

83. $f(x) = \sin x - \cos x; \quad x_1 = -\frac{\pi}{4}$

84. $f(x) = x^3 - 6x^2 + 12x - 6; \quad x_1 = 3$

85. $f(x) = \begin{cases} -\sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}; \quad x_1 = a \ (a \neq 0)$

86. $f(x) = \sqrt[3]{x}; \quad x_1 = a \ (a \neq 0)$

87. $f(x) = -x^3 + 9x^2 - 19x + 19; \quad x_1 = 3$

88. The following rule for approximating the square root of a has been known since ancient times.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Use Newton's method to derive this rule. (**Hint:** Start with the equation $x^2 - a = 0$.)

89. Generalizing Exercise 88, use Newton's method to derive a rule for approximating $\sqrt[k]{a}$, $k \geq 3$.

90. Using the approach you have taken in the previous two exercises, derive the following formula approximating $1/a$.

$$x_{n+1} = x_n (2 - a x_n)$$

91–96 Use the formulas you derived in Exercises 88–90 to approximate the given number to five decimal places.

91. $\sqrt{2}$

92. $\sqrt{50}$

93. $\sqrt[3]{10}$

94. $\sqrt[5]{30}$

95. $\frac{1}{7}$

96. $\frac{1}{19}$

4.5 Technology Exercises

97. Use a graphing utility to approximate π by generating the first 10 iterations of Newton's method for solving the equation $\sin x = 0$ with an appropriate starting value.

98. Repeat Exercise 97 for the equation $(x-5)^{50} = 0$ with the starting value of $x_1 = 6$. What do you find? Graph $f(x) = (x-5)^{50}$, and see if the graph gives insight into why things went wrong.

99–100 Perform the first two iterations of Newton's method with each of the given starting values in an attempt to find the positive root of $f(x)$; then use a graphing utility to come up with better approximations. What do you find? Graph $f(x)$, and see if the graph gives insight into why things went wrong.

99. $f(x) = x^3 - 2x - 1; \quad x_1 = 0.9, \quad x_1 = 0.8, \quad x_1 = -0.4$

100. $f(x) = x^4 - 6x^3 + 9.5x^2 - 1.5x - 4.9375;$
$x_1 = 3, \quad x_1 = 2.9, \quad x_1 = 0$

Solution

The given information consists only of the two angles and the two speeds, but in the figure we have already labeled other quantities that might help us relate the angles and speeds and arrive at Snell's Law. Specifically, we have let p denote the horizontal distance between the object B and the point directly beneath the observer at A. And since the point C of refraction (where the rays of light bend) is unknown, we have given its horizontal displacement from A the label x, meaning the horizontal distance between C and B is $p-x$. The vertical distances a and b are fixed, but the lengths of the hypotenuses, d_1 and d_2, will vary as x varies.

At this point, it may very well be unclear how to arrive at Snell's Law from what we have. But we haven't yet applied Fermat's Principle, and we can deduce many relationships between the labeled quantities. To begin with, since distance = rate · time, the time it takes light to travel from B to C is d_2/v_2 and the time it takes to travel from C to A is d_1/v_1. So the total time is expressed as follows.

$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

We can express the two hypotenuses as functions of x by noting that $a^2 + x^2 = d_1{}^2$ and $b^2 + (p-x)^2 = d_2{}^2$, so

$$T(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(p-x)^2}}{v_2}$$

and the domain of T is $[0, p]$.

The actual distance x must be the value of x that minimizes T, so our next step is to find T'.

$$\begin{aligned} T'(x) &= \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{p-x}{v_2\sqrt{b^2+(p-x)^2}} \\ &= \frac{x}{v_1 d_1} - \frac{p-x}{v_2 d_2} && \text{Substitute } d_1 \text{ and } d_2 \text{ for their formulas.} \\ &= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} && \sin\theta_1 = \frac{x}{d_1} \text{ and } \sin\theta_2 = \frac{p-x}{d_2} \end{aligned}$$

Note that $T_+'(0) < 0$ and $T_-'(p) > 0$, so by Darboux's Theorem (Section 3.1), there is a point $x \in [0, p]$ for which $T'(x) = 0$. And by the First Derivative Test, that point must minimize T. Rewriting $T'(x) = 0$, we have developed the formula for Snell's Law.

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

4.6 Exercises

1. Find two integers whose sum is 120 and whose product is as large as possible. (**Hint:** If you denote the first number by x, then the second number is $120 - x$. Now write a formula for the product, and use calculus to find the maximum.)

2–14 Use the strategy suggested in Exercise 1 to find two numbers satisfying the given requirements.

2. The sum is S and the product is a maximum.

3. The difference is 36 and the product is as small as possible.

4. Two positive numbers whose product is 144 and the sum is a minimum.

5. Two positive numbers whose product is n^2 and the sum is a minimum.

6. Two positive numbers whose product is 162 and the sum of twice the first and the second is a minimum.

7. Two positive numbers that are reciprocals of each other and their sum is a minimum.

8. Two positive integers so that the square of the first number plus the second number is 243 and their product is a maximum.

9. The sum of twice the first and three times the second is 480 and their product is a maximum.

10. The product of two positive integers is 32 and the sum of twice the first plus the second is a minimum.

11. Two positive numbers whose product is 16 and the sum of whose squares is a minimum.

12. Two positive numbers whose sum is 1 and the sum of whose cubes is a minimum.

13. Two nonnegative numbers whose sum is 1 and the sum of whose cubes is a maximum.

14. Repeat Exercises 12 and 13 using fourth powers instead of cubes.

15. Modify Example 2 by inscribing a rectangle in the region bounded by the x-axis and the parabola $y = k - x^2$ $(k > 0)$.

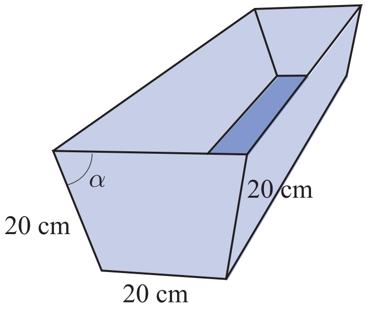
16. A vertex of a rectangle is at the origin; the opposite vertex sits in the first quadrant and on the line $2y + x = 4$. Find the dimensions that maximize the area of such a rectangle.

17. Repeat Exercise 16 with the opposite vertex sitting on the graph of $y = 32 - x^3$.

18. From among all lines through the point $(3,1)$, find the one forming with the coordinate axes a right triangle of minimum area.

19. Repeat Exercise 18, this time finding the line forming a triangle whose hypotenuse is of minimum length.

20. Suppose that when constructing a trough similar to the one in Exercise 35 of Section 3.8, both the shorter base and the legs of its cross-section are 20 centimeters long. Find the base angle α that maximizes the volume of the trough.



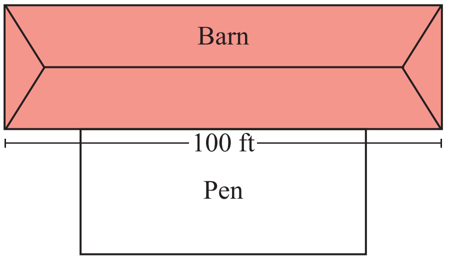
21. Find the coordinates of the point on the graph of $y = \sqrt{x}$ that is closest to the point $(1,0)$.

22. Find the coordinates of the point on the graph of $y = x^3$ that is closest to the point $(-4,0)$.

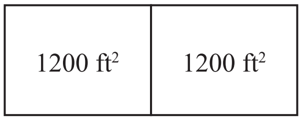
23. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $2x^2 + 6y^2 = 12$.

24. Find the equation of the line tangent to the graph of $y = 1 - x^2$ that forms with the coordinate axes the triangle of minimum area in the first quadrant.

25. A farmer has 120 feet of fencing to construct a rectangular pen up against the straight side of a barn, using the barn for one side of the pen. The length of the barn is 100 feet. Determine the dimensions of the rectangle of maximum area that can be enclosed under these conditions. (**Hint:** Be mindful of the domain of the function you are maximizing.)

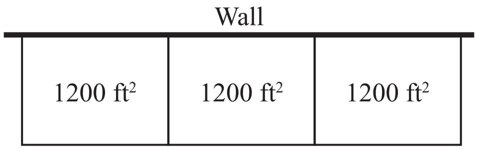


26. A farmer needs to construct two adjoining rectangular pens of identical areas, as shown. If each pen is to have an area of 1200 square feet, what dimensions will minimize the cost of fencing?

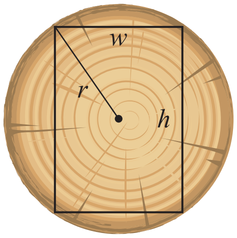


27. Repeat Exercise 26 if the pens are constructed against a straight wall that serves as a side for each.

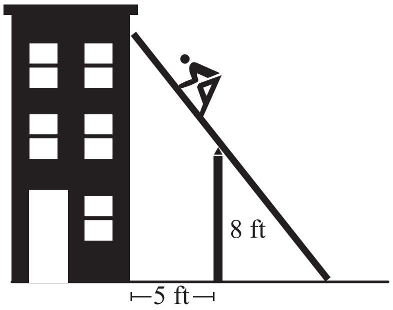
28. Repeat Exercise 27 if three identical adjoining pens are to be constructed, as shown.



29. A supporting beam with a rectangular cross-section is to be cut from a log that has an approximately circular cross-section with a radius of r inches. Knowing that the strength of such a beam is directly proportional to the width multiplied by the second power of the height of its cross-section, find the strongest beam that can be cut under these conditions.



30. An 8-foot fence stands 5 feet from a tall building. A contractor needs to reach the building with a ladder from the outside of the fence. Find the minimum length of the ladder that can do the job.



31. A 30 in. piece of wire is cut and the pieces are bent into a circle and a square, respectively. Where should we cut in order to minimize the sum of the areas of these two shapes?

32. Repeat Exercise 31, this time producing an equilateral triangle and a square.

33. Repeat Exercises 31 and 32, this time maximizing the sum of the two areas.

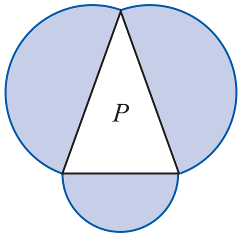
34. Prove that among all rectangles that can be inscribed in a circle, the square has the greatest perimeter.

35. Prove that among all isosceles triangles of a given area, the equilateral triangle has the minimum perimeter.

36. Find the dimensions of the rectangle whose perimeter is P units and area is a maximum.

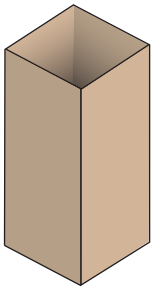
37. Find the dimensions of the rectangle whose area is A units and perimeter is a minimum.

38. The perimeter of an isosceles triangle is P inches. Find the side lengths so as to minimize the sum of areas of the semicircles drawn onto the sides of the triangle.



39. Suppose we want to construct a can in the shape of a right circular cylinder with no top whose surface area is to be S square inches. What dimensions will maximize the volume?

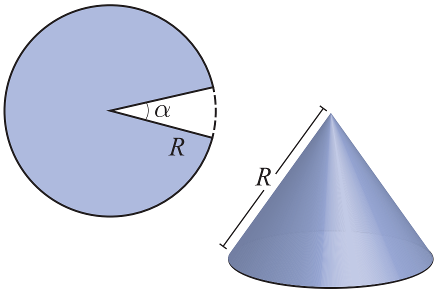
40. If we want to make a rectangular box with a square bottom and no top that holds 32 cubic inches, and the construction material costs 3 cents per square inch, what are the dimensions and the cost of the least expensive box that can be made?



41. If the box to be constructed in Exercise 40 is to hold the same volume, but we need to construct a top from an expensive, heat-resistant material that costs 21 cents per square inch, how does the new requirement change the cost and dimensions of the least expensive box?

42. Determine the dimensions and maximum volume of the rectangular box with no top and a square base if its surface area is A square inches.

43. A cone is to be constructed by cutting out a sector of central angle α of a disk of radius R and gluing the cut lines together to form a cone. Find the value of α that maximizes the volume of the cone.

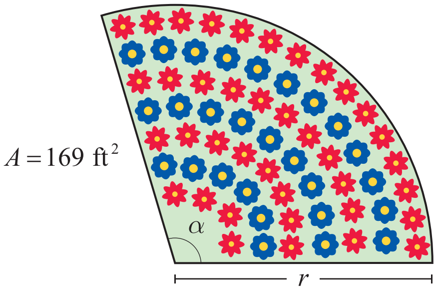


44. The pages of a children's book are to contain 54 square inches of printed matter and illustrations, with margins of 1 inch along the sides and $1\frac{1}{2}$ inches along the top and bottom of each page. Find the dimensions of the page that will require the minimum amount of paper.

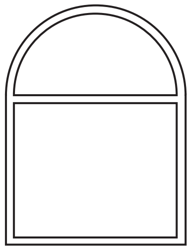
45. A poster is to contain 150 square inches of printed matter, surrounded by margins that are 3 inches wide on the top and bottom, and 2 inches on each side. Find the dimensions for the poster that minimize its total area.

46. The sum of squares of lengths of the sides of a right triangle is 64 square inches. Find the side lengths that maximize the area of the triangle.

47. A flower bed is planned in the form of a circular sector. Find the central angle and radius if it is to cover 169 square feet, and its perimeter is to be a minimum.



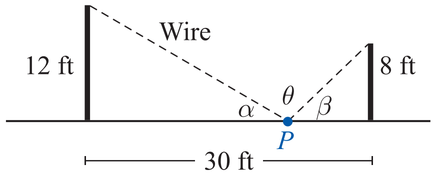
48. The shape of a Norman window can be approximated by a rectangle with a semicircle on top. What dimensions will admit the maximum amount of light if the perimeter of the window is to be P inches?



49. In Exercise 103 of Section 3.6, find the optimum distance s that maximizes the viewing angle.

50. An office building is located right on a riverbank, which is straight. A small power plant is on the opposite bank, 1500 feet downstream from the point directly opposite the office building. The river is 300 feet wide. If we want to connect the power plant and the building by cable, which costs \$1700 per foot to lay down underwater and \$800 per foot underground, what is the least expensive path for the cable?

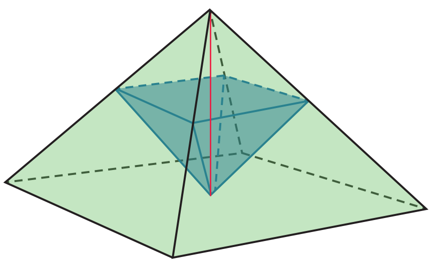
51. Two antennas standing 30 feet apart are to be stayed with a single wire. The wire runs from the top of the first antenna, is secured to the ground somewhere between the antennas, and is finally attached to the top of the second antenna. If the height of the first antenna is 12 feet, while that of the second is 8 feet, find the point along the line segment connecting the bases where the wire needs to be staked to the ground if the length of the wire is to be minimal.



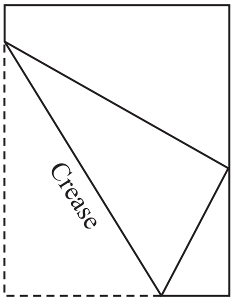
52. If we denote the heights of the antennas in Exercise 51 by h_1 and h_2, respectively, and the distance between them is d, prove that the wire has minimal length if and only if $\alpha = \beta$.

53. In Exercise 51, find the location of P that maximizes the angle θ.

54.* An inverted square pyramid is to be inscribed into a larger square pyramid of volume V, so that the two have a common axis, and the vertex of the inscribed pyramid coincides with the center of the outer pyramid's base. Find the ratio of the pyramids' altitudes so that the volume of the inscribed pyramid is maximal.



55.* The lower left corner of a letter-sized paper, which is 8.5 in. by 11 in., is folded over to reach the right edge of the paper. Find a way that this can be done so as to produce a crease of minimum length.



56. Find the radius of the base and the height of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

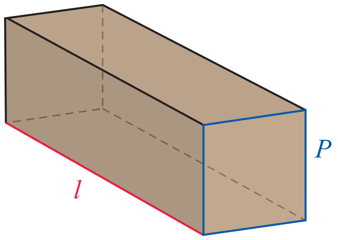
57. Repeat Exercise 56, but inscribe a right circular cone instead of a cylinder in the sphere of radius R.

58. Find the radius of the base and the height of the right circular cylinder of largest volume that can be inscribed in a circular cone if the height of the cone is H and its base has radius R.

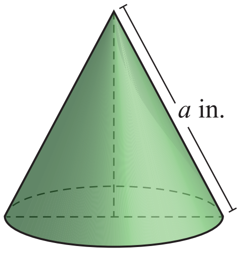
59. Repeat Exercise 58, but find the extremum of the surface area of the cylinder instead of its volume.

60. The sum of the height and the radius of the base of a circular cylinder is 12 inches. Find their lengths if the volume of the cylinder is to be a maximum.

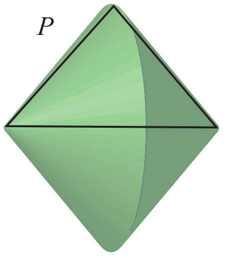
61. Suppose that we want to send a parcel in the shape of a square-based rectangular solid, and the Standard Post service limits the sum of the length and girth (girth = the perimeter of the base) to 130 inches. Find the dimensions of the package of the greatest volume under these conditions.



62. Find the maximum volume a right circular cone can have if its slant height is a inches.



63. An isosceles triangle of perimeter P is rotated around its base. What base length will produce the solid of maximum volume?



64. A lighthouse is 2 miles off a straight shoreline, and a grocery store is 10 miles down the coast. If the lighthouse keeper can row at 2.4 mph and walk at 4 mph, where should he land in order to make the best time to the store to get supplies? What if he is picked up by a golf cart that can drive at 9.9 mph?

65. Repeat Exercise 64 if the lighthouse keeper uses a motorboat whose top speed is 20.1 mph, and will be picked up by a car that will drive at the posted speed limit of 45 mph.

66. Repeat Exercise 64 if the lighthouse and the store are both on the shore of a circular lake of diameter d at the endpoints of the diameter.

67. At noon on a certain day, a plane is 200 miles south of another airliner and flying north at 550 mph, while the second plane is flying southwest at 600 mph. How much later after this instant is their distance a minimum?

68. A straight two-lane highway intersects a straight interstate at a right angle. A car exits the interstate and starts moving away from it on the two-lane highway at 50 mph. At the same instant, another car, moving at 75 mph on the interstate, is approaching the same intersection, but is still 10 miles from it. When will their distance be a minimum and what will this distance be?

69. The position of an object connected to a spring is given by $d(t) = \sin 3t + \cos 3t$, where d is measured in feet, and t in seconds. Find when the absolute value of its velocity first reaches its maximum and the value of the maximum velocity.

70. In Exercise 69, find when the absolute value of the acceleration first reaches its maximum and the value of this acceleration.

71. Ignoring air resistance, the range r of a projectile fired from the ground in a flat area with an initial velocity of v_0 can be calculated by $r = \left(v_0^{\,2}/g\right)\sin 2\theta,$ where g is the gravitational acceleration and θ is the launch angle relative to the horizontal. Find the launch angle that maximizes the range if the initial velocity is a given constant.

72. The luminance E_l at distance d from a light source is directly proportional to the light intensity F_l (also called luminous flux) and inversely proportional to the square of distance: $E_l = F_l/\left(4\pi d^2\right).$ Suppose two light bulbs are 3 meters apart, with respective light intensities of $F_{l,1} = 1700$ lumens (lm) and $F_{l,2} = 1000$ lm. Where between these light bulbs will the sum of their luminance levels be a minimum?

73. Management and Power, Inc. has found that its seminar on management techniques attracts 800 people when the seminar fee is set to \$600. They estimate that for each \$15 discount in the charge, an additional 50 people will attend the seminar. Find the amount that Management and Power, Inc. should charge for the seminar to maximize the revenue, and find the maximum revenue.

74. A blueberry farmer owns 1056 plants, each producing p pounds on average during a regular season. He estimates that for each additional dozen of new plants planted on his farm, average production per plant is going to drop by a half percent. What would be the optimum number of plants on the farm in order to maximize production, and what is the optimum production level?

75. The manager of a 115-unit apartment complex finds that all units are rented at a price of \$1500 per month. Research shows that for each \$20 increase in rent, one additional unit remains vacant. How much should he charge for rent in order to bring in maximum revenue, and how many units are rented then?

76. A moving company sends a truck on a 2000-mile round-trip to move two households. The hourly fuel consumption of the truck is approximated by $2 + \frac{1}{280.1}v^2$ gallons, where v is assumed to be a constant speed somewhere between 35 and 70 miles per hour. If a gallon of diesel fuel costs \$4.50 and the driver is paid \$35 an hour, what speed will minimize the company's transportation costs?

77. Cool Wheels, a manufacturer of die-cast model cars, has a monthly overhead cost of \$6000, material costs of \$2 per toy car, and each has associated labor costs of \$0.40. When producing and marketing 2500 cars a month, each sells for \$30.75. When producing more, it was found that for each additional 100 units, the market conditions cause the price to drop by a dollar. In addition, labor costs go up by 5 cents for each additional 100 units because of expensive overtime pay. Find the production level and selling price that maximize the profit under these conditions.

78. Suppose it costs a candy company \$3 to produce and distribute a box of Chi-Can chipotle candy bars, and the number of boxes sold at x dollars a box is approximated by $n = \dfrac{80}{x-11} + 15(50-x).$ What sale price will bring the maximum profit?

79. Prove that when the company in Exercise 78 maximizes its profit, the marginal cost equals the marginal revenue.

80. Suppose that $R(x) = 2x^3 - 15x^2$ and $C(x) = 3x^3 - 25x^2 + 21x$ are the weekly revenue and cost functions for a particular commodity, where x represents units of 100 individual products and where the model is thought to be accurate up to approximately $x = 10$. What is the profit zone, and what level of production will maximize the profit? (See Example 4.)

81. The cost of manufacturing x units of a commodity is given by $C(x) = x^3 - 15x^2 + 12{,}000x.$ Find the value of x that minimizes the average cost of production. (See Example 5.)

4.6 Technology Exercises

82–83 Use the graphing and symbolic differentiation capabilities of a computer algebra system to solve the problem.

82. Suppose we have a small supply of craft paint, enough for 1 square foot, and we want to use it to paint a regular tetrahedron and a cube from a children's toy set. What should the dimensions of these solids be if we want to maximize the total volume? How about minimizing the volume?

83. Repeat Exercise 82 for a tetrahedron and a sphere.

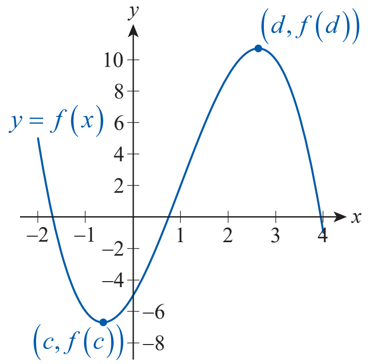


Figure 5

Solution

We begin by observing that f' is positive on the interval (c,d), negative on the intervals $(-\infty,c)$ and (d,∞), and that $f'(c) = f'(d) = 0$. This tells us that f is increasing on (c,d), decreasing on $(-\infty,c)$ and (d,∞), and that f has a relative minimum at $x = c$ and a relative maximum at $x = d$. Further, f' has a critical point at 1 and appears to be differentiable there (the graph of f' is nicely smooth), so $f''(1) = 0$; moreover, since f' is increasing to the left of 1 and decreasing to the right of 1, it must be the case that f changes concavity as it passes through the point $(1,2)$.

Putting all these observations together, the graph of f must be something along the lines of the one in Figure 5, though the actual values of $f(c)$ and $f(d)$ can be nothing more than a rough guess.

The values of f' from the original graph tell us approximately how fast the graph of f rises or falls near a given point—for instance, since $f'(1) = 8$, the "slope" of f at $(1,2)$ should be 8.

4.7 Exercises

1–8 Verify by differentiating that $F(x)$ is an antiderivative of $f(x)$.

1. $f(x) = \frac{1}{\sqrt{x}} + \frac{1}{x^2}, \quad F(x) = 2\sqrt{x} - \frac{1}{x}$

2. $f(x) = 2(x-1)(x+5), \quad F(x) = \frac{2}{3}x^3 + 4x^2 - 10x$

3. $f(x) = -x(x+2)(x-4), \quad F(x) = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + 4x^2 + \frac{5}{3}$

4. $f(x) = 6\cos 3x, \quad F(x) = 2\sin 3x$

5. $f(x) = 5\sec^2(5x+1), \quad F(x) = \tan(5x+1) + 5$

6. $f(x) = \frac{x^2+1}{\sqrt{x}}, \quad F(x) = \frac{\sqrt{x}}{5}(10 + 2x^2)$

7. $f(x) = \frac{2x}{x^2+7}, \quad F(x) = \ln(x^2+7)$

8. $f(x) = \pi^{2x}, \quad F(x) = \frac{\pi^{2x}}{2\ln\pi}$

9–20 Find an antiderivative of the function.

9. $f(x) = 1$

10. $g(x) = 2x + 2$

11. $h(x) = 4x^3 - x$

12. $u(x) = x^5 + x^3 + \pi$

13. $v(x) = \sec^2 x + 3x$

14. $k(x) = \frac{2}{x}$

15. $f(x) = 5e^x$

16. $m(x) = \frac{1}{2\sqrt{x}}$

17. $u(t) = -\frac{4}{t^3}$

18. $v(s) = \frac{1}{6s^{2/3}}$

19. $w(z) = \frac{1}{\sqrt{1-z^2}}$

20. $g(s) = \frac{1}{1+s^2} + 1 + s^2$

21–32 Find the general antiderivative of $f(x)$; then find the particular antiderivative $F(x)$ that satisfies $F(1) = 1$.

21. $f(x) = 2x - 3$

22. $f(x) = 3x^2 + \frac{1}{2}$

23. $f(x) = \frac{1}{\sqrt{x}}$

24. $f(x) = 1$

25. $f(x) = 0$

26. $f(x) = -\frac{1}{x}$

27. $f(x) = x^3 - \frac{1}{x^2}$

28. $f(x) = \frac{-1}{3x^{2/3}}$

29. $f(x) = (\ln 10)10^x$

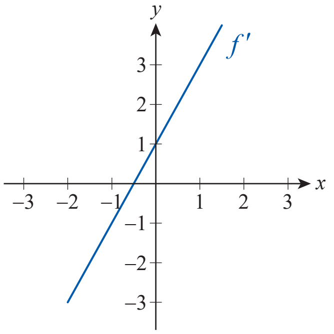
30. $f(x) = \sin x$

31. $f(x) = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}x\right) + 1$

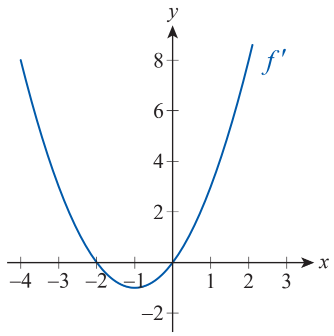
32. $f(x) = \frac{2x}{x^2+4}$

33–36 Given the graph of f' and the knowledge that f passes through the point $(3,1)$, sketch a possible graph for f.

33.



34.



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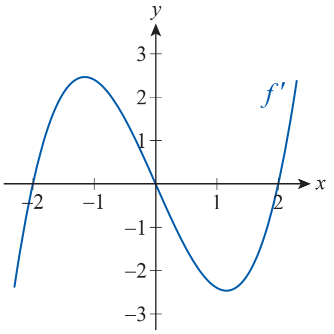

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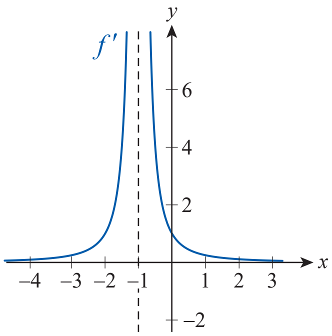


37–60 Find the general antiderivative of the given function, and check your answer by differentiation. (If necessary, rewrite the function before antidifferentiation.)

37. $f(x) = 6x^2 - 4x + 1.5$

38. $g(x) = 5x^3 - \pi x$

39. $h(x) = 3x^5 - 10x^4 + x^2 + 7$

40. $u(x) = -7x^4 + \frac{1}{2}x^3 + 6x^2 - 8x + \frac{5}{2}$

41. $v(x) = 3(x+6)(2x+1)$

42. $k(x) = -x(x+3)(7x-5)$

43. $h(x) = x^3\sqrt{x}$

44. $m(x) = \frac{3}{\sqrt{x}} + 2x\sqrt[3]{x}$

45. $n(x) = \frac{x^3+7x}{x^2}$

46. $f(t) = \frac{t^2-t}{\sqrt{t}+1}$

47. $a(y) = \left(\sqrt[3]{y^4} - 1\right)^2$

48. $w(z) = \frac{2}{z} + \frac{2}{\sqrt{z}}$

49. $g(t) = e^{3t} - 3\sec t \tan t$

50. $s(t) = 2 \cdot 10^{1.5t}$

51. $t(\theta) = \theta + \cos\theta$

52. $c(\theta) = \theta^2 + \csc^2\theta$

53. $v(x) = (\csc x - \cot x)\csc x$

54. $t(x) = -\sec^2 x\left(\cos^2 x + \sin^2 x\right)$

55. $w(x) = \frac{\cos x}{\cos^2 x - 1}$

56. $u(x) = \frac{2\tan 2x}{2\cos^2 x - 1}$

57. $a(x) = \frac{5}{1+9x^2}$

58. $b(x) = \frac{1}{\sqrt{1-4x^2}}$

59. $c(x) = \frac{4}{|5x|\sqrt{25x^2-1}}$

60. $d(x) = \frac{-3}{\sqrt{4-36x^2}}$

61–76 Find $f(x)$ that satisfies the specified conditions. (When no initial conditions are specified, find the general antiderivative.)

61. $f''(x) = \pi, \quad f'(1) = 0, \quad f(1) = 0$

62. $f''(x) = 1 - 4x, \quad f'(-1) = 1, \quad f(-1) = -4$

63. $f'''(x) = 0, \quad f''(2) = 2, \quad f'(2) = 2, \quad f(2) = 2$

64. $f'''(x) = x + 1, \quad f''(0) = 1, \quad f'(0) = 2, \quad f(0) = 3$

65. $f''(x) = \sqrt[3]{x}, \quad f'(1) = 0, \quad f(1) = \frac{1}{7}$

66. $f''(x) = x + \frac{1}{\sqrt{x}}, \quad f'(4) = 6, \quad f(4) = 0$

67. $f'''(x) = \sqrt{x} + 1, \quad f''(0) = 1, \quad f'(0) = -1, \quad f(0) = 7$

68. $f''(x) = \sqrt[3]{x}(x-3), \quad f'(0) = 0, \quad f(0) = 0$

69. $f'(x) = \dfrac{4}{1+4x^2}, \quad f\left(\dfrac{1}{2}\right) = \pi$

70. $f'(x) = \dfrac{-1}{\sqrt{1-3x^2}}, \quad f\left(\dfrac{\sqrt{3}}{3}\right) = 0$

71. $f'''(x) = -\cos 2x, \quad f''(0) = 1, \quad f'(0) = 1, \quad f(0) = -1$

72. $f'''(x) = \cos x - \sin x$

73. $f''(x) = 2^{5x}$

74. $f'''(x) = e^x + e, \quad f''(0) = 1, \quad f'(0) = 2, \quad f(0) = 3$

75. $f''(x) = \cos x - e^{2x}, \quad f'(0) = -2, \quad f(0) = 1$

76. $f'''(x) = \sin 10x + 10x + 10, \quad f''(0) = 0,$
$f'(0) = 3.5, \quad f(0) = -0.5$

77–85 Use $-32\ \text{ft/s}^2$ for the acceleration caused by gravity ($-9.81\ \text{m/s}^2$ in the metric system). Ignore air resistance. (**Hint:** See Example 4.)

77. A soccer ball is kicked upward from a height of 3 feet with an initial velocity of 48 feet per second. How high will it go?

78. A student drops a pen from a classroom window on the fourth floor of the mathematics building. If the window is 48 ft above ground level, how long is the pen in the air and with what speed does it hit the ground?

79. A hiker throws a pebble into a canyon that is 350 meters deep, with a downward initial velocity of $10\ \text{m/s}$. For how many seconds is the pebble in the air and what is the speed of impact?

80. A baseball is thrown upward from a height of 1.5 meters with an initial velocity of 30 meters per second. How high will it go, and for how long is it going to rise?

81.* With what initial velocity do we need to throw a tennis ball vertically upward in order for it to reach the top of a 60 ft campus flagpole?

82. An air rifle shoots a pellet at 1200 feet per second. What is the horizontal range of the rifle, that is, how far from where the pellet is shot will it hit the ground, if we shoot horizontally from a height of 5 feet?

83. A golf ball is hit horizontally at 40 meters per second from the top of a slight hill that is 1.5 meters high. If the terrain around the hill is nearly flat, approximately how far will the golf ball fly?

84. Prove that the position function of an object thrown vertically from an initial height of h_0 feet with an initial velocity of v_0 feet per second is $h(t) = -16t^2 + v_0 t + h_0$.

85. Repeat Exercise 84 using the metric system (meters and seconds) to arrive at the formula $h(t) = -4.905t^2 + v_0 t + h_0$.

86. The acceleration due to gravity on the lunar surface is approximately $-5.25\ \text{ft/s}^2$. How high would the soccer ball of Exercise 77 fly on the moon?

87. Find out what would happen in the situation described in Exercise 83 under lunar conditions. (See Exercise 86 for the acceleration due to gravity on the moon.)

88. The rate of growth of a rabbit population in a certain state park, where food supply is limited and predators are present, is proportional to $e^{-0.1t}$, where t is time measured in months. If the initial population size is 300 rabbits, which grows to 400 in three months, find the population size in a year. (**Hint:** Let $P(t)$ stand for population size, and use $\dfrac{d}{dt}P(t) = ke^{-0.1t}$.)

89. The rate of growth of a population of a certain virus in a medical experiment is proportional to $\sqrt[3]{t}$, where t is time measured in days. If the initial population size is 1000, which grows to 1500 in a day, find the population size in five days. (See and appropriately modify the hint given in Exercise 88.)

90.* A modern Formula One car is able to come to a complete stop from $200\ \text{km/h}$ (124.3 mph) using a braking distance of only about 65 meters. Assuming constant deceleration (which is not fully realistic), what multiple of g is this? (**Hint:** $1\ \text{m/s} = 3.6\ \text{km/h}$)

91. The Bugatti Veyron, the fastest production grand tourer from 2010 to 2017, can go from 0 to $100\ \text{km/h}$ in 2.5 seconds. Find its position function when accelerating from a standstill and the distance covered during the first 1.5 seconds. What is the car's acceleration time from 0 to 60 mph? (Use the simplifying assumption that acceleration is constant. Also see the hint provided in Exercise 90.)

92. Jerry the mouse is running toward his hole at a steady speed of $11\ \text{ft/s}$. Still 20 feet from his destination, he is discovered by Tom the cat, who is 2 feet behind Jerry at that moment. If Tom can reach his top speed of $40\ \text{ft/s}$ in 3 seconds, will he be able to catch Jerry? (Suppose the locations of Tom, Jerry, and the mousehole remain collinear throughout the pursuit.)

93.* Assume that an airplane needs to reach a liftoff speed of 180 mph and that it can achieve the same on a runway that is 0.8 miles long. Assuming constant acceleration during takeoff, what would this acceleration be?

94. The acceleration function of a particle moving along the x-axis is $a(t) = 3\sqrt{t} - \dfrac{1}{\sqrt{t}}$ units/s^2. If it starts at the origin with an initial velocity of 2 units per second, find the position function of the particle. Where will it be in 5 seconds?

95. Repeat Exercise 94 for the acceleration function $a(t) = (2-t)\sqrt{t}$, if the particle starts from rest at the point $(3,0)$. Where will it be in 5 seconds, and when will its instantaneous velocity be zero?

96. It follows from our discussions in Section 3.6 as well as the present section that an antiderivative of $-1/\sqrt{1-x^2}$ can be written as $-\sin^{-1}x$. Use the graphs of inverse trigonometric functions provided in Section 3.6 to argue that $\cos^{-1}x$ is also an antiderivative of $-1/\sqrt{1-x^2}$. (It follows that the general antiderivative of $-1/\sqrt{1-(kx)^2}$, $|kx|<1$ is $(1/k)\cos^{-1}(kx)+C$. See also Exercise 67 of Section 3.6.)

Concept Check

97–103 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

97. If $f(x)$ has an antiderivative on an interval I, then it has infinitely many antiderivatives on the same interval.

98. All polynomials have antiderivatives on the entire real line $\mathbb{R}$.

99. It is possible for a function to have an unique antiderivative on an interval I.

100. Whenever F_1 and F_2 are both antiderivatives of f on an open interval, then $F_1 - F_2$ is a constant function.

101. If a function has an antiderivative on the interval $(-a, a)$ for some $a > 0$, then it has exactly one antiderivative whose graph goes through the origin.

102. Every antiderivative of a polynomial function of degree n has degree $n + 1$.

103. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$ on an interval I, then $F(x)\cdot G(x)$ is an antiderivative of $f(x)\cdot g(x)$ on the same interval.

Example 4 Finding the Area Bounded by the *x*-Axis and the Graph of a Function

Find the area under the graph of $f(x)=x^2$ and above the *x*-axis on the interval $[0,1]$.

Solution

For variety, we will construct an expression for U_n (the underestimate based on n subintervals) and evaluate $\lim_{n\to\infty} U_n$. In Exercise 77 you will show that the same answer is obtained by evaluating $\lim_{n\to\infty} O_n$.

If we divide $[0,1]$ into n subintervals of equal width, each one has width $\Delta x = 1/n$. The minimum value of f on $[x_{i-1}, x_i]$ occurs at $x_i^* = x_{i-1}$, that is, at the left endpoint of each subinterval. So $x_1^* = 0$, $x_2^* = 1/n$, $x_3^* = 2/n$, and in general $x_i^* = (i-1)/n$. So we have the following Riemann sum.

$$U_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$

We have already simplified the sum $\sum_{i=1}^{n}(i-1)^2$ in part c. of Example 3.

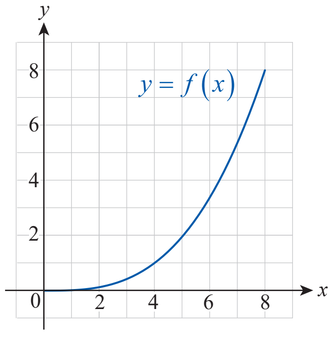
$$U_n = \frac{1}{n^3}\left(\frac{2n^3-3n^2+n}{6}\right) = \frac{2n^3-3n^2+n}{6n^3}$$

Hence, $\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{2n^3-3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

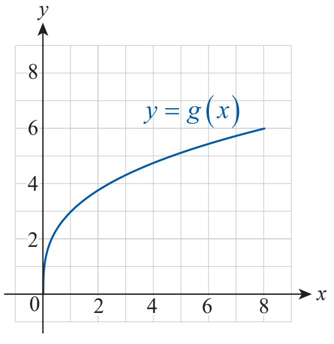
5.1 Exercises

1–2 Use $(O_4+U_4)/2$ to estimate the area under the graph of the function and above the *x*-axis on the interval $[0,8]$.

1. $f(x)=\left(\frac{x}{4}\right)^3$

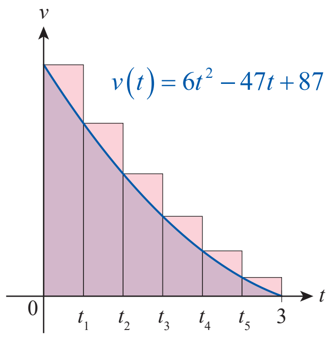


2. $g(x)=3\sqrt[3]{x}$



3–4. Repeat Exercises 1–2 using eight rectangles.

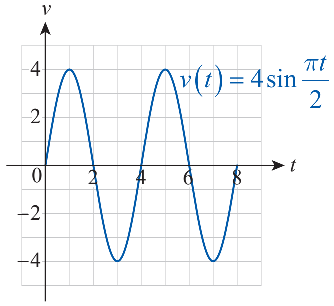
5. The figure below shows the upward velocity (in feet per second) of a model rocket during its rise. Use the method of Example 1 to estimate how high the rocket rose by calculating O_6. Is your estimate an over- or underestimate?



6. The velocity of an object undergoing simple harmonic motion is given by the graph below (time is measured in seconds, distance in feet). Using subintervals of width $\frac{1}{4}$,

a. give an overestimate for the total distance covered from $t = 1$ s to $t = 6$ s;

b. estimate the total displacement from $t = 1$ s to $t = 6$ s.



7. The given table contains the velocity data recorded by an automotive testing device during an acceleration test.

Time (s)	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
***v* (m/s)**	3	6.6	9.8	13	16.1	19.1	21.6	23.8	25.8	27.6	28.5	29.1

a. Use 12 subintervals to give over- and underestimates of the distance covered by the car during the acceleration run (i.e., find O_{12} and U_{12}).

b. Approximate the above distance using 6 subintervals of equal width and choosing the midpoint of each as the sample point (we shall call the resulting quantity M_6).

c. Compare M_6 with $(O_{12} + U_{12})/2$. Which one is greater? Explain why this is the case.

8. In order to estimate the length of the runway, a passenger on an airplane jotted down some velocity data during takeoff from the on-board entertainment screen. From the resulting table given below, calculate $(O_8 + U_8)/2$ to find his estimate.

Time (s)	6	12	18	24	30	36	42	48
***v* (mph)**	30	79	115	150	180	204	223	230

9–14 Use four rectangles to estimate the area between the graph of the given function and the *x*-axis on the given interval. Construct three estimates for the function: the first using the left endpoints of the subintervals as the sample points, the second using the right endpoints of the subintervals, and the third using the midpoints of the subintervals. Can you tell which are guaranteed to be underestimates or overestimates? (**Hint:** Consider the increasing/decreasing and concavity features of the graph. It is helpful to make a sketch.)

9. $f(x) = \sqrt{x}$ on $[0, 4]$

10. $f(x) = \frac{x^3}{16}$ on $[0, 4]$

11. $f(x) = \frac{1}{x}$ on $[1, 5]$

12. $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

13. $f(x) = \cos\frac{x}{2}$ on $[0, \pi]$

14. $f(x) = e^{2-x}$ on $[0, 2]$

15–24 Write the given sum using sigma notation.

15. $3 + 6 + 9 + \cdots + 99$

16. $1 + 2 + 9 + 28 + \cdots + (25^3 + 1)$

17. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{10{,}000}$

18. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{50}$

19. $a_{-3}+a_{-1}+a_1+a_3+a_5+\cdots+a_{77}$

20. $b_0+b_3+b_6+b_9+b_{12}+\cdots+b_{297}$

21. $f\left(\frac{3}{n}\right)+f\left(\frac{6}{n}\right)+f\left(\frac{9}{n}\right)+\cdots+f\left(\frac{3(n-1)}{n}\right)+f(3)$

22. $g(c_0)+g(c_5)+g(c_{10})+g(c_{15})+\cdots+g(c_{650})$

23. $f(x_0^*)\Delta x+f(x_1^*)\Delta x+f(x_2^*)\Delta x+\cdots+f(x_n^*)\Delta x$

24. $s(t_1^*)\Delta t+s(t_2^*)\Delta t+s(t_3^*)\Delta t+\cdots+s(t_{n-1}^*)\Delta t$

25–30 Assuming that $\sum_{i=0}^{n} a_i = 36$ and $\sum_{i=0}^{n} b_i = 100$, find the given sum.

25. $\sum_{i=0}^{n}(b_i-a_i)$

26. $\sum_{i=0}^{n}(2a_i+3b_i)$

27. $\sum_{i=0}^{n}(5b_i+1)$

28. $\sum_{i=0}^{n}\left(\frac{a_i}{6}-\frac{b_i}{2}\right)$

29. $\sum_{j=0}^{n}\left(\frac{4a_j}{3}-\frac{b_j}{4}+2\right)$

30. $\sum_{k=1}^{n}\left(\frac{4}{3}-2a_k+\frac{b_k}{2}\right)$

31–42 Find the value of the sum. Use a summation formula when possible.

31. $\sum_{i=2}^{4}\frac{1}{i-1}$

32. $\sum_{j=-2}^{2}\sqrt{j+2}$

33. $\sum_{i=1}^{10}(5i-2)$

34. $\sum_{i=1}^{n}(1-3i)$

35. $\sum_{j=1}^{n}\frac{4j+5n}{2}$

36. $\sum_{k=1}^{n}\frac{6k^2+2k}{3}$

37. $\sum_{j=1}^{30}(2j^2-4j+1)$

38. $\sum_{i=1}^{100}(2i-1)(3-i)$

39. $\sum_{j=1}^{n}(3j+1)^2$

40. $\sum_{i=1}^{n}\left(i^3-2i^2+\frac{1}{n}\right)$

41. $\sum_{j=1}^{n}2j^2(j-2)$

42. $\sum_{k=0}^{n}k(k+1)(k+2)$

43–48 Write out the first few terms as well as the last few terms of the sum. Find a way to simplify and use your observation to evaluate the sum. (Sums of this type are called *collapsing sums.*)

43. $\sum_{i=1}^{10}\left(\frac{1}{i}-\frac{1}{i+1}\right)$

44. $\sum_{k=3}^{n}\left[\frac{1}{k^3}-\frac{1}{(k+1)^3}\right]$

45. $\sum_{j=1}^{n}\left(\sqrt{j}-\sqrt{j+1}\right)$

46. $\sum_{k=1}^{n+1}\ln\frac{k}{k+1}$

47. $\sum_{j=2}^{n+3}\left(e^j-e^{j+1}\right)$

48. $\sum_{k=1}^{2n+1}\left[\sin k\pi-\sin((k+1)\pi)\right]$

49–52 A *geometric sum* (or *geometric progression*) is a sum of the following form.

$$a+ar+ar^2+\cdots+ar^n=\sum_{i=0}^{n}ar^i, \quad r\neq 1$$

(Notice that each term is a constant multiple of the preceding term; this constant is called the *common ratio* and is denoted by *r*.)

Use the formula $\sum_{i=0}^{n}ar^i = a\frac{1-r^{n+1}}{1-r}$ to evaluate the sum.

49. $\sum_{i=0}^{10}3^i$

50. $\sum_{j=0}^{8}5\left(\frac{1}{2}\right)^j$

51. $\sum_{k=0}^{99}(-1)^k\left(\frac{2}{3}\right)^k$

52. $\sum_{n=0}^{1000}4.9(-3.9)^n$

53. Prove the formula for the sum of the first $n+1$ terms of a geometric progression given in the directions preceding Exercises 49–52. (**Hint:** Let S denote the sum, recognize $S-rS$ as a collapsing sum, evaluate, and solve for S.)

54–57 Sometimes, sums become easier to manage (the general term becomes simpler) after an appropriate shift in the index. For example, $\sum_{i=1}^{n+1}(i-1)^2$ can be rewritten as $\sum_{i=0}^{n} i^2$. Perform an appropriate shift in the indexing of the given sum to simplify its general term.

54. $\sum_{i=5}^{25} 2(i-4)^3$

55. $\sum_{j=3}^{n} \frac{1}{j-2}$

56. $\sum_{k=0}^{n} n(k-3)^2$

57. $\sum_{l=4}^{20} \cos((2l+2)\pi)$

58–63 Follow the lead of Examples 2 and 4 in using the limit process to find the area under the graph of $f(x)$ and above the x–axis on the given interval. (In Exercises 58 and 59, use a formula from geometry to check your answer.)

58. $f(x)=\frac{1}{2}x+1$ on $[0,4]$

59. $f(x)=5-x$ on $[1,3]$

60. $f(x)=x^2$ on $[1,2]$

61. $f(x)=x-x^3$ on $[0,1]$

62. $f(x)=x^2+3x$ on $[-2,2]$

63. $f(x)=(1-x^2)(1+x)$ on $[0,1]$

64–67 Identify the region whose area is the given limit. Do not evaluate the limit. (**Hint:** For guidance, see Example 4 and Exercises 58–63.)

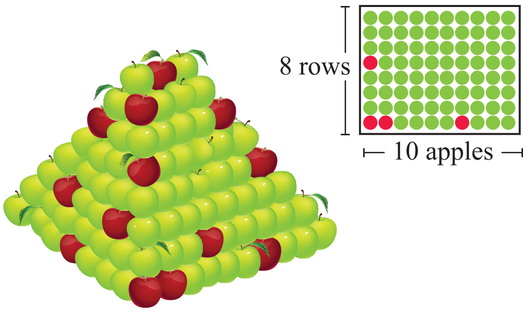
64. $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{i}{n}\right)^2$

65. $\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3+\frac{6i}{n}\right]$

66. $\lim_{n\to\infty}\frac{2}{n}\sum_{i=0}^{n-1}\sqrt{1+\frac{2i}{n}}$

67. $\lim_{n\to\infty}\frac{\pi}{2n}\sum_{i=0}^{n-1}\sin\frac{\pi i}{2n}$

68. A fruit vendor stacks apples in a rectangular, pyramid-like pile. If the foundational layer consists of 8 rows of 10 apples, and the top layer is a single row of apples, find how many apples are in the stack. Generalize to the case of an $m \times n$ bottom layer of fruit.



69. In statistics, the standard deviation of a data set $x_1, x_2, ..., x_n$ is defined to be the square root of the average of the squares of deviations of the data from their mean $\bar{x}$.

$$s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}}$$

Rewrite the definition of s using sigma notation, and use summation facts to derive the following formula for the variance of the data set, which is the square of the standard deviation.

$$s^2=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\bar{x}^2$$

70. Find the distance covered by the pebble in Exercise 79 of Section 4.7 from $t=2$ seconds to $t=5$ seconds. (**Hint:** Find the velocity function first.)

71. Assuming constant acceleration, use the method of Exercise 70 to find the distance covered by the Bugatti in Exercise 91 of Section 4.7 from $t=1$ second to $t=3$ seconds.

72. Assuming constant deceleration, use the method of Example 2 to find the distance covered by the braking race car in Exercise 90 of Section 4.7 from $t=1$ second to $t=2$ seconds. (See the hint given in Exercise 70.)

73. The velocity function of a moving object is given by $v(t)=9-0.5t^2$ m/s from $t=0$ s to $t=3$ s. Find the distance covered by the object during this time.

74. Repeat Exercise 73 for $v(t)=4-0.5t^3$ on the interval $[0,2]$.

75. Use geometry to show that the shaded area under the curve $s(t)$ in Example 2 is 128 square units. (**Hint:** Divide the region into a rectangle and a right triangle, and use well-known area formulas. Alternatively, you may want to use the area formula for a trapezoid.)

76. Show that you can obtain the same answer in Example 2 by evaluating $\lim_{n\to\infty} U_n$, that is, by choosing $t_i^* = t_{i-1}$ for every index value i.

77. Show that you can obtain the same answer in Example 4 by evaluating $\lim_{n\to\infty} O_n$, that is, by choosing $x_i^* = x_i$ for every index value i.

78. Show that you can obtain the same answer in Example 2 by choosing t_i^* to be the midpoint of the i^{th} interval for every index value i.

79. Use an elementary argument to prove the following summation formula.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(**Hint:** Letting $S = \sum_{i=1}^{n} i$, add to S its terms in "reverse order"; that is, calculate $2S$ as $2S = \sum_{i=1}^{n} i + \sum_{j=0}^{n-1}(n-j)$, and notice that, after rearranging terms, this latter sum equals $(1+n)+(2+(n-1))+(3+(n-2))+\cdots$
$= (n+1)+(n+1)+(n+1)+\cdots$.

Use this observation to complete the argument. Note that this argument is attributed to C. F. Gauss, who discovered it as a barely nine-year-old elementary school student.)

80.* Use mathematical induction to establish the summation formula of Exercise 79.

81.* Use mathematical induction to establish the following summation formula.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

82.* Use mathematical induction to establish the following summation formula.

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

83.* Prove the summation formula of Exercise 82 by making use of the following identity.

$$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$$

84.* Inscribe a regular n-gon in a circle of radius r. Use radii to divide the n-gon into n isosceles triangles, and add the areas of the triangles to obtain the area of the inscribed n-gon. Finally, let $n \to \infty$ to obtain the area formula for the circle.

85–87 *Double summations* are important in many areas of mathematics, statistics, computer science, and the sciences in general. They have the form $\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}$.

Evaluate the given double sum.

85. $\sum_{i=1}^{4}\sum_{j=1}^{5}(i+j)$

86. $\sum_{i=1}^{5}\sum_{j=1}^{6} ij$

87. $\sum_{i=1}^{n}\sum_{j=1}^{m} ij$

5.1 Technology Exercises

88–91 Use a graphing utility to express the area under the graph of $f(x)$ and above the x-axis on the indicated interval as a limit. Then use technology to evaluate the limit to find the area.

88. $f(x) = x^6$ on $[0,1]$

89. $f(x) = \sin x$ on $[0,\pi]$

90. $f(x) = e^x$ on $[1,2]$

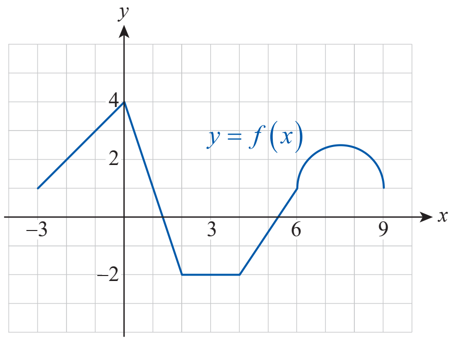
91. $f(x) = x + \cos^2(\pi x)$ on $\left[0, \frac{1}{2}\right]$

5.2 Exercises

1. Use the given graph along with appropriate formulas from geometry to evaluate each of the indicated definite integrals. (Note that the graph of f consists of linear pieces and a semicircle.)

   a. $\int_{-3}^{1} f(x)\,dx$ **b.** $\int_{-3}^{9} f(x)\,dx$

   c. $\int_{0}^{6} |f(x)|\,dx$ **d.** $\int_{0}^{9} [f(x)-2]\,dx$



2. Calculate the total distance traveled by the stone of Example 1 by evaluating $\int_0^4 |v(t)|\,dt$.

3. Suppose that in Example 1, George shoots the stone upward while standing near the edge of a deep canyon, and this time pulls the slingshot a bit harder, achieving a velocity function of $v(t) = 80 - 32t$ ft/s. What is the height of the stone, relative to its initial height, at $t = 4$ seconds? How about at $t = 6$ seconds? 10 seconds?

4–13 Use the given partition and sample points to approximate the definite integral of $f(x)$ on the indicated interval. (Note that the subintervals do not always have to be of equal width, and the sample points may be unevenly spaced.)

4. $f(x) = \frac{1}{3}x + 1, \quad x_0 = 0 < 1 < 2 < 3 < 4 < 5 < 6 = x_6, \quad x_i^* = x_i$

5. $f(x) = x^2 + x + 2, \quad x_0 = -1 < 0 < 1 < 2 < 3 = x_4, \quad x_i^* = x_{i-1}$

6. $f(x) = -x - \frac{3}{2}, \quad x_0 = -2 < -1.5 < -0.9 < 0 < 1 = x_4, \quad x_1^* = -1.8,\ x_2^* = -1,\ x_3^* = -0.4,\ x_4^* = 0.5$

7. $f(x) = \frac{1}{x^2}, \quad x_0 = 1 < 2 < 3 < 4 = x_3, \quad x_i^* = \frac{x_{i-1} + x_i}{2}$

8. $f(x) = \frac{1}{1+x^2}, \quad x_0 = -3 < -2 < -1 < 0 < 1 < 2 < 3 = x_6, \quad x_i^* = \frac{x_{i-1} + x_i}{2}$

9. $f(x) = x^3 - x, \quad x_0 = 0 < 0.3 < 0.5 < 1 < 1.5 = x_4, \quad x_1^* = 0.25,\ x_2^* = 0.5,\ x_3^* = 1,\ x_4^* = 1.2$

10. $f(x) = \sin x, \quad x_0 = 0 < \frac{\pi}{6} < \frac{\pi}{4} < \frac{\pi}{3} < \frac{\pi}{2} < \frac{2\pi}{3} < \frac{3\pi}{4} < \frac{5\pi}{6} < \pi = x_8, \quad x_i^* = x_{i-1}$

11. $f(x) = \ln(x+1), \quad x_0 = -0.5 < 1 < 2 < 2.5 = x_3, \quad x_1^* = 0,\ x_2^* = e - 1,\ x_3^* = 2$

12. $f(x) = 10^{-x}, \quad x_0 = 0 < 0.05 < 0.15 < 1 = x_3, \quad x_1^* = 0.01,\ x_2^* = 0.1,\ x_3^* = 1$

13. $f(x) = \sqrt{x}, \quad x_0 = 0 < \frac{1}{25} < \frac{4}{25} < \frac{9}{25} < \frac{16}{25} < 1 = x_5, \quad x_i^* = x_i$

14–27 Use the concept of the definite integral to find the total area between the graph of $f(x)$ and the x–axis, by taking limits of the associated Riemann sums. When setting up the Riemann sums, make your choice between the left-endpoint, right-endpoint, and midpoint strategies. (**Hint:** Extra care is needed on those intervals where $f(x) < 0$. Remember that the definite integral represents a signed area.)

14. $f(x) = 2x + 4$ on $[0, 2]$

15. $f(x) = x - 1$ **on** $[0, 5]$

16. $f(x) = \frac{3 - x}{2}$ on $[0, 5]$

17. $f(x) = x^2$ **on** $[1, 3]$

18. $f(x)=x^2-1$ on $[-1,1]$

19. $f(x)=x^2-4x$ on $[0,5]$

20. $f(x)=\frac{x^2}{2}+2$ on $[-2,2]$

21. $f(x)=3x^2-3$ on $[-1,1]$

22. $f(x)=x^2-2x-3$ on $[-1,4]$

23. $f(x)=x^3$ on $[0,1]$

24. $f(x)=4x^3-32$ on $[0,2]$

25. $f(x)=x^3+3x^2+1$ on $[0,3]$

26. $f(x)=\begin{cases}1-(x-1)^2 & \text{if } 0\le x\le 3\\ x-6 & \text{if } 3\le x\le 4\end{cases}$

27. $f(x)=\begin{cases}x^3 & \text{if } 0\le x\le 2\\ 8x-2x^2 & \text{if } 2\le x\le 4\end{cases}$

28. Generalize Exercise 13 to n subintervals and find the definite integral $\int_0^1 \sqrt{x}\,dx$ by letting $n\to\infty$. (**Hint:** Let $x_i^*=i^2/n^2$.)

29. Use the same approach as in Exercise 28 to find $\int_0^2 \sqrt[3]{x}\,dx$. (**Hint:** Let $x_i^*=2i^3/n^3$.)

30–33 Express the integral as a limit of Riemann sums. (Do not attempt to evaluate the limit.)

30. $\int_1^3 \frac{1}{x}\,dx$

31. $\int_0^4 \left(x^2-\log_2 x\right)dx$

32. $\int_{-a}^{a} \frac{1}{x^2+1}\,dx$

33. $\int_2^b \sqrt[4]{x}\,dx$

34–45 Sketch the region whose (signed) area is represented by the definite integral, and then use appropriate formulas from geometry to evaluate the integral.

34. $\int_{-1}^{5} 3\,dx$

35. $\int_{2.5}^{12} (-2)\,dx$

36. $\int_4^2 (1-x)\,dx$

37. $\int_8^3 \left(4-\frac{1}{2}x\right)dx$

38. $\int_0^4 |2x-3|\,dx$

39. $\int_{-1}^{5} \left(5-|2x|\right)dx$

40. $\int_0^{10} \left(|x-2|-|7-x|\right)dx$

41. $\int_{-5}^{0} \sqrt{25-x^2}\,dx$

42. $\int_{-a/2}^{a} \sqrt{a^2-x^2}\,dx, \quad a>0$

43. $\int_{-2}^{5} \left(2-[\![x]\!]\right)dx$

44. $\int_{-3}^{8} [\![3x-1]\!]\,dx$

45. $\int_{-4}^{6} \left(x-[\![x]\!]\right)dx$

46. Use Riemann sums resulting from midpoint estimates to prove $\int_a^b x\,dx=\left(b^2-a^2\right)/2$. (**Hint:** Notice that after using $(b+a)(b-a)=b^2-a^2$, each Riemann sum becomes a collapsing sum.)

47. Provide an alternate proof for Exercise 46 by making a sketch and using areas of triangles.

48.* Mimic the argument used in Exercise 46, but using $x_i^*=\sqrt{\left(x_{i-1}^2+x_{i-1}x_i+x_i^2\right)/3}$, to prove the formula $\int_a^b x^2\,dx=\left(b^3-a^3\right)/3$.

49. The Dirichlet function is defined as follows.

$$\xi(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

Prove that $\xi(x)$ is not integrable. (**Hint:** For a given n, form a Riemann sum by choosing each sample point x_i^* to be rational, then see what happens if each x_i^* is irrational. Use your observation to argue that $\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$ does not exist.)

50. Prove that the function $f(x)=\begin{cases}1/x^2 & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ is not integrable on $[0,1]$.

(**Hint:** By examining the first term of each R_n, show that $\lim_{n\to\infty} R_n$ does not exist.)

51. Repeat Exercise 50 for $g(x)=\begin{cases}1/x & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ on $[0,1]$.

(**Hint:** Show that arbitrarily large Riemann sums can be constructed by choosing appropriate x_i^*'s.)

52–59 Decide whether the function is integrable on the indicated interval. If not, say why. (Do not evaluate the integral.)

52. $f(x)=\frac{1}{\sqrt{x+2}}$ on $[-1,1]$

53. $g(x)=\frac{2}{x}$ on $[-2,2]$

54. $h(x)=\frac{-3}{x-1}$ on $[0,5]$

55. $F(x)=\frac{x}{|x|}$ on $[-3,4]$

56. $G(x)=x\cdot[\![x]\!]$ on $[-2,2]$

57. $H(x)=\begin{cases}\frac{\sin x}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ on $[-1,1]$

58. $u(x)=\begin{cases}\cos\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ on $[-1,1]$

59. $v(x)=\begin{cases}3.14 & \text{if } x \text{ is rational}\\ \pi & \text{if } x \text{ is irrational}\end{cases}$ on $[0,2]$

60–65 Match the given property of the definite integral to the relevant illustration (labeled A–F).

60. $\int_a^b k\,dx=k(b-a)$ (Property 3)

61. $\int_a^b kf(x)\,dx=k\int_a^b f(x)\,dx$ (Property 4)

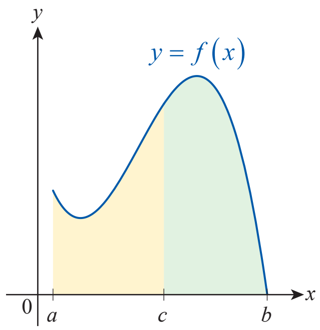
62. $\int_a^b [f(x)\pm g(x)]\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx$ (Property 5)

63. $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx$ (Property 6)

64. If $f(x)\leq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx\leq\int_a^b g(x)\,dx$. (Property 7)

65. If $m=\min\limits_{a\leq x\leq b} f(x)$ and $M=\max\limits_{a\leq x\leq b} f(x)$, then $m(b-a)\leq\int_a^b f(x)\,dx\leq M(b-a)$. (Property 8)

A.



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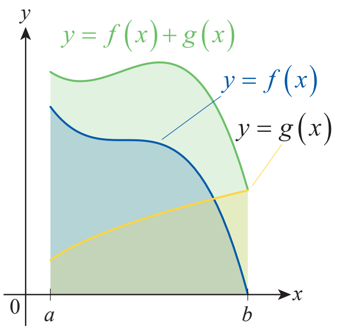

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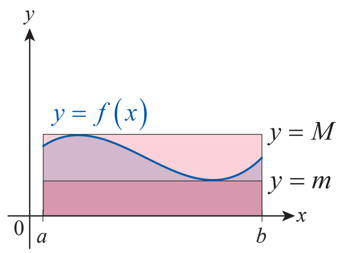

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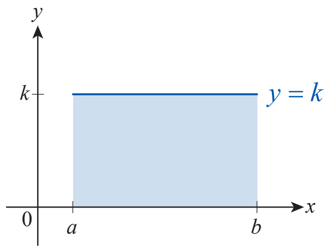

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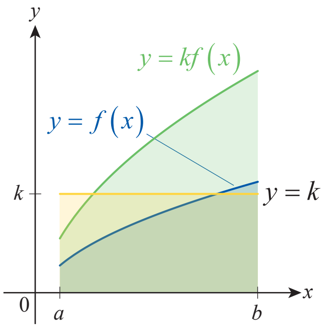
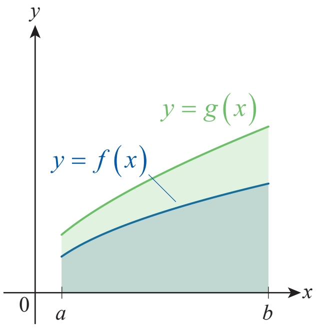


F.



66–75 Use the properties of the definite integral to find the given integral, if possible, given that $\int_a^b f(x)\,dx=3$, $\int_c^b f(x)\,dx=-1$, and $\int_a^b g(x)\,dx=-5$.

66. $\int_a^b [f(x)-g(x)]\,dx$

67. $\int_a^c [2f(x)+1]\,dx$

68. $\int_c^a 10f(x)\,dx$

69. $\int_a^a f(x)g(x)\,dx$

70. $\int_a^b \left[4f(x)+\frac{g(x)}{10}\right]dx$

71. $\int_b^a \frac{\sqrt{2}}{2}g(x)\,dx$

72. $\int_a^b [f(x)+2g(x)-2]\,dx$

73. $\int_a^b [f(x)]^2\,dx$

74. $\int_a^b \frac{5}{g(x)}\,dx$

75. $\int_a^b \left[\frac{f(x)}{3}-\pi g(x)\right]dx$

76–83 Use the results from Exercises 46 and 48, along with the properties of the definite integral and formulas from geometry, to evaluate the given integral.

76. $\int_0^2 (3x-1)\,dx$

77. $\int_{\sqrt{2}}^{-1}\left(1-\frac{\sqrt{2}}{2}x\right)dx$

78. $\int_{-1}^{4}(x^2+5)\,dx$

79. $\int_1^4 (2x^2-x)\,dx$

80. $\int_0^3\left(t^2+\frac{t}{4}+4\right)dt$

81. $\int_0^1 (2\sqrt{x}+x)\,dx$

82. $\int_2^0\left(\frac{\sqrt[3]{x}}{4}-x^2\right)dx$

83. $\int_{-2}^{2}(u-3\sqrt[3]{u})\,du$

84. Suppose that f is an even function, g is odd, and both are integrable on $[-a,a]$. Use the properties of the definite integral to prove the following statements.
$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ and $\int_{-a}^{a} g(x)\,dx = 0$

85–90 Suppose that f is an even function, g is odd, both are integrable on $[-2,2]$, and we know that $\int_0^2 f(x)\,dx = 1$, while $\int_0^2 g(x)\,dx = 2.5$. If possible, find the integral.

85. $\int_{-2}^{2}[f(x)+g(x)]\,dx$

86. $\int_{-2}^{2}[2f(x)-3g(x)]\,dx$

87. $\int_{-2}^{2} g(x)\,dx$

88. $\int_{-2}^{2} f(x)g(x)\,dx$

89. $\int_{-1}^{1}[f(x)]^2\,dx$

90. $\int_0^2 |g(x)|\,dx$

91. Use Property 8 of the definite integral to prove the validity of the following upper and lower estimates: $12 \le \int_0^4 \sqrt{x^2+9}\,dx \le 20.$

92–96 Use an argument similar to the one you gave in Exercise 91 to give upper and lower estimates for the given definite integral.

92. $\int_{-1}^{4}\sqrt{5+x}\,dx$

93. $\int_2^3 \sqrt{3-x}\,dx$

94. $\int_4^5 \frac{1}{x-2}\,dx$

95. $\int_0^6\left(\frac{x^2}{32}-\frac{x}{4}+\frac{3}{2}\right)dx$

96. $\int_1^{\sqrt{3}} \arctan x\,dx$

97. Use Property 7 of the definite integral to prove the following inequalities.

a. $\int_0^1 \sqrt{1-x}\,dx \le \int_0^1 \sqrt{1-x^2}\,dx$

b. $\int_0^{\pi/2} \cos x\,dx \le \int_0^{\pi/2} \frac{\sin x}{x}\,dx$

98. Prove Property 4 of the definite integral. (**Hint:** Write a typical Riemann sum for f on $[a,b]$; use the Constant Multiple Rule for Finite Sums, followed by properties of limits.)

99. Prove Property 6 of the definite integral in general; that is, prove that the property $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ holds irrespective of the order of the points a, b, and c. (**Hint:** The standard case of $a < c < b$ is discussed in the text. To start you off with the remaining cases, assume, for example, that $a < b < c$. By an argument analogous to the one given in the text, we see that $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$ Observe by Property 2 that $\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx,$ and rearrange the terms. Handle the remaining cases in a similar fashion.)

100. Prove Property 7 of the definite integral. (**Hint:** For a particular partition of $[a,b]$ and choice of sample points, argue that $\sum_{i=1}^{n} f(x_i^*)\Delta x \le \sum_{i=1}^{n} g(x_i^*)\Delta x,$ and take the limits as $n \to \infty$.)

101. Use Property 7 to prove that the definite integral of a nonnegative function is nonnegative: If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0.$ Then state and prove the analogous statement for nonpositive functions.

102. Prove Property 8 of the definite integral. (**Hint:** Use Property 7 with the constant function $g(x) = M$. The other inequality can be handled in a similar manner.)

103. Use Properties 4 and 7 to prove the following statement: If f is integrable on $[a,b]$, then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$ (**Hint:** Let $k = 1$ or $k = -1$ so that $\left|\int_a^b f(x)\,dx\right| = k\cdot\int_a^b f(x)\,dx.$ Use the fact that $k\cdot f(x) \le |f(x)|,$ along with Properties 4 and 7.)

104–115 Find the average value of the function over the given interval. (**Hint:** Instead of using Riemann sums, try using the results from Exercises 46 and 48 along with formulas from geometry and the properties of the definite integral.)

104. $f(x) = 3x-1$ on $[0,4]$

105. $g(x) = -1-\frac{1}{2}x$ on $[-2,2]$

106. $h(x) = x^2-2$ on $[-1,5]$

107. $F(x) = -3x^2+7x+12$ on $[-2,3]$

108. $G(x) = 9x-x^3$ on $[-4,4]$

109. $H(x) = x^3 - 2x^2 - 1$ on $[0,2]$

110. $k(x) = |x-4| - 2$ on $[0,7]$

111. $m(x) = |x| + |x+1|$ on $[-3,2]$

112. $u(x) = \sqrt{1-(x-1)^2}$ on $[0,2]$

113. $v(x) = [\![x]\!]$ on $\left[\frac{1}{2}, 3\right]$

114. $t(x) = \sqrt{x} - 1$ on $[0,5]$

115. $w(x) = \sqrt[3]{x} - x$ on $[-1,8]$

116–119 Recognize the given limit as a Riemann sum of a function over an interval and then use geometry to evaluate it.

116. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(2 - \frac{i}{n}\right)$

117. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{2i}{3n} + 4\right)$

118. $\lim_{n\to\infty} \sum_{i=1}^{n-1} \left(\frac{2}{n} + \frac{4i}{n^2}\right)$

119. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{4 - \left(\frac{2i}{n}\right)^2}$

120. Prove that if $f(x)$ is an increasing nonnegative function on $[a,b]$, then for every n, $L_n \le \int_a^b f(x)\,dx \le R_n$. Then state and prove the analogous statement for a decreasing function $g(x)$ on the same interval.

121. Prove that L_n corresponding to $f(x)$ of Exercise 120 is increasing as $n \to \infty$, while R_n is decreasing. Then state and prove the analogous statement for $g(x)$.

122. Use geometry and a fundamental trigonometric identity to find $\int_0^{\pi} \sin^2 x\,dx$. (**Hint:** Start out by comparing the given integral with $\int_0^{\pi} \cos^2 x\,dx$.)

123. Use the result of Exercise 122 to evaluate $\int_0^{\pi} \left(2\sin^2 x + x^2 - 3x\right)dx$.

124.* Suppose that the nonnegative function $R(x)$ has the property that $R(x) = 0$ whenever x is rational. If R is integrable on the interval $[a,b]$, prove that $\int_a^b R(x)\,dx = 0$.

Concept Check

125–130 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

125. If f and g are both integrable on $[a,b]$, then $\int_a^b f(x)\cdot g(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$.

126. The integral $\int_a^b f(x)\,dx$ is numerically equal to the area between the graph of $f(x)$ and the x-axis.

127. A Riemann sum for $f(x)$ on $[a,b]$ can be based upon a division of $[a,b]$ into subintervals of unequal width.

128. If $|f(x)|$ is integrable on $[a,b]$, then so is $f(x)$.

129. If $f(x)$ is positive and increasing on $[a,b]$, then $\int_a^b f(x)\,dx \ge f(a)(b-a)$.

130. If $\int_a^b f(x)\,dx < 0$, then $f(x) \le 0$ on $[a,b]$.

Example 7 Using the Fundamental Theorem of Calculus, Part II, to Evaluate a Definite Integral

Recall that the difference between the functions F and G in Example 4 was determined to be $\int_5^{10} \frac{x-2}{x+1}\,dx$. Evaluate this integral.

Solution

As always, an antiderivative of the integrand will make the evaluation of the integral an easy task. In this case, though, it will take a bit more thought to arrive at an antiderivative—it's not immediately clear what sort of function has a derivative of $(x-2)/(x+1)$.

We will learn many techniques for systematically developing antiderivatives in coming sections, but in this case rewriting the integrand as follows will suffice.

$$\frac{x-2}{x+1} = \frac{(x+1)-3}{x+1} = 1 - \frac{3}{x+1}$$

The same result can be obtained by dividing $x-2$ by $x+1$.

$$\begin{array}{r} 1 \\ x+1 \,\big)\, \overline{x-2} \\ \underline{-(x+1)} \\ -3 \end{array}$$

$$\frac{x-2}{x+1} = 1 - \frac{3}{x+1}$$

Also, since $\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}$ (you should verify this),

$$\begin{aligned} \int_5^{10} \frac{x-2}{x+1}\,dx &= \int_5^{10}\left(1 - \frac{3}{x+1}\right)dx \\ &= \left[x - 3\ln(x+1)\right]_5^{10} \\ &= (10 - 3\ln 11) - (5 - 3\ln 6) \\ &= 5 - 3\ln 11 + 3\ln 6 \\ &= 5 + 3\ln\frac{6}{11}. \end{aligned}$$

5.3 Exercises

1–8 Find every point c in the given interval at which $f(x)$ takes on its average value.

1. $f(x) = x^3;\quad [0,2]$

2. $f(x) = \frac{x(6-x)}{2};\quad [0,6]$

3. $f(x) = \frac{-x^4}{4} + 4;\quad [-2,2]$

4. $f(x) = e^x;\quad [0,1]$

5. $f(x) = \sin x;\quad [0,\pi]$

6. $f(x) = x - \sqrt{x+1};\quad [0,8]$

7. $f(x) = \csc^2 x;\quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

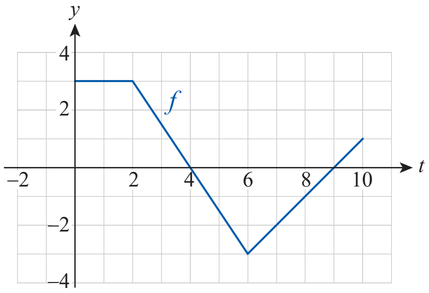
8. $f(x) = \frac{x^2+2}{x^2};\quad [1,3]$

9–10 Let $F(x)=\int_0^x f(t)\,dt$. Use the graph of f to answer the questions. (Note that the graph in Exercise 10 consists of linear and parabolic pieces.)

9. **a.** Evaluate $F(2), F(4), F(6), F(8)$, and $F(10)$.

b. Give a formula for $F(x)$. (**Hint:** It will be a piecewise-defined function.)

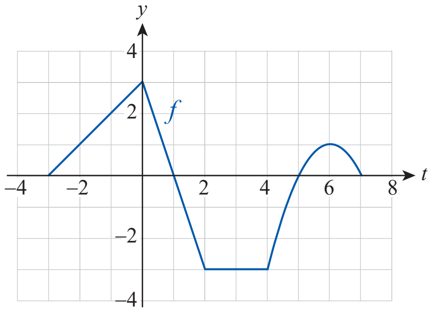
c. Sketch the graph of $F(x)$.



10. **a.** Evaluate $F(0), F(2), F(4)$, and $F(7)$.

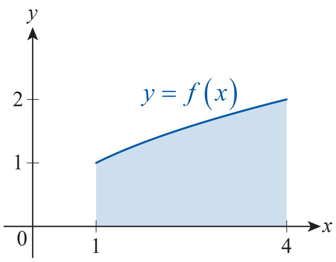
b. Give a formula for $F(x)$. (**Hint:** It will be a piecewise-defined function.)

c. Sketch the graph of $F(x)$.

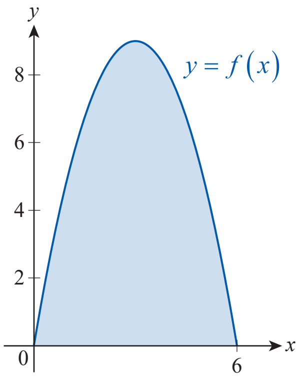


11–16 Find the area between the graph of $f(x)$ and the x-axis on the indicated interval.

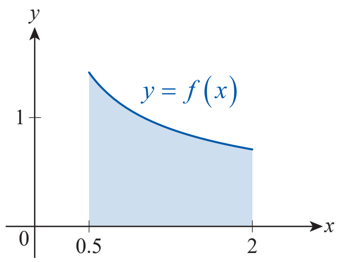
11. $f(x)=\sqrt{x}$ on $[1,4]$



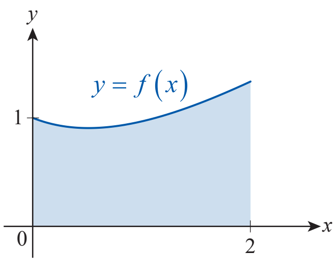
12. $f(x)=6x-x^2$ on $[0,6]$



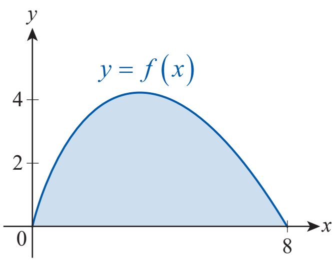
13. $f(x)=\dfrac{1}{\sqrt{x}}$ on $[0.5,2]$



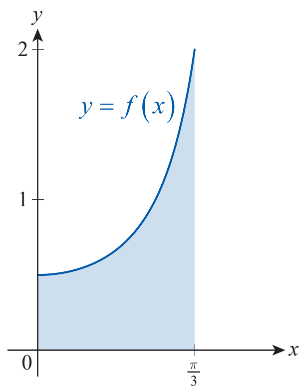
14. $f(x)=e^{-x}+0.6x$ on $[0,2]$



15. $f(x)=-2.5x^{4/3}+5x$ on $[0,8]$



16. $f(x)=0.5\sec^2 x$ on $\left[0,\dfrac{\pi}{3}\right]$



17–32 Use Part I of the Fundamental Theorem of Calculus to find the derivative of the given function.

17. $F(x)=\int_0^x \frac{1}{3}\left(t^2+\sqrt{t}\right)dt$

18. $F(x)=\int_{1/2}^x \ln s\, ds$

19. $G(x)=\int_{-4}^x \frac{t^4}{t^4+4}dt$

20. $G(x)=\int_2^x \sqrt[3]{u^2-u}\, du$

21. $y=\int_x^1 \sin\sqrt{t+1}\, dt$

22. $y=\int_x^0 t\arccos t\, dt$

23. $y=\int_{-5}^{3x}\left(t^2+3\right)e^{t-2}\, dt$

24. $y=\int_0^{x^2}\sec^{2/3}\sqrt{t}\, dt$

25. $y=\int_0^{\sin x}\left(t^2+e^t\right)dt$

26. $y=\int_{\sqrt{x}}^1 \log t\, dt$

27. $y=\int_x^{\pi x}\sin t\, dt$

28. $y=\int_{\sqrt{x}}^{x^2}\cos\left(z^2\right)dz$

29. $F(x)=\int_0^{\cos^{-1}x}\sqrt{1+\sqrt{1+\sec^2 t}}\, dt$

30. $G(x)=\int_{x-c}^{x+c}\sin t\, dt$

31. $H(x)=\int_{\ln x}^x \ln t\, dt$

32. $K(x)=\int_x^{x^2}\sqrt{1+t^4}\, dt$

33–38 Find a formula for $F(x)$ that is free of the integral symbol. Then differentiate it to verify Part I of the Fundamental Theorem of Calculus.

33. $F(x)=\int_1^x 2\, dt$

34. $F(x)=\int_{-3}^x (5-t)\, dt$

35. $F(x)=\int_x^1 \left(t^2+t\right)dt$

36. $F(x)=\int_x^8 \frac{w+2}{\sqrt[3]{w}}dw$

37. $F(x)=\int_1^{\sqrt{x}} \frac{1}{s^2}ds$

38. $F(x)=\int_0^{\tan x}\left(1+u^2\right)du$

39–65 Use Part II of the Fundamental Theorem of Calculus to evaluate the definite integral.

39. $\int_{-2}^4 (-5)\, dx$

40. $\int_0^{1/\pi} 3\pi^2\, dx$

41. $\int_2^9 (4x+3)\, dx$

42. $\int_{-2.5}^6 (1-5u)\, du$

43. $\int_{-2}^4 \left(1.5x^2-x+3\right)dx$

44. $\int_0^3 (5s-1)(2+s)\, ds$

45. $\int_1^7 \left(2.4x^3-4x^2+1\right)dx$

46. $\int_{-1}^1 \left(2x^2+1\right)^2 dx$

47. $\int_1^2 \left(1-\frac{2}{x}\right)dx$

48. $\int_1^3 \left(\frac{1}{x^2}+\frac{2}{x}+3\right)dx$

49. $\int_{-2}^{-1} \frac{2x^5-4x^2}{x^3}dx$

50. $\int_0^3 \frac{2x^2-\sqrt{x}}{4}dx$

51. $\int_1^2 \left(x\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$

52. $\int_2^4 \frac{5x^2-3x+2}{\sqrt{x}}dx$

53. $\int_{1/8}^1 \left(2\sqrt[3]{t}-\sqrt[3]{\frac{2}{t}}\right)dt$

54. $\int_0^1 \frac{x+3\sqrt{x}}{\sqrt[5]{x}}dx$

55. $\int_0^{\pi/2} \left(\frac{\sin x}{2}-\sqrt{x}\right)dx$

56. $\int_0^{\pi/3} \frac{2}{\cos^2\theta}d\theta$

57. $\int_{\sqrt{2}/2}^1 \frac{3}{\sqrt{1-t^2}}dt$

58. $\int_{\sqrt{3}/3}^1 \frac{-5}{1+x^2}dx$

59. $\int_0^{\pi/3} \left(e^x+\sec x\tan x\right)dx$

60. $\int_{-3}^3 2^x\, dx$

61. $\int_{-2}^4 \left|x(x-2)\right|dx$

62. $\int_1^3 f(x)\, dx$, where $f(x)=\begin{cases}\sin\frac{\pi x}{4} & \text{if } 0<x\le 2\\ (x-2)^2+1 & \text{if } 2<x\le 3\end{cases}$

63. $\int_{\pi/4}^{3\pi/4} (1-\csc\theta\cot\theta)\, d\theta$

64. $\int_{\pi/4}^{\pi/2} \frac{2}{1-\cos^2 x}dx$

65. $\int_{-1}^1 g(x)\, dx$, where $g(x)=\begin{cases}\sqrt{x+1} & \text{if } -1<x\le 0\\ e^x & \text{if } 0<x\le 1\end{cases}$

66–69 Recognize the given limit as a Riemann sum of a function over an interval and then use the Fundamental Theorem of Calculus to evaluate it.

66. $\lim_{n\to\infty}\sum_{i=1}^n \frac{\sqrt{i}}{n^{3/2}}$

67. $\lim_{n\to\infty}\sum_{i=1}^n \frac{2}{n}\left(\frac{2i}{n}\right)^4$

68. $\lim_{n\to\infty}\sum_{i=1}^n \frac{\pi}{2n}\cos\frac{\pi i}{2n}$

69. $\lim_{n\to\infty}\sum_{i=1}^{n-1} \frac{e-1}{n+i(e-1)}$

70–78 Find the area of the region between the graph of the given function and the x-axis on the indicated interval.

70. $y = \cos x$ on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

71. $y = -x^3$ on $[-2,2]$

72. $y = -x^2 + 1$ on $[-1,2]$

73. $y = \frac{1}{x}$ on $\left[\frac{1}{e}, e^2\right]$

74. $y = -2|x-3| + 6$ on $[0,10]$

75. $y = -x^3 + 7x^2 - 10x$ on $[0,6]$

76. $y = 2\sqrt{x} - x$ on $[0,9]$

77. $y = \frac{1-2x}{2x+1}$ on $[0,1]$

78. $y = x^4 - x^2$ on $[-1,1]$

79–87 Use the method of Example 7 to evaluate the definite integral.

79. $\int_3^4 \frac{x}{x-2}\,dx$

80. $\int_5^7 \frac{x+5}{x-4}\,dx$

81. $\int_0^{e-1} \frac{2x-5}{x+1}\,dx$

82. $\int_{-3/2}^0 \frac{3x-1}{2x+4}\,dx$

83. $\int_0^1 \frac{3x^2+4}{x^2+1}\,dx$

84. $\int_0^2 \frac{5x^2-1}{2x^2+4}\,dx$

85. $\int_4^6 \frac{3x^2+2x-9}{x^2-3}\,dx$

86. $\int_0^2 \frac{2x^2+4x+11}{x^2+x+5}\,dx$

87. $\int_{-1}^1 \frac{x^3+5x^2+4x+1}{x^3+2x^2+1}\,dx$

88–90 The function $v(t)$ gives the velocity, in units per second, of a particle moving along the x-axis, having started from the origin. Find **a.** the position of the particle at $t = t_0$ and **b.** the total distance traveled by the particle in the time interval $[0, t_0]$.

88. $v(t) = 1 - (t-1)^2;\quad t_0 = 4$

89. $v(t) = \frac{t-1}{2(t+1)};\quad t_0 = 3$

90. $v(t) = t(t-3)(t-5);\quad t_0 = 6$

91. Find a formula for $f(x)$ if $\int_0^x f(t)\,dt = \sin 2x + x$.

92.* Repeat Exercise 91 if $\int_0^{x^2} f(t)\,dt = x^3$.

93.* Let $f(x) = x - [\![x]\!]$ and $F(x) = \int_0^x f(t)\,dt$. Prove that F is continuous, briefly discuss its graph, and sketch it on paper.

94. Show that the piecewise-defined function $F(x) = \begin{cases} -\frac{1}{2}x^2 & \text{if } x \le 0 \\ \frac{1}{2}x^2 & \text{if } x > 0 \end{cases}$ is an antiderivative of $f(x) = |x|$. Then find an easier formula for $F(x)$ and use the Fundamental Theorem of Calculus to evaluate $\int_a^b |x|\,dx$.

95. Write a paragraph entitled "Differentiation and Integration as Inverse Operations." Quote the Fundamental Theorem of Calculus and include concrete examples.

96.* Use the properties of the definite integral to show directly that if $f(x)$ is integrable on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on the same interval. (**Hint:** Argue that there is an M so that $|f(x)| \le M$ on $[a,b]$ and use the result of Exercise 103 of Section 5.2.)

97. Let $l(x)$ be defined as the integral function of $1/x$, that is, $l(x) = \int_1^x (1/t)\,dt$. Show that $l(x) = \ln x$. (**Hint:** See the discussion in Example 4.)

98.* Use the definition from Exercise 97 to show the following well-known property of logarithms: For positive $a, b \in \mathbb{R}$, $l(a \cdot b) = l(a) + l(b)$. (**Hint:** Use the definition to show that $l'(ax) = l'(x)$, which implies $l(ax) = l(x) + C$ for some constant C. Argue that $l(a) = C$. Finally, let $x = b$.)

99.* Use the definition from Exercise 97 to show that $l(1/x) = -l(x)$.

100. Taking a cue from Exercises 97–99, let

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$$

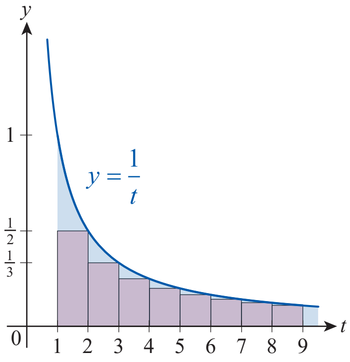
be the definition of the natural logarithm function—that is, let all our knowledge of the natural logarithm function be determined by this particular definite integral. Note that, by the Fundamental Theorem of Calculus (which applies, since $1/t$ is continuous on the interval $0 < t < \infty$), the natural logarithm is a differentiable function and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

a. Prove that $\ln 1 = 0$ and that $\lim_{x\to\infty} \ln x = \infty$. (**Hint:** Construct a Riemann sum based on the given figure to show that

$$\int_1^x \frac{1}{t}\,dt > \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots$$
$$> \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

for sufficiently larger x.)



Note that these two facts, along with the fact that $\ln x$ is a continuous function, implies that $\ln x$ takes on every positive real value over the interval $1 < x < \infty$ (and also, given the result of Exercise 99, every negative real value over the interval $0 < x < 1$).

b. Prove that $\ln x$ is one-to-one and hence has an inverse function. (**Hint:** Prove that $\ln x$ is strictly increasing.) Given this fact, define e^x to be the inverse of $\ln x$; that is, define e^x by $e^x = \ln^{-1} x$. In particular, define $e = \ln^{-1}(1)$.

c. Use L'Hôpital's Rule to prove $\lim_{u\to 0} \frac{u}{\ln(1+u)} = 1$.

d. Use the result from part c. to prove $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$. (**Hint:** Let $u = e^h - 1$ and note that $u \to 0$ as $h \to 0$.)

101. Archimedes (287–212 BC) discovered that the area under a parabolic arch is two-thirds the length of the base times its height. Sketch the graph of $y = h - ax^2$, the general parabolic arch with vertex at $(0, h)$ and use the FTC to verify Archimedes' formula. (Note the interesting parallel between Archimedes' formula and that of the area of an isosceles triangle of the same base and height.)

102. The marginal cost of production of baby toys at a small company has been determined to be $C'(x) = 200/\left(3\sqrt[3]{x}\right)$ dollars. How much will it cost to increase production from 400 to 500 toys?

Concept Check

103–108 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

103. If $f(x)$ is continuous on $[a,b]$ and $c \in [a,b]$ is the point guaranteed by the Mean Value Theorem for Definite Integrals, then $y = f(x)$ and the constant function $y = f(c)$ both have the same definite integral on $[a,b]$.

104. When evaluating a definite integral using the Fundamental Theorem of Calculus, we can use *any* of the antiderivatives of the integrand.

105. If $f(x)$ is a continuous, odd function on $\mathbb{R}$ and $F(x) = \int_{-a}^{x} f(t)\,dt$ for some $a > 0$, then $F(x)$ has a zero at $x = a$.

106. If $f(x)$ is integrable on $\mathbb{R}$, then $\int_a^x f(t)\,dt$ and $\int_b^x f(t)\,dt$ have the same derivative for all $a, b \in \mathbb{R}$.

107. $\frac{d}{dx}\int_a^{x^3} (t+1)^3\,dt = \left(x^3+1\right)^3$

108. If f is continuous on $[a,b]$ and F is any antiderivative of f, then the area of the region bounded by the graph of f and the x-axis is $F(b) - F(a)$.

c. This integral is difficult to get a handle on in its original form, but it is more tractable if we multiply the numerator and denominator by 3^x.

$$\int \frac{dx}{3^x+3^{-x}} = \int \frac{1}{3^x+3^{-x}}\left(\frac{3^x}{3^x}\right)dx \qquad \text{Multiply the integrand by 1.}$$

$$= \int \frac{3^x}{3^{2x}+1}dx \qquad \text{Set } u=3^x\text{; so } du=3^x(\ln 3)dx.$$

$$= \frac{1}{\ln 3}\int \frac{1}{u^2+1}du \qquad 3^{2x}=(3^x)^2=u^2$$

$$= \frac{1}{\ln 3}\tan^{-1}u + C$$

$$= \frac{1}{\ln 3}\tan^{-1}(3^x) + C$$

5.4 Exercises

1–15 Evaluate the integral, definite or indefinite, as indicated. (**Hint:** See Examples 1 and 2 and the subsequent table of integrals.)

1. $\int (12x^5+7.5x^4-x^3+2)dx$
2. $\int (-3x^4+0.8x^3-6x^2+4x-\pi)dx$
3. $\int 2(x+1)(5x-2)dx$
4. $\int_{-1}^{1} -x(x+4)(2x-1)dx$
5. $\int_0^1 x\sqrt[3]{x}\,dx$
6. $\int \frac{x^4-3\sqrt{x}}{x^2}dx$
7. $\int \frac{x^2-2x}{\sqrt{x}+\sqrt{2}}dx$
8. $\int \left(\sqrt[3]{x^2}+\frac{1}{\sqrt{x}}\right)^2 dx$
9. $\int (\pi \sec x - \tan x)\sec x\,dx$
10. $\int (e^{3x}+2^{2x/3})dx$
11. $\int \frac{\cot 2x}{2\sin x\cos x}dx$
12. $\int_0^{\sqrt{3}/2} \frac{2}{1+4x^2}dx$
13. $\int_0^{1/2} \frac{2}{\sqrt{1-4x^2}}dx$
14. $\int \frac{3}{|4x|\sqrt{16x^2-1}}dx$
15. $\int \frac{-7}{\sqrt{1-25x^2}}dx$

16–36 Perform the suggested substitution to evaluate the given indefinite integral.

16. $\int 6x(3x^2+5)^7\,dx;\quad u=3x^2+5$
17. $\int x^3\sqrt{x^4+2}\,dx;\quad u=x^4+2$
18. $\int \frac{2x}{4x^2+1}dx;\quad u=4x^2+1$
19. $\int \frac{2}{4x^2+1}dx;\quad u=2x$
20. $\int 4e^{2t+3}dt;\quad u=2t+3$
21. $\int 4e^{2t+3}dt;\quad u=e^{2t+3}$
22. $\int \cos 5\theta\,d\theta;\quad u=5\theta$
23. $\int \frac{5^{\arctan x}}{1+x^2}dx;\quad u=\arctan x$
24. $\int \frac{\sin 2x}{\sqrt{\sin^2 x+1}}dx;\quad u=\sin^2 x+1$
25. $\int \frac{\sec^2(1+\sqrt{s})}{\sqrt{s}}ds;\quad u=1+\sqrt{s}$
26. $\int \frac{1}{3s^{2/3}\sqrt{1-s^{2/3}}}ds;\quad u=\sqrt[3]{s}$
27. $\int \frac{1}{x^2}\sec^2\frac{1}{x}dx;\quad u=\frac{1}{x}$

28. $\int \sqrt{x}\cos^3\left(x^{3/2}\right)dx;\quad u = x^{3/2}$ (**Hint:** Use $u = x^{3/2}$ and $\cos^3 u = \cos^2 u\cos u = \left(1-\sin^2 u\right)\cos u$, and then perform another substitution $w = \sin u$.)

29. $\int \sqrt{1+\cot^2 x}\cot x\csc^2 x\,dx;\quad u = 1+\cot^2 x$

30. $\int (x+1)(x-7)^8\,dx;\quad u = x-7$

31. $\int \frac{z}{2z-1}dz;\quad u = 2z-1$

32. $\int z\sqrt{z^2-5}\,dz;\quad u = z^2-5$

33. $\int x\sqrt{x-5}\,dx;\quad u = x-5$

34. $\int \frac{\left(\sqrt{x}-2\right)\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx;\quad u = 1+\sqrt{x}$

35.* $\int \sqrt{1-x^2}\,dx;\quad x = \sin u$

36. $\int \sin^4 x\cos^3 x\,dx;\quad u = \sin x$

37–84 Use an appropriate substitution to evaluate the indefinite integral.

37. $\int 3(3x-2)^7\,dx$

38. $\int -2x\sqrt{9-x^2}\,dx$

39. $\int (4x+3)\left(2x^2+3x\right)^{20}dx$

40. $\int -2x^3\sqrt{9-x^2}\,dx$

41. $\int \cot z\,dz$

42. $\int (z-2)\left(3z^2-12z\right)^{99}dz$

43. $\int x^2\left(x^3-5\right)^{19}dx$

44. $\int 6x^3\left(x^4+2\right)^{14}dx$

45. $\int e^{\sin x}\cos x\,dx$

46. $\int e^x\csc^2\left(e^x\right)dx$

47. $\int (5-s)^{37}\,ds$

48. $\int s\sqrt{s^2+1}\,ds$

49. $\int x^3\sqrt[3]{x^4+11}\,dx$

50. $\int \sqrt[3]{5x+9}\,dx$

51. $\int \frac{2x+1}{\left(x^2+x-7\right)^2}dx$

52. $\int \frac{x^2}{x^3-1}dx$

53. $\int \frac{-x^3}{\sqrt{2+x^4}}dx$

54. $\int \frac{4x^3+10x}{x^4+5x^2+6}dx$

55. $\int \left(1+\frac{1}{t}\right)^3\frac{1}{t^2}dt$

56. $\int \frac{3t+9}{\sqrt{2t^2+12t}}dt$

57. $\int \frac{2}{t\ln 2t}dt$

58. $\int \frac{\ln x}{x}dx$

59. $\int xe^{x^2-3}\,dx$

60. $\int x\sin\left(x^2\right)dx$

61. $\int \cos \pi x\,dx$

62. $\int \frac{x}{\cos^2\left(x^2\right)}dx$

63. $\int \sin^3 2t\cos 2t\,dt$

64. $\int 3x\sec^2\left(x^2+1\right)dx$

65. $\int \frac{\csc^2 v}{e^{\cot v-1}}dv$

66. $\int \frac{5^{\sqrt{3v}}}{\sqrt{3v}}dv$

67. $\int \frac{\sqrt{x}}{\left(30-x^{3/2}\right)^2}dx$

68. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

69. $\int \frac{\sin 2x}{\sin^2 x+2}dx$

70. $\int \frac{\tan\sqrt{x}\sec\sqrt{x}}{\sqrt{x}}dx$

71. $\int \cos x\cos(\sin x)\,dx$

72. $\int \frac{\ln\left(x^3\right)}{x}dx$

73. $\int \frac{\sin 2x}{1-\cos 2x}dx$

74. $\int \frac{e^{2t}}{e^t-2}dt$

75. $\int \tan^3 2x\sec 2x\,dx$

76. $\int \frac{z^3}{1-z^2}dz$

77. $\int \frac{\sqrt{1+\sqrt{w}}}{\sqrt{w}}dw$

78. $\int \sqrt{1+\sqrt{w}}\,dw$

79. $\int (x-5)(x+1)^{11}\,dx$

80. $\int 2x\sqrt{x+2}\,dx$

81. $\int \frac{x+2}{4x+1}dx$

82. $\int \frac{x^2+x+1}{x-2}dx$

83. $\int \frac{(2\ln x+5)(1-\ln x)^3}{2x}dx$

84. $\int \sqrt{2+\sqrt{x}}\,dx$

85–90 Find the function that satisfies the given conditions.

85. $\frac{df}{dx} = 4x\sqrt{4x^2+4};\quad f(0) = 1$

86. $\frac{dg}{ds} = \frac{2\ln s}{s};\quad g(1) = 5$

87. $\frac{dy}{dt} = \frac{\sin 2t}{\sin^2 t+e};\quad y(0) = 0$

88. $y'(x) = \frac{3\sqrt{x}}{\left(1-x^{3/2}\right)^2};\quad y(0) = 0$

89. $y''(x) = \cos 4x;\quad y'(0) = 1;\quad y(0) = 0$

90. $\frac{d^2y}{dt^2} = 2\cot t\csc^2 t;\quad y'\left(\frac{\pi}{2}\right) = 0;\quad y\left(\frac{\pi}{2}\right) = 0$

91. A particle that started at the origin and is moving along the x-axis has a velocity function given by

$$v(t) = \frac{1+\sqrt{t+1}}{\sqrt{t+1}} \text{ units/s}.$$

What is the particle's position at $t = 3$ seconds?

92. A particle is undergoing simple harmonic motion along the y-axis around the equilibrium $y = 0$, while its acceleration is given by

$$a(t) = 4\pi^2 \sin\frac{\pi(1+8t)}{4} \text{ units/s}^2.$$

Find the particle's position at $t = 1.5$ seconds and the total distance covered by the particle from $t = 0$ seconds to $t = 1.5$ seconds.

Concept Check

93–98 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

93. If f is defined on the interval $[a,b]$ and has an antiderivative, then the indefinite integral of f on $[a,b]$ is a number.

94. If f is defined on the interval $[a,b]$ and has an antiderivative, then the indefinite integral of f on $[a,b]$ is a function.

95. Two different elements of $\int f(x)\,dx$ can only differ by a constant; that is, if both $F(x)$ and $G(x)$ are elements of the set, then there is a constant C such that $F(x) = G(x) + C$.

96. The Substitution Rule can be interpreted as the Chain Rule in reverse.

97. $\int (\cos x + 1)^2\,dx = \frac{(\cos x+1)^3}{3} + C$

98. $\int \frac{1}{x^2+x+1}\,dx = \ln\left|x^2+x+1\right| + C$

Example 4 Finding the Area Bounded by the Graphs of Equations

Find the area of the region bounded by the graphs of the equations $y = 0$, $3x - 5y = 12$, and $y = \sqrt{x}$.

Solution

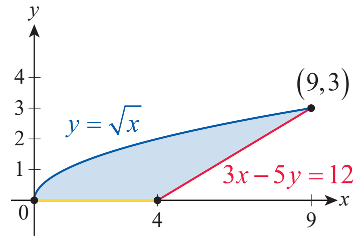


Figure 7

Two edges of the described region are lines, and the remaining edge is the graph of the function $y = \sqrt{x}$, which we can also think of as the upper half of the parabola $x = y^2$. As always, if it's possible to sketch a picture of what we are doing, such a sketch is bound to be helpful—in this case, it's easy to graph the region we're discussing (see Figure 7).

Note that we need two integrals if we want to express the area of the region as an integral in x, since the lower function changes at $x = 4$. On the interval $[0,4]$, the lower function is $g(x) = 0$ (corresponding to the equation $y = 0$), and on the interval $[4,9]$ the lower function is $g(x) = \frac{3}{5}(x-4)$, which we obtain by solving $3x - 5y = 12$ for y. The upper function is $f(x) = \sqrt{x}$ for the entire interval $[0,9]$. So the total area of the described region is

$$A = \int_0^4 \sqrt{x}\,dx + \int_4^9 \left[\sqrt{x} - \frac{3}{5}(x-4)\right]dx.$$

This is certainly doable, and you are asked to evaluate the above integrals in Exercise 52. But if we think of the region as being composed of horizontal strips, we see that the left edge can be described as a single function of y, and the same is true for the right edge—we don't have to divide the interval of integration into subintervals. Specifically, the left edge of the region is the function $g(y) = y^2$ (corresponding to the equation $x = y^2$) and the right edge is the function $f(y) = \frac{5}{3}y + 4$ (obtained by solving $3x - 5y = 12$ for x). So, each horizontal differential element of area can be written as

$$dA = \left[f(y) - g(y)\right]dy = \left(\frac{5}{3}y + 4 - y^2\right)dy$$

and thus the area is calculated as follows.

$$\begin{aligned} A = \int_0^3 dA &= \int_0^3 \left(\frac{5}{3}y + 4 - y^2\right)dy \\ &= \left[\frac{5}{6}y^2 + 4y - \frac{1}{3}y^3\right]_0^3 \\ &= \frac{15}{2} + 12 - 9 = \frac{21}{2} \end{aligned}$$

5.5 Exercises

1–51 Evaluate the definite integral. Whenever possible, take advantage of symmetry.

1. $\int_0^1 2(2x+1)^5\,dx$

2. $\int_0^2 2x\sqrt{4-x^2}\,dx$

3. $\int_0^2 (x^2-1)(x^3-3x)^8\,dx$

4. $\int_0^4 (x-2)(2x^2-8x)^{49}\,dx$

5. $\int_1^2 w^2(w^3+4)^{99}\,dw$

6. $\int_1^{10} 2x^3(x^4-1)^{49}\,dx$

7. $\int_1^3 (2-x)^6\,dx$

8. $\int_0^2 x\sqrt{x^2+1}\,dx$

9. $\int_0^1 x^3\sqrt[3]{x^4+1}\,dx$

10. $\int_0^3 \sqrt{2x+1}\,dx$

11. $\int_1^4 \frac{2x+1}{\left(x^2+x+1\right)^2}\,dx$

12. $\int_0^1 \frac{z^2}{z^3+2}\,dz$

13. $\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx$

14. $\int_1^3 \frac{8x^3+20x}{x^4+5x^2+6}\,dx$

15. $\int_1^2 \left(1+\frac{1}{t^2}\right)^2 \frac{1}{t^3}\,dt$

16. $\int_1^2 \frac{6x+10.5}{\sqrt{2x^2+7x}}\,dx$

17. $\int_e^{e^2} \frac{1}{s\ln\left(s^3\right)}\,ds$

18. $\int_{10}^{100} \frac{\log x}{x\ln 10}\,dx$

19. $\int_1^{\sqrt{2}} (\ln 2)x\cdot 2^{x^2-1}\,dx$

20. $\int_0^{\sqrt{\pi}} 4x\cos\frac{x^2}{2}\,dx$

21. $\int_0^1 \sin \pi x\,dx$

22. $\int_{\sqrt{\pi/4}}^{\sqrt{3\pi/4}} \frac{-x}{\sin^2\left(x^2\right)}\,dx$

23. $\int_0^{\pi/4} \sin^2 2x\cos 2x\,dx$

24. $\int_0^1 x\sec^2\left(2x^2-1\right)dx$

25. $\int_{\pi^2/16}^{9\pi^2/16} \frac{\cot\sqrt{x}\csc\sqrt{x}}{\sqrt{x}}\,dx$

26. $\int_{-\pi}^{\pi} \sin x\sin(\cos x)\,dx$

27. $\int_1^e \frac{\ln\left(2x^2\right)}{x}\,dx$

28. $\int_0^1 \frac{e^{2t}}{e^t+1}\,dt$

29. $\int_{-2}^2 \frac{t^3}{t^2+1}\,dt$

30. $\int_0^{\pi/4} \frac{\csc^2\theta}{e^{\cot\theta}}\,d\theta$

31. $\int_{\pi^2/16}^{\pi^2/4} \frac{\csc^2\left(\frac{\pi}{4}+\sqrt{t}\right)}{\sqrt{t}}\,dt$

32. $\int_0^2 \frac{e^{\sqrt{2x}}}{\sqrt{2x}}\,dx$

33. $\int_0^1 \frac{\sqrt{v}}{\sqrt{v^{3/2}+1}}\,dv$

34. $\int_0^2 \frac{1}{\sqrt{x}\left(2+\sqrt{x}\right)^2}\,dx$

35. $\int_0^{\sqrt{3}} \frac{e^{\arctan x}}{1+x^2}\,dx$

36. $\int_0^1 \frac{1}{x^{2/3}\left(1+x^{2/3}\right)}\,dx$

37. $\int_0^{\pi/2} \frac{\sin 2x}{\cos^2 x+1}\,dx$

38. $\int_0^1 -e^{-x}\sec\left(e^{-x}-1\right)\tan\left(e^{-x}-1\right)dx$

39. $\int_4^9 \frac{\sqrt{2+\sqrt{t}}}{\sqrt{t}}\,dt$

40. $\int_0^4 \sqrt{2+\sqrt{t}}\,dt$

41. $\int_{-1}^1 \frac{x^2+x+1}{x-2}\,dx$

42. $\int_1^e \frac{(\ln x+1)(2\ln x+3)^2}{x}\,dx$

43. $\int_2^3 (x+3)(x-2)^7\,dx$

44. $\int_{-1}^0 x\sqrt[3]{x-1}\,dx$

45. $\int_{-1}^1 x\sqrt{4-x^2}\,dx$

46. $\int_4^{e+3} \frac{2x+5}{x-3}\,dx$

47. $\int_{-4}^0 \frac{x^2-x+3}{x+5}\,dx$

48. $\int_1^{e^2} \frac{(2\ln x+3)(\ln x-1)^2}{x}\,dx$

49. $\int_1^9 \frac{\left(2\sqrt{x}+1\right)\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

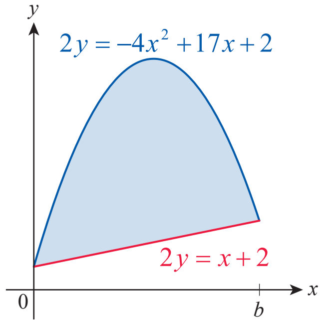
50. $\int_0^{\pi^{2/3}} \sqrt{x}\sin^2\left(x^{3/2}\right)\cos^3\left(x^{3/2}\right)dx$

51. $\int_0^{\pi/4} \tan x\sec^3 x\,dx$

52. Evaluate $A=\int_0^4\sqrt{x}\,dx+\int_4^9\left[\sqrt{x}-\frac{3}{5}(x-4)\right]dx$ from Example 4.

53–56 Find the area of the region bounded by the graphs of the given equations, as shown.

53.



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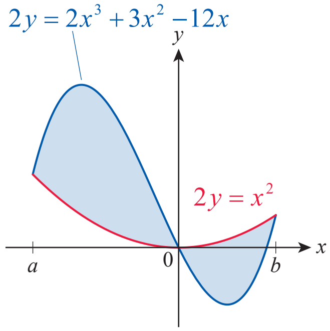

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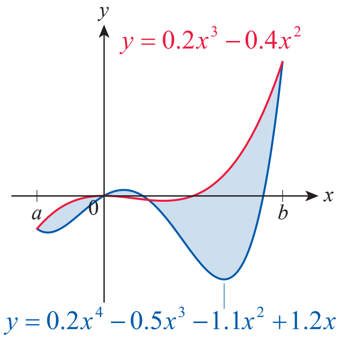

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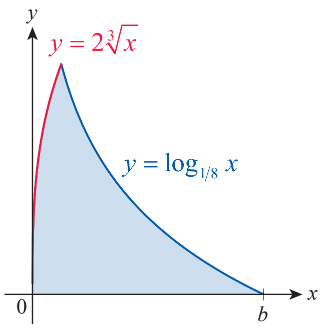


57–96 Find the area of the region bounded by the graphs of the given equations. Be careful to find intersection points, if applicable, and to identify the upper and lower functions on each interval. If convenient or necessary, divide the region into horizontal rather than vertical strips and integrate with respect to *y*. Whenever possible, take advantage of symmetry.

57. $y = x^2, \quad y = 2x$

58. $y = x^2, \quad y = 2$

59. $y = 4x - x^2, \quad y = x$

60. $y = 1 - x^4, \quad y = |x| - 1$

61. $y = |x^3 - 1|, \quad 3y = 5x + 11$

62. $y = x^2, \quad y = x^4$

63. $y = 2x - x^2, \quad y = x^3$

64. $y = \sqrt{x}, \quad y = 2 - x$

65. $x = y^2, \quad y = x^3, \quad x \ge 0$

66. $y = \sqrt[3]{x}, \quad y = \sqrt[7]{x}$

67. $2xy - y = 3 - 2x, \quad y = x, \quad y = 0$

68. $x + 30y = 2y^3 + 5, \quad 9y - 31 = x + y^3$

69. $y = \sqrt{x+1}, \quad y = x^2 - 1$

70. $y = x^3, \quad y = x$

71. $y = \frac{1}{x}, \quad y = \frac{1}{x^2}, \quad x = e$

72. $y = x^3, \quad y = \frac{3}{2} - \frac{x}{2}, \quad y = 0$

73. $y = 4x^2 - x^4, \quad y = -5x^2$

74. $y = 3x^2 - x - 4, \quad y = x^2 + 3x + 2$

75. $y^2 - x = 2, \quad y = x$

76. $1 - y^2 = x, \quad (1-y)^2 = x$

77. $3y - x = 3y^2, \quad 6y^3 = x + 6y^2$

78. $y = x^3 - 3x^2 + 2x, \quad y = x^2 - x$

79. $y = x^3 - 6x^2 + 5, \quad y = -x^3 + 12x - 11$

80. $y = x^4 - \frac{x^3}{2} - 4x^2, \quad y = -\frac{x^3}{2}$

81. $y = 2\sqrt[3]{x+1}, \quad y = 2 - x, \quad y = 0$

82. $y = \sqrt{x}, \quad y = \frac{1}{x^2}, \quad x = 4$

83. $x = y^3 - 10y^2 + 20y, \quad x = 4y^2 - 33y + 40$

84. $x = y^4 - 3y^3 - y^2, \quad x = 5y^2 - 8y$

85. $y = \sqrt[4]{x}, \quad y = -\frac{1}{16}x^2 + 18, \quad y = 0$

86. $(y-1)^2 = \frac{1}{x}, \quad x - 2 = 4y, \quad y = 0$

87. $y = \cos x, \quad y = \sin 2x, \quad x = 0, \quad x = \pi$

88. $y = \frac{2}{x^2+1}, \quad y = x^2$

89. $y = \sin x, \quad y = \sqrt{2} - \sin x, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}$

90. $y = \cot x, \quad y = 2\cos x, \quad 0 < x < \pi$

91. $y = x + 1, \quad y = \sqrt{18 - (x+1)^2}, \quad y = 0$

92. $y = \arctan x, \quad y + x = 1 + \frac{\pi}{4}, \quad x = 0$

93. $y = \cos\left(\frac{\pi}{2}x\right), \quad y = x^4 - 1$

94. $y = 2\sqrt{2}\sin x, \quad y = \csc^2 x, \quad 0 < x < \pi$

95. $y = \sin^2 x, \quad y = \cos^2 x, \quad x = 0, \quad x = \pi$

96. $y = \tan^2 x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{4}$

97. Use the Substitution Rule to prove the following property of the definite integral.

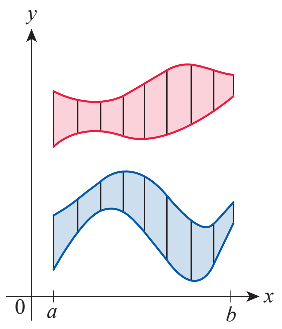
$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

Note that the above is often referred to as the *translation invariant property* of the definite integral. Using a generic $f(x)$, make a sketch of both integrals and explain the reason for the name of this property.

98. Use Exercise 97 to explain why the following definite integrals are equal.

$$\int_{-1}^{2} 4x\sqrt[3]{x+2}\,dx = \int_1^4 (4x-8)\sqrt[3]{x}\,dx$$

99. Italian mathematician Bonaventura Cavalieri (1598–1647), who can be considered as one of the early forerunners of modern calculus, discovered what we today call *Cavalieri's Principle*. Use our discussion preceding Example 3 to prove the following version of Cavalieri's Principle: Suppose two plane regions are included between the lines $x = a$ and $x = b$, and are bounded by graphs of integrable functions. If they have the property that any vertical line intersects both regions in line segments of the same length, then the regions have equal areas.



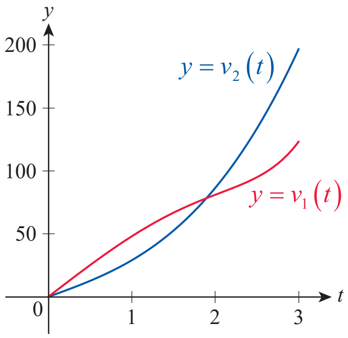
100. Consider the region bounded by the graphs of the equations $y = \sqrt{x}$, $x = 9$, and the x-axis. Find the vertical line $x = a$ that bisects the region in two subregions of equal area.

101. The graphs below show the velocities of two bikes at a motorcycle race right after the start (velocity is measured in km/h). Use the figure to answer the following questions.

a. Which bike is ahead initially?

b. What happens at the instant when the curves intersect?

c. Do the curves suggest that a pass happened, and if so, approximately when?



102. Suppose that the function $B(t) = 85\cdot(1.1163)^t$ approximates the birth rate of a rabbit population on an isolated island, while the death rate is $D(t) = 21\cdot(1.0811)^t$ (t is measured in months). Find the area between the graphs of these two functions on the interval $[0,12]$. Use your own words to give a real-life interpretation to this number.

5.5 Technology Exercises

103–107 Use a graphing utility to plot the graphs of $f(x)$ and $g(x)$ on the same screen. After choosing the appropriate viewing window, identifying intersection points, and finding the region bounded by the curves, use the integration features of your technology to find the area of the region. (**Hint:** As in Examples 3 and 4, be sure to identify the upper and lower functions on each subinterval and integrate accordingly. As a final step, you may want to check your answer by evaluating $\int_a^b |f(x)-g(x)|\,dx$, where a and b are the first and last of the intersection points. Do you obtain the same answer?)

103. $f(x) = 35x - 9$,
$g(x) = 6x^3 - 4.95x^2 - 3.04x - 22.2525$

104. $f(x) = 3\sin x$, $g(x) = 0.3x$

105. $f(x) = e^x$, $g(x) = \frac{1}{2}x + 2$

106. $f(x) = 2x^4 - 8x^3 - 6.5x^2 + 29x - 12$,
$g(x) = 2x^3 - 4x^2 - 3.5x + 2.5$

107. $f(x) = \dfrac{0.8^x \sin 2x}{2}$, $g(x) = \frac{1}{2}\sqrt{x}$

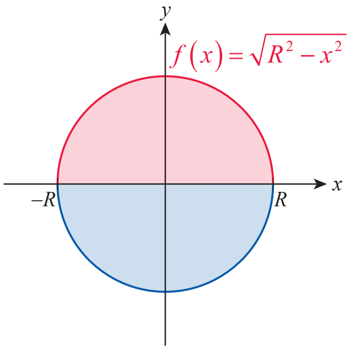


Figure 19

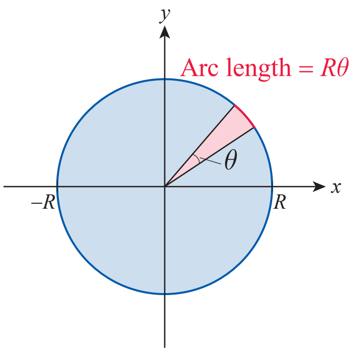


Figure 20

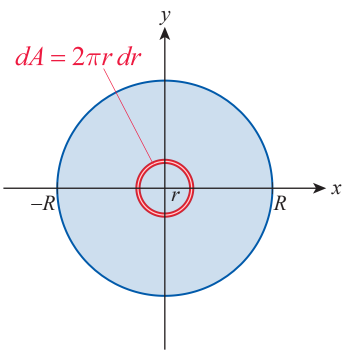


Figure 21

Example 8 Using Integration in Multiple Ways to Derive the Formula for Area of a Circle

You know that the area of a circle of radius R is $A = \pi R^2$, and one straightforward way to arrive at this formula is to integrate the function $f(x) = \sqrt{R^2 - x^2}$ over the interval $[-R, R]$ and then double the result (see Figure 19). However, this is not the easiest way (finding an antiderivative of f is best done using a technique we will learn in Chapter 7). Here are two other approaches.

1. The area of a triangle is half the product of its base and height. A thin sector of our circle is approximately a triangle (and the approximation gets better and better as the angle approaches 0), with the base of the triangle equal to $R\,d\theta$ and the height equal to R (see Figure 20).

$$dA = \left(\frac{1}{2}\right)(R)(R\,d\theta) = \frac{1}{2}R^2\,d\theta$$

$$A = \int dA = \int_0^{2\pi} \frac{1}{2}R^2\,d\theta = \frac{1}{2}R^2\,\theta\Big]_{\theta=0}^{\theta=2\pi} = \pi R^2$$

2. If we assume the formula for a circle's circumference is known, then we can decompose our given circle into a sequence of thin concentric rings with each one having area approximately equal to its circumference times dr (see Figure 21).

$$dA = 2\pi r\,dr$$

$$A = \int dA = \int_0^R 2\pi r\,dr = 2\pi \cdot \frac{r^2}{2}\Big]_{r=0}^{r=R} = \pi R^2$$

Example 9 Using Integration to Derive the Formula for Volume of a Sphere

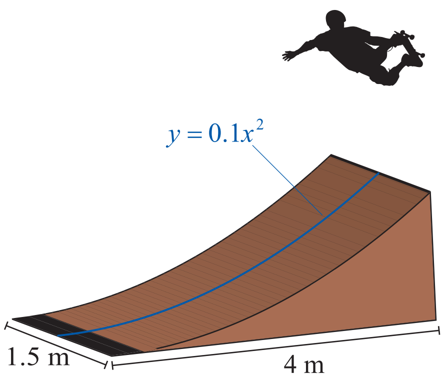
Assume the formula for the surface area of a sphere of radius r is known: $A = 4\pi r^2$ (we will soon find this formula on our own). Then an alternative way of finding the volume of a sphere of radius R is to decompose it into a sequence of thin concentric shells, each with volume approximately equal to its surface area times dr.

$$dV = 4\pi r^2\,dr$$

$$V = \int dV = \int_0^R 4\pi r^2\,dr = 4\pi \cdot \frac{r^3}{3}\Big]_{r=0}^{r=R} = \frac{4}{3}\pi R^3$$

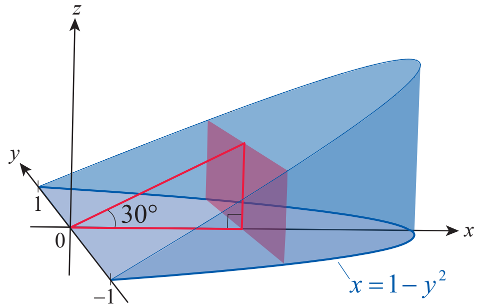
6.1 Exercises

1. Find the volume of a skateboard ramp with a rectangular base of 4 meters by 1.5 meters, if each vertical cross-section is congruent to the "parabolic triangle" bounded by the x-axis, the vertical line $x = 4$, and the graph of $y = 0.1x^2$ (longitudinal units are meters).

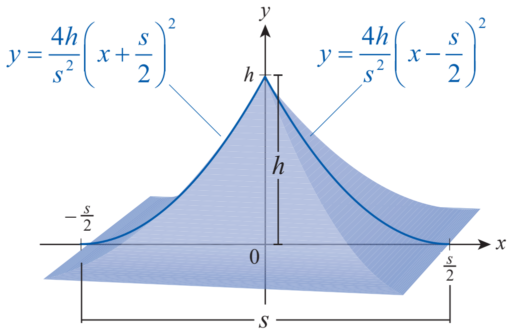


2. Mimic Example 2 to derive the formula for the volume of the right circular cone of height h, if the radius of its base is r.

3. Let's modify Example 3 by assuming that the cylinder is parabolic, that is, the base of the wedge is bounded by the graph of $x = 1 - y^2$ and the y-axis. Moreover, suppose that the second plane makes a 30° angle with the xy-plane. Find the volume of the curved wedge under these conditions.

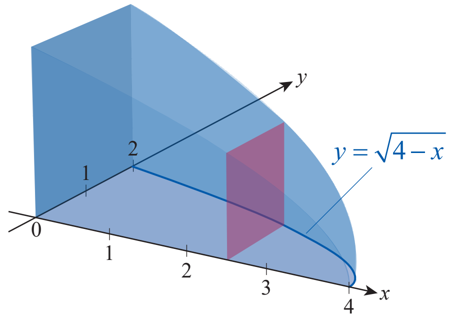


4. Suppose we "carve out" the faces of the pyramid of Example 2 so that the "side view" (i.e., the perpendicular cross-section that contains the altitude and is parallel to a pair of base edges) becomes the region bounded by the graphs of $y = \frac{4h}{s^2}\left(x - \frac{s}{2}\right)^2$, $y = \frac{4h}{s^2}\left(x + \frac{s}{2}\right)^2$, and the x-axis $(-s/2 \le x \le s/2)$. Find the volume of the resulting "concave pyramid."

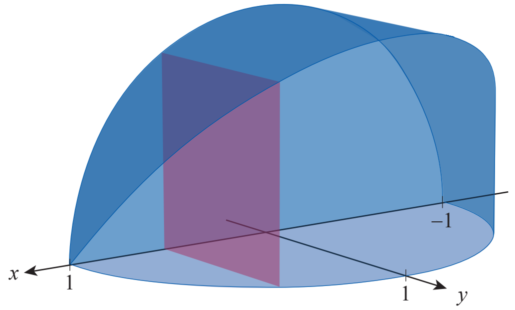


5–14 The base of a solid S is described in the xy-plane along with its cross-sections in a certain direction. Find the volume of S.

5. The base of S is the region bounded by the graph of $y = \sqrt{4-x}$ and the coordinate axes. The cross-sections perpendicular to the x-axis are squares.



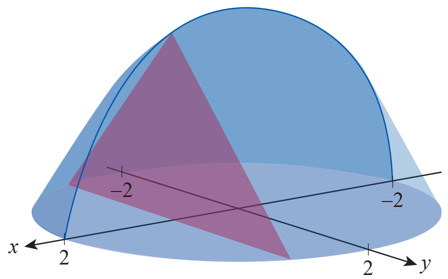
6. The base of S is a half disk of radius 1, and the cross-sections perpendicular to the diameter are squares.



7. Repeat Exercise 6 if the cross-sections are equilateral triangles.

8. The base of S is a disk of radius 2, and the cross-sections perpendicular to the base are rectangles, each with a height that equals twice the width.

9. The base of S is a disk of radius 2, and the cross-sections perpendicular to the base are isosceles triangles, each with a height half as long as its base.



10. The base of S is bounded by $y = x^3$, the y-axis, and the line $y = 8$. Cross-sections perpendicular to the y-axis are squares.

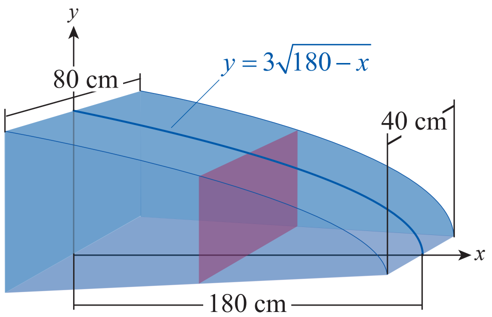
11. The base of S is the region bounded by $y = x^4$ and $y = x^2$, $0 \le x \le 1$, and each cross-section perpendicular to the x-axis is an isosceles right triangle with the right angle's vertex sitting on $y = x^2$.

12. The base of S is bounded by the x-axis and $y = \sin x$, $0 \le x \le \pi$. Each of its cross-sections perpendicular to the x-axis is an isosceles triangle of altitude 1.

13. The base of S is bounded by $y = \cos x$ and $y = \left(\frac{2}{\pi}x\right)^2 - 1,\ -\pi/2 \le x \le \pi/2$. Each of its cross-sections perpendicular to the x-axis is an isosceles triangle of altitude 2.

14. The base of S is bounded by $y = \sqrt{4-x},\ y = 0$, and $x = 0$. Each of its cross-sections perpendicular to the x-axis is a rectangle of perimeter 10.

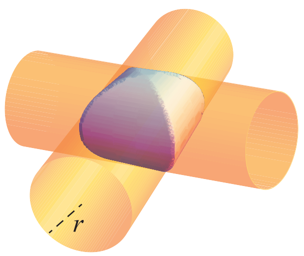
15.* A solid modeling the nose of a race car has a base in the shape of an isosceles trapezoid with a height of 180 cm and base lengths of 80 cm and 40 cm, respectively. Suppose the vertical cross-section through the axis of symmetry of the base (i.e., the cross-section that "cuts the model in half" longitudinally) is the region bounded by the graph of $y = 3\sqrt{180-x}$ and the coordinate axes. All vertical cross-sections perpendicular to the axis of symmetry of the base are rectangles. Find the volume of this model.



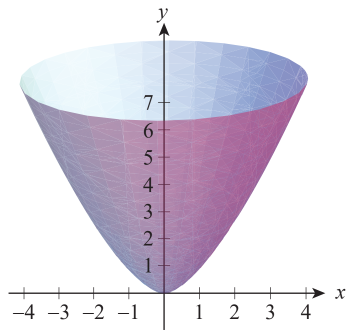
16. Find the volume of water remaining in a spherical reservoir of radius r if the water's depth is $r/3$.

17. A piece in a wooden toy set is a sphere of radius 2 cm, with a cylindrical hole of radius 1 cm drilled through the center. Find the volume of this piece.

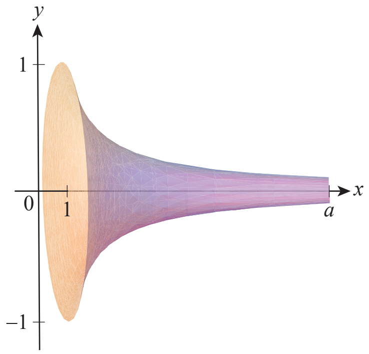
18.* Two plastic pipes (with circular cross-sections) of radius r inches cross at right angles. Find the volume of the solid region that is common to both pipes. (**Hint:** Find cross-sections that are squares.)



19. Find the volume of the "bowl" that results from rotating the graph of $y = x^{3/2},\ -4 \le x \le 4$, around the y-axis.



20. Find the volume of the solid that results from rotating the graph of $y = 1/x,\ 1 \le x \le a$, around the x-axis (this solid is known as *Gabriel's horn*). What can you say if $a \to \infty$?



21–28 Find the volume of the solid that results from rotating the region between the graph of the given equation and the x-axis about the x-axis over the indicated interval.

21. $y = -\frac{1}{3}x + 3;\quad 0 \le x \le 3$

22. $y = 2x + \frac{2}{3};\quad 0 \le x \le 1$

23. $y = x^2 + \frac{1}{2};\quad 0 \le x \le 2$

24. $y = x^3;\quad -1 \le x \le 3$

25. $y = -\sqrt{5-x};\quad 1 \le x \le 4$

26. $y = x^{5/3};\quad -1 \le x \le 1$

27. $y = \sqrt{9-x^2};\quad 0 \le x \le 3$

28. $y = \sec x;\quad 0 \le x \le \pi/4$

29–36 Find the volume of the solid that results from rotating the region bounded by the graphs of the equations about the *y*-axis.

29. $y=\frac{1}{2}x-1, \quad x=0, \quad y=0$

30. $2y+6x=9, \quad x=0, \quad y=0$

31. $y=(x-1)^2, \quad x=1, \quad x=4, \quad y=0$

32. $y=x^3, \quad x=2, \quad y=0$

33. $y=\sqrt[3]{x}+1, \quad x=0, \quad y=3$

34. $y=x^{3/5}-2, \quad x=0, \quad y=-2, \quad y=6$

35. $y=\sqrt{4-x^2}, \quad x=0, \quad y=0$

36.* $y=\csc^{-1}x, \quad x=0, \quad y=\pi/4, \quad y=\pi/2$

37–48 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about **a.** the *x*-axis and **b.** the line $y=-1$.

37. $y=\frac{1}{2}x, \quad y=0, \quad x=2$

38. $y-3x-1=0, \quad y=0, \quad x=1$

39. $y=x^3, \quad y=0, \quad x=1$

40. $y=2\sqrt{x}, \quad y=0, \quad x=4$

41. $y=\frac{x^4}{4}, \quad y=x^2, \quad x=0, \quad x=2$

42. $y=\frac{2}{x+1}, \quad y=0, \quad x=0, \quad x=2$

43. $y=x^{5/3}, \quad y=0, \quad x=0, \quad x=1$

44. $y=e^x, \quad y=0, \quad x=0, \quad x=1$

45. $y=\sqrt[3]{x}, \quad y=0, \quad x=1, \quad x=8$

46. $y=\frac{1}{2}x^{2/3}, \quad y=\frac{1}{2}x^{3/2}, \quad x=0, \quad x=1$

47. $y=\sqrt{1-(x-1)^2}, \quad y=0, \quad x=2$ (**Hint:** When evaluating $\int_0^2 \sqrt{1-(x-1)^2}\,dx$, use a well-known formula from geometry.)

48. $y=\cos x, \quad y=0, \quad x=0, \quad x=\pi/2$
(**Hint:** When evaluating the integral, use the identity $\cos^2 x=(1+\cos 2x)/2$ or the identity $\cos^2 x+\sin^2 x=1$.)

49–57. In each of Exercises 39–47, rotate the region about **a.** the *y*-axis and **b.** the line $x=-2$, and find the volume of the resulting solid. (For Exercise 54, use the antiderivative $\int(\ln x)^2\,dx = 2x-2x\ln x+x(\ln x)^2$. For Exercise 57, see the hint given in Exercise 47.)

58–71 Find the volume of the solid that results from rotating the region bounded by the graphs of the equations about the indicated line. Use any of the methods discussed in this section.

58. $y=-2x+5, \quad y=-1, \quad x=0;$ about $x=-1$

59. $y-4x-3=0, \quad y=0, \quad x=0, \quad x=3;$ about $x=-3$

60. $y=x^2-\sqrt{8x}, \quad y=0, \quad x=0, \quad x=2;$ about the *x*-axis

61. $x=2y(y-5), \quad x=0;$ about the *y*-axis

62. $x\sqrt{1+y^2}=1, \quad x=0, \quad y=0, \quad y=\sqrt{3};$ about the *y*-axis

63. $x=y^2, \quad y=x^2;$ about $y=1$

64. $y=\sqrt[4]{2x+1}, \quad y=0, \quad x=4;$ about the *x*-axis

65. $y=(x-2)\sqrt{x^2+1}, \quad y=0, \quad 0\le x\le 2;$ about the *x*-axis

66. $y=\sqrt{-x\cos(x^2+\pi)}, \quad y=0, \quad x=0, \quad x=\sqrt{\pi/2};$ about the *x*-axis

67. $x=\sqrt{y}+2, \quad 4x=y^2, \quad y=0, \quad y=4;$ about the *y*-axis

68. $y=\sin^{-1}x, \quad y=0, \quad x=1;$ about the *y*-axis (See the hint given in Exercise 48.)

69. $y=\cos x, \quad y=\sin x, \quad x=0, \quad x=\pi/4, \quad y=0;$ about the *x*-axis

70. $x=e^y+e^{-y}, \quad x=0, \quad y=0, \quad y=1;$ about $x=-e$

71. $y=\csc x, \quad y=\cot x, \quad x=\pi/4, \quad x=\pi/2;$ about the *x*-axis

72–79 The given integral represents the volume of a solid of revolution. Describe the solid. (Do not evaluate the integral.)

72. $\int_0^{\sqrt{3}} 3\pi y^2\,dy$

73. $\int_0^4 \pi x\,dx$

74. $\int_0^2 \pi(4-y^2)\,dy$

75. $\int_0^1 (\pi e^2-\pi e^{2x})\,dx$

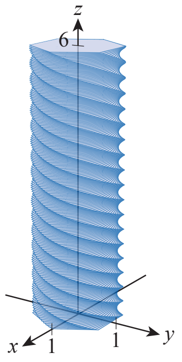
76. $\int_0^1 \pi(\arctan y)^2\,dy$

77. $\pi\int_0^1 (\sqrt{x}-x^8)\,dx$

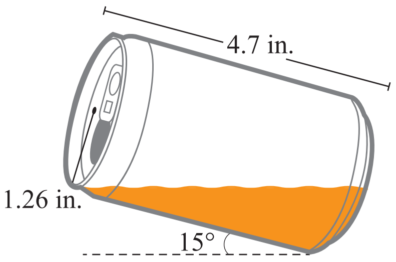
78. $\int_0^{\pi} \pi\sin^2 x\,dx$

79. $\pi\int_1^2 \left[(\log_2 x)^2-(x-1)^2\right]dx$

80. A regular hexagon of side length 1 lies in the xy-plane so that its center of symmetry coincides with the origin. Suppose the hexagon moves 6 units vertically upward so that its center rides on the line perpendicular to the xy-plane (the line is actually the z-axis), the hexagon at any time is parallel to the xy-plane and makes three revolutions around its center (at a constant angular speed) as it moves. Find the volume of the resulting corkscrew-type solid.

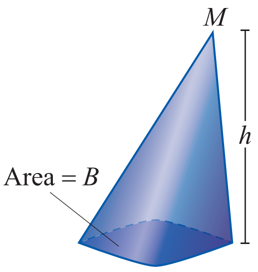


81.* A student is drinking from a standard soda can. Find the volume of the remaining soda when the can's axis of symmetry makes a 15° angle with the horizontal direction. Express your answer in milliliters. (**Hint:** Approximate the soda can with a right circular cylinder of radius 1.26 inches and height of 4.7 inches. Assume that when the can is held as described in the problem, the soda is level with the lowest point of the top rim of the can.)



82.* Let D denote the depth of water in a bowl that has a tiny hole in the bottom. According to Torricelli's Law, water drains through the hole at the rate of $dV/dt = -m\sqrt{D}$, where m is a positive constant. Find the rate at which the water level is decreasing if the bowl is generated by rotating around the y-axis the graph of **a.** $y = cx^2$ and **b.** $y = cx^4$, $c > 0$. Which of the two rates do you think can be used as a "water clock," and why?

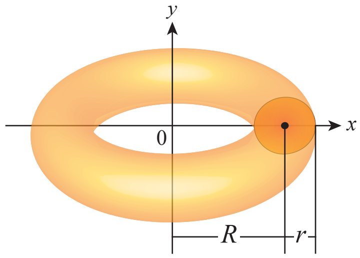
83. Although we typically work with circular cones, in a more general sense, cones don't have to be circular. In fact, if C is a simple closed curve in a plane S, and M is a point not in the same plane, then the solid generated by a line passing through M and moving along C, is said to be a cone (the region bounded by C is the base, M is the vertex of the cone, the distance between M and S is its height). Use the results and methods of this section to prove that the volume of a cone of base area B and height h is $V = \frac{1}{3}Bh$. (**Hint:** Mimic Example 2, using the fact that by similarity, in the general case, we have $\frac{A(y)}{B} = \frac{y^2}{h^2}$, that is, $A(y) = \frac{B}{h^2} \cdot y^2$.)



84.* Consider the circle in the xy-plane defined by the following equation.

$$(x-R)^2 + y^2 = r^2 \quad (R > r)$$

By rotating the region bounded by this circle around the y-axis, prove that the volume of the generated solid, a torus, is $V = 2\pi^2 Rr^2$.



85.* Rotate about the x-axis the region between the graphs of $y = \sqrt{4-x}$ and $y = c$ $(0 \le c \le 2)$ over the interval $[0,4]$. Find the value of c that minimizes the volume of the resulting solid. For what c-value is the volume maximal?

6.1 Technology Exercises

86. Use a graphing utility to repeat Exercise 85 for the region bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$ rotating about the line $y = c$ $(0 \le c \le 1)$.

87.* With the aid of a graphing utility, revisit Exercise 81, assuming this time that the angle between the can's axis of symmetry and the horizontal direction is 10°.

88–92 Use a graphing utility to find (or approximate) the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the indicated axis.

88. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = -1$, $x = 1$; about the x-axis

89. $y = \arcsin x$, $y = 0$, $x = 0$, $x = 1$; about the x-axis

90. $y = e^x$, $x = 0$, $y = 1$, $y = e^2$; about the y-axis

91. $y = \dfrac{1}{\log x}$, $y = 0$, $x = 10$, $x = 100$; about the x-axis

92. $x = e^{1-y^2}$, $y = 0$, $y = 1$, $x = 0$; about the y-axis

93–94 Use a graphing utility to sketch the region bounded by the graphs of the given equations. Approximate the intersection points, then find an approximation for the volume of the solid generated by rotating the region about the x-axis.

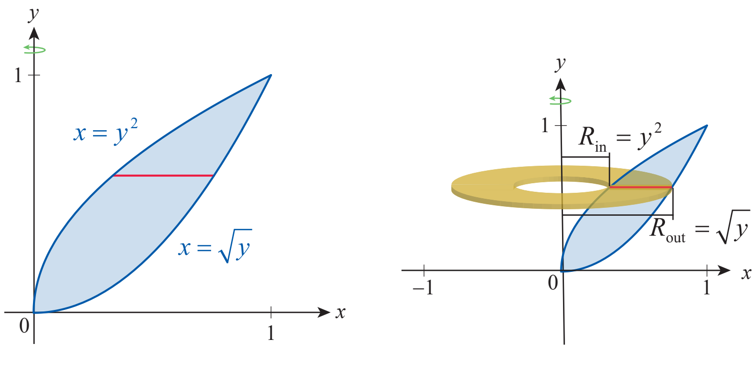
93. $y = x\cos^2 x$, $y = 2x(2-x)$

94. $y = x + 3$, $y = e^x + e^{-x} - 2$

Method 2: The Washer Method

In this example, the washer method applies just as well. The setup differs, of course, as we need to describe the inner and outer radii of a sequence of vertically stacked washers. Using this formulation, the outer radius R_{out} is the function $\sqrt{y}$ and the inner radius R_{in} is the function y^2. Note that our integration will be with respect to y over the interval $0 \le y \le 1$; Figure 13 illustrates the region and a typical washer cross-sectional cut of the region. The volume computation is as follows.

$$V = \pi\int_0^1 \left(R_{\text{out}}^2 - R_{\text{in}}^2\right)dy = \pi\int_0^1 \left(y - y^4\right)dy$$
$$= \pi\left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 = \frac{3\pi}{10}$$



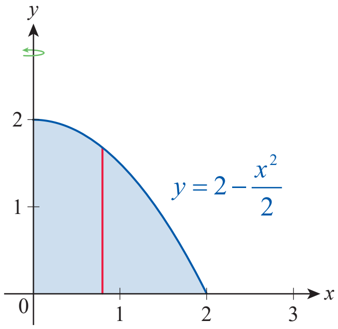
(a) (b)

Figure 13 The Washer Method

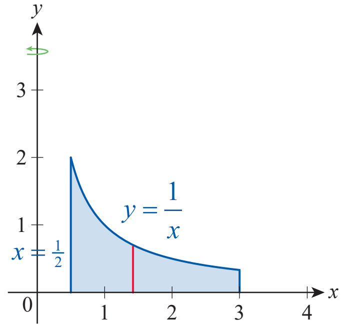
6.2 Exercises

1–6 Use the shell method to find the volume of the solid obtained by revolving the shaded region about the indicated line.

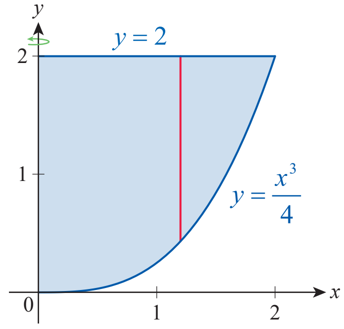
1.



2.



3.



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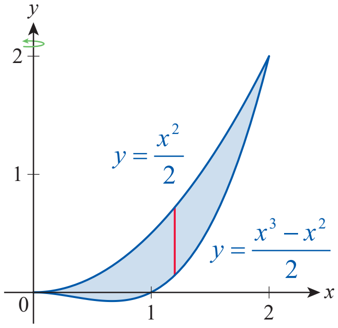
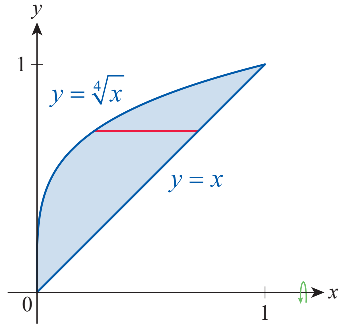
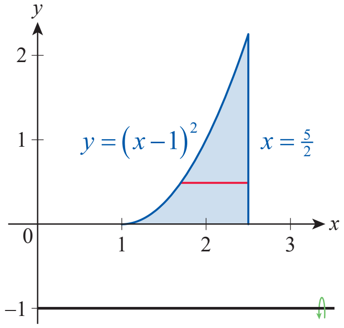


5.

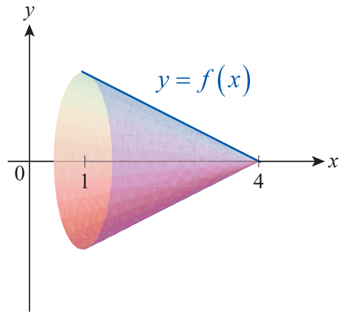


6.

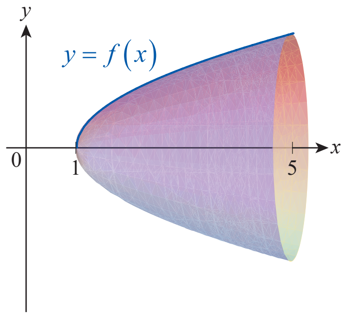


7–8 Use the shell method to find the volume of the solid that is generated by revolving the region bounded by $y = 0$ and the graph of f about the x-axis over the indicated interval.

7. $f(x) = 2 - 0.5x; \quad 1 \le x \le 4$



8. $f(x) = \sqrt{x-1}; \quad 1 \le x \le 5$



9–26 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the y-axis.

9. $y = -2x + 6, \quad x = 0, \quad y = 0$

10. $y - x = 0, \quad y = 0, \quad y = 6$

11. $2y - 5x + 8 = 0, \quad y = 0, \quad x = \frac{8}{5}, \quad x = 3$

12. $3y - x = 0, \quad x = 0, \quad x = 3, \quad 3y + 2x - 9 = 0$

13. $y = x^2, \quad x = 0, \quad y = 9$

14. $y = x^2 - 1, \quad y = 2x - 1$

15. $y = x^2 - 2x + 3, \quad x = 0, \quad x = 3, \quad 3y = 2x + 12$

16. $y = \sqrt{4-x}, \quad x = 0, \quad y = 0$

17. $y = x^{3/5}, \quad y = 0, \quad x = 1$

18. $y = \sqrt{4-x^2}, \quad x = 0, \quad y = 0$

19. $y = (x-2)^3, \quad x = 2, \quad y = 8$

20. $y = \sqrt[3]{x} + 1, \quad x = 0, \quad x = 1, \quad y = 0$

21. $y = \cos^2(x^2), \quad y = \sin^2(x^2), \quad x = 0, \quad x = \frac{\sqrt{\pi}}{2}$

22. $y = \sec^2(x^2), \quad x = 0, \quad x = \frac{\sqrt{\pi}}{2}, \quad y = 0$

23. $y = \frac{1}{(x+1)^2}, \quad x = 0, \quad x = 1, \quad y = 0$

24. $y = \frac{1}{3\sqrt{x}}, \quad x = 1, \quad x = 4, \quad y = 0$

25. $y = \frac{e^x}{\pi x}, \quad x = 1, \quad x = 2, \quad y = 0$

26. $y = \begin{cases} \frac{1 - \cos x}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = 2\pi$

27–38 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the x-axis.

27. $7y + 4x = 28, \quad x = 0, \quad y = 0$

28. $y - x + 1 = 0, \quad y + 2x - 8 = 0, \quad y = 0$

29. $y = \sqrt{x}, \quad y = x - 2, \quad y = 0$

30. $y = 2x^2 - 8, \quad x = 0, \quad y = 0$

31. $y = \sqrt[3]{3-x}, \quad y = \frac{1}{2}x, \quad y = 0$

32. $x = \frac{1}{y+2}, \quad x = 0, \quad y = 0, \quad y = 2$

33. $2x\sqrt{y+1} = 1, \quad x = 0, \quad y = 0, \quad y = 3$

34. $y\sqrt{y^3+1} = x, \quad x = 0, \quad y = 0, \quad y = 2$

35. $\frac{1}{y^2+1} = x, \quad x = 0, \quad y = 0, \quad y = 4$

36. $x = \frac{1}{y(y^2+1)}, \quad x = 0, \quad y = 1, \quad y = \sqrt{3}$

37. $y = \sqrt{\ln(1-x)+1}, \quad x = 0, \quad y = 0$

38. $y = \sqrt{\arcsin x}, \quad x = 0, \quad y = \sqrt{\frac{\pi}{2}}$

39–44 Revolve the region bounded by the graphs of the equations about the given line and use the shell method to find the volume of the resulting solid.

39. $y = 5 - 2x, \quad x = 0, \quad y = 0$

a. About $x = -2$ **b.** About $y = -1$

c. About $x = 2.5$ **d.** About $y = 7$

40. $y = \frac{3}{2}x, \quad x = 2, \quad y = 0$

a. About $x = -1$ **b.** About $y = -2$

c. About $x = 2$ **d.** About $y = 3$

41. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

a. About $x = -1$ **b.** About $y = -\frac{1}{2}$

c. About $x = 6$ **d.** About $y = 2$

42. $y = x, \quad y = x^3, \quad x = 0, \quad x = 1$

a. About $x = -3$ **b.** About $y = -\frac{1}{2}$

c. About $x = \frac{3}{2}$ **d.** About $y = 2$

43. $y = 4x - x^2, \quad y = 0$

a. About $x = -1$ **b.** About the x-axis

c. About $x = 2$ **d.** About $y = -1$

44. $y = \sqrt{4x - x^2}, \quad y = 0$

a. About the x-axis **b.** About $y = -1$

c. About $x = 2$ **d.** About $y = 2$

45–50 Using the shell method, find a formula for the volume of the solid that results when the region bounded by the graphs of the equations is revolved about the indicated axis. Do not evaluate the integral.

45. $y = 2e^{-x} - 1, \quad x = 0, \quad x = \ln 2, \quad y = 0;$ about the y-axis

46. $y = 2e^{-x} - 1, \quad x = 0, \quad x = \ln 2, \quad y = 0;$ about the x-axis

47. $y = \arcsin x, \quad x = 0, \quad x = 1, \quad y = 0;$ about the y-axis

48. $y = \arcsin x, \quad x = 0, \quad x = 1, \quad y = 0;$ about the x-axis

49. $y = \tan x, \quad x = 0, \quad y = \sqrt{3};$ about the y-axis

50. $y = \tan x, \quad x = 0, \quad y = \sqrt{3};$ about the x-axis

51–59 The given integral represents the volume of a solid of revolution. Describe the solid. (Do not evaluate the integral.)

51. $4\pi\int_0^2 \left(2x - x^2\right) dx$

52. $2\pi\int_0^{\sqrt{3}} \left(3y - y^3\right) dy$

53. $4\pi\int_0^1 y\sqrt{1 - y^2}\, dy$

54. $4\pi\int_2^4 x\sqrt{1 - (x-3)^2}\, dx$

55. $2\pi\int_0^1 (1+x)\left(1 - x^4\right) dx$

56. $2\pi\int_0^1 (y+2)\left(e - e^y\right) dy$

57. $\pi\int_{\pi/2}^{\pi} (2x - \pi)\sin x\, dx$

58. $2\pi\int_0^{1/2} 3y\, dy + 2\pi\int_{1/2}^1 y\left(\frac{1}{y^2} - 1\right) dy$

59. $\pi\int_{\pi/4}^{3\pi/4} (2x+2)\csc x\, dx$

60. Use the washer method to solve the problem posed in Example 3. (**Hint:** Completing the square for $f(y) = -y^2 + y$ yields $f(y) = x = \frac{1}{4} - \left(y - \frac{1}{2}\right)^2$. Express y in terms of x to obtain the graphs bounding the given region. Since the axis of rotation is $y = 2$, you will obtain $R_{\text{in}} = 2 - \left(\frac{1 + \sqrt{1 - 4x}}{2}\right)$ and $R_{\text{out}} = 2 - \left(\frac{1 - \sqrt{1 - 4x}}{2}\right)$. Simplify the integral $V = \pi\int_0^{1/4} \left(R_{\text{out}}^2 - R_{\text{in}}^2\right) dx$ and verify that it equals $\pi/2$.)

61–66 Use the shell method or the disk/washer method to find the volume of the solid obtained by revolving the region bounded by the graphs of the equations about the given axis. Choose the method that seems to work best.

61. $y(x+1) = 1, \quad y = 0, \quad x = 0, \quad x = 1;$ about the x-axis

62. $y = 1 - (x-1)^4, \quad x = 0, \quad y = 0;$ about the y-axis

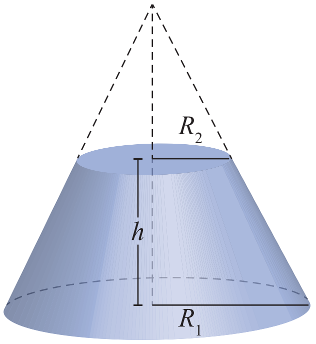
63. $y = x^3 - 6x^2 + 8x, \quad y = 0, \quad x = 0, \quad x = 2;$ about the x-axis

64. $y = 2x + \frac{1}{x^2}, \quad y = 0, \quad x = 1, \quad x = 2;$ about the y-axis

65. $y = 3 - \frac{x^2}{3}, \quad y = 0;$ about the x-axis

66. $y = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = \pi;$ about the y-axis

67. The solid that remains after "chopping off" the upper part of a right circular cone by a plane parallel to its base is called a frustum of a cone. Suppose that the radii of the base and top are R_1 and R_2, respectively, while the height is h.



a. Use the shell method to prove that the volume of the cone frustum is $V = \frac{\pi h}{3}\left(R_1^2 + R_1 R_2 + R_2^2\right)$. (**Hint:** One approach is to rotate the region between the line segment $y = \frac{R_1 - R_2}{h}x,\ 0 \le x \le h$, and the line $y = -R_2$ about the said line and use the volume of the resulting solid to find that of the frustum.)

b. Now use the method of disks to establish the above formula. Which method do you prefer?

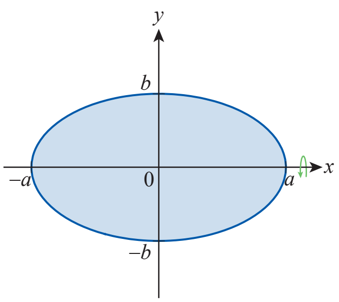
68. Use the shell method to prove the volume formula for a sphere of radius R: $V = \frac{4}{3}\pi R^3$.

69. Use the shell method to find the volume of the wooden toy piece of Exercise 17 of Section 6.1.

70. How deep is the water in a bowl-shaped hemispherical tank of radius r when the tank is filled to exactly $\frac{14}{27}$ of its full capacity?

71.* Use the shell method to find the formula for the volume of the torus of Exercise 84 of Section 6.1.

72.* The graph of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse centered at the origin, with its axes of symmetry of lengths $2a$ and $2b$, coinciding with the coordinate axes. By rotating the region bounded by this ellipse about the x-axis, use the shell method to find a formula for the volume of the resulting ellipsoid.



Concept Check

73–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

73. If V is the volume of the solid obtained by revolving the region bounded by $y = f(x)$ and the x-axis $(a \le x \le b)$ about the x-axis and k is a positive constant, then the volume of the solid generated by $y = kf(x)$ over the same interval is kV.

74. The volume of the solid generated by revolving about the y-axis the region bounded by $y = x^2$, the y-axis, and $y = b$ is directly proportional to b.

75. The volume of a solid of revolution can be interpreted as the limit of Riemann sums.

76. The shell method always results in a less complicated integral than the disk method.

77. If R is the region bounded by $y = f(x)$ and $y = g(x)$, while R_1 is bounded by $y = f(x) + 1$ and $y = g(x) + 1$, then the solids obtained by revolving about the x-axis the regions R and R_1, respectively, have equal volume.

6.2 Technology Exercises

78–81 Use a graphing utility to sketch the region bounded by the graphs of the equations. Then use the shell method, along with the integration capabilities of your technology, to find the volume of the solid generated by rotating the region about the given line.

78. $y = x^2 \cos^2 x, \quad y = 0, \quad x = 0, \quad x = \pi/2;$ about the y-axis

79. $x^{2/3} + y^{5/3} = 1, \quad x = 0, \quad y = 0;$ about the y-axis

80. $x = \left(\frac{\pi}{2} - y\right)^3 \cos y, \quad y = 0, \quad x = 0;$ about $y = -1$

81. $x^3 = y(y-2)^2, \quad y = 0, \quad x = 0;$ about $y = -2$

6.3 Exercises

1–4 Use integration to determine the length of the given line segment. Then use the distance formula to check your answer.

1. $y = 3x - 5; \quad 1 \le x \le 7$
2. $x = 2\sqrt{2}y + 1; \quad \sqrt{2} \le y \le \sqrt{18}$
3. $y = \frac{1}{2}x + 1; \quad 2 \le x \le 8$
4. $x = 3 - \frac{1}{3}y; \quad 0 \le y \le 6$

5–16 Determine the arc length L of the curve defined by the equation over the given interval.

5. $y = \frac{5 + 2x^{3/2}}{3}; \quad 1 \le x \le 8$
6. $y = x^{3/2} - \frac{1}{3}\sqrt{x}; \quad 1 \le x \le 4$
7. $y = \frac{x^2}{8} - \ln x; \quad 1 \le x \le e$
8. $y = \frac{x^3}{3} + \frac{1}{4x}; \quad 1 \le x \le 3$
9. $yx^3 - x^8 = \frac{1}{60}; \quad \frac{1}{2} \le x \le 1$
10. $8x^2y = x^6 + 2; \quad 1 \le x \le 3$
11. $y = \sqrt{1 - x^2}; \quad 0 \le x \le 1$
12. $y = \sqrt[3]{x}\left(x^{4/3} - \frac{9}{20}\right); \quad 1 \le x \le 8$
13. $y = \frac{e^x}{4} + e^{-x}; \quad 0 \le x \le 1$
14. $y = \int_{1/2}^{x} \sqrt{\cos 2t}\, dt; \quad 0 \le x \le \frac{\pi}{4}$
15. $x = \int_{1/3}^{y} \sqrt{\frac{1}{t^2} - 1}\, dt; \quad \frac{1}{e} \le y \le 1$
16. $x = \ln(\cos y); \quad 0 \le y \le \frac{\pi}{4}$

17–25 Set up, but do not evaluate, an integral defining the arc length of the graph of the equation over the given interval. Then find the corresponding arc length function $s(x)$ or $s(y)$ as appropriate.

17. $y = x^2 + 1; \quad 0 \le x \le 2$
18. $3x - y^2 = y + 2; \quad 1 \le y \le 4$
19. $x + y = y^2; \quad 0 \le y \le 1$
20. $y = \frac{1}{x - 2}; \quad 3 \le x \le 4$
21. $y = \frac{1}{x^2}; \quad 1 \le x \le 2$
22. $y = 2\ln x; \quad 1 \le x \le e$
23. $y = e^{2 - x}; \quad 0 \le x \le 2$
24. $y = \cos x; \quad \frac{\pi}{2} \le x \le \frac{3\pi}{2}$
25. $y = \sin^{-1} x; \quad -\frac{1}{2} \le x \le \frac{1}{2}$

26–35 Find the surface area of the solid obtained by revolving the indicated curve about the x-axis.

26. $y = \frac{1}{2}x - 1; \quad 2 \le x \le 5$
27. $x = 5 - y; \quad 1 \le x \le 3$
28. $x = \sqrt[3]{y}; \quad 0 \le y \le 1$
29. $y = 2\sqrt{x}; \quad 0 \le x \le 4$
30. $x = y^2 + 3; \quad 1 \le y \le \sqrt{2}$
31. $y = x^3 + \frac{1}{12x}; \quad \frac{1}{2} \le x \le 1$
32. $6y\sqrt{x} = 12x^2 - x; \quad 1 \le x \le 2$
33. $y = e^x + \frac{e^{-x}}{4}; \quad 0 \le x \le \frac{1}{2}$
34. $y = \sqrt{4x - x^2}; \quad 0 \le x \le 4$
35. $x^2 + y^2 = 2x; \quad 1 \le x \le 2, \quad 0 \le y \le 1$

36–41 Find the surface area of the solid obtained by revolving the indicated curve about the y-axis.

36. $3y = x + 1; \quad 1 \le y \le 3$
37. $4y = x^2; \quad 0 \le y \le 2$
38. $y = \sqrt[3]{4x}; \quad 0 \le y \le 1$
39. $2y = x^2 - 1; \quad 0 \le y \le 1$
40. $20xy^{-1/3} = 20y^{4/3} - 9; \quad 0 \le y \le 1$
41. $12xy = 4y^4 + 3; \quad 1 \le y \le 2$

42–45 Find the surface area of the solid obtained by revolving the given curve about the indicated line. (**Hint:** Remember that $A = \int 2\pi r\, ds$, where r denotes the radius from the axis of revolution to the surface and ds is the arc length differential.)

42. $y = x - 1;\quad 1 \le x \le 3;\quad$ about $y = -2$

43. $y = \frac{1}{2}x - 1;\quad 0 \le y \le 2;\quad$ about $x = 1$

44. $12xy = 6y^4 + 2;\quad 1 \le y \le 2;\quad$ about $x = -\frac{1}{4}$

45. $2y = e^x + e^{-x};\quad 0 \le x \le 1;\quad$ about $y = -1$

46–53 Set up, but do not evaluate, an integral defining the surface area of the solid obtained by revolving the given curve about the indicated axis.

46. $y = \cos x;\quad \frac{\pi}{2} \le x \le \frac{3\pi}{2};\quad$ about the x-axis

47. $y = \sin^{-1} x;\quad 0 \le x \le \frac{\sqrt{3}}{2};\quad$ about the x-axis

48. $y = \ln x;\quad 1 \le x \le e;\quad$ about the y-axis

49. $y = \frac{1}{x^2};\quad 2 \le x \le 4;\quad$ about the x-axis

50. $x^2 - 7 = y^2 + 4y;\quad 0 \le y \le 2;\quad$ about the y-axis

51. $y = \sqrt[3]{x};\quad 1 \le x \le 8;\quad$ about $y = -3$

52. $x(y-5) = 1;\quad 6 \le y \le 8;\quad$ about the y-axis

53. $y = x^4 - 1;\quad 1 \le x \le \sqrt[4]{2};\quad$ about $x = -2$

54. By generalizing Exercise 11, prove that the circumference of a circle of radius R is $C = 2\pi R$.

55. Recall that the equation of the astroid of Exercise 28 of Section 3.5 is $x^{2/3} + y^{2/3} = 10$. Rotate the graph about the x-axis and find the surface area of the resulting solid. Do you get the same answer if you rotate about the y-axis? Why?

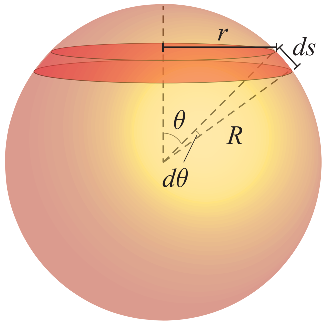
56. The shape of a clothesline stretched between two trees at a campsite can be approximated by the equation $y = \frac{1}{10}x^2 + 1,\ -1.5 \le x \le 1.5$ (distance is measured in yards). Find a formula for the length of the clothesline.

57. A particle is moving in the two-dimensional coordinate system so that its position (the x- and y-coordinates) as a function of time is given by $x = t/2$ and $y = t^{3/2},\ 0 \le t \le 4$.

a. Sketch the path of the particle. (**Hint:** Paying attention to the domain of the variable t, use t to express y in terms of x.)

b. Find the distance traveled by the particle. (**Hint:** Use the formula you found in part a. Alternatively, you may observe that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, and use the notation of Example 4.)

58. Show that the surface area of a sphere of radius R is $4\pi R^2$ by the following steps.



a. Show that the distance r in the diagram above is given by $r = R\sin\theta$ and note that the arc length differential is given by $ds = R\,d\theta$.

b. Determine the appropriate limits of integration and apply the surface area formula $A = \int 2\pi r\, ds$.

59.* Suppose that a pair of parallel planes intersect a sphere, as in the illustration provided for Exercise 58. However this time assume that the fixed distance between the planes is D units. Modify your argument in Exercise 58 to prove that the surface area of the zone of the sphere that falls between the planes is $A = 2\pi RD$. (Notice that this area depends only on the distance between the two parallel planes, not their actual location!)

60. Recall that the lateral surface area of a circular cone of slant height s and base radius r is $A = \pi rs$ (circumference of base $\cdot$ slant height/2). Use the surface area integral of this section to verify this formula. (**Hint:** Rotate the line segment $y = rx/\sqrt{s^2 - r^2},\ 0 \le x \le \sqrt{s^2 - r^2}$ around the y-axis.)

61. Find the surface area of the solid generated by revolving the region bounded by the graphs of $x^2 = 5y$ and $5y + 36 = 5x^2$ about the y-axis.

62. Use the methods of this section to find the surface area of the torus that is obtained by revolving the circle $x^2 + (y-2)^2 = 1$ about the x-axis.

63.* Generalize your solution to Exercise 62 to obtain the surface area of the torus of Exercise 84 in Section 6.1.

64.* If we rotate the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ about the x-axis, the resulting solid is called an ellipsoid. Find the surface area of this solid. (**Hint:** Handle the integral of type $\int\sqrt{k^2-u^2}\,du$ by substituting $u=k\sin\theta$. Next, for the integral of type $\int\cos^2 t\,dt$ use the formula $\cos^2 t=\left(1+\cos 2t\right)/2$.)

65.* Generalize Exercise 64 for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ($a>b$) to finds the surface area of the resulting ellipsoid.

6.3 Technology Exercises

66–74. Use the formulas you found and the integration capabilities of a graphing utility to evaluate the arc length integrals in Exercises 17–25.

75–82. Use the formulas you found and the integration capabilities of a graphing utility to evaluate the surface area integrals in Exercises 46–53.

83. Use a graphing utility to find the length of the clothesline in Exercise 56.

84. Rotate the parabola of Exercise 56 about the vertical axis. Find the surface area of the resulting paraboloid. (For example, a large satellite dish might have this shape.)

6.4 Exercises

1–3 Find the moment M_0 about the origin and the center of mass $\bar{x}$ for the point masses located on the x-axis.

1. $m_1 = 4, x_1 = -3;\quad m_2 = 5, x_2 = -1;\quad m_3 = 2, x_3 = 6$

2. $m_1 = 2, x_1 = -5;\quad m_2 = 10, x_2 = -2;$
$m_3 = 8, x_3 = 1;\quad m_4 = 3, x_4 = 7$

3. $m_1 = 3.5, x_1 = -10;\quad m_2 = 5, x_2 = -2;$
$m_3 = 2, x_3 = 5.5;\quad m_4 = 2.5, x_4 = 11$

4–6 Find the moments M_x, M_y about the coordinate axes and the center of mass for the system of point masses.

4. $m_1 = 4, P_1(-6,-8);\quad m_2 = 5, P_2(1.5,2);$
$m_3 = 2, P_3(3,4)$
(What do you notice about $\bar{x}$?)

5. $m_1 = 2, P_1(-2,6);\quad m_2 = 4, P_2(-1,-5);$
$m_3 = 7, P_3(2,0);\quad m_4 = 8, P_4(3,4)$

6. $m_1 = 2.5, P_1(-9,0);\quad m_2 = 3, P_2(-4.5,0.25);$
$m_3 = 2, P_3(0,-2.5);\quad m_4 = 6, P_4(2.25,5);$
$m_5 = 1.5, P_5(5.5,-3.5)$

7. Tyler and Christina are sitting on the ends of a 14-foot seesaw, unable to balance it because of their weight difference. However, Tyler's little sister, Lisa, is quick to come to the rescue. If Tyler and Christina weigh 110 and 80 pounds, respectively, while little Lisa is 35 pounds, where should she sit in order for the seesaw to balance? (**Note:** Strictly speaking, when you are multiplying weight by distance, you are calculating torque rather than moment, but the technique in obtaining balance is the same.)

8. The design of a certain front-wheel-drive family sedan allows for 54% of its total mass to rest on the front axle, while 46% is resting on the rear axle. The distance between the front and rear axles (the wheelbase of the car) is 2.65 meters. How far behind the front axle is the car's center of mass?

9. An experimental rocket is fired toward the north and is on track to hit the target when a midair explosion breaks it apart. The tail section, which is twice as heavy as the nosepiece, is found 200 yards southwest from the intended target. Where is the nosepiece likely to be found? (**Hint:** Even after breaking apart, the center of mass of the rocket will arrive in the target area.)

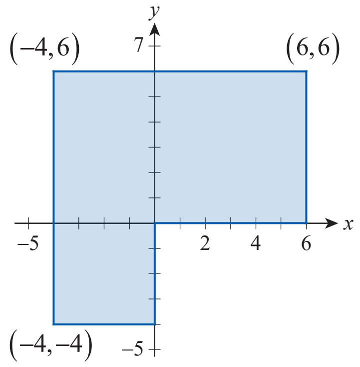
10–11 A thin rod of length l, with the given continuously varying density is placed into the coordinate system so that it lies horizontally on the x-axis with its left endpoint coinciding with the origin. Find its center of mass.

10. $l = 2.25$ m, $\quad \rho = 1 + \sqrt{x}$ kg/m

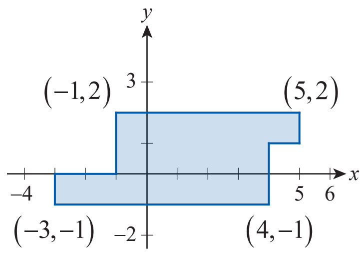
11. $l = 110$ cm, $\quad \rho = 0.6 + 0.01x$ g/cm

12–15 Use the indicated coordinates to determine the center of mass of the given region. (**Hint:** Divide the region into appropriate subregions and treat the centers of mass of the subregions as point masses.)

12.



13.



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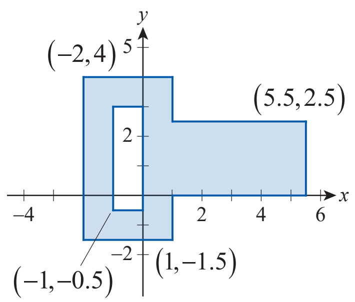

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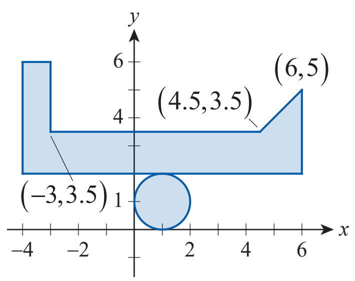


16. Show that if the plate of Example 3 has constant density, then its center of mass is located at $\left(\frac{3}{4}, \frac{3}{10}\right)$.

17–40 Find the centroid of the plane region bounded by the given curves. If possible, use symmetry to simplify your calculations. (In Exercise 31, use the formula $\cos^2 x = (1+\cos 2x)/2$ before integrating.)

17. $y = x^2 + 4x, \quad y = 0$

18. $y = \sqrt{4-x}, \quad x = 0, \quad y = 0$

19. $y = 2x^2, \quad 32x + y^2 = 0$

20. $y = \dfrac{x^2}{4}, \quad x = 2, \quad y = 0$

21. $y = \dfrac{\sqrt{x}}{2} + 1, \quad y = \dfrac{1}{4}x + 1$

22. $y = x^2, \quad y = x^3$

23. $y = 3 - x, \quad x = 0, \quad y = 0$

24. $x + 2y = 13, \quad 2x - y + 4 = 0, \quad x = 0$

25. $y = \dfrac{1}{x}, \quad x = 1, \quad x = 3, \quad y = 0$

26. $y = \sqrt{2x}, \quad y = x$

27. $y = x^2 - 9, \quad y = 0$

28. $y = x^2 - 9, \quad x \ge 0, \quad y = 0$

29. $y = x^3, \quad y = 4x, \quad x \ge 0$

30. $y = \sqrt{x}, \quad y = x^2$

31.* $y = \cos^2 x, \quad y = 0, \quad 0 \le x \le \pi$

32. $xy = 1 \quad x = 1, \quad y = 3$

33. $y = \dfrac{1}{x^2}, \quad y = -\dfrac{1}{x^2}, \quad 1 \le x \le 2$

34. $y = x^{2/3}, \quad x = 0, \quad y = 1$

35. $y = (x+2)^2, \quad y = (x-2)^2, \quad y = 0$

36. $y = 2\sqrt{x+4}, \quad 2y = 8 - x, \quad y = 0$

37. $y = x + 1, \quad y(x+1) = 1, \quad y = 0, \quad x = 1$

38. $y = \sqrt{3-x}, \quad y = 2\sqrt{-x}, \quad y = 0$

39. $4x^2 + 9y^2 = 49, \quad x \ge 0, \quad y \ge 0$

40. $x^4(1-x^2) = y^2, \quad x \ge 0, \quad y \ge 0$

41–50 Find the center of mass of the plane region of varying density that is bounded by the given curves.

41. $3y = 6 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = x$

42. $3y = 6 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = 1 + y$

43. $y = x^{3/2}, \quad y = \sqrt{x}; \quad \rho(x,y) = \sqrt{x}$

44. $y = \sqrt{1-x^2}, \quad x \ge 0, \quad y = 0; \quad \rho(x,y) = x$

45. $y = 4 - x^2, \quad y = 2 - x; \quad \rho(x,y) = 2 + x^2$

46. $y = \dfrac{3}{x^3}, \quad x = \dfrac{1}{2}, \quad x = 1, \quad y = 0; \quad \rho(x,y) = 1 - x^3$

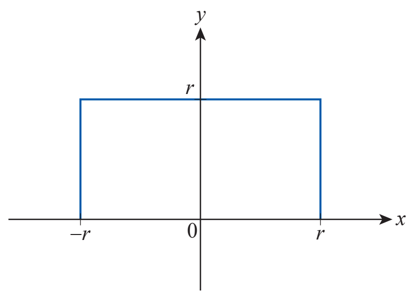
47. $y = \sqrt[4]{x}, \quad y = \sqrt{x}; \quad \rho(x,y) = 1 - y$

48. $xy^2 = 3, \quad x = 0, \quad y = \dfrac{1}{3}, \quad y = 1; \quad \rho(x,y) = y^3 - y^2$

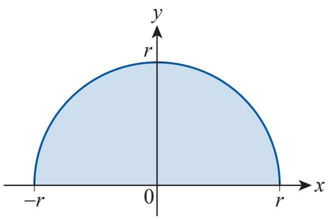
49. $y + 2x = 5, \quad x = y + 1, \quad y = 0; \quad \rho(x,y) = 5 - x$

50. $x = \sqrt{y}, \quad xy = 1, \quad x = 0, \quad y = 2; \quad \rho(x,y) = y$

51. Find the center of mass of the wire if we modify Example 4 by bending the wire to form three sides of a rectangle as shown in the figure below.



52. Find the centroid of the half disk shown in the figure below. Assume constant density.



53. Find the centroid of the first quadrant of the half disk in Exercise 52.

54.* Find the center of mass of the half disk of Exercise 52 if its density function is $\rho(x,y) = y$. (**Hint:** To eliminate the radical, use the substitution $x = r\sin\theta$ where applicable.)

55. Find the center of mass of the wire in Example 4 if it has a variable density of $\rho(\theta) = |\cos\theta|$.

56.* Prove that the center of mass of a triangle is the intersection of its medians. (**Hint:** Recall what you learned in geometry about the intersection point of the medians.)

57. Prove that the distance of the center of mass of a triangle from each side is one third of the corresponding altitude. (**Hint:** Try to prove the result for right triangles first.)

58.* The graph of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse centered at the origin. Find the centroid of the plane region bounded by the first quadrant of this ellipse and the coordinate axes.

59. Find a formula for the centroid of the region bounded by $y=\sqrt[n]{x}$, $y=0$, and $x=1$ as a function of n. What can you say if $n\to\infty$?

60. Use Pappus' Theorem for volumes to answer Exercise 52 provided that you know the formula for the volume of the sphere: $V=\frac{4}{3}\pi r^3$.

61. Use Pappus' Theorem for volumes to find the centroid of a right triangle. (**Hint:** Place a right triangle into the coordinate system so that its vertices coincide with the origin and the points $(h,0)$ and $(0,R)$. Then rotate the triangle and use the volume formula for the right circular cone, along with Pappus' Theorem.)

62. Use Pappus' Theorem to find the volume of the solid resulting from revolving the triangle with vertices $(-3,1)$, $(4,1)$, and $(2,7)$ about the line $y=-2$.

63. Rotate the half disk of Exercise 52 about the line $y=r$. Use Pappus' Theorem to find the volume of the resulting solid.

64. Rotate the region of Exercise 24 about the line $y=-4$. Use Pappus' Theorem to find the volume of the resulting solid.

65. Rotate the region of Exercise 28 about the line $y=1$. Use Pappus' Theorem to find the volume of the resulting solid.

66. Rotate the region of Exercise 30 about the line $y=3$. Use Pappus' Theorem to find the volume of the resulting solid.

67. Use Pappus' Theorem to find the volume of the solid generated by revolving the rectangle with vertices $(-1,4)$, $(5,4)$, $(5,8)$, and $(-1,8)$ about the line $y=-x-2$.

68. Use the result of Exercise 40 along with Pappus' Theorem to find the volume of the solid obtained by revolving the right loop of the curve $x^4\left(1-x^2\right)-y^2=0$ about the y-axis.

69. Use the result of Example 4 along with Pappus' Theorem to find the surface area of the solid in Exercise 63.

70. Use Pappus' Theorem for surface areas to verify that the lateral surface area of a right circular cone of slant height s and base radius r is $A=\pi rs$. (**Hint:** Appropriately place a line segment of length s in the coordinate system and rotate to obtain the cone.)

71. Generalize your solution to Exercise 70 to verify the formula for the lateral surface area of a frustum of a cone (for the definition, see the discussion preceding Example 3 in Section 6.3).

Concept Check

72–75 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

72. The center of mass of a thin triangular plate always coincides with the intersection of its medians (this point is also called the centroid of the triangle).

73. The center of mass of an object is always a point on the object itself.

74. Pappus' Theorem makes integration unnecessary for calculating volumes.

75. If a child is 50% heavier than another child, then in order for them to balance on a seesaw, he or she has to sit 50% closer to the pivot point.

Multiplying both ends of this last equation by either ρg or δ (depending on the context), we obtain the following result.

$$F = \rho g\bar{y}A = \delta\bar{y}A$$

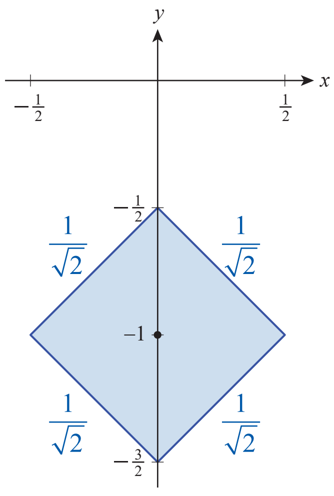


Figure 10

Example 8 Using the Centroid and Area of a Vertical Plate to Find the Fluid Force Exerted on It

Calculate the fluid force on the square cover plate of Example 7 using its centroid and area.

Solution

The centroid of the plate is 1 m from the water's surface, and the area of the square plate is $\frac{1}{2}$ m^2.

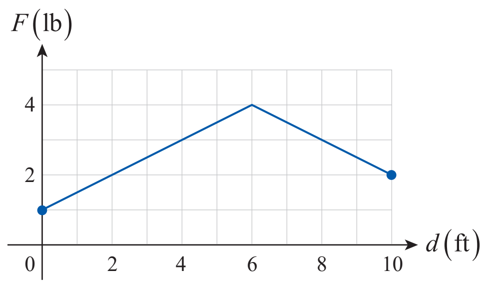
$$F = \rho g\bar{y}A = \left(1000\ \text{kg}/\text{m}^3\right)\left(9.81\ \text{m}/\text{s}^2\right)\left(1\ \text{m}\right)\left(\frac{1}{2}\ \text{m}^2\right)$$
$$= 4905\ \frac{\text{kg}\cdot\text{m}}{\text{s}^2} = 4905\ \text{N}$$

6.5 Exercises

1. A dad is pushing his child on a sled with a constant, horizontal force a distance of 100 m. The child and sled together have a mass of 35 kg and the coefficient of friction is $\mu = 0.1$. Find the work done against friction. Use $g \approx 9.81\ \text{m}/\text{s}^2$. (**Hint:** For a refresher on friction and coefficient of friction, see Example 5 of Section 1.5.)

2. An object of mass 10 kg is pulled 8 m up a 30° ramp. If the coefficient of friction is $\mu = 1/\left(4\sqrt{3}\right)$, find the work done during the process. (**Hint:** Work needs to be done against both friction and gravity. The normal force between the object and the surface of the ramp is $F_\perp = mg\cos 30°$.)

3. The graph of a variable force is shown below as it moves an object in a straight line a distance of 10 ft. Find the work done during this process.



4. A 15 cm long unstressed spring requires a force of 3 N to be held stretched to 20 cm. How much work will be done in stretching the spring an additional 5 cm?

5. A force of 50 lb stretches an 8 ft spring by $\frac{1}{2}$ ft. Find the work done in stretching the spring **a.** from its original length to 10 ft and **b.** from 11 ft to 13 ft.

6. A particle is moving along the y-axis from the origin to the point $(0,4)$ under the influence of the variable force $F(y) = \frac{1}{2}y^2 + \sqrt{y}$ (units are in meters and newtons, respectively). Find the work done by the force.

7. If a 3 lb force compresses a 5 ft spring by 18 in., how much work is done in **a.** compressing it from its original length to 3 ft and **b.** stretching it to 6.75 ft?

8. When a mass of 100 g is hung on a vertical spring, the spring is stretched by 4 cm. How much work is done in stretching the spring an additional 5 cm?

9. A bumping post at a railway company has a spring constant of $3.8\times10^5\ \text{N}/\text{cm}$. Find the work done in compressing this spring by 2 cm.

10. Suppose that $12.5\ \text{ft}\cdot\text{lb}$ of work is needed to stretch a spring from its unstressed length of 1.5 ft to 2 ft. Find the spring constant.

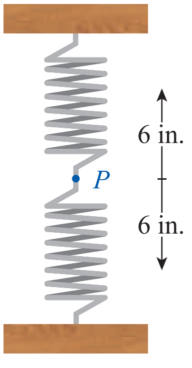
11. If the unstressed length of a spring is 20 cm and 0.12 J of work is required to stretch it from 22 cm to 24 cm, how much work is required to stretch it from 24 cm to 30 cm?

12. Suppose the work done in stretching a spring from 8 in. to 10 in. is $\frac{1}{2}$ ft · lb, and an additional 2 ft · lb of work will stretch it by another 4 in. Use this information to find the unstressed length of the spring.

13. We will call a spring linear if it obeys Hooke's Law (recall that this can only be expected between reasonable limits). Prove that the work done in stretching (or compressing) a linear spring by x units from its original length is $W = \frac{1}{2}kx^2$, where k is the spring constant.

14. Suppose that an elastic rope with an unstressed length of 20 ft behaves in the following, nonlinear manner: the force required to stretch it by x ft is $F = kx^{5/4}$ lb. If a force of 3 lb stretches the rope to 21 ft, how much work is done in stretching it from 21 ft to 22 ft?

15. Two identical springs with a spring constant of 5 lb/ft are attached to each other at point P, with their other ends fastened to the top and bottom of a wooden box, so that they are in a vertical, unstressed position. Find the work done in moving point P vertically up or down by 6 in. (Ignore the weight of the springs.)

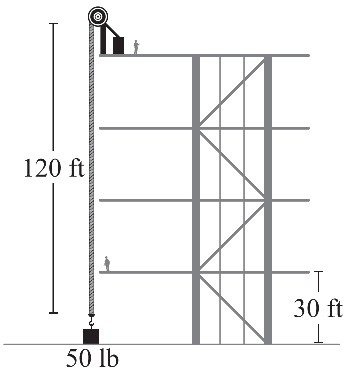


16. A 15 ft chain that weighs 4 lb/ft is hanging from a cylindrical drum so that its other end touches the ground.

a. How much work does it take to wind it up completely?

b. How much work does it take to wind up only two-thirds of the chain?

17. A 120 ft cable is supporting a 50 lb piece of equipment at a construction site. In order to lift the equipment by 30 ft, the cable is wound on a cylindrical drum. If the cable weighs 3 lb/ft, find the work done during this process.



18. A cable starts to unwind from a cylindrical drum at $t = 0$, at a rate of 0.5 m/s. If the cable weighs 20 N/m, find the work done by gravity from $t = 0$ to $t = 12$ s. (**Hint:** Let x denote the length of cable already unwound at time t.)

19. Find the work done in lifting 300 kg of coal from a 400 m deep mine by a rope that weighs 35 N/m. (**Note:** The original term *horsepower* was coined by James Watt after actually watching ponies lift coal from a mine and calculating their work done in unit time.)

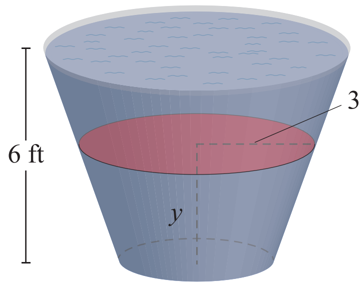
20. A 5 lb bucket is used to draw water for livestock from a 60 ft deep well. The bucket weighs 50 lb when full, but is leaking water at 0.1 lb/s. If it is being pulled at a rate of 1 ft/s, find the work done in getting it to the surface. (Ignore the weight of the rope).

21. A crane is lifting a leaky container full of fresh liquid mortar at a construction site. If the container weighs 40 kg and is able to hold 250 kg of mortar, but is leaking at a rate of 0.6 kg/s and is being lifted at 1.5 m/s for 10 s, find the work done by the crane.

22. A cylindrical tank of depth 5 ft and radius 4.087 ft holds about the same volume as the tank in Example 5. If such a tank is filled with gasoline, determine the work required to pump all the gasoline to the top of the tank, and compare the answer to part a. of Example 4 and to Example 5.

23. Find the work required to pump the gasoline out of the tank of Example 4 over its top if the tank stands on its base. Compare your answer to part a. of Example 4 and explain.

24. A tank in the shape of an inverted cone frustum has cross-sections of radius $3+\frac{1}{3}y$ feet at an altitude of y feet above the base. If its height is 6 feet and it is filled with water, how much work is done to pump all of the water out over the top of the tank? (For the weight density of water, see Example 6.)



25. A rectangular tank of base 5 ft by 8 ft and of height 10 ft is filled with oil. The weight density of oil is 50 lb/ft^3.

a. Find the work required to pump the oil out of the tank through an outlet on the top.

b. Find the work required if the tank is only half full at the start of pumping.

c. Is the answer you gave in part b. half of that given in part a.? Why?

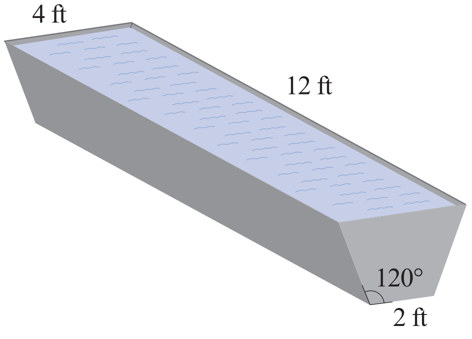
d.*How long will it take for a $\frac{1}{2}$-horsepower pump to empty the tank? What if we start with the tank half full? (One horsepower (hp) is $550 \text{ ft}\cdot\text{lb/s}$.)

26. A cistern in the form of an inverted rectangular pyramid with a square base of side length 2 m and a depth of 4 m is full of water. The weight density of water is $\delta = 9810 \text{ N/m}^3$.

a. How much work is required to pump all of the water out over the top edge of the cistern?

b. How long will it take for a 1 hp electric pump motor to do the job? One horsepower is 746 watts (W), where $1 \text{ W} = 1 \text{ J/s}$.

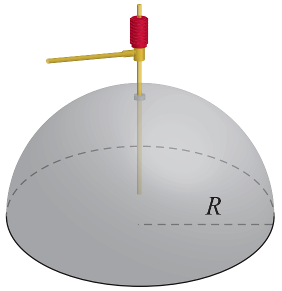
27. The two ends of a watering trough are isosceles trapezoids sitting on the shorter base which is 2 ft, with the legs making 120° angles with that base. The opening of the trough is a 4 ft by 12 ft rectangle. Starting with a full trough, find the energy expended (i.e., total work done) in pumping all of the water out of the trough over its top edge. (For the weight density of water, see Example 6.)



28. Answer Exercise 27 if the ends of the trough are semicircular with a radius of 2 ft.

29. A water trough with the same top opening as that in Exercise 27 has vertical cross-sections in the form of an equilateral triangle. Find the work required to fill it with water through an opening on its bottom.

30. How much work is necessary to pump all fluid of weight density δ out of a hemispherical container of radius R through an opening in its top?



31. Find the work required if the tank in Exercise 30 is inverted.

32. The shape of a kerosene tank can be approximated by rotating the graph of $y = x^4$, $-2 \le x \le 2$, about the y-axis (units in meters). Find the work required to fill up the empty tank through an opening at its lowest point $x = 0$. The weight density of kerosene is 8016.24 N/m^3.

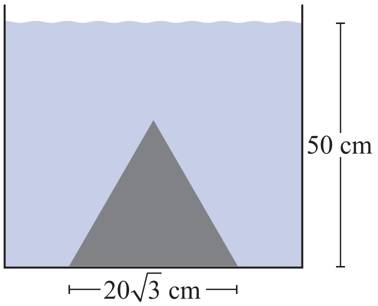
33. According to Archimedes' Principle, the buoyant force acting on a body immersed in fluid is equal to the weight of the fluid it displaces. Use this principle to find the work required to completely immerse in water a cube of negligible weight if its edges are 1 ft. (**Hint:** Keep the top horizontal during immersion.)

34. Repeat Exercise 33 for a buoy that is a circular cone of base radius R and height h, when it is immersed vertically, vertex first in a fluid of weight density δ.

35–43 Use the integral formula for fluid force to answer the question.

35. A 2 ft by 3 ft rectangular plate is positioned vertically on the bottom of an 8 ft pool. Find the fluid force exerted on one side of the plate if it sits **a.** on its long edge and **b.** on its short edge.

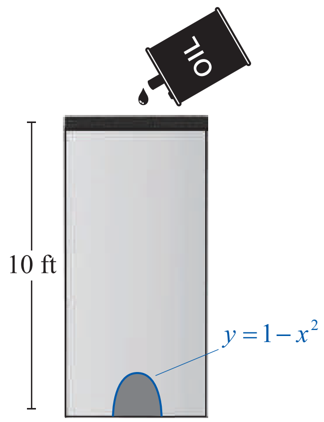
36. An equilateral triangle of side length $20\sqrt{3}$ cm is standing on its base that is 50 cm underwater. Find the fluid force against one face of the triangle. Use $\delta = 9.81 \cdot 10^{-3}\ \text{N/cm}^3$ for the weight density of water.



37. Find the force against one face of the triangle from Exercise 36 if the upper third of its vertical altitude is sticking out of the water.

38.* Answer Exercise 36 if the triangle is tilted, making a 30° angle with the horizontal.

39. Suppose that a vertical, parabolic gate is installed on a vertical side of a 10 ft deep gas tank. The gate is given by the equation $y = 1 - x^2$, so that $y = 0$ coincides with the bottom edge of the tank (units in feet). If the gate is designed to withstand a maximum force of 600 lb, verify that it won't break when gas is stored in the tank. Will the gate break if the tank is filled up with oil? The weight densities of gas and oil are $44\ \text{lb/ft}^3$ and $50\ \text{lb/ft}^3$, respectively.

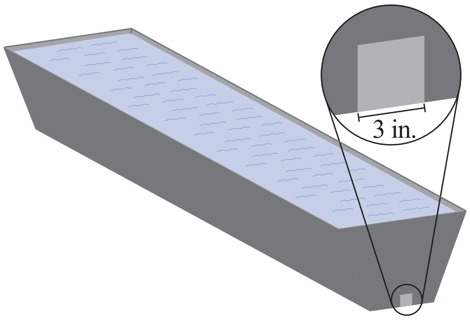


40. Find the force exerted on the end of the full trough in Exercise 27.

41. Find the force exerted on the end of the full trough in Exercise 29.

42.* Find the force exerted on one end of a trough with the same ends and opening as the one in Exercise 29, but with its ends tilted outward by 45°.

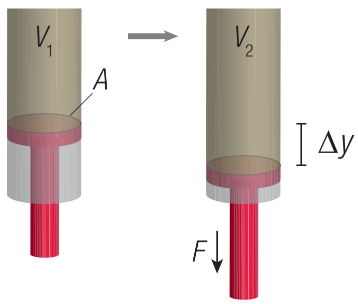
43. Suppose that an outlet near the bottom of the trough in Exercise 27 is covered by a 3 in. by 3 in. square plate, so that the lower edge of the plate coincides with that of the trough. What force is pressing against the plate?



44–47 Consider a certain gas of pressure P and volume V_1 confined in a cylinder, closed on one end by a moveable piston. If A is the area of the piston, the force acting on the piston is $F = PA$. Thus, as the gas expands, pushing the piston by a small increment of Δy, the work done on the piston is $\Delta W = F\Delta y = PA\Delta y = P\Delta V$. Integrating, we obtain the work done by the gas as it expands from a volume of V_1 to V_2.

$$W = \int_{V_1}^{V_2} P\,dV$$

(**Note:** This argument can be "reversed" to obtain the work required to compress the gas.)



In Exercises 44–47, apply the above formula.

44. Assume that 0.5 ft^3 of gas in a cylinder under an initial pressure of $300\ \text{lb/ft}^2$ expands to 1.5 ft^3. Assuming that $PV = c$ (a constant) during this process, find the work done by the gas on the piston.

45. Suppose that 3 L of gas under initial pressure of 150 kPa in a cylinder is compressed to 1 L. Find the work done by the piston. (Assume, as in Exercise 44 that $PV = c$.)

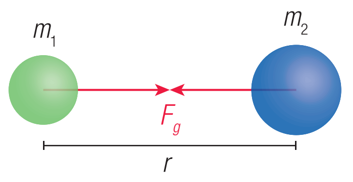
46. Repeat Exercise 44 under the assumption that $PV^{1.4} = c$. (This happens when heat loss is negligible. We say that in this case, P and V are related adiabatically.)

47. Repeat Exercise 45 under the assumption that $PV^{1.4} = c$.

48–55 According to Newton's Law of Gravitation, two masses m_1 and m_2 attract each other by a force that is directly proportional to the product of their masses and inversely proportional to the square of their distance (or rather the square of the distance between their respective centers of gravity):

$$F_g = \frac{m_1 m_2 G}{r^2},$$

where G is the universal gravitational constant. Its value in metric units is $6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.



48. Use the above formula to show that the acceleration caused by gravity on a free-falling mass near Earth's surface is approximately

$$g = \frac{MG}{R^2},$$

where M and R are the mass and radius of Earth, respectively (we are assuming that Earth is perfectly spherical).

49. Show that if an object of weight w is launched to a height h above the surface of Earth, then Earth attracts it by a force of

$$F_g(h) = \frac{R^2 w}{(R+h)^2}.$$

(**Hint:** Use the fact that $w = mg$ along with the result of Exercise 48.)

50. Find the work done against gravity in moving a 3-pound object to an altitude of 500 miles above the surface of Earth. Assume that the radius of Earth is approximately 4000 miles. (**Hint:** Integrate the variable force obtained in Exercise 49 between appropriate limits.)

51. Find the work done against gravity in moving a 2-ton satellite to an altitude of 200 miles above the surface of Earth. Express your answer in the units foot-pounds. (**Hint:** Ignore the work done to accelerate the spacecraft, air resistance, as well as the weight of the launching vehicle and fuel.)

52. How much energy is expended (i.e., work done) in lifting a rocket of mass 10 metric tons to an altitude of 300 km above Earth? Assume the radius of Earth is approximately 6371 km. (See the hint given in Exercise 51.)

53. Calculate the work done in Exercise 52 if the rocket is launched from the moon. The acceleration due to gravity on the moon is $1.6\ \text{m/s}^2$, about $\frac{1}{6}$ of that on Earth. The moon's radius is approximately 1737 km.

54. Find the work done in moving a spacecraft of mass 110 metric tons to an altitude of 300 miles above the surface of Earth. Use 1 mile ≈ 1600 m and $g \approx 9.81\ \text{m/s}^2$. Express your answer in megajoules. (**Hint:** Ignore the work done to accelerate the spacecraft, air resistance, as well as the weight of the launching vehicle and fuel. See Exercise 52 for Earth's radius.)

55. Find the limit of the work done in moving a weight w to a distance d above Earth's surface as $d \to \infty$.

56–57 The magnitude of the force acting between two point charges q_1 and q_2 at a distance of r units from each other, is described by Coulomb's Law, as follows.

$$F = k\frac{|q_1 q_2|}{r^2}$$

Notice the analogy between Coulomb's Law and the law of gravity! The charges are analogous to the masses, while k is analogous to the universal gravitational constant. The approximate value of k is $8.98755 \times 10^9\ \text{Nm}^2/\text{C}^2$. (A coulomb (C) is the SI unit for electric charge. Note that all units are SI; there is no British system of electrical units.) It is also worth noting that Coulomb's Law only gives the magnitude of the force, since electric forces can be attractive or repulsive, while the force of gravity is always attractive.

In Exercises 56–57, use Coulomb's Law.

56. Two like electrical charges of 10^{-4} C each, are 50 cm apart, with one of them fixed. Find the work done in bringing the other charge to a distance of 20 cm from the fixed charge.

57. If two point charges of opposite sign attract each other by a force of 200 N when 3 cm apart, find the work done in moving them from 2 cm apart to 8 cm apart.

58–64 The centroid can be used in some circumstances to simplify work calculations, as well as fluid force calculations. For instance, when the 5 lb rope in Example 3 is extended 50 ft down the well, it behaves, from a center of mass perspective, like a 5 lb point mass 25 ft down the well. The work required to lift such a point mass to the surface is $(5 \text{ lb})(25 \text{ ft}) = 125 \text{ ft} \cdot \text{lb}$. Added to the $500 \text{ ft} \cdot \text{lb}$ of work required to lift the 10 lb bucket 50 ft, we get the total work required: $625 \text{ ft} \cdot \text{lb}$.

In Exercises 58–64, use this centroid argument to find a second solution.

58. Exercise 16

59. Exercise 17

60. Exercise 18

61. Exercise 19

62. Exercise 25

63. Exercise 28 (**Hint:** Use Exercise 52 of Section 6.4.)

64. Exercise 29

65–68 Use the technique of Example 8 to find a second solution.

65. Exercise 33

66. Exercise 34

67. Exercise 36

68. Exercise 41

69. Prove the following statement: If two cylindrical tanks hold the same volume, but the height of tank B is half of the height of tank A, then the work required to pump all liquid out of a full tank A over its top is twice the work required to do the same for tank B. Do the proof in two ways: **a.** by using integration and **b.** by using the centroid approach.

6.5 Technology Exercises

70–71 Use a graphing utility to solve the problem.

70. Suppose that the rocket in Exercise 52 runs out of fuel after 1.9×10^7 kJ of useful energy is expended. How high will the rocket go?

71. Suppose that in Exercise 25 part d., the pump stops due to an electrical failure 3 minutes after starting the job. What is the fluid level at that instant?

b. $\int_0^{1/3} \frac{dx}{1-x^2} = \left[\tanh^{-1} x\right]_0^{1/3} = \tanh^{-1}\left(\frac{1}{3}\right)$

Note that we used $\tanh^{-1} x$ as the antiderivative (as opposed to $\coth^{-1} x$) because, in the given interval of integration, $x^2 < k^2$ $(k = 1)$. Further, if we apply the result of Exercise 74, we can rewrite our answer as follows.

$$\tanh^{-1}\left(\frac{1}{3}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{2}\ln 2$$

6.6 Exercises

1–4 Find the value of the function.

1. a. $\cosh 0$
   b. $\sinh(-2)$
   c. $\sinh\frac{\pi}{2}$
2. a. $\tanh 0$
   b. $\cosh 1$
   c. $\sinh(\ln 3)$
3. a. $\sinh^{-1} 0$
   b. $\cosh^{-1} 1$
   c. $\sinh^{-1} 2$
4. a. $\tanh^{-1} 0$
   b. $\operatorname{sech}^{-1} 1$
   c. $\operatorname{csch}^{-1}(-3)$

5–10 Use the given equation to classify the hyperbolic function as even or odd. Then use the definition of the function to prove your assertion.

5. $\sinh(-x) = -\sinh x$
6. $\cosh(-x) = \cosh x$
7. $\tanh(-x) = -\tanh x$
8. $\coth(-x) = -\coth x$
9. $\operatorname{sech}(-x) = \operatorname{sech} x$
10. $\operatorname{csch}(-x) = -\operatorname{csch} x$

11–22 Sketch the graph of the equation on a piece of paper. (**Hint:** Study the graphs of the six hyperbolic functions and their inverses in the text.)

11. $y = 1 - \sinh x$
12. $y = \cosh 2x - 3$
13. $y = -\tanh(x-1)$
14. $y = 2 - 2\operatorname{sech}(x-1)$
15. $y = -\frac{1}{2}\operatorname{csch} x + 1$
16. $y = -\frac{1}{4}\coth x - 2$
17. $y = \frac{1}{3}\sinh^{-1}(x-3)$
18. $y = 2\cosh^{-1}(1-x)$
19. $y = -\tanh^{-1}\left(\frac{x}{3}\right)$
20. $y = -3\operatorname{csch}^{-1}(x+2)$
21. $y = \operatorname{sech}^{-1}\left(1-\frac{x}{2}\right) - 1$
22. $y = 2\coth^{-1}\left(\frac{x}{2}+1\right)$

23–42 Verify the given identity.

23. $e^{kx} = \cosh kx + \sinh kx$
24. $e^{-kx} = \cosh kx - \sinh kx$
25. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
26. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
27. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
28. $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
29. $\sinh 2x = 2\sinh x \cosh x$
30. $\cosh 2x = \cosh^2 x + \sinh^2 x$
31. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$
32. $\coth^2 x = 1 + \operatorname{csch}^2 x$
33. $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
34. $\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$
35. $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$
36. $\coth 2x = \frac{1 + \coth^2 x}{2\coth x}$
37. $\sinh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}}$
38. $\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$
39. $\tanh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}}$
40. $(\cosh x + \sinh x)^2 = \cosh 2x + \sinh 2x$
41. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{N}$
42. $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx, \quad n \in \mathbb{N}$

43–50 Verify the differentiation or integration formula.

43. $\frac{d}{dx}(\sinh x) = \cosh x$

44. $\int \sinh x\,dx = \cosh x + C$

45. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

46. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

47. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

48. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

49. $\int \tanh x\,dx = \ln(\cosh x) + C$

50. $\int \coth x\,dx = \ln|\sinh x| + C$

51–62 Find the given derivative.

51. $\frac{d}{dx}(\sinh^2 x)$

52. $\frac{d}{dt}(\operatorname{sech}^2 2t)$

53. $\frac{d}{dx}(\tanh^2 x)$

54. $\frac{d}{dx}\left(\cosh^2 \frac{2x+5}{3}\right)$

55. $\frac{d}{dx}(e^{-x}\sinh 2x)$

56. $\frac{d}{dx}\left[\operatorname{sech}(1-x^2)\right]$

57. $\frac{d}{dx}\sqrt{1+\tanh 2x}$

58. $\frac{d}{dx}\left[\ln(\cosh(2x+1))\right]$

59. $\frac{d}{dt}\left[(t^3+1)\cosh(t^3+1)\right]$

60. $\frac{d}{dz}\left[\frac{1}{z}\tanh(z^2)\right]$

61. $\frac{d}{dy}\left[\cosh(\ln(2y+1))\right]$

62. $\frac{d}{dx}\left[\frac{\sinh(\ln x)}{x+1}\right]$

63–71 Evaluate the given integral.

63. $\int \frac{1+\sinh^2 x}{\cosh x}\,dx$

64. $\int \operatorname{sech}^2\left(\frac{x}{3}+5\right)dx$

65. $\int \frac{\operatorname{sech}\frac{1}{z}\tanh\frac{1}{z}}{z^2}\,dz$

66. $\int \frac{\coth\sqrt{x}}{\sqrt{x}}\,dx$

67. $\int \tanh(2-w)\,dw$

68. $\int x\cosh(x^2)\,dx$

69. $\int_0^1 \operatorname{csch}^2(2-t)\,dt$

70. $\int_0^3 \operatorname{sech}^2 u\sqrt{\tanh u}\,du$

71. $\int_0^2 \frac{\sinh t}{\sqrt{4-\cosh^2 t}}\,dt$

72. By differentiating, verify the given identity.

$$\frac{d}{dx}\left[\tan^{-1}(\sinh x)\right] = \frac{d}{dx}\left[\sin^{-1}(\tanh x)\right] = \operatorname{sech} x$$

73–74 Given that the hyperbolic functions can be expressed in terms of exponential functions, it's not surprising that their inverses can be expressed in terms of logarithms. For example, if we let $y = \sinh^{-1} x$, then $x = \sinh y$ and hence

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$e^y - 2x - e^{-y} = 0$$

$$e^{2y} - 2xe^y - 1 = 0 \qquad \text{Multiply through by } e^y.$$

$$(e^y)^2 - 2xe^y - 1 = 0 \qquad \text{Express as a quadratic in } e^y.$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \qquad \text{Solve for } e^y.$$

$$e^y = x + \sqrt{x^2+1} \qquad x - \sqrt{x^2+1} < 0 \text{ but } e^y > 0, \text{ so discard } x - \sqrt{x^2+1}.$$

$$y = \ln\left(x + \sqrt{x^2+1}\right). \qquad \text{Take the natural logarithm of both sides.}$$

Use the procedure above to verify the identity.

73. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \ge 1$

74. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$

(**Hint:** Begin by setting $y = \tanh^{-1} x$; then write the equation as $\tanh y = x$, square both sides, and apply the identity $\tanh^2 y = 1 - \operatorname{sech}^2 y$. Solve the result for $\cosh y$, apply $\cosh^{-1}$ to both sides, and apply the result of the previous exercise. Then apply some logarithmic properties.)

75–77 Given a function f, let $1/f$ denote its reciprocal—that is, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$. The following is a useful relationship between the functions f^{-1} and $(1/f)^{-1}$, assuming both of these inverse functions exist.

$$\left(\frac{1}{f}\right)\left(f^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{f\left(f^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

$$\Rightarrow \left(\frac{1}{f}\right)^{-1}(x) = f^{-1}\left(\frac{1}{x}\right)$$

Applied to inverse hyperbolic functions, this fact indicates the following relationships.

$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$ $\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$ $\coth^{-1} x = \tanh^{-1}\frac{1}{x}$

Use these relationships to verify the equality.

75. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right); \quad x \neq 0$

76. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); \quad 0 < x \leq 1$

77. $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right); \quad |x| > 1$

78–81 Prove the given formula for the derivative of an inverse hyperbolic function.

78. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$

79. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

80. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}, \quad |x| < 1$

81. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}, \quad |x| > 1$

82–88 Find the given derivative.

82. $\frac{d}{dx}\left[\frac{1}{2}\sinh^{-1}(5x)\right]$

83. $\frac{d}{dx}\left[\cosh^{-1}\left(\frac{7x}{2}\right)\right]$

84. $\frac{d}{dx}\left[\operatorname{sech}^{-1}(\sin x)\right]$

85. $\frac{d}{dx}\left[\tanh^{-1}(\cos 2x)\right]$

86. $\frac{d}{dx}\left[\cosh^{-1}(\sec x)\right]; \quad 0 < x < \frac{\pi}{2}$

87. $\frac{d}{dx}\left[\coth^{-1}\left(\frac{1}{2x}\right)\right]$

88. $\frac{d}{dx}\left(x \cdot 2^{\sinh^{-1} x}\right)$

89–102 Evaluate the given integral.

89. $\int \frac{dx}{\sqrt{16x^2+9}}$

90. $\int \frac{x}{\sqrt{25x^4-4}}\,dx$

91. $\int \frac{dx}{-x^2+6x-8}$

92. $\int \frac{x}{\sqrt{x^4-4x^2+3}}\,dx$

93. $\int \frac{-\cos x}{\sqrt{\sin^2 x + \sin^4 x}}\,dx$

94. $\int_0^{1/8} \frac{2}{1-4x^2}\,dx$

95. $\int_0^5 \frac{10}{\sqrt{1+25x^2}}\,dx$

96. $\int \frac{dx}{\sqrt{9+e^{2x}}}$

97. $\int_{-3/2}^{-1} \frac{-1}{(x+2)\sqrt{-3-4x-x^2}}\,dx$

98. $\int_1^2 \frac{-1}{x\sqrt{1+x^2}}\,dx$

99. $\int \frac{dx}{x\sqrt{1+x^3}}$

100. $\int \frac{dx}{\sqrt{x}\sqrt{1+x}}$

101. $\int_{1/8}^{1/4} \frac{dx}{x\sqrt{1-4x^2}}$

102. $\int_7^9 \frac{-4}{-x^2+6x-5}\,dx$

103–110 Evaluate the given limit.

103. $\lim_{x\to\infty} \frac{\cosh x}{x^2}$

104. $\lim_{x\to 0} \frac{\sinh x}{x}$

105. $\lim_{x\to 0} \frac{\sinh^{-1} x}{x}$

106. $\lim_{x\to 0} \frac{\tanh x}{x}$

107. $\lim_{x\to\infty} \frac{\cosh x}{e^x}$

108. $\lim_{x\to\infty} x\coth^{-1} x$

109. $\lim_{x\to 0} \frac{1-\operatorname{sech} x}{\sinh x}$

110. $\lim_{x\to 0}\left(\operatorname{csch} x - \coth x\right)$

111. To exhibit another nice analogy between trigonometric and hyperbolic functions, prove that the area of both the circular and hyperbolic sectors corresponding to the parameter u in Figures 4 and 5 equals $A = u/2$. (**Hint:** For the circular sector, use the fact that the arc length of the sector is u units, while the radius is 1. For the hyperbolic sector, notice that

$$A(u) = \frac{\cosh u \sinh u}{2} - \int_1^{\cosh u} \sqrt{x^2-1}\,dx,$$

and show by differentiating that $A'(u) = \frac{1}{2}$.)

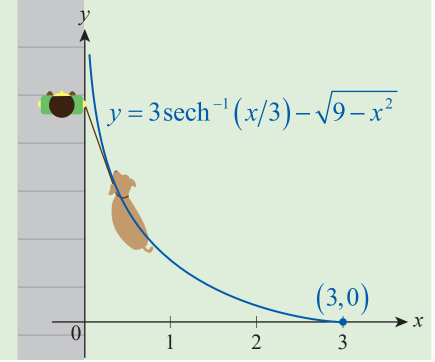
112. Prove the following interesting property of the hyperbolic cosine function: The area under the graph of $y = \cosh x$ $(0 \leq x \leq c)$ is equal to its arc length over the same interval.

113. A flexible chain or cable suspended between two fixed points forms a curve called a *catenary* (from the Latin word "catenarius," meaning "related to a chain"). The equation of a catenary is $y = a\cosh(x/a)$. Note that the clothesline of Exercise 56 in Section 6.3 is much better approximated by the equation $y = 5\cosh(x/5) - 4$. Use this equation to answer the following questions.

a. What is the slope of the clothesline at each of its endpoints? Compare your answer to the slope predicted by the parabolic model.

b. What is the length of the clothesline? Is your answer close to that given to Exercise 83 in Section 6.3? (Galileo observed that the parabolic model is almost exact when the angle of elevation is less than 45°.)

114.* Suppose that a dog is 3 yards away from a sidewalk, held on a tight leash by its owner, such that the leash is initially perpendicular to the sidewalk. The dog's owner starts walking on the sidewalk at a steady, slow pace, pulling the dog while it is offering slight resistance, thereby keeping the leash tight. The curve of the dog's path in this situation is called a *tractrix* (from the Latin word "trahere," meaning "pull" or "drag"), and can be given by the equation $y = 3\operatorname{sech}^{-1}(x/3) - \sqrt{9 - x^2}$. (We assume that the owner started from the origin, walking along the positive y-axis, while the dog's initial position was $(3, 0)$.)



a. Find how far the owner has to walk in order to bring the dog within 1 yd of the sidewalk.

b. Prove that the dog's velocity vector at any time is pointing toward its owner. (For this reason, the tractrix is also called a "curve of pursuit.")

Concept Check

115–118 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

115. $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

116. $|\operatorname{sech} x| \le 1$ for all x

117. The value of $\operatorname{csch} 0$ is infinity.

118. The function $f(x) = \operatorname{sech}^{-1} x$ is defined on $[0, 1]$.

6.6 Technology Exercises

119–120 Use a graphing utility to solve the problem.

119. The Gateway Arch of St. Louis (constructed in 1963–1965) was designed by Eero Saarinen so that its central curve (the curve tracing the centroids of the triangular cross-sections) is an inverted, "flattened" catenary that is described by the equation $y = -68.7672\cosh(0.0100333x) + 693.8597$ (units in feet).

a. How tall is the central curve of the arch?

b. How wide is the central curve at ground level?

c. What is the slope of the central curve at an altitude of 600 ft above ground level?

120. Suppose that the clothesline of Exercise 113 became loose, and resembles the parabola $y = \frac{1}{2}x^2 + 0.1$, $-1.5 \le x \le 1.5$ (in particular, it is sagging so that its lowest point is just 0.1 yd above the ground). However, we know that in reality, the clothesline can be much better approximated by a catenary. Find an equation for such a catenary, and graph both curves on the same screen. Do they seem to "overlap," or are they distinguishable? Then graph both the parabolic model and the catenary from Exercise 113 on the same screen. Comparing the two sets of graphs, what can you conclude?

121–122 Use a graphing utility to graph the given function and to find its derivative and any relative extrema.

121. $f(x) = (x^2 - x)\tanh x$

122. $g(x) = x^2(\sinh x - 3\tanh x)$

7.1 Exercises

1–4 Evaluate the integral using integration by parts with the suggested choice for u and dv.

1. $\int xe^x\,dx;\quad u=x,\quad dv=e^x\,dx$
2. $\int 4x\cos 2x\,dx;\quad u=x,\quad dv=2\cos 2x\,dx$
3. $\int 4x^3\ln x\,dx;\quad u=\ln x,\quad dv=4x^3\,dx$
4. $\int \arctan x\,dx;\quad u=\arctan x,\quad dv=dx$

5–36 Evaluate the integral. (**Hint:** For some integrals, you may use an alternative method of integration in addition to or instead of integration by parts.)

5. $\int (t+1)e^{5t}\,dt$
6. $\int x^2e^{x^3}\,dx$
7. $\int 2x\sin x\,dx$
8. $\int \arcsin x\,dx$
9. $\int x\ln\left(x^2\right)dx$
10. $\int \frac{dx}{x\ln\left(x^2\right)}$
11. $\int x^2\ln x\,dx$
12. $\int \frac{\ln\left(x^2\right)}{x}dx$
13. $\int (s-3)e^{s-3}\,ds$
14. $\int (s-3)e^{(s-3)^2}\,ds$
15. $\int (2x+7)e^{3x-1}\,dx$
16. $\int \sqrt{z}\ln\left(z^2\right)dz$
17. $\int x\sqrt{x-2}\,dx$
18. $\int (3x+1)\sqrt[3]{x-1}\,dx$
19. $\int \sqrt[3]{3x}\ln x\,dx$
20. $\int \log_3 x\,dx$
21. $\int \frac{\ln\sqrt{x}}{\sqrt{x}}dx$
22. $\int (\theta+1)\sin\theta\,d\theta$
23. $\int x\sec x\tan x\,dx$
24. $\int \sec^2 x\tan x\,dx$
25. $\int x\csc^2 x\,dx$
26. $\int x\csc^2\left(x^2\right)dx$
27. $\int \frac{3x-1}{e^x}dx$
28. $\int x^{-2}e^{1/x}\,dx$
29. $\int (t+2)\cosh t\,dt$
30. $\int \frac{x}{\sqrt{2x-5}}dx$
31. $\int x\sqrt{x+1}\,dx$
32. $\int (3-4x)\sinh 2x\,dx$
33. $\int (3x+2)\operatorname{sech}^2 x\,dx$
34. $\int 2\operatorname{csch}^2 x\coth x\,dx$
35. $\int x\sqrt{2x-5}\,dx$
36. $\int \theta\cos\left(\theta^2\right)d\theta$

37–48 Evaluate the integral. If necessary, use integration by parts more than once.

37. $\int \theta^2\cos\theta\,d\theta$
38. $\int t^3e^t\,dt$
39. $\int e^x\cos x\,dx$
40. $\int \sin x\cos x\,dx$
41. $\int 9t^2\sin 3t\,dt$
42. $\int 2t^2e^{5t+1}\,dt$
43. $\int e^{3x}\sin 2x\,dx$
44. $\int 2e^x\sin x\cos x\,dx$
45. $\int x^2\sinh x\,dx$
46. $\int \sec^3 x\,dx$
47. $\int \cos(\ln x)\,dx$
48. $\int (\ln x)^3\,dx$

49–56 Combine the method of integration by parts with substitution to evaluate the integral.

49. $\int e^{\sqrt{x}}\,dx$
50. $\int 2x^3\sin\left(x^2+1\right)dx$
51. $\int \frac{\arctan\left(1/\sqrt{x}\right)}{2\sqrt{x}}dx$
52. $\int t\ln(2-t)\,dt$
53. $\int \frac{\arccos\sqrt{x}}{\sqrt{x}}dx$
54. $\int \sin(2x)e^{\sin x}\,dx$
55. $\int 9x^2(\ln x)^2\,dx$
56. $\int \frac{\cos(1/\theta)}{\theta^3}d\theta$

57–64 Evaluate the definite integral. (Use integration by parts only when necessary.)

57. $\int_{1/2}^{1}\arccos x\,dx$
58. $\int_{2}^{2e}\ln\frac{x}{2}\,dx$
59. $\int_0^{\pi}e^{2\theta}\sin\theta\,d\theta$
60. $\int_0^{\pi}t^2\sin t\,dt$
61. $\int_{\sqrt{3}/3}^{\sqrt{3}}\arctan\frac{1}{x}\,dx$
62. $\int_0^1 x^3e^x\,dx$
63. $\int_0^1\ln\left(t^2+1\right)dt$
64. $\int_1^e\frac{\ln x^2}{x^2}dx$

65–70 Integration by parts can often be used to evaluate integrals involving inverses of functions, and in fact leads to a general formula, as follows.

$$\int f^{-1}(x)\,dx=\int yf'(y)\,dy \qquad \text{Let } y=f^{-1}(x)\text{, so } x=f(y)\text{ and } dx=f'(y)\,dy.$$

$$=yf(y)-\int f(y)\,dy \qquad u=y \quad dv=f'(y)\,dy \quad du=dy \quad v=f(y)$$

$$=xf^{-1}(x)-\int f(y)\,dy$$

For instance, if we let $f(x)=e^x$, then $y=f^{-1}(x)=\ln x$.

$$\begin{aligned}\int \ln x\,dx&=\int f^{-1}(x)\,dx\\&=xf^{-1}(x)-\int e^y\,dy\\&=x\ln x-e^y\\&=x\ln x-e^{\ln x} \qquad y=\ln x\\&=x\ln x-x\end{aligned}$$

In Exercises 65–70, use this method to evaluate the given indefinite integral. (**Hint:** In starred exercises, show that $\cosh\left(\sinh^{-1} x\right) = \sqrt{1+x^2}$ and that $\cosh\left(\tanh^{-1} x\right) = 1/\sqrt{1-x^2}$ as part of the process toward the answers.)

65. $\int \sin^{-1} x\,dx$

66. $\int \cos^{-1} x\,dx$

67. $\int \tan^{-1} x\,dx$

68. $\int \log_2 x\,dx$

69.* $\int \sinh^{-1} x\,dx$

70.* $\int \tanh^{-1} x\,dx$

71. Use integration by parts to find the area of the region bounded by the graphs of $y = 6\tan^{-1}(2x)$, $y = 0$, and $x = \sqrt{3}/2$.

72. Use integration by parts to find the area of the region bounded by the graphs of $y = \sin(\ln x)$ and $y = 0$ $(1 \le x \le e^{\pi})$.

73. Consider the region bounded by the graphs of $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$.

a. Find the centroid of the region.

b. Use the shell method to find the volume of the solid generated by revolving the region about the y-axis.

74. Repeat Exercise 73 for the region bounded by the graphs of $y = x\cos x$ and $y = 0$ $(0 \le x \le \pi/2)$.

75. Consider the region bounded by the graphs of $y = \arcsin x$, $x = 0$, and $y = \pi/2$.

a. Find the centroid of the region.

b. Rotate the region about the x-axis and use the shell method to find the volume of the resulting solid.

76. Use the shell method to find the volume of the solid obtained by revolving the region bounded by the graphs of $y = 2^x$, $y = 0$, $x = 0$, and $x = 1$ about the line $x = -1$.

77. Find the centroid of the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

78. The definite integral $\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$, $n \in \mathbb{N}$ is called a *Fourier coefficient.* Use integration by parts to verify that its value is $(-1)^{n+1}\frac{2}{n}$. (The theory of Fourier series is very important in applied mathematics. You will be introduced to infinite series of functions in Chapter 10.)

79. Use integration by parts to prove that if $f(x)$ is continuously differentiable on $[-\pi, \pi]$, then the limit of the Fourier coefficients is 0.

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0$$

80. Use integration by parts to prove the formula

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C.$$

81–87 Use integration by parts to prove the given reduction formula for $n \in \mathbb{N}$.

81. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

82. $\int x^n \cos(kx)\,dx = \frac{x^n}{k}\sin(kx) - \frac{n}{k}\int x^{n-1}\sin(kx)\,dx$

83. $\int x^n \sin(kx)\,dx = -\frac{x^n}{k}\cos(kx) + \frac{n}{k}\int x^{n-1}\cos(kx)\,dx$

84. $\int x^n e^{kx}\,dx = \frac{1}{k}x^n e^{kx} - \frac{n}{k}\int x^{n-1}e^{kx}\,dx, \quad k \ne 0$

85.* $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \quad n \ne 1$

86. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, n \ne 1$

87. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

88–95 Use the above reduction formulas to evaluate the integrals.

88. $\int \sin^5 x\,dx$

89. $\int x^3 \cos 2x\,dx$

90. $\int x^3 \sin(\pi x)\,dx$

91. $\int \cos x\sin^3 x e^{\sin x}\,dx$

92. $\int \sec^6 x\,dx$

93. $\int (\ln x)^3\,dx$

94. $\int \tan^4 x\,dx$

95. $\int_0^1 x^4 e^x\,dx$

96. Use the appropriate reduction formula to evaluate the definite integral $\int_0^{\pi/2} \sin^6 x\,dx$. Can you conjecture a possible formula for $n = 8, 10, 12, \ldots$?

97. Use integration by parts to prove the formula

$$\int \sin(kx)e^{lx}\,dx = \frac{l\sin(kx) - k\cos(kx)}{k^2 + l^2}e^{lx} + C.$$

98. Use your solution to Exercise 97 to find a similar formula for $\int \cos(kx)e^{lx}\,dx$.

99–102 In Exercise 91, you had to use a reduction formula to evaluate $\int u^3 e^u \, du$. Note that you can obtain the same answer by using the *method of undetermined coefficients*, as follows. Assuming that the answer has the form

$$Au^3e^u + Bu^2e^u + Cue^u + De^u + E,$$

differentiating yields

$$u^3e^u = Au^3e^u + (3A+B)u^2e^u + (2B+C)ue^u + (C+D)e^u.$$

By equating coefficients we obtain

$$A = 1, \quad 3A + B = 0, \quad 2B + C = 0, \quad \text{and} \quad C + D = 0.$$

Solving the above system yields $B = -3$, $C = 6$, and $D = -6$ (while $E = C_1$ is arbitrary).

In Exercises 99–102, use the method of undetermined coefficients to evaluate the given integral. (Note that this method will become important in Section 7.2.)

99. $\int 8x^3e^{2x} \, dx$

100. $\int (x^4 - x)e^x \, dx$

101. $\int 13e^{3x}(\sin 2x) \, dx$ (**Hint:** Anticipate the answer in the form stated below.)

$$Ae^{3x}\sin 2x + Be^{3x}\cos 2x$$

102. $\int 5\sin 2x \cos 3x \, dx$ (**Hint:** Anticipate the answer in the form stated below.)

$$A\sin 2x\cos 3x + B\cos 2x\cos 3x$$
$$+ C\cos 2x\sin 3x + D\sin 2x\sin 3x + E$$

$$\int \left(\frac{3}{x-1} + \frac{1}{x^2+x+1} \right) dx$$

$$= 3\ln|x-1| + \int \left[\frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \right] dx$$

$x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4}$
$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$$= 3\ln|x-1| + \int \left[\frac{\frac{4}{3}}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1} \right] dx$$

Multiply top and bottom by $\frac{4}{3}$.

$$= 3\ln|x-1| + \int \left[\frac{\frac{4}{3}}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} \right] dx$$

$\frac{4}{3}\left(x+\frac{1}{2}\right)^2 = \left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2$
$= \left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2$

$$= 3\ln|x-1| + \left(\frac{4}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\int \frac{du}{u^2+1}$$

$u = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$
$du = \frac{2}{\sqrt{3}}dx$

$$= 3\ln|x-1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$\int \frac{du}{u^2+1} = \tan^{-1} u + C$

7.2 Exercises

1–9 Use the guidelines discussed in this section to write the form of the partial fraction decomposition of the given rational function. Do not solve for the coefficients in your decomposition.

1. $\frac{2}{x(x-1)}$

2. $\frac{5x}{(x+4)(x-2)(x+7)}$

3. $\frac{2x+5}{(x+1)(x-3)^3}$

4. $\frac{3x-1}{(2x+3)(x^2+2)}$

5. $\frac{x-4}{(x^2+x+2)^3}$

6. $\frac{14x-3}{(3x-1)(x^2+1)^2}$

7. $\frac{3}{2x^2-5x-3}$

8. $\frac{2x+1}{5x^3-11x^2+7x-1}$

9. $\frac{4x^2-1}{x^5+x^4+2x^3+2x^2+x+1}$

10–35 Use the partial fractions method to evaluate the given integral.

10. $\int \frac{dx}{x(x+2)}$

11. $\int \frac{4}{(3x-1)x}dx$

12. $\int \frac{x-4}{(x+2)(x-1)}dx$

13. $\int \frac{2}{(x+3)(2x+5)}dx$

14. $\int \frac{5x+2}{(x+1)(3x-1)(x+3)}dx$

15. $\int \frac{2x}{(x-2)^2(x+1)}dx$

16. $\int \frac{x}{(x+2)^3}dx$

17. $\int \frac{2-z}{z(z+1)^2}dz$

18. $\int \frac{dx}{x(x^2+2)}$

19. $\int \frac{dx}{x^2+3x+2}$

20. $\int \frac{2-z}{z^2-1}dz$

21. $\int \frac{2}{x-x^3}dx$

22. $\int \frac{s-3}{s(s-1)(s+3)}ds$

23. $\int \frac{t-1}{t^3+t^2+t+1}dt$

24. $\int \frac{x^2+1}{(x^2+2x+3)^2}dx$

25. $\int \frac{x^2-9}{x^4+3x^3}dx$

26. $\int \frac{5x^3-5x-40}{x^4+x^3+4x^2+4x}dx$

27. $\int \frac{11x-12}{x(x^2+x-6)}dx$

28. $\int \frac{3x^4-5x^3+15x^2-8x+20}{x^5-2x^4+4x^3-8x^2+4x-8}dx$

29. $\int \frac{32}{x^4 - 4x^3 - 2x^2 + 12x + 9}\,dx$

30. $\int \frac{2x^4 + 4x^2 - x + 2}{\left(1 + x^2\right)^3}\,dx$

31. $\int \frac{16v}{v^4 - 16}\,dv$

32. $\int \frac{-x^2 - 1}{x^4 + 5x^2 + 6}\,dx$

33. $\int \frac{3x}{x^3 - 1}\,dx$

34. $\int \frac{x}{(x + a)(x - b)}\,dx$

35. $\int \frac{a}{x\left(x^2 + b\right)}\,dx$

36–41 Use the Heaviside cover-up method to evaluate the given integral.

36. $\int \frac{6x^2 - 19x - 12}{x\left(x^2 - x - 6\right)}\,dx$

37. $\int \frac{16(x + 2)}{(x - 7)\left(x^2 - 1\right)}\,dx$

38. $\int \frac{2v^2 + 13v + 6}{v^3 - v^2 - 10v - 8}\,dv$

39. $\int_0^1 \frac{-5t^2 + 8t + 19}{t^3 + 2t^2 - 5t - 6}\,dt$

40. $\int_1^2 \frac{6s^3 - 38s^2 + 48s + 12}{s^4 - 6s^3 + 5s^2 + 12s}\,ds$

41. $\int_{-1}^1 \frac{-x^3 + 4x^2 - 11x - 6}{\left(x^2 - 4\right)\left(x^2 - 9\right)}\,dx$

42. If $k \in \mathbb{R}$, use partial fractions to prove the formula
$$\int \frac{1}{x^2 - k^2}\,dx = -\frac{1}{2k}\ln\left|\frac{x + k}{x - k}\right| + C.$$

43. Complete the square in the denominator of Example 1 and use the formula in Exercise 42 to arrive at the same answer as in Example 1.

44–46 Use your approach taken in Exercise 42 to establish the given formula for $k, l \in \mathbb{R}$.

44. $\int \frac{dx}{x(kx + l)} = \frac{1}{l}\ln\left|\frac{x}{kx + l}\right| + C$

45. $\int \frac{dx}{x^2(kx + l)} = -\frac{1}{lx} - \frac{k}{l^2}\ln\left|\frac{x}{kx + l}\right| + C$

46. $\int \frac{x}{(kx + l)^2}\,dx = \frac{1}{k^2}\left(\frac{l}{kx + l} + \ln|kx + l|\right) + C$

47–61 Use any of the techniques seen in Examples 3–7 to evaluate the given integral (definite or indefinite, as indicated). Whenever applicable, use the formulas from Exercises 42, 44–46. (**Note:** If the integrand is not a proper rational function, be sure to divide first.)

47. $\int \frac{x^2 + 1}{x^3 + 1}\,dx$

48. $\int \frac{3x^3 - 4x^2 + 2}{x^2 - x}\,dx$

49. $\int_5^8 \frac{dx}{x^2 - 16}$

50. $\int \frac{2}{s\left(s^2 + 1\right)^2}\,ds$

51. $\int_1^2 \frac{dx}{x(x + 2)}$

52. $\int_0^{\sqrt{7}} \frac{x^3}{x^2 + 9}\,dx$

53. $\int \frac{3x^3}{x^3 - 1}\,dx$

54. $\int_1^2 \frac{dx}{x^2(x + 2)}$

55. $\int \frac{3x^2 - 8x + 2}{(x - 3)\left(x^2 - 2x + 2\right)}\,dx$

56. $\int_0^2 \frac{x^2 - 5x}{x^2 + x + 3}\,dx$

57. $\int_0^3 \frac{v}{(v + 5)^2}\,dv$

58. $\int \frac{16x^5}{(x - 1)^2(3 - x)^4}\,dx$

59. $\int \frac{a}{(x - b)(x - c)}\,dx$

60. $\int \frac{z^2 + 8z + 9}{\left(z^2 + 2z + 3\right)^2}\,dz$

61. $\int \frac{x^4 + 1}{x^3 + 4x}\,dx$

62–73 Combine integration by substitution and the partial fractions method to evaluate the given integral. (When applicable, also use integration by parts.)

62. $\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx$

63. $\int \frac{\sin x \cos x}{(\cos x - 1)(\cos x + 2)}\,dx$

64. $\int \frac{2\sec^2 x}{\tan^2 x - 1}\,dx$

65. $\int \frac{6}{x\left[(\ln x)^2 - \ln\left(x^3\right)\right]}\,dx$

66. $\int \frac{4e^x}{e^{2x} + 2e^x - 3}\,dx$

67. $\int \frac{\sqrt{x}}{\left(1 + \sqrt{x}\right)^2}\,dx$

68. $\int \frac{\sqrt[3]{x}}{\left(1 - \sqrt[3]{x}\right)^2}\,dx$

69. $\int \frac{dx}{x\sqrt{1 + \sqrt{x}}}$

70. $\int e^x \ln\left(e^{2x} + 2\right)dx$

71. $\int \frac{\sin(\ln x)}{x\left[\cos^2(\ln x) + \cos(\ln x)\right]}\,dx$

72. $\int \frac{e^{(3/2)x}}{e^x + 1}\,dx$

73. $\int \frac{\left[\ln\left(w^2\right)\right]\left[\arctan(\ln w)\right]}{w}\,dw$

74. Use the disk method to find the volume of the solid generated by revolving the graph of $f(x) = 2/\sqrt{x^2 - 3x - 10}$, $6 \le x \le 10$, about the x-axis.

75. Use the shell method to find the volume of the solid obtained by revolving the region bounded by $g(x) = 1/(-x^2 + 2x + 8)$, $y = 0$, $x = 0$, and $x = 2$ about the line $x = -1$.

76. Find the centroid of the region bounded by the graphs of $f(x) = \dfrac{-4x - 1}{x^2 - x - 2}$, $y = 0$, $x = 0$, and $x = 1$.

77.* Suppose that we are looking for a function $y = y(t)$ whose rate of change $y'(t)$ is directly proportional to $(c^4 - y^4)/y^2$ (c is a constant). In other words, $y(t)$ then satisfies the equation $\dfrac{dy}{dt} = m(c^4 - y^4)/y^2$ for some constant m. (Such an equation, containing a derivative of an unknown function, is called a differential equation. You will learn more about differential equations in Chapter 8. The equation in this exercise is used in physical chemistry.) Use partial fractions to find an implicit formula for $y(t)$. (**Hint:** As a first step, rewrite the equation in differential form, $y^2/(c^4 - y^4)\,dy = m\,dt$, and then use partial fractions to integrate.)

78.* If the ability of the environment to support a population is limited and, thus, the population cannot grow larger than a certain size, the model $P'(t) = mP(t)$ and its solution $P(t) = P(0)e^{mt}$ is no longer adequate to describe the population growth. Instead, the so-called *logistic model* $dP/dt = mP(L - P)$ has been proposed, where L is the upper limit of the population size. If the world's population in 1940 was 2.3 billion, which grew to 6.9 billion by 2010, and supposing that Earth cannot support more than $L = 15$ billion people, what will be the world's population by 2050? (See the hint given in Exercise 77.)

79.* Assuming that the rate at which a disease is spreading after an infected person enters a community of N susceptible people is proportional to the product of the number of already infected individuals by the number of still-healthy people, and letting $I(t)$ stand for the number of individuals already infected, this latter function satisfies $dI/dt = kI(N - I)$, with $I(0) = 1$. Use your approach taken in the previous two exercises to find a formula for $I(t)$.

7.2 Technology Exercises

80–81 Use a computer algebra system to find the partial fraction decomposition of the given rational function.

80. $f(x) = \dfrac{3x^7 + 20x^6 + 81x^5 + 123x^4 - 61x^3 - 1033x^2 - 2056x - 2401}{x^8 + 7x^7 + 31x^6 + 66x^5 + 78x^4 - 78x^3 - 203x^2 - 245x + 343}$

81. $f(x) = \dfrac{3x^8 + 17x^7 + 47x^6 + 97x^5 + 156x^4 + 207x^3 + 194x^2 + 107x + 42}{x^6 + 6x^5 + 16x^4 + 26x^3 + 27x^2 + 20x + 12}$

We have seen this situation arise before; since the integral we are trying to evaluate appears on both sides of the equation, we can solve for it.

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x\tan x + \ln\left|\sec x + \tan x\right|\right) + C$$

7.3 Exercises

1–36 Evaluate the given indefinite or definite integral involving powers of sines and cosines.

1. $\int \sin^2 x\cos x\,dx$ **2.** $\int \sin x\cos^3 x\,dx$

3. $\int \cos^5 x\,dx$ **4.** $\int \cos x\sin x\,dx$

5. $\int \sin^4 x\,dx$ **6.** $\int \sin^{-4} x\cos x\,dx$

7. $\int_0^{\pi/3} \cos^{-2} x\sin x\,dx$ **8.** $\int_0^{\pi/2} \sin^3 2t\,dt$

9. $\int_{-\pi}^{\pi} \cos^6\left(\frac{\theta}{2}\right)d\theta$ **10.** $\int \frac{2\cos 4\alpha}{\sqrt{\sin 4\alpha}}\,d\alpha$

11. $\int \sin^7 x\cos^8 x\,dx$ **12.** $\int \sin^2 x\cos^2 x\,dx$

13. $\int \sin^4\left(\frac{\theta}{4}\right)\cos^4\left(\frac{\theta}{4}\right)d\theta$ **14.** $\int_0^{\pi} \sin^5 x\,dx$

15. $\int_0^{\pi} \sin^2 x\,dx$ **16.** $\int \cos^3 3t\sin^4 3t\,dt$

17. $\int \sqrt{\cos x}\sin^3 x\,dx$

18. $\int_0^{3\pi} \sin^3\left(\frac{t}{3}\right)\cos^2\left(\frac{t}{3}\right)dt$

19. $\int \sin^3 x\cos^5 x\,dx$ **20.** $\int_0^{\pi/2} \sin^7 x\,dx$

21. $\int_{\pi^2/4}^{\pi^2} \frac{8\cos^2\sqrt{x}\sin^2\sqrt{x}}{\sqrt{x}}\,dx$

22. $\int \frac{\sin 2x + \sin^3 x}{\cos x}\,dx$

23. $\int 16\sin^2 x\cos^4 x\,dx$ **24.** $\int \cos^6 x\,dx$

25. $\int \sin^7 x\cos^3 x\,dx$ **26.** $\int \sin x\sin 5x\,dx$

27. $\int_0^{\pi/2} \cos 2x\cos 3x\,dx$ **28.** $\int_0^{\pi} \cos(-4x)\sin 6x\,dx$

29. $\int \sin 8x\sin(-7x)\,dx$ **30.** $\int \sqrt{\frac{1+\cos 2\theta}{2}}\,d\theta$

31. $\int_0^{\pi/3} \frac{\sin 3x}{\sqrt{\cos x}}\,dx$ **32.** $\int_0^{\pi/2} \sqrt{1-\cos x}\,dx$

33. $\int \frac{\cos x}{\sqrt{1-\sin x}}\,dx$ **34.** $\int \frac{\cos x}{\sqrt{1-\cos x}}\,dx$

35. $\int \sin x\sqrt{1-\sin x}\,dx$ **36.** $\int \frac{\cos^2 x}{\sqrt{1+\sin x}}\,dx$

(**Hint:** In Exercises 35 and 36, use the identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.)

37–57 Evaluate the given indefinite or definite integral involving powers of tangents and secants. Note that the integrals involving cotangents and cosecants can be handled by rules analogous to those discussed in this section.

37. $\int \tan^5 x\,dx$ **38.** $\int \cot^4 x\,dx$

39. $\int \sec^4 x\,dx$ **40.** $\int \tan^3 x\sec^3 x\,dx$

41. $\int \frac{\tan\theta}{\sqrt{\sec\theta}}\,d\theta$ **42.** $\int \frac{\cot^5 x}{\csc x}\,dx$

43. $\int \tan x\sec^6 x\,dx$ **44.** $\int \frac{\sec^2 x}{\sqrt[3]{\tan x}}\,dx$

45. $\int \csc^4 t\cot^{3/2} t\,dt$ **46.** $\int \tan^4 x\sec^4 x\,dx$

47. $\int \tan^2 x\sec x\,dx$ **48.** $\int \csc^3 x\,dx$

49. $\int_0^{\pi/3} \tan^5 x\sec^3 x\,dx$ **50.** $\int_{\pi/3}^{2\pi/3} \csc x\cot^2 x\,dx$

51. $\int_{\pi/6}^{\pi/2} \csc^4 x\,dx$ **52.** $\int \frac{\csc^4 z}{\cot z}\,dz$

53. $\int \frac{\sec^4 2\alpha}{\cot^3 2\alpha}\,d\alpha$ **54.** $\int \sec^6 3\beta\cot^3 3\beta\,d\beta$

55. $\int \csc^4 4x\cot 4x\,dx$

56. $\int_{\sqrt[3]{\pi/4}}^{\sqrt[3]{\pi/2}} s^2\csc^2\left(s^3\right)\cot^4\left(s^3\right)ds$

57. $\int_0^{\pi/4} \frac{\cot^2 t - 1}{\csc^2 t}\,dt$

58–63 The given integral does not directly fit any of the cases discussed in this section. Use trigonometric identities and familiar integration rules to evaluate it.

58. $\int \sin x\cot 2x\,dx$ **59.** $\int x\sec^2 x\,dx$

60. $\int \sec^3 x\cot x\,dx$ **61.** $\int z\tan^2 2z\,dz$

62. $\int \csc^4 x \cos^3 x\, dx$ **63.** $\int \frac{\sin 3x}{\sec x}\, dx$

64. Verify the following reduction formula for $m, n \in \mathbb{N}$. (**Hint:** Use integration by parts.)

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx$$

65. Use Exercises 3, 24, and 64 to evaluate the following integrals.

a. $\int \sin^2 x \cos^5 x\, dx$ **b.** $\int \sin^2 x \cos^6 x\, dx$

66. Find the area of the region between the x-axis and the graph of $f(x) = \sin^2 x \cos^3 x$ from $x = 0$ to $x = \pi/2$.

67. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \tan x + \cot x$, $y = 0$, $x = \pi/6$, and $x = \pi/3$ about the x-axis.

68. Repeat Exercise 67 for the graphs of $y = \cos x + \sec x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$.

69. Find the centroid of the region bounded by the graphs of $y = x + \sin x$, $y = 0$, and $x = \pi$.

70. A particle is starting from the origin and moving along the x-axis so that its velocity at t seconds is $v(t) = \pi \tan^2(\pi t/18)\sin(\pi t/18)$ units per second $(0 \le t \le 8)$. Find its position at $t = 6$ seconds.

71–73 Use the product-to-sum identities of this section to verify the given formula for $m, n \in \mathbb{N}$.

71. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$

72. $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$

73. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$

Concept Check

74–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. $\int \sin^2 x\, dx - x = C - \int \cos^2 x\, dx$

75. $\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \cos^2 x\, dx$

76. According to the text, the best way to evaluate $\int \sec^4 x \tan^4 x\, dx$ is by using integration by parts.

77. According to the text, the best way to evaluate $\int \sec^3 x \tan^4 x\, dx$ is by using integration by parts.

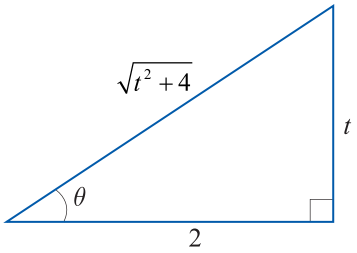


Figure 6 $\tan\theta = t/2$

Note that our antiderivative is a function of θ, but the limits of integration are still in terms of t. In order to evaluate the expression, we sketch the triangle in Figure 6 and write everything in terms of t.

$$\frac{1}{12}\sin\theta\Big]_{t=0}^{t=3} = \frac{1}{12}\cdot\frac{t}{\sqrt{t^2+4}}\Big]_{t=0}^{t=3}$$

$$= \frac{1}{12}\cdot\frac{3}{\sqrt{13}} = \frac{1}{4\sqrt{13}}$$

7.4 Exercises

1–6 Choose the substitution(s) that are helpful in evaluating the integral. (Do not actually evaluate the integral. There may be more than one correct answer.)

1. $\int x\sqrt{x-1}\,dx$

a. $x = \tan\theta$ **b.** $x = \sec\theta$
c. $x = \sin\theta$ **d.** $\theta = x - 1$

2. $\int \frac{dx}{\sqrt{x^2+1}}$

a. $x = \sin\theta$ **b.** $\tan x = \theta$
c. $x = \tan\theta$ **d.** $x = \sec\theta$

3. $\int x\sqrt{9-x^2}\,dx$

a. $x = 3\sec\theta$ **b.** $\theta = 3\sin x$
c. $x = 3\sin\theta$ **d.** $\theta = 9 - x^2$

4. $\int \frac{dz}{z^2\sqrt{z^2-9}}$

a. $z = \sin t$ **b.** $z = \sec t$
c. $z = 3\sec t$ **d.** $z = 3\cosh t$

5. $\int \frac{dx}{x^2\sqrt{4x^2+9}}$

a. $x = 2\tan t$ **b.** $x = \frac{3}{2}\tan t$
c. $x = 3\tan t$ **d.** $x = \frac{3}{2}\sec t$

6. $\int \frac{\sqrt{9-(x-1)^2}}{x-1}\,dx$

a. $x = 3\sin\theta + 1$ **b.** $x = 3\sin\theta$
c. $x - 1 = 3\sec t$ **d.** $x - 1 = 3\tan t$

7–12 Choose the correct answer. (You should be able to identify the correct answer without actually evaluating the integral.)

7. $\int \frac{dx}{\sqrt{x^2+1}}$

a. $\sqrt{\arctan x} + C$

b. $2\sqrt{x^2+1} + C$

c. $\ln\left|\sqrt{x^2+1} + x\right| + C$

d. $\ln\sqrt{x^2+1} + C$

8. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

a. $\frac{x^2}{2}\arcsin\frac{x}{2} + C$

b. $\frac{2x^3}{3}\sqrt{9-x^2} + C$

c. $-\sqrt{9-x^2} + C$

d. $\frac{9}{2}\arcsin\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$

9. $\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

a. $\frac{t}{2}\sqrt{1-t^2} - \frac{\tan^{-1} t}{2} + C$

b. $\frac{\sin^{-1} t}{2} - \frac{t}{2}\sqrt{1-t^2} + C$

c. $-\sqrt{1-t^2} + C$

d. $\frac{1}{2}\left(t\sqrt{1-t^2} + \ln\left|t + \sqrt{t^2-1}\right|\right) + C$

10. $\int \frac{\sqrt{1-x^2}}{x}\,dx$

a. $\frac{1}{x\arcsin x}+C$

b. $\sqrt{1-x^2}+\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+C$

c. $\frac{2\left(\sqrt{1-x^2}\right)^{3/2}}{3x}+C$

d. $\sqrt{x^2-1}+\arctan\frac{1}{\sqrt{x^2-1}}+C$

11. $\int \frac{dx}{\sqrt{x^2-4}}$

a. $\frac{1}{2}\arcsin\frac{x}{2}+C$ **b.** $2\sqrt{x^2-4}+C$

c. $\ln\left|x+\sqrt{x^2-4}\right|+C$ **d.** $\frac{2\left(\sqrt{x^2-4}\right)^{3/2}}{3}+C$

12. $\int \frac{dx}{\left(x^2+9\right)^{3/2}}$

a. $\frac{x}{9\sqrt{x^2+9}}+C$ **b.** $\frac{-2}{\sqrt{x^2+9}}+C$

c. $\frac{1}{3}\left(\arctan\frac{x}{3}\right)^{3/2}+C$ **d.** $\ln(x+9)^{3/2}+C$

13–48 Use the three trigonometric substitutions discussed in this section to evaluate the given indefinite or definite integral. (**Note:** Not all integrals require trigonometric substitution.)

13. $\int \frac{3}{\sqrt{x^2+9}}\,dx$

14. $\int x\sqrt{x-1}\,dx$

15. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

16. $\int \frac{\sqrt{9-x^2}}{2x}\,dx$

17. $\int \frac{t^2}{\sqrt{25-t^2}}\,dt$

18. $\int \frac{ds}{s\sqrt{s^2-4}}$

19. $\int \frac{dx}{x^2\sqrt{x^2-36}}$

20. $\int \frac{\sqrt{x^2-2}}{x^3}\,dx$

21. $\int \frac{-x}{\sqrt{1-x^2}}\,dx$

22. $\int \frac{x^2}{\sqrt{x^2+25}}\,dx$

23. $\int \frac{2-x}{\sqrt{4+x^2}}\,dx$

24. $\int \frac{x^2-2x+5}{\sqrt{1-x^2}}\,dx$

25. $\int_{5/2}^{3} \frac{dz}{2z^2\sqrt{z^2-4}}$

26. $\int_0^5 \frac{z}{\sqrt{z^2+4}}\,dz$

27. $\int_0^{\sqrt{5}} w^2\sqrt{5-w^2}\,dw$

28. $\int_3^5 \frac{\sqrt{x^2-9}}{x}\,dx$

29. $\int \frac{dt}{t^2\sqrt{t^2+9}}$

30. $\int \frac{s^2}{\sqrt{s^2-25}}\,ds$

31. $\int \frac{\sqrt{4-9x^2}}{x}\,dx$

32. $\int \frac{dy}{y\sqrt{25+9y^2}}$

33. $\int \frac{\left(4-w^2\right)^{3/2}}{w^6}\,dw$

34. $\int_1^2 \frac{dx}{4x^2\sqrt{4x^2+1}}$

35. $\int_{1/2}^{2} \frac{dy}{y^2\sqrt{16-y^2}}$

36. $\int \sqrt{1-x^2}\,dx$

37. $\int \sqrt{9x^2+16}\,dx$

38. $\int x^2\sqrt{9-x^2}\,dx$

39. $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

40. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

41. $\int \frac{x}{\sqrt{3x^2+1}}\,dx$

42. $\int \frac{x^2}{\left(x^2+16\right)^{3/2}}\,dx$

43. $\int \frac{-x}{\left(1-x^2\right)^{3/2}}\,dx$

44. $\int \frac{dt}{\sqrt{9t^2-4}}$

45. $\int \frac{dx}{x^2\sqrt{25-9x^2}}$

46. $\int \frac{dx}{\sqrt{9x^2+4}}$

47. $\int_{-2}^{2} \frac{dt}{\left(t^2+5\right)^{3/2}}$

48. $\int \frac{dx}{\left(x^2+1\right)^2}$

49–60 Complete the square and use applicable substitutions from this section to evaluate the given integral.

49. $\int \frac{dx}{\sqrt{x^2+x+3}}$

50. $\int \frac{dv}{\left(9v^2-18v+5\right)^{3/2}}$

51. $\int \frac{x^2}{\sqrt{6x-x^2}}\,dx$

52. $\int_1^3 \frac{2}{\sqrt{4x-x^2}}\,dx$

53. $\int_2^3 \frac{dx}{\sqrt{-2x^2+8x-4}}$

54. $\int \frac{dx}{\sqrt{x^2-6x+10}}$

55. $\int_2^3 \frac{dx}{\left(4x^2-8x+3\right)^{3/2}}$

56. $\int \frac{v^2}{\sqrt{3+2v-v^2}}\,dv$

57. $\int \sqrt{7+6x-x^2}\,dx$

58. $\int \sqrt{4x^2-16x+25}\,dx$

59. $\int \frac{2}{\left(x^2-10x+29\right)^2}\,dx$

60. $\int \frac{ds}{\left(s^2-8s+17\right)^{3/2}}$

61–63 Use an appropriate substitution followed by a trigonometric substitution to evaluate the integral.

61. $\int e^x\sqrt{1-e^{2x}}\,dx$

62. $\int \frac{\sqrt{x}}{1+x}\,dx$

63. $\int \frac{\cot x\csc x}{\sqrt{\sin^2 x+1}}\,dx$

64–71 Use an appropriate trigonometric substitution to find a general formula for the expression. (Assume $a > 0$.)

64. $\int \sqrt{x^2+a^2}\,dx$

65. $\int \sqrt{x^2-a^2}\,dx$

66. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

67. $\int \frac{dx}{x\sqrt{x^2-a^2}}$

68. $\int \sqrt{a^2-x^2}\,dx$

69. $\int \frac{dx}{\sqrt{x^2+a^2}}$

70. $\int \frac{-dx}{x\sqrt{a^2-x^2}}$

71. $\int \frac{-dx}{x\sqrt{x^2+a^2}}$ $(x>0)$

72–74. Use hyperbolic substitutions to find alternative general formulas for Exercises 69–71.

75. Evaluate Exercise 42 by using the hyperbolic substitution $x = 4\sinh t$, and then generalize your result to arrive at both a trigonometric and a hyperbolic formula for

$$\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx.$$

76–77 Combine integration by parts and trigonometric substitution to evaluate the integral.

76. $\int t\arcsin t\,dt$

77. $\int t\arccos t\,dt$

78. Find the area of the region between the graph of $y = \frac{1}{\left(x^2+2\right)^{3/2}}$ and the x-axis from $x=-1$ to $x=1$.

79. Repeat Exercise 78 for the curve $y = \frac{1}{x^2\sqrt{x^2-3}}$ on the interval $[2,3]$.

80. Find the area enclosed by the unit circle $x^2+y^2=1$ and the parabola $y=\sqrt{2}x^2$.

81. Rotate the region bounded by the graph of $y=\frac{\sqrt{x^2-9}}{x^3}$, $3 \le x \le 5$, about the y-axis. Use the shell method to find the volume of the resulting solid.

82. Repeat Exercise 81 for the curve $y=\frac{27x^2}{\left(9x^2+4\right)^{3/2}}$ on the interval $[0,1]$.

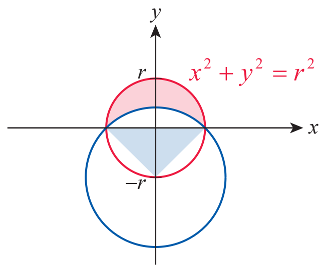
83. Use the method of disks to determine the volume of the solid obtained by revolving the graph of the curve $y=\frac{\sqrt[4]{16-4x^2}}{x^2}$, $1 \le x \le 2$, about the x-axis.

84. Find the arc length of the prototypical parabola $y=x^2$ between the origin and the point $(2,4)$.

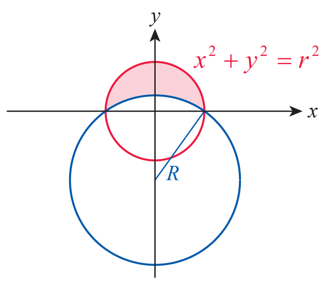
85. Find the arc length for the graph of $y=\ln x$ between $x=\sqrt{3}$ and $x=2\sqrt{2}$.

86. A cylindrical fuel tank of radius 10 in. is positioned so its axis is horizontal. Find the fluid force acting on one end of the tank if it is partially filled with diesel fuel so that the top 4 in. of the tank are empty. Use $55\ \text{lb}/\text{ft}^3$ for the weight density of diesel fuel.

87. In an attempt to square the circle, Hippocrates of Chios showed about 2500 years ago that the area of the red shaded region (called a *lune*) in the figure below is equal to the area of the shaded triangle (which in turn is half of a square). Given that the bigger circle is centered at $(0,-r)$, use calculus to prove Hippocrates' result.



88.* Find a more general formula for the area of the lune in the case where the radius of the bigger circle is R.



Concept Check

89–92 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

89. According to the text, the best substitution for $\int \frac{1}{\sqrt{x^2-1}}\,dx$ is $x=\sin\theta$.

90. Substituting $x=\sin\theta$, we obtain $\int 2x\sqrt{1-x^2}\,dx = \int 2\sin\theta\cos\theta\,d\theta = \int \sin 2\theta\,d\theta$.

91. A straightforward way to evaluate $\int_0^{1/2} \frac{1}{1-x^2}\,dx$ is to find $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$ by substituting $x=\sin\theta$ and then squaring the answer.

92. According to this textbook, the only way to evaluate the integral $\int \frac{1}{x^2-4}\,dx$ is to substitute $x = 2\sec\theta$.

7.4 Technology Exercises

93–96 Use a graphing utility to revisit the given exercise. Do you get the same answer that you obtained by hand?

93. Exercise 22 **94.** Exercise 37 **95.** Exercise 54 **96.** Exercise 58

7.5 Exercises

1–20 Use the integration guidelines listed in this section to evaluate the given indefinite or definite integral. (**Hint:** Whenever possible, try to simplify before integrating.)

1. $\int \frac{dx}{5x(x+5)}$

2. $\int \frac{2x}{(2x+3)^2}\,dx$

3. $\int \frac{2\,dx}{x\sqrt{x^2+4}}$

4. $\int \sin^2 x \sec^2 x\,dx$

5. $\int \frac{\tan^2 2x+1}{\csc^2 2x}\,dx$

6. $\int \frac{\cos^2\theta\csc^2\theta}{\tan\theta}\,d\theta$

7. $\int \frac{dx}{x^2+4x+8}$

8. $\int \frac{3x^3-12x^2+15x+2}{x^2-4x+5}\,dx$

9. $\int \frac{x(\sqrt{x}+2)}{\sqrt{x}}\,dx$

10. $\int \frac{1-\cos 2x}{\sin x}\,dx$

11. $\int \frac{x}{\sqrt{4-x^4}}\,dx$

12. $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

13. $\int \sqrt{x^2+2x+5}\,dx$

14. $\int \frac{\sqrt{4t+1}}{4t+5}\,dt$

15. $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{\arctan u}{u^2}\,du$

16. $\int \sqrt{e^2+e^x}\,dx$

17. $\int_{\pi/4}^{\pi/2} \frac{1+\cos z}{1-\cos z}\,dz$

18. $\int \frac{t^2}{t^6-9}\,dt$

19. $\int \frac{2}{\sqrt{z+2}+\sqrt{z}}\,dz$

20. $\int \sqrt{x}\cos\sqrt{x}\,dx$

21–40 Transform the integral into a form that you can integrate by using a table of integrals or any of the techniques discussed in the previous sections. Then evaluate the integral.

21. $\int \sqrt{\frac{x+2}{x-2}}\,dx$

22. $\int \frac{\cos x}{\sin x(2\sin x+7)}\,dx$

23. $\int \frac{dx}{x\sqrt{2x^2+9}}$

24. $\int \frac{dx}{\sqrt{e^{2x}-9}}$

25. $\int \frac{3x^2}{x^6(2x^3+1)}\,dx$

26. $\int \frac{4x}{(x^4+4)^2}\,dx$

27. $\int \frac{\tan u}{\cos^2 u\sqrt{5+2\tan u}}\,du$

28. $\int \frac{dx}{\sqrt{3e^x-16}}$

29. $\int \ln\sqrt{x}\,dx$

30. $\int \sin 2x(2\sin x+3)^3\,dx$

31. $\int \frac{\cot x}{\sqrt{2\sin x-\sin^2 x}}\,dx$

32. $\int t^2\sqrt{t^6-4}\,dt$

33. $\int \frac{\sqrt{3x-2}}{x}\,dx$

34. $\int \frac{\sqrt{2x+1}}{x}\,dx$

35. $\int \frac{x^2-2x+1}{\sqrt{1+6x-3x^2}}\,dx$

36. $\int x\sqrt{x^2+2x+5}\,dx$

37. $\int_{\pi/3}^{2\pi/3} \csc^5 t\,dt$

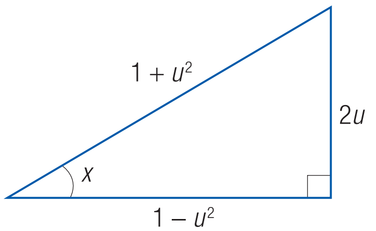
38. $\int \frac{\sin 2x}{\sqrt{2\sin x+9}}\,dx$

39. $\int_0^1 \arccos\sqrt{z}\,dz$

40. $\int e^{2t}\tan^{-1}e^t\,dt$

41–45 If the integrand is a rational function of $\sin x$ and $\cos x$, we can turn it into a rational function of u by the substitution $x = 2\arctan u$. The figure reflects the aforementioned substitution (as you will determine in Exercise 41). Using the figure along with differentiation, we obtain the following.

$$dx = \frac{2\,du}{1+u^2},\quad \sin x = \frac{2u}{1+u^2},\quad \text{and}\quad \cos x = \frac{1-u^2}{1+u^2}$$



After integrating the resulting rational function, we express our answer in terms of the original variable x by using $u = \tan(x/2)$.

41. By using the identity $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$, show that the figure indeed reflects the substitution $x = 2\arctan u$.

Use the above substitution technique to evaluate the given integral.

42. $\int \frac{dx}{3-\sin x}$

43. $\int \frac{dx}{\cos x-\sin x+2}$

44. $\int \frac{\sin x}{1+\cos x-\sin x}\,dx$

45. $\int \frac{du}{2\cos u+3}$

46–55 The given integration formula can be found in most tables of integrals. Verify it by an appropriate integration technique learned from this chapter.

46. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

47. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

48. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

49. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50. $\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right)+C$

51. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

52. $\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C$

53. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

54. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

55. $\int \frac{du}{\left(u^2-a^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

7.5 Technology Exercises

56–59 Use a computer algebra system to solve the given exercise. If the answer looks different from what you obtained by hand, prove that the answers are equivalent.

56. Exercise 6 **57.** Exercise 13

58. Exercise 27 **59.** Exercise 34

60–65 Compare a computer algebra system's answer to the given exercise with those obtained by substitution and/or integration tables. Use the differentiation feature of your technology to prove that both answers are correct.

60. Exercise 5 **61.** Exercise 16

62. Exercise 21 **63.** Exercise 24

64. Exercise 30 **65.** Exercise 36

66–68 Use a computer algebra system to find $F(x)$ that satisfies the given condition. (This problem type is called an initial value problem. We have already seen similar problems in Section 4.7, but you will learn more about them in Chapter 8.)

66. $F(x) = \int \frac{dx}{x^2-2x+4};\quad F(0)=0$

67. $F(x) = \int 3x\sqrt{x^2-4x+5}\,dx;\quad F(1)=0$

68. $F(x) = \int x^2\arccos x\,dx;\quad F(0)=1$

69–71 Use a computer algebra system to give an approximate solution to the given equation. (These are examples of *integral equations*.)

69. $\int_0^x \frac{dt}{\sqrt{t^2+2}} = 2$ **70.** $\int_1^x \frac{\tan^{-1}t}{t^2}\,dt = 1$

71. $\int_x^{\pi/3} \frac{\sqrt{1-\sin^2 t}}{\tan t}\,dt = 2$

72. Paper-and-pencil skills are important, even when powerful software is at your disposal. Use an appropriate substitution to evaluate

$$\int \frac{(1+x)e^x}{\sqrt{x^2e^{2x}+1}}\,dx,$$

and then use a computer algebra system to check your answer. What do you find?

73–78 Even with all of our integration techniques, tables, and computer algebra systems, we are far from being able to find antiderivatives for all elementary functions (roughly speaking, these are finite combinations of the types of functions you have studied so far). Even more surprisingly, many elementary functions do not even have elementary antiderivatives! Some of these appear relatively "easy," so that you might even be tempted to try and integrate them. For example, $f(x)=e^{x^2}$ does not have an elementary antiderivative. Integrals such as $\int e^{x^2}\,dx$ are called *nonelementary integrals*. Working with them requires infinite series (see Chapter 10) or the numerical methods that we shall learn in Section 7.6.

Try to evaluate the nonelementary integral using a computer algebra system. What answer do you get? (Answers will vary.)

73. $\int e^{x^2}\,dx$ **74.** $\int \frac{e^x}{x}\,dx$

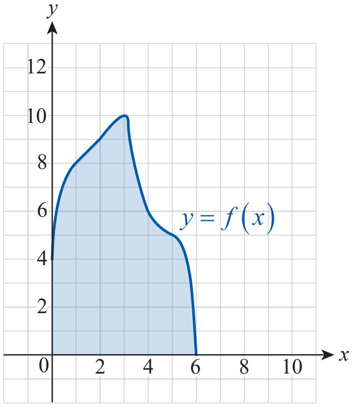
75. $\int \cos\left(x^2\right)dx$ **76.** $\int \frac{\sin x}{x}\,dx$

77. $\int \ln(\ln x)\,dx$ **78.** $\int \frac{1}{\ln x}\,dx$

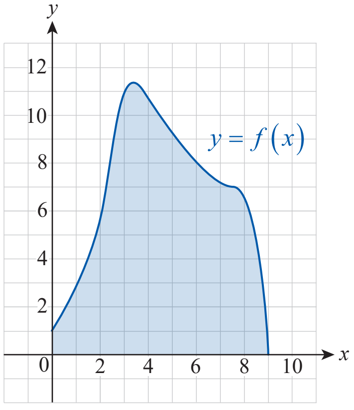
7.6 Exercises

1–2 The function $f(x)$ is given by its graph. Use the Trapezoidal Rule and Simpson's Rule, respectively, to approximate the shaded area $\int_a^b f(x)\,dx$ by **a.** T_6 and **b.** S_6.

1.



2.



3–17 Use the Trapezoidal Rule and Simpson's Rule with $n = 8$ to approximate the integral. Then find the exact value and compare your answers.

3. $\int_0^8 x^4\,dx$

4. $\int_1^5 \frac{1}{x}\,dx$

5. $\int_1^5 \frac{1}{x^2}\,dx$

6. $\int_0^4 \sqrt{x}\,dx$

7. $\int_0^4 x^3\,dx$

8. $\int_0^{2\pi} |\sin x|\,dx$

9. $\int_{-2}^6 \sqrt[3]{x+2}\,dx$

10. $\int_0^2 e^x\,dx$

11. $\int_1^5 \ln x\,dx$

12. $\int_{-2}^6 \left(4 - \frac{1}{2}x\right)dx$

13. $\int_{-4}^4 \left(16 - x^2\right)dx$

14. $\int_{-4}^4 \sqrt{16 - x^2}\,dx$

15. $\int_0^4 x\sqrt{x^2+2}\,dx$

16. $\int_0^{16} \frac{1}{\sqrt{x+1}}\,dx$

17. $\int_0^8 \frac{x}{\sqrt{x^2+1}}\,dx$

18–20 Use **a.** the Trapezoidal Rule and **b.** Simpson's Rule to approximate the definite integral for the indicated value of n.

18. $\int_0^4 \sqrt[4]{x}\,dx; \quad n = 4$

19. $\int_{-\pi/2}^{\pi/2} \cos x\,dx; \quad n = 6$

20. $\int_0^5 \sqrt{x^4+4}\,dx; \quad n = 10$

21–23 Some texts discuss the *Midpoint Rule* as a numerical integration method. The idea is simply forming a Riemann sum by choosing the midpoint of each subinterval as the sample point.

Use the Midpoint Rule with $n = 8$ to approximate the integral and compare your answers to those in Exercises 3–5.

21. $\int_0^8 x^4\,dx$

22. $\int_1^5 \frac{1}{x}\,dx$

23. $\int_1^5 \frac{1}{x^2}\,dx$

24–29 Use the formula discussed in the text to find an error estimate for the Trapezoidal Rule when it is used to approximate the integral of $f(x)$ over the given interval. Use $n = 10$.

24. $f(x) = x^2, \quad [0,2]$

25. $f(x) = \cos 2x, \quad [\pi/2, 3\pi/2]$

26. $f(x) = \sqrt{x}, \quad [1,4]$

27. $f(x) = x^4, \quad [0,1]$

28. $f(x) = \frac{1}{\sqrt{x+1}}, \quad [0,3]$

29. $f(x) = \frac{1}{x}, \quad [1,11]$

30–35. Repeat Exercises 24–29, this time providing an error estimate for Simpson's Rule with $n = 10$. Compare your error bounds to those you gave for the Trapezoidal Rule.

36–41 Determine a number of subintervals that will guarantee an approximation of the given definite integral to within an error of 0.001 using the Trapezoidal Rule. (Answers will vary.)

36. $\int_0^1 \frac{x^4}{2}\,dx$

37. $\int_0^\pi \sin^2 x\,dx$

38. $\int_1^2 \frac{2-x^2}{x^2}\,dx$

39. $\int_0^1 e^{x^2}\,dx$

40. $\int_1^3 x\ln x\,dx$

41. $\int_0^{\pi/2} x\sin x\,dx$

42–47. For the integrals in Exercises 36–41, find a number of subintervals that will guarantee an approximation of the same accuracy (within an error of 0.001), this time using Simpson's Rule. Compare your answers to those you gave for the Trapezoidal Rule.

48. Use Simpson's Rule with $n = 6$ to approximate $\ln 2 = \int_1^2 \frac{1}{x}\,dx$, and find the percentage error of your approximation (the actual error divided by the true value expressed as a percentage).

49. Repeat Exercise 48 to approximate $\pi = \int_0^1 \frac{4}{x^2+1}\,dx$.

50. Use Simpson's Rule with $n = 24$ to approximate the area of the region bounded by the graphs of $y = \sqrt{1+x^4}$, $x = -6$, $x = 6$, and the x-axis. (Notice that this problem leads to a nonelementary integral.)

51. Use the shell method along with the Trapezoidal Rule to find the approximation T_7 for the volume of the solid obtained by revolving the region between the graph of $y = e^x/x^2$ $(1 \le x \le 2)$ and the x-axis about the y-axis.

52. Combine the disk method with Simpson's Rule to find the approximation S_6 for the volume of the solid obtained by revolving the region between the graph of $y = e^{x^2}$ $(0 \le x \le 1)$ and the x-axis about the x-axis.

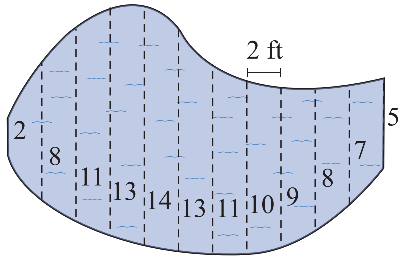
53. The following table summarizes acceleration data for a Ford Mustang Boss 302 Laguna Seca. Use Simpson's Rule to estimate the total distance traveled by "the Boss" during its timed 0–120 mph run. (**Hint:** Sketching a graph similar to the one in Example 2 is useful. Be sure to identify which area you can approximate and how it yields the answer to the problem.)

Acceleration Times

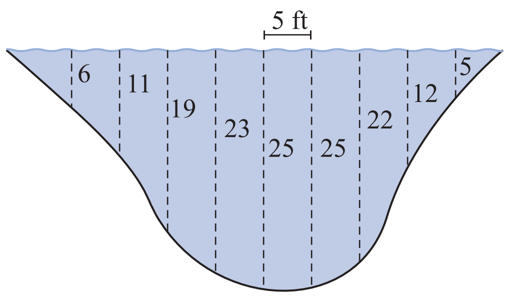
Miles per Hour	Seconds
0–120	13.0
0–110	10.9
0–100	9.1
0–90	7.6
0–80	6.3
0–70	5.2
0–60	4.1
0–50	3.3
0–40	2.4
0–30	1.7
0–20	1.1
0–10	0.4

Source: *Road & Track*

54. Use the Trapezoidal Rule to estimate the amount of water needed to raise the water level by two inches in a pool with the shape shown in the figure. At 2-foot intervals, the distances across the pool (in feet) are as indicated in the diagram.



55. The figure shows the vertical cross-section of the Lazee river where the Dinkatown ferry docks. The depth of the river is indicated at 5-foot intervals in the diagram. If the river flows at 5 ft/s, use Simpson's Rule to estimate the amount of water passing by the dock every second.



56. Prove that Simpson's Rule actually gives the exact answer for definite integrals of all polynomials of degree 3 or less.

7.6 Technology Exercises

57–60 Write a short program for a computer algebra system to approximate the given integral using the Trapezoidal Rule with $n = 100$. Then ask the software to evaluate the integral directly and compare your approximation with the exact value.

57. $\int_0^2 \sqrt{x}\sqrt{x+2}\,dx$

58. $\int_0^6 \frac{dx}{3+x^2}$

59. $\int_0^{20} \frac{dx}{1+x^3}$

60. $\int_0^{10} \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx$

61–64. Modify the program you wrote for Exercises 57–60 to find S_{100} for the integrals. Compare your answers with the exact values you found in Exercises 57–60.

65–70 Use the program you wrote for Exercises 61–64 to find the Simpson approximation S_{50} for the given nonelementary integral. To get a feel for how close your answer is to the exact value of the integral, use the graph of the fourth derivative of the integrand, and identify an error bound as small as you can by using the theorem from this section. (Answers may vary slightly. **Hint:** You may check your error bound by using the command specific to your CAS to obtain the "exact value" of the integral. Note that the CAS uses numerical algorithms to obtain that answer and, thus, it isn't exact either. It is, however, more accurate than what we can obtain by Simpson's Rule with $n = 50$.)

65. $\int_0^1 e^{x^2}\,dx$

66. $\int_1^3 \frac{e^x}{x}\,dx$

67. $\int_0^\pi \cos x^2\,dx$

68. $\int_1^\pi \frac{\sin x}{x}\,dx$

69. $\int_2^4 \ln(\ln x)\,dx$

70. $\int_e^5 \frac{1}{\ln x}\,dx$

71–73 With the help of a computer algebra system and Simpson's Rule, approximate the solution of the given integral equation. (Use $n = 100$).

71. $\int_0^x e^{\sqrt{t}}\,dt = 3$

72. $\int_0^x \tan t^2\,dt = 1$

73. $\int_0^x \sqrt{1+t^4}\,dt = 4$

Example 9 Using the Direct Comparison Test

Determine whether $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.01}}\,dx$ converges.

Solution

For $x \geq 1$, we have the following.

$$\sqrt{x^2 - 0.01} \leq x$$
$$\frac{1}{\sqrt{x^2 - 0.01}} \geq \frac{1}{x}$$

You will show in Exercise 17 that the integral $\int_1^{\infty} (1/x)\,dx$ diverges, so $\int_1^{\infty} \left(1/\sqrt{x^2 - 0.01}\right) dx$ diverges as well.

7.7 Exercises

1–8 Decide whether the given integral is an improper integral. If so, explain why and identify its type. (Do not evaluate the integral.)

1. $\int_1^{\infty} 2^{-x}\,dx$

2. $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

3. $\int_0^{1} x^{-1/3}\,dx$

4. $\int_{-1}^{1} \frac{1}{x^2}\,dx$

5. $\int_0^{1} \frac{1}{(x+1)^{4/5}}\,dx$

6. $\int_{-\infty}^{-10} \frac{1}{x}\,dx$

7. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

8. $\int_2^{\infty} \frac{1}{(x-2)^{3/2}}\,dx$

9–16 Use the definitions from this section to write the given improper integral in terms of limits. (Do not evaluate the integral.)

9. $\int_2^{\infty} \frac{1}{x^2 - 1}\,dx$

10. $\int_0^{1} \frac{1}{x^2 - 1}\,dx$

11. $\int_{-\infty}^{-1} \frac{-3}{x}\,dx$

12. $\int_{-\infty}^{\infty} \frac{2}{\sqrt{x^2 + 2}}\,dx$

13. $\int_{-\infty}^{0} e^x\,dx$

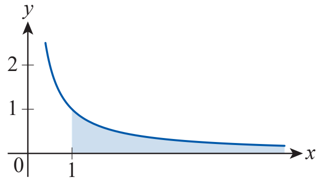
14. $\int_0^{2} \frac{1}{(x-1)^2}\,dx$

15. $\int_0^{\infty} \frac{2}{x}\,dx$

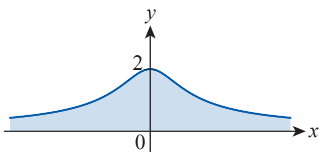
16. $\int_{-\infty}^{\infty} \frac{5}{x^2}\,dx$

17–20 Determine whether the improper integral pictured is convergent or divergent. If it is convergent, find its value.

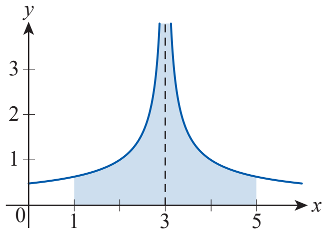
17. $\int_1^{\infty} \frac{dx}{x}$



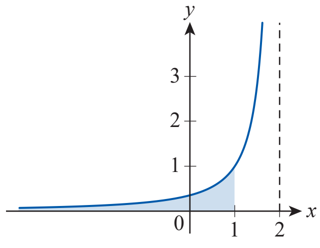
18. $\int_{-\infty}^{\infty} \frac{2}{\sqrt{1+x^2}}\,dx$



19. $\int_1^{5} \frac{dx}{(x-3)^{2/3}}$



20. $\int_{-\infty}^{1} \frac{dx}{(2-x)^{3/2}}$



21–64 Identify the type of the improper integral and determine whether it is convergent or divergent. If it is convergent, find its value.

21. $\int_1^\infty \frac{1}{x^{3/2}}\,dx$

22. $\int_{-\infty}^{-1} \frac{-3}{x}\,dx$

23. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

24. $\int_{-\infty}^{1} 2e^x\,dx$

25. $\int_0^\infty xe^{-x}\,dx$

26. $\int_0^\infty \cos x\,dx$

27. $\int_1^2 \frac{dx}{(x-1)^3}$

28. $\int_1^2 \frac{dx}{\sqrt[3]{x-1}}$

29. $\int_0^\infty \frac{6\,dx}{x^2+9}$

30. $\int_0^\infty \frac{6\,dx}{x^2-9}$

31. $\int_0^2 \frac{dx}{(x-1)^2}$

32. $\int_2^\infty \frac{dx}{(x-2)^2}$

33. $\int_{-\infty}^\infty \frac{2e^x}{e^{2x}+1}\,dx$

34. $\int_0^2 \frac{t}{\sqrt{4-t^2}}\,dt$

35. $\int_0^\infty \frac{dt}{(t+1)\sqrt{t}}$

36. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

37. $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$

38. $\int_0^\infty e^{-\theta}\cos\theta\,d\theta$

39. $\int_0^\infty \frac{e^x}{x}\,dx$

40. $\int_0^{16} \frac{dt}{\sqrt[4]{16-t}}$

41. $\int_0^9 \frac{dt}{9-t}$

42. $\int_{-\infty}^\infty \frac{dx}{9x^2+1}$

43. $\int_4^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

44. $\int_0^\infty t^2e^{-t}\,dt$

45. $\int_e^\infty \frac{dz}{z\ln^2 z}$

46. $\int_e^\infty \frac{\ln z}{z^2}\,dz$

47. $\int_0^e \frac{\ln z}{z^2}\,dz$

48. $\int_0^e \frac{\ln z}{z}\,dz$

49. $\int_0^\infty \frac{dx}{2\sqrt{x}(x+1)}$

50. $\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$

51. $\int_{-\infty}^\infty \frac{v}{(v^2+4)^2}\,dv$

52. $\int_0^2 \frac{4v}{\sqrt{16-v^4}}\,dv$

53. $\int_{-4}^0 \frac{dx}{\sqrt{|x+2|}}$

54. $\int_1^\infty \frac{\arctan x}{1+x^2}\,dx$

55. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$

56. $\int_3^\infty \frac{dx}{x^2-2x}$

57. $\int_2^3 \frac{dx}{x^2-2x}$

58. $\int_0^\infty \frac{dx}{(x+2)(x+3)^2}$

59. $\int_0^{\pi/2} \sec x\,dx$

60. $\int_0^\infty \frac{dx}{x(x+2)^2}$

61. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

62. $\int_0^{\pi/2} \tan\theta\,d\theta$

63. $\int_1^\infty \frac{dt}{t\sqrt{3t+1}}$

64. $\int_0^\infty \frac{x-2}{x^2+x+2}\,dx$

65. Classify the integrals of the form $\int_1^\infty (1/x^p)\,dx$ according to convergence for all possible values of p. (**Hint:** Consider the three cases of $p > 1$, $p = 1$, and $p < 1$.)

66. Repeat Exercise 65 for the integrals of the form $\int_0^1 (1/x^p)\,dx$.

67–74 Use the Direct Comparison Test to determine whether the integral converges.

67. $\int_1^\infty \frac{dx}{\sqrt{x^4+2x+3}}$

68. $\int_2^\infty \frac{\ln x}{\sqrt{x^2-1}}\,dx$

69. $\int_1^\infty \frac{dx}{e^{2x}+x^{3/2}}$

70. $\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$

71. $\int_1^\infty \frac{\ln x}{x}\,dx$

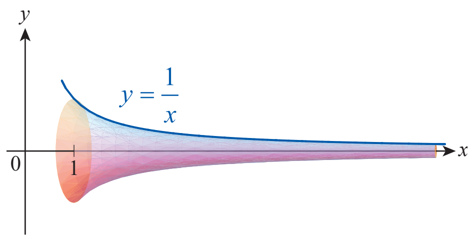
72. $\int_1^\infty \frac{\ln x}{x^{5/2}}\,dx$

73. $\int_1^\infty \frac{dx}{x^{1/2}-\frac{1}{2}}$

74. $\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}\ln x}\,dx$

75. Rotate the infinite region bounded by the graphs of $y = x^2/\sqrt{1-x^2}$, $y = 0$, and $x = 1$ about the y-axis. Use the method of shells to find the volume of the resulting unbounded solid.

76. If the infinite region between the graph of $y = 1/x$ and the x-axis ($x \geq 1$) is revolved about the x-axis, we obtain the solid nicknamed *Gabriel's horn* (see figure). Use improper integrals to show that Gabriel's horn has a finite volume and infinite surface area. Note that this means that, at least theoretically, Gabriel's horn can be filled with a finite amount of paint, but it would take an infinite amount to paint its surface! Can you find a mathematical explanation for this conclusion? (**Hint:** To show that the surface area is infinite, use the Direct Comparison Test.)



Gabriel's Horn

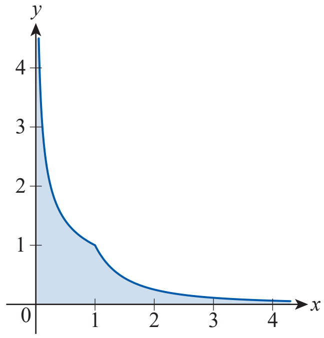
77. Show that the process in Exercise 76 results in a solid of finite volume when revolving about the x-axis, but in a solid of infinite volume when revolving about the y-axis.

78. Find the area between the graph of $y = 2/(x^2 - 1)$ and its horizontal asymptote for $x \geq 2$.

79. Find the total area between the graph of $y^2 = x^2/(16 - 16x^2)$ and its vertical asymptotes. (**Hint:** Be sure to consider both branches of the graph.)

80.* The figure below shows the graph of the following piecewise-defined function.

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ \dfrac{1}{x^2} & \text{if } 1 < x < \infty \end{cases}$$



Show that $\int_0^{\infty} f(x)\,dx = 3.$

81. Sketch the graph of $x^{2/3} + y^{2/3} = 1$ (hypocycloid with four cusps) and find its perimeter.

82. For $a > 0$, find a formula for the integral

$$\int_1^{\infty} \frac{dx}{x\sqrt{x^2 + a^2}}.$$

83. In probability theory, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is called the density function of the standard normal distribution. Show that its mean is 0 and its variance is 1; that is,

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx = 0 \text{ and } \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1.$$

(**Hint:** For the first integral, adapt your solution to Exercise 36. For the second, use integration by parts along with L'Hôpital's Rule.)

84. Find the function in the one-parameter family $y = c/(x^2 + 1)$ so that

$$\int_{-\infty}^{\infty} \frac{c}{x^2 + 1}\,dx = 1.$$

(The function you just found is another probability density function, called the *Cauchy density function*.)

85–87 In certain cases, substitution can help turn an improper integral into a proper one. For example, as part of your solution to Exercise 35, substituting $u = \sqrt{t}$ you may find

$$\int_0^1 \frac{dt}{(t+1)\sqrt{t}} = 2\int_0^1 \frac{du}{u^2 + 1},$$

the latter being a proper integral.

Use the above idea to turn the given improper integral into a proper one and evaluate.

85. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{x}\sqrt{1-x}}$

86. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

87. $\displaystyle\int_1^{\infty} \frac{1}{1 + x^2}\,dx$ (**Hint:** Substitute $u = 1/x$.)

88–93 The Laplace transform of a function $f(t)$, denoted $L\{f(t)\}$, is defined by the improper integral

$$L\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)\,dt,$$

as long as it converges. The Laplace transform is very useful in physics and engineering, most notably, for solving certain linear ordinary differential equations.

Find the Laplace transform of the given function (we assume s is appropriately restricted so that the Laplace transform converges).

88. $L\{1\}$

89. $L\{t\}$

90. $L\{t^2\}$

91. $L\{e^{at}\}$

92. $L\{\sin kt\}$

93. $L\{\cos kt\}$

94. Recognizing a pattern from Exercises 88–90, conjecture and use induction to prove a general formula for $L\{t^n\}$ $(n \in \mathbb{N})$. (**Hint:** You may want to calculate $L\{t^3\}$ to firm up your conjecture.)

95–97 Suppose you invest money at an annual interest rate of r, which is compounded continuously, with a goal of having N dollars in t years. The amount you invest today to achieve that goal is called the present value (denoted PV) of the N dollars that is still t years out in the future. This present value can be calculated by the formula

$$PV = Ne^{-rt},$$

since investing PV dollars today will yield $(PV)e^{rt} = (Ne^{-rt})e^{rt} = N$ dollars in t years.

Using Riemann sums, it is straightforward to derive the formula for the present value of an annuity, a terminating income stream of fixed payments over a finite time period, say T years:

$$PV = \int_0^T A(t)e^{-rt}\,dt,$$

where $A(t)$ is the amount paid out annually (we assume continuous payment).

In Exercises 95–97, use this formula to generate the required improper integral.

95. If the annual interest rate is 5%, find the present value of a perpetual annuity (one paying dividends forever) that continuously pays \$10,000 every year.

96. How much should we invest at an annual interest rate of 4% if we want a never-ending income stream of continuous annual payments of \$500?

97. Suppose we expect an investment to generate profits at $p(t) = 6500\sqrt{t}e^{0.015t}$ dollars annually forever (t stands for the number of years elapsed). If the annual interest rate is 6%, find the present value of this income stream.

98. Use an improper integral to find the work done in propelling a 500 kg satellite out of Earth's gravitational field. (**Hint:** Approximate the radius of Earth by 6371 km. For the magnitude of the force of gravity at great altitudes, see Exercise 49 in Section 6.5.)

99. If an object is leaving the surface of Earth with a velocity v_0 big enough that its kinetic energy $E_{kin} = \frac{1}{2}mv_0^2$ is equal to the work required to propel it out of Earth's gravitational field, then the object will never return, but rather travel "infinitely far away" into outer space. Use Exercise 98 to find the value of the escape velocity on the surface of Earth. (**Hint:** As in Exercise 98, approximate the radius of Earth by 6371 km. See also Exercise 55 of Section 6.5 and the Chapter 6 Project.)

100.* The *gamma function*, which after an argument shift becomes an extension of the factorial function $f(n) = n!$, is defined as follows.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

Note that here x can be any real (or even complex) number. The gamma function is especially important in the fields of combinatorics, probability, and statistics.

Prove each of the following statements.

a. The above improper integral converges for all $n > 0$. (**Hint:** Show that for $n > 0$, $0 < \frac{x^{n-1}}{e^x} \le \frac{1}{x^2}$ for all appropriately large x-values and use the Direct Comparison Test.)

b. $\Gamma(1) = 1$

c. $\Gamma(n+1) = n\Gamma(n)$
(**Hint:** Use integration by parts.)

d. $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$
(**Hint:** Use mathematical induction.)

Concept Check

101–104 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

101. If $\int_1^\infty f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) = L$ with $L \ne 0$ or $\lim_{x\to\infty} f(x)$ doesn't exist.

102. If $f(x)$ is continuous on $[1,\infty)$, positive, and decreasing, and if for any $M > 0$ there is a $b > 1$ such that $\int_1^b f(x)\,dx > M$, then $\int_1^\infty f(x)\,dx$ diverges.

103. The integral $\int_1^\infty \frac{\ln(x^3)}{x}\,dx$ diverges.

104. For any positive $a \in \mathbb{R}$, $\int_a^\infty \frac{dx}{x^{1+a}}$ converges.

7.7 Technology Exercises

105–106 The function, defined in terms of an improper integral, is important for its applications within or outside of mathematics, for example in number theory, statistics, probability, physics, or engineering. Use a graphing utility to sketch the graph and observe important features. (Answers will vary.)

105. $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$ (the error function)

106. $\operatorname{Li}(x) = \displaystyle\int_2^x \frac{dt}{\ln t}$ (the logarithmic integral function)

107–108 Attempt to use a computer algebra system to evaluate the improper integral (from Exercises 67 and 74) and see what happens. (**Note:** Theoretical results such as the Direct Comparison Test are extremely useful, even when powerful mathematical software is at our disposal. As we mentioned during our discussion in the text, sometimes concluding the fact of convergence is more important than the actual value of the integral; and the Direct Comparison Test provides a fast and trouble-free way to do just that.)

107. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+2x+3}}$

108. $\displaystyle\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}\ln x}\,dx$

8.1 Exercises

1–6 Rewrite the first-order differential equation in the form $dy/dx = f(x, y)$. (Do not attempt to solve the equation.)

1. $\sqrt{x}\dfrac{dy}{dx} = \sqrt{y}$

2. $dy = 3y\,dx + e^{2x}\,dx$

3. $\sqrt{1-y^2}\,dx = \dfrac{dy}{x}$

4. $\dfrac{\tan y}{x} = \cos x\dfrac{dx}{dy}$

5. $xy = y - x^2y'$

6. $3x^2y = xe^x - xy'$

7–12 Verify that the function (or family of functions) is a solution of the given differential equation.

7. $xy' = 5y; \quad y = Cx^5$

8. $y' - \dfrac{3x^2}{y} = 0; \quad y = \sqrt{2x^3+5}$

9. $x^2\,dy - dx = 0; \quad y = 6 - \dfrac{1}{x}$

10. $3y' + 2xy = 0; \quad y = Ce^{-x^2/3}$

11. $y' + y = e^{5x}; \quad y = Ce^{-x} + \dfrac{e^{5x}}{6}$

12. $x\,dy - (y - x)\,dx = 0; \quad y = Cx - x\ln x$

13–20 Solve the differential equation with the given initial condition.

13. $y' = x^2 + \cos x; \quad y(0) = 5$

14. $y' = \dfrac{1}{x^2+1} - 2; \quad y(0) = 0$

15. $xy' = 1 + x; \quad y(1) = 0$

16. $y'\sqrt{x} = \sqrt{x} + 1; \quad y(1) = 5$

17. $dx = x^2\,dy; \quad y(2) = 3$

18. $\sqrt{1-x^2}\,y' = x; \quad y(0) = 2$

19. $\sqrt{1-x^2}\,y' = 1; \quad y(0) = 2$

20. $y' = \dfrac{1}{x^2+x}; \quad y(1) = -4$

21–28 Determine whether the differential equation is separable. (Do not solve the equation.)

21. $y' = (2x-1)e^y$

22. $y' = y^2 + 4$

23. $xy' + 4y^2 = 0$

24. $y' = x + y$

25. $y' = \sqrt{x}\,y - ye^x$

26. $y' = (x+y)^2$

27. $y' = 3x^2y - x\sqrt{y}$

28. $x^2\,dy - (y^2 + yx)\,dx = 0$

29–40 Solve the separable differential equation.

29. $y' = 2x(y^2+1)$

30. $y' = y(x^2+4)$

31. $y' = 3xy + 6x$

32. $y' = 9y^2x^2$

33. $dy - 3x^2y\,dx = 0$

34. $y' = \sec y \sec^2 x$

35. $y'\sqrt{4-x^2} = y$

36. $y' = \sqrt{\dfrac{1-y^2}{x}e^{\sqrt{x}}}$

37. $e^{1/x}\dfrac{dx}{dt} = -2tx^2$

38. $(x+2)\,dy = (x+5)\,dx$

39. $dy = \dfrac{y+6}{x}dx$

40. $x^2y\,dy = (y^2+1)\,dx$

41–43 Solve the given separable equation, treating y as the independent variable and solving for $x = x(y)$.

41. $x\,dy - (y+1)\,dx = 0$

42. $dx - 3(x^2+1)\,dy = 0$

43. $y' = \dfrac{2x}{y+2}$

44–53 Solve the given initial value problem.

44. $y' = -\dfrac{x}{y}; \quad y(1) = 1$

45. $y' = \dfrac{x^2-1}{y^2}; \quad y(0) = 2$

46. $y' = \dfrac{4x^3+2x}{2y}; \quad y(2) = 5$

47. $y' = e^{2x-5y}; \quad y(0) = 0$

48. $x^3y' = (x-1)y^2; \quad y(1) = -1$

49. $y'\sec x = y^2; \quad y\left(\dfrac{\pi}{2}\right) = \dfrac{1}{3}$

50. $\dfrac{dy}{dx} = 4x^3e^{-y}; \quad y(0) = 1$

51. $\dfrac{y'}{x} = y^2 + y; \quad y(0) = -2$

52. $y' = \dfrac{1+y^2}{1+x^2}; \quad y(1) = -2$

53. $\dfrac{dy}{1+\sin t} = 2e^{-y}\cos t\,dt; \quad y(0) = 0$

54. A 50-gallon tank is filled with brine (water nearly saturated with salt; used as a preservative) holding 12 pounds of salt in solution. A salt solution containing 0.5 pounds of salt per gallon is added to the tank at the rate of 1 gallon per minute. The contents of the tank are continuously and thoroughly mixed and drained out at the same rate. What is the amount of salt in the tank after an hour?

55. A tank contains 2000 gallons of diesel fuel. A fuel mixture containing a lubricity additive is pumped into the tank through two inlets. The mixture flowing in through the first inlet contains 0.48 oz of additive per gallon and is being pumped in at a rate of 25 gal/min. Meanwhile, the mixture being allowed in by the second inlet at a rate of 10 gal/min contains 10.4 oz of additive per gallon. The mixture in the tank is continuously and thoroughly mixed and drained out at the rate of 35 gal/min. If there should be 16 oz of additive for every 120 gallons of diesel fuel, how long will it take to reach the right mixture?

56. To freshen the air, a small window is opened in a room initially containing 0.12% carbon dioxide. Fresh air with 0.04% carbon dioxide is pouring in at a rate of $6 \text{ m}^3/\text{min}$, and we assume that the uniform mixture is leaving the room at the same rate. If the dimensions of the room in meters are $4 \times 6 \times 3$, how long will it take to cut the initial carbon dioxide content down to half?

57. The cane sugar in fruit juice converts into dextrose under certain conditions. At any time, the rate of this process is proportional to the amount of cane sugar that is yet to be converted. If 100 grams of cane sugar is added to a certain fruit juice and we know that 12 grams are converted into dextrose during the first hour, how much dextrose will be present in the juice after 3 hours?

58. Suppose that an ice cube melts so that its volume $V(t)$ decreases at a rate proportional to its surface area.

a. Find a differential equation satisfied by $V(t)$.

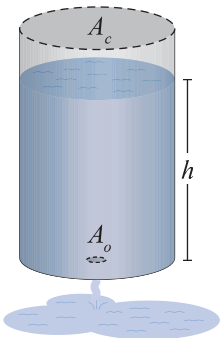
b. If an ice cube of side length 1 inch loses a third of its volume in 2 minutes, use your model to predict how long it will take for it to completely melt away. (Consider the cube melted away when your model predicts less than 1 percent remaining.)

59–61 Suppose a container is filled with fluid to a height of h. According to Torricelli's Law, when viscosity and friction are ignored, the speed v of efflux of the fluid through a small, sharp-edged opening through the bottom of the container equals the speed that the fluid would acquire when falling freely from a height of h, as follows.

$$v = \sqrt{2hg}$$

59.* If A_c is the horizontal cross-sectional area of a vertical cylindrical tank, while A_o denotes the area of the hole at the bottom of the tank, prove that the rate at which the fluid level is falling in the container is described by the following differential equation.

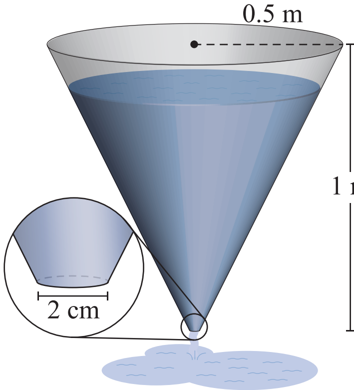
$$\frac{dh}{dt} = -\frac{A_o}{A_c}\sqrt{2hg}$$



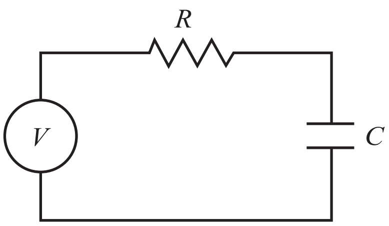
(**Hint:** First find the rate of fluid leaving the tank dV/dt; then use the fact that $V = A_c h$.)

60. If a cubic tank of side length 1 m is initially full of water but is draining through a circular orifice of diameter 2 cm that is on the bottom of the tank, what is the water level in the tank 2 minutes later? (**Hint:** Use the formula from Exercise 59.)

61.* Answer the question of Exercise 60 if the container is an inverted right circular cone of height 1 m and base radius 0.5 m. The opening on the bottom is the same as that in Exercise 60. (**Hint:** First, convince yourself that even though $A_c = A_c(h)$ now depends on h, the formula $dh/dt = -(A_0/A_c)\sqrt{2hg}$ remains in effect. Next, adapt your solution for Exercise 60. Ignore the geometrical change to the cone caused by the presence of the orifice.)



62.* The figure below shows a series circuit containing a resistor and a capacitor (this is called an RC circuit). Find a differential equation for the charge $q(t)$ if the impressed voltage V on the circuit is constant. (**Hint:** Use Ohm's Law, as well as the fact that the voltage drop on the capacitor is $\frac{1}{C}q$. For a statement of Ohm's Law, see the discussion preceding Example 5 in Section 8.2.)



63. Suppose that the air resistance encountered by a falling body is proportional to its velocity v.

a. Use Newton's Second Law of Motion (see Section 3.7) to find a differential equation satisfied by a falling body of mass m.

b. Solving your equation, find a formula for the terminal velocity of a body if it is falling from rest.

64. When starting from rest, the acceleration of a sailboat is proportional to the difference between the boat's velocity and that of the wind. Suppose that two minutes after starting from rest in 18 mph wind, a sailboat is moving at 8 mph.

a. Find a differential equation satisfied by the boat's velocity function.

b. Find the boat's velocity function.

c. How fast is the boat moving 4 minutes after it starts?

65–73 Determine the orthogonal trajectories of the family of curves, where a is an arbitrary nonzero constant. (If technology is available, sketch several curves from both families and visually check orthogonality.)

65. $y = ax^2$

66. $y = ax^3$

67. $y = \dfrac{ax}{x+2}$

68. $y = ae^x$

69. $y = \dfrac{x}{ax+1}$

70. $y = \dfrac{1}{a+x}$

71. $x^2 + ay^2 = 1$

72. $y = a\cos x$

73. $y = \dfrac{a}{1+x^2}$

Concept Check

74–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. A differential equation is an equation that relates an unknown function and at least one of its derivatives.

75. The equation $x^3y' + 1 = xy - x + y$ is separable.

76. A separable equation in the variables x and y always has a solution in the form $y = f(x)$.

77. This is the first time in this text that we are solving initial value problems.

$$I(t) = e^{-(R/L)t}\left[\int \frac{e^{(R/L)t}V}{L}\,dt + C\right]$$
$$= e^{-(R/L)t}\left[\left(\frac{V}{L}\right)\left(\frac{L}{R}\right)e^{(R/L)t} + C\right] = \frac{V}{R} + Ce^{-(R/L)t}$$

The current I begins at 0 when $t = 0$, so it must be the case that $C = -V/R$ and hence the particular solution is as follows.

$$I(t) = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} = \frac{V}{R}\left[1 - e^{-(R/L)t}\right]$$

As $t \to \infty$, $\left[1 - e^{-(R/L)t}\right] \to 1$, so the *steady-state* current in the circuit is simply V/R. The inductance in the circuit is most significant immediately after voltage is applied—it is then that the term $1 - e^{-(R/L)t}$ has greatest effect, as shown in Figure 4.

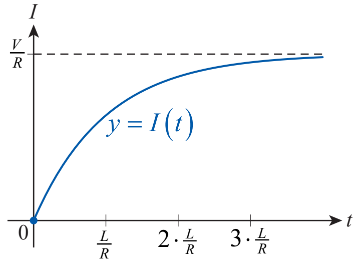


Figure 4

The length of time L/R is referred to as the circuit's *time constant* and is a measure of how its inductance and resistance interact to affect the current. In Exercise 44, you will show that $I(t)$ attains slightly more than 95% of its steady-state value after three time constants.

8.2 Exercises

1–6 Classify the differential equation as linear or nonlinear. (Do not attempt to solve the equation.)

1. $xy' - \frac{e^x}{x+1}y = \sqrt{x} - \frac{2y}{x}$
2. $xy' - \frac{e^x}{x+1}y = \sqrt{y} - \frac{2y}{x}$
3. $\frac{y}{x} = x^2y'$
4. $yy' + (x+1)y = x^2$
5. $\tan x = \frac{2x + y'}{y}$
6. $y' - xy^2 = x$

7–10 Decide if the equation is linear in the dependent variable y. If not, check whether it is linear when x is considered to be the dependent variable.

7. $2\,dy - x\,dx = y\left(1 - 3\sqrt{x}\right)dx$
8. $x\,dy = \cos y\,dy - y\,dx$
9. $y\,dx = (5x + 2y - 4)\,dy$
10. $y\,dy - \left(1 + x^2\right)y\,dx = e^x\,dx$

11–25 Solve the linear differential equation. (**Hint:** In some cases, x has to be the dependent variable in order for the equation to be linear.)

11. $y' + 3\frac{y}{x} = 0$
12. $xy' - 2y = x^3e^x$
13. $xy' + 4y = 1$
14. $y' + 2xy = 4x$
15. $x\frac{dy}{dx} = y + 2x^2 - 3x + 5$
16. $(x-1)y' = 4(x-1)^3 - y$
17. $dx + \frac{8x}{y}dy = y\,dy$
18. $y' - y\tan x = e^{\sin x}$
19. $y' = \sin x + \cos x - y$
20. $yx' - x = y^3e^y$
21. $\left(x^4 + 1\right)y' = x^4 - 4x^3y + 1$
22. $\cot x\,dy + y\,dx = \csc x\,dx$
23. $x(\ln x)y' + y = \ln x$
24. $dx - xy\,dy = y\,dy$
25. $\frac{dx}{dt} + x\tan t = t\tan t + \sec t + 1$

26–33 Solve the given initial value problem.

26. $\frac{dy}{dx}+4y=16x; \quad y(0)=0$

27. $\frac{dy}{dx}+3x^2y=x^2; \quad y(0)=2$

28. $(1+x^2)y'+2xy=1; \quad y(0)=10$

29. $(1+x^2)y'-2xy=1+x^2; \quad y(0)=10$

30. $t\frac{dS}{dt}-S=2t; \quad S(1)=7$

31. $(1+x)y'-xy=\frac{e^x}{1+x}; \quad y(0)=0$

32. $y'\sin x+y\cos x=\frac{x}{\csc x}; \quad y(\pi/2)=-3$

33. $y\,dx=(4\ln y-2x)\,dy; \quad x(1)=4$

34–37 Find a first-order linear differential equation in standard form that has the given general solution. (**Hint:** Identify the integrating factor and "reverse" the solution technique discussed in the text.)

34. $y=\frac{e^x}{x^2}+\frac{1}{x}+Cx^{-2}$

35. $y=\sin x+\frac{C}{x}$

36. $y=\frac{x^3+C}{e^x}$

37. $y=1+\frac{x+C}{\ln x}$

38–41 A first-order differential equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)y^{\alpha}$$

is called the **Bernoulli equation** (named after Jakob Bernoulli), where α is any real number. You should check that by introducing the new dependent variable $u=y^{1-\alpha}$ ($\alpha\neq 0$, $\alpha\neq 1$) and noting that

$$\frac{du}{dx}=(1-\alpha)y^{-\alpha}\frac{dy}{dx},$$

we can turn a Bernoulli equation into the following standard-form linear one.

$$\frac{du}{dx}+(1-\alpha)P(x)u=(1-\alpha)Q(x)$$

In Exercises 38–41, use the above substitution method to solve the given Bernoulli equation.

38. $\frac{dy}{dx}+2y=x\sqrt{y}$

39. $y'+\frac{y}{x}=y^2$

40. $y'+2y=e^xy^2$

41. $y'+\frac{y}{x}=2y^{3/2}$

42. Explain why it is no loss of generality to always let the constant of integration be 0 when determining the integration factor in the solution of a linear differential equation.

43. Find the time when the additive in the tank of Example 4 reaches a maximum.

44. Show that $I(t)$ in Example 5 attains slightly more than 95% of its steady-state value after three time constants.

45. A 50-gallon tank is filled with brine (water nearly saturated with salt; used as a preservative) holding 12 pounds of salt in solution. A salt solution containing 0.5 pounds of salt per gallon is added to the tank at the rate of 1 gallon per minute. The contents of the tank are continuously and thoroughly mixed and drained out at 5 quarts per minute. What is the amount of salt in the tank after an hour? (Compare with Exercise 54 of Section 8.1.)

46. A tank contains 2000 gallons of diesel fuel. A fuel mixture containing a lubricity additive is pumped into the tank through two inlets. The mixture flowing in through the first inlet contains 0.48 oz of additive per gallon and is being pumped in at a rate of 25 gal/min. Meanwhile, the mixture being allowed in by the second inlet at a rate of 10 gal/min contains 10.4 oz of additive per gallon. The mixture in the tank is continuously and thoroughly mixed and drained out at the rate of 20 gal/min. If there should be 16 oz of additive for every 120 gallons of diesel fuel, how long will it take to reach the right mixture? (Compare with Exercise 55 of Section 8.1.)

47. Suppose that V_0 gallons of gasoline contain a_0 pounds of a seasonal additive. A gasoline mixture containing a_1 pounds of additive per gallon is added to the tank at the rate of $r_1 \text{ gal/min}$. The gasoline solution in the tank is continuously and thoroughly mixed and drained out at a rate of $r_2 \text{ gal/min}$. Set up the initial value problem whose solution is $y(t)$, the amount of additive in the mixture at time *t*.

48. In learning theory, the rate of memorization, or learning, is considered to be proportional to the amount of material yet to be memorized. On the other hand, the amount forgotten is proportional to the amount already learned. If T stands for the total amount of material to be memorized, and $M(t)$ is the amount memorized at time *t*, find a differential equation satisfied by $M(t)$. What type of differential equation did you obtain?

49. Find $I(t)$ in a simple RL circuit if 5 volts are applied at time $t = 0$, $I(0) = 0$, the inductance is 0.2 henries and the resistance is 10 ohms.

50. Find a formula for $I(t)$ in Exercise 49 if the impressed voltage is $V(t) = 1.2t$.

51. Suppose that 150 volts are impressed on an RC circuit with resistance of 50 ohms and capacitance of 10^3 farads. If $q(0) = 0$, find the charge $q(t)$ on the capacitor. What happens to the charge as $t \to \infty$? (**Hint:** See Exercise 62 of Section 8.1.)

52.* Find the current in the RL circuit of Example 5 if the impressed voltage (also called electromotive force) is $V(t) = V \sin(\omega t)$.

53. Antibiotics are taken by a patient at the rate of d milligrams per day. Assume that the drug is removed by the body from the bloodstream at a rate proportional to the amount present.

a. Find the differential equation satisfied by $A(t)$, the amount of antibiotics present in the bloodstream.

b. Solve for $A(t)$, assuming $A(0) = 0$, and determine what happens as $t \to \infty$.

54.* Find the current induced by a discharging capacitor in an RC circuit if $V(t) = 0$ and $I(0) = I_0$. (**Hint:** Consider the differential equation satisfied by $q(t)$, and start by differentiating both sides of the equation.)

Concept Check

55–58 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

55. A linear differential equation cannot be separable.

56. The equation $y\,dx + 3x\,dy - 2y\,dy = y^4\,dy$ is linear.

57. The only way to solve a linear differential equation is by the use of an integrating factor.

58. The integrating factor for a standard-form linear differential equation is the function $I(x) = e^{\int P(x)\,dx}$.

8.2 Technology Exercises

59–69. Use a computer algebra system to solve the equations in Exercises 15–25. Compare the results with the ones you obtained by hand.

70. A skydiver jumps out of a plane at 2000 meters and deploys his chute after 10 seconds of free fall. The total mass of the diver and his gear is 80 kilograms. Assume that air resistance is proportional to velocity both before and after deploying the chute, with respective constants of proportionality of 8 and 100. Use a computer algebra system to create a model to find how long after the jump the skydiver will land. (Distance is measured in meters, time in seconds.)

Example 5 Using Euler's Method

Use Euler's method to approximate a solution to the IVP $y' = y - 2x$ with the condition $y(0) = 1$.

Solution

We have $x_0 = 0$, $y_0 = 1$, and $f(x, y) = y - 2x$. We need to choose a *step size h* in order to apply the iterative approximations $x_n = x_0 + nh$ and $y_n = y_{n-1} + h(y_{n-1} - 2x_{n-1})$. We will compare the results with $h = 0.5$ and $h = 0.1$.

$h = 0.5$

n	x_n	y_n
0	0	1
1	0.5	1.5
2	1.0	1.75
3	1.5	1.625
4	2.0	0.9375

Table 1

$h = 0.1$

n	x_n	y_n	n	x_n	y_n
0	0	1	11	1.1	1.3470
1	0.1	1.1	12	1.2	1.2617
2	0.2	1.19	13	1.3	1.1479
3	0.3	1.269	14	1.4	1.0027
4	0.4	1.3359	15	1.5	0.8230
5	0.5	1.3895	16	1.6	0.6053
6	0.6	1.4285	17	1.7	0.3458
7	0.7	1.4514	18	1.8	0.0404
8	0.8	1.4565	19	1.9	–0.3156
9	0.9	1.4422	20	2.0	–0.7272
10	1.0	1.4064			

Table 2

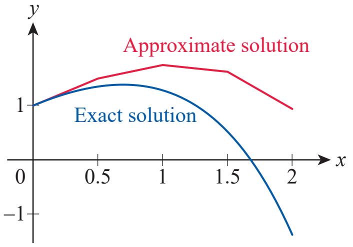


Figure 8 $h = 0.5$

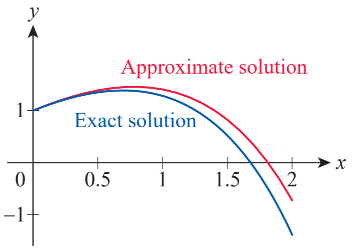


Figure 9 $h = 0.1$

The tables show the results of calculating (x_n, y_n) up to $x = 2$ with the two different step sizes, and Figures 8 and 9 compare the graphs of these estimated points (in red) with the exact solution $y = 2x + 2 - e^x$ (which you will determine when you solve the first-order linear equation $y' = y - 2x$ in Exercise 25).

8.3 Exercises

1–6 Match the differential equation with its slope field (labeled A–F).

1. $y' = x$

2. $y' = 1 - yx$

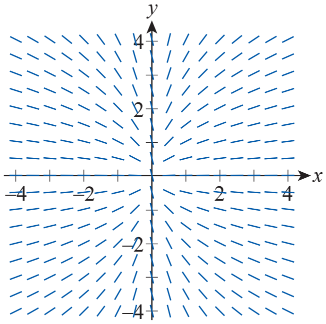
3. $y' = \dfrac{y}{2x}$

4. $y' = \dfrac{x^2 y}{3}$

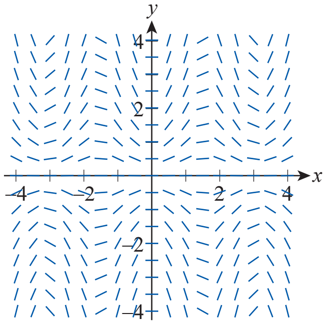
5. $y' = \sqrt{x^2 + y^2}$

6. $y' = y \sin 2x$

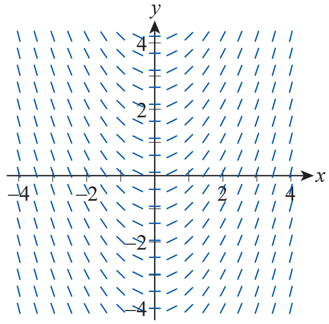
A.



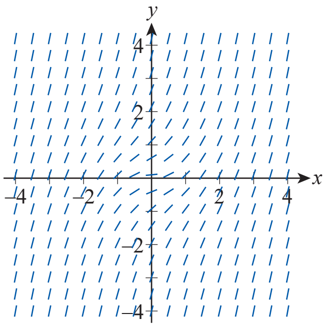
B.



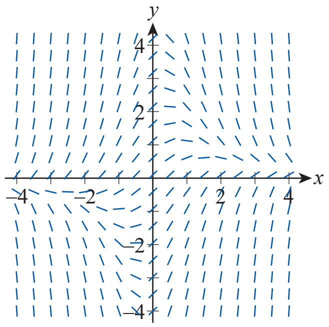
C.



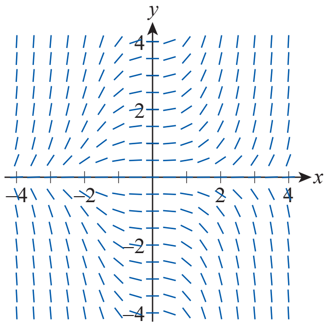
D.



E.

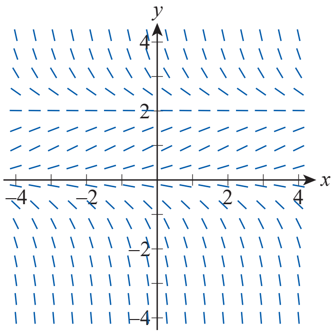


F.

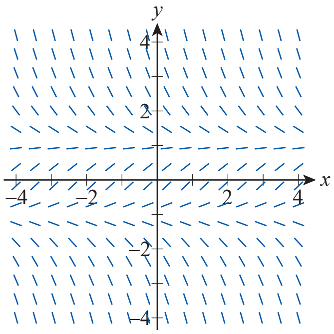


7–10 An autonomous equation and its slope field are given. Find any equilibrium solutions and classify them as stable or unstable.

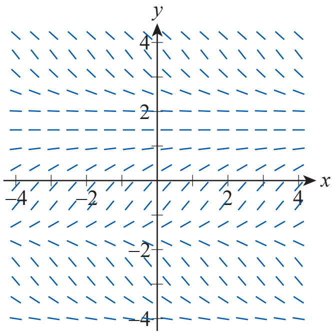
7. $y' = y - \dfrac{y^2}{2}$



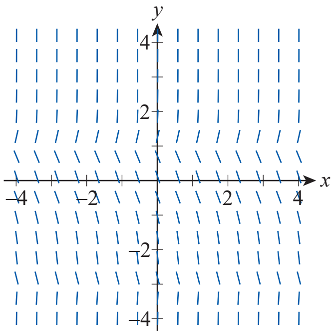
8. $y' = \dfrac{1-y^2}{\sqrt{1+y^2}}$



9. $y' = \cos y\left(1-\sin y\right), \quad -4 \le y \le 4$



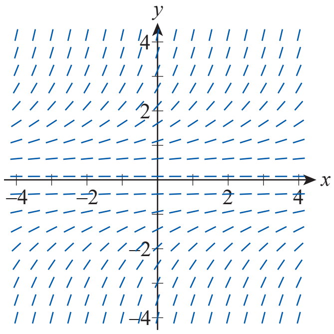
10. $y' = \left(y+3\right)\left(y^3-1\right)$



11–12 An autonomous equation and its slope field are given. Sketch the graphs of the particular solutions satisfying the specified initial conditions.

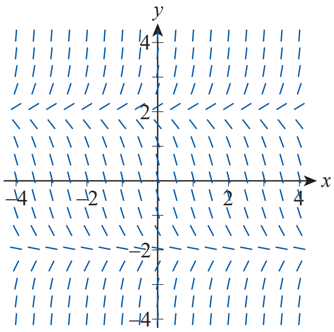
11. $y' = \dfrac{1}{4}y^2$

a. $y\left(0\right) = -1$ **b.** $y\left(1\right) = 1$



12. $y' = y^2 - 4$

a. $y\left(0\right) = 0$ **b.** $y\left(4\right) = 4$



13–16 Graph by hand the slope field of the given differential equation. If applicable, find and classify each equilibrium solution as stable or unstable.

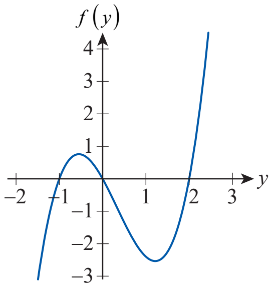
13. $y' = \frac{1}{3}y$

14. $y' = \left(2 - \frac{1}{2}y^2\right)y$

15. $y' = -y(1-y)(2-y)$

16. $y' = x - 3y$

17. Create a rough sketch of the slope field of the differential equation $y' = f(y)$, where the graph of f is given below. Classify equilibria as stable or unstable.



18. A can of soda that was forgotten on the kitchen counter and warmed up to 22 °C was put back in the refrigerator whose interior temperature is kept at a constant 3 °C. If the soda's temperature after 5 minutes is 17 °C, what will its temperature be after 20 minutes? Sketch the slope field resulting from your model; include the equilibrium and the soda's temperature curve.

19. A cake is removed from a 320 °F oven and is kept at a room temperature of 72 °F to cool down. The cake's temperature after 4 minutes is 190 °F. Use Newton's Law of Cooling to model the cooling process, and make a rough sketch of the slope field for your model, highlighting the equilibrium and the cake's temperature curve.

20. Recall from Exercise 79 in Section 7.2 that the spread of a disease in a community of N people can be modeled by the logistic differential equation $dI/dt = kI(N-I)$, where $I(t)$ stands for the number of persons already infected. Sketch the slope field and graph of this model, assuming a community of 200 people with one sick person initially, and five more catching the disease three days later.

21. An owl population grows logistically with a carrying capacity of 500 owls and constant of proportionality $k = 0.3$ per year.

a. Find the population size $P(t)$ as a function of time if initially 100 owls are present in the ecosystem.

b. How long does it take for the owl population to reach 300?

22. Use the method of Example 4 to answer the question in Exercise 63b of Section 8.1.

23. Repeat Exercise 22, this time assuming that the air resistance is proportional to $\sqrt{v}$.

24. Find the terminal velocity in Example 4 by solving the equation $\frac{dv}{dt} = g - \frac{k}{m}v^2$ and finding $\lim_{t\to\infty} v(t)$. (**Hint:** Begin with the two steps below.)

$$\frac{dv}{dt} = g\left(1 - \frac{k}{mg}v^2\right)$$

$$\frac{mg}{k}\left(\frac{dv}{\frac{mg}{k} - v^2}\right) = g\,dt$$

25. Verify that $y = 2x + 2 - e^x$ is the exact solution of the initial value problem of Example 5, that is, $y' = y - 2x$ with the condition $y(0) = 1$.

26–31 For the initial value problem, **a.** use Euler's method with the indicated step sizes to approximate the given value of y and **b.** solve the IVP by conventional methods and compare your approximations with the exact answer.

26. $y' = 3y;\quad y(0) = 2;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.125$

27. $y' = 2y + x;\quad y(0) = 1;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.1$

28. $y' = xy;\quad y(0) = 1;$
approximate $y(2)$ with (i) $h = 0.4$ (ii) $h = 0.2$

29. $y' = x^2 - y;\quad y(0) = 3;$
approximate $y(1.5)$ with (i) $h = 0.3$ (ii) $h = 0.15$

30. $y' = 2x - 2y + 1;\quad y(0) = -1;$
approximate $y(1)$ with (i) $h = 0.25$ (ii) $h = 0.1$

31. $y' = 1 + y^2;\quad y(0) = 0;$
approximate $y\left(\frac{\pi}{3}\right)$ with (i) $h = \frac{\pi}{12}$ (ii) $h = \frac{\pi}{24}$

32–33. The following formula is called the improved Euler's method, or Heun's method.

$$x_n = x_{n-1} + h = x_0 + nh \text{ and}$$

$$y_n = y_{n-1} + h\frac{f\left(x_{n-1}, y_{n-1}\right) + f\left(x_n, y_n^*\right)}{2}, \text{ where}$$

$$y_n^* = y_{n-1} + hf\left(x_{n-1}, y_{n-1}\right)$$

(Note that the quotient appearing in this formula can be interpreted as the average slope between x_{n-1} and x_n. The derivation of Heun's method is left for a textbook on differential equations or numerical methods.)

Use Heun's method to redo Exercises 26–27 with $h = 0.25$ and compare your results with your earlier answers to illustrate the accuracy of the improved Euler's method.

8.3 Technology Exercises

34–37. Use a graphing utility to create the slope fields you sketched in Exercises 13–16. If applicable, visually check the location of, and classify, any equilibria.

38–43. Use a graphing utility to improve your approximations in Exercises 26–31, using Euler's method with 20 equal increments. Then graph the results along with the exact solutions to visually check the accuracy of the method.

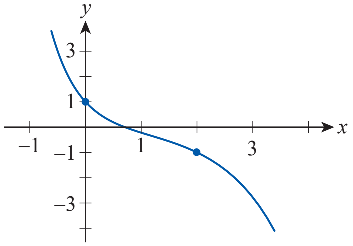


Figure 4
Boundary Value Problem Solution

$$c_1 = \frac{-e^4 - 1}{e^6 - 1} \quad \text{and} \quad c_2 = \frac{e^6 + e^4}{e^6 - 1}$$

The graph of $y(x) = c_1e^x + c_2e^{-2x}$ for these particular choices of c_1 and c_2 is shown in Figure 4. Note that the graph does indeed satisfy the boundary conditions.

b. The boundary conditions $y(0) = 0$ and $y(2) = 0$ can only be imposed on the general solution $y(x) = c_1e^x + c_2e^{-2x}$ by letting $c_1 = 0$ and $c_2 = 0$, resulting in the trivial solution $y(x) = 0$.

8.4 Exercises

1–18 Find the general solution of the given differential equation.

1. $y'' + y' - 2y = 0$
2. $y'' - 4y = 0$
3. $3y'' - 5y' - 2y = 0$
4. $2y'' + 9y' - 5y = 0$
5. $y'' + 2y' + y = 0$
6. $2y'' - 12y' + 18y = 0$
7. $y'' + y' + \frac{y}{4} = 0$
8. $y'' + y = 0$
9. $y'' - 4y' + 13y = 0$
10. $y'' - 2y' + 5y = 0$
11. $25y'' - 10y' + y = 0$
12. $3y'' - \sqrt{3}y' + y = 0$
13. $3\frac{d^2y}{dx^2} - \sqrt{13}\frac{dy}{dx} + y = 0$
14. $\frac{d^2S}{dt^2} - 12S = 0$
15. $2\frac{d^2w}{dt^2} + 2\frac{dw}{dt} + w = 0$
16. $y'' + k^2y = 0$
17. $y'' - k^2y = 0$
18. $y'' - k^2y' = 0$

19–26 Solve the given second-order initial value problem.

19. $y'' - 3y = 0;\quad y(0) = 5;\quad y'(0) = 0$
20. $y'' - 3y' - 4y = 0;\quad y(0) = 1;\quad y'(0) = -1$
21. $y'' - y' - 2y = 0;\quad y(0) = 4;\quad y'(0) = 5$
22. $y'' - 2y' + y = 0;\quad y(0) = -1;\quad y'(0) = 0$
23. $y'' - 2y' + 5y = 0;\quad y(0) = 1;\quad y'(0) = -3$
24. $y'' - 6y' + 18y = 0;\quad y(0) = 0;\quad y'(0) = 6$
25. $4y'' - 12y' = -9y;\quad y(0) = -1;\quad y'(0) = \frac{1}{2}$
26. $y'' - 10y' + 26y = 0;\quad y(0) = -2;\quad y'(0) = -7$

27–34 Solve the boundary value problem, if possible.

27. $y'' - 5y' = 0;\quad y(0) = 5 - 5e^5;\quad y(1) = 0$
28. $9y'' - 6y' + y = 0;\quad y(0) = -1;\quad y(1) = 0$
29. $y'' - y = 0;\quad y(0) = 1;\quad y(1) = e$
30. $y'' + 0.2y' + 0.01y = 0;\quad y(0) = 0;\quad y(2) = 4e^{-1/5}$
31. $y'' + 9y = 0;\quad y(0) = 0;\quad y\left(\frac{2\pi}{3}\right) = 1$
32. $y'' + 4y' = 0;\quad y(0) = -5;\quad y(3) = -5$
33. $y'' - 2y' + 17y = 0;\quad y(0) = 2;\quad y\left(\frac{\pi}{8}\right) = 1$
34. $4y'' - 4y' + 5y = 0;\quad y(0) = 1;\quad y\left(\frac{\pi}{2}\right) = 1$

35–38 Our techniques from this section easily generalize to higher-order homogeneous linear equations with constant coefficients. For example, the characteristic equation of

$$y''' - 3y'' + y' - 3y = 0$$

is the cubic polynomial

$$r^3 - 3r^2 + r - 3 = 0$$

with characteristic roots $r_1 = 3$ and $r_{2,3} = \pm i$. These give rise to the following general solution.

$$y = c_1e^{3x} + c_2\cos x + c_3\sin x$$

In Exercises 35–38, use this generalized technique to find the general solution of the differential equation.

35. $y''' - 2y'' - 3y' = 0$
36. $y''' - 4y'' + 5y' - 2y = 0$
37. $y''' - y'' + 2y' - 2y = 0$
38. $y^{(4)} - 3y'' - 4y = 0$

39–42 The following exercises offer a glimpse into one way of handling certain nonhomogeneous linear equations. A more general version of the theorem below is proved in differential equations texts.

Theorem: If y_p is any given particular solution of the nonhomogeneous linear equation

$$ay'' + by' + cy = F(x) \qquad \text{(1a)}$$

on an interval I and y_c (also called the complementary function) is the general solution of the associated homogeneous equation

$$ay'' + by' + cy = 0 \qquad \text{(1b)}$$

on the same interval, then the general solution of (1a) on I can be written as follows.

$$y = y_c + y_p$$

In other words, the general solution of (1a) is the sum of any one of its particular solutions and the general solution of the associated homogeneous equation (1b).

For example, you can easily verify that $y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x$ is a particular solution of the equation

$$y'' + y' = \cos x \qquad \text{(2)}$$

while the general solution of its associated homogeneous equation $y'' + y' = 0$ is $y_c = c_1 + c_2e^{-x}$. Thus, the general solution of (2) is as follows.

$$y = y_c + y_p = c_1 + c_2e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$

But how can we find a particular solution y_p in order to put the theorem to work? One method that works well for certain equations is one we have already seen in Exercises 99–102 of Section 7.1—the method of undetermined coefficients. In the above case, knowing that all derivatives of the sine and cosine functions are again of the same type, we might guess that a particular solution of (2) has the following form.

$$y_p = A\cos x + B\sin x$$

Substituting this into (2), we obtain

$$(B - A)\cos x - (A + B)\sin x = \cos x,$$

which yields the coefficients $A = -\frac{1}{2}$ and $B = \frac{1}{2}$, and thus,

$$y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x.$$

In Exercises 39–42, use this theorem and the method of undetermined coefficients to find the general solution of the equation with the indicated initial "guess" for y_p.

39. $y'' + y' - 2y = x;$ guess $y_p = Ax + B$

40. $y'' + 2y = x^2 + 1;$ guess $y_p = Ax^2 + Bx + C$

41. $y'' + 4y' - 5y = \sin x;$ guess $y_p = A\cos x + B\sin x$

42. $y'' + 4y = e^{3x};$ guess $y_p = Ae^{3x}$

43–44 Use the method described in the directions for Exercises 39–42 to solve the given initial value problem.

43. $y'' + y = x^2;\quad y(0) = -1;\quad y'(0) = 1;$
guess $y_p = Ax^2 + Bx + C$

44. $y'' - 2y' = 4\sin 2x;\quad y(0) = 1;\quad y'(0) = -2;$
guess $y_p = A\cos 2x + B\sin 2x$

45. Show that the only solution of the boundary value problem

$$y'' + 4y = 0;\quad y(0) = 0;\quad y\left(\frac{\pi}{4}\right) = 0$$

is the trivial solution $y = 0$.

46. Show that the boundary value problem

$$y'' + 9y = 0;\quad y(0) = 0;\quad y\left(\frac{2\pi}{3}\right) = 0$$

has infinitely many solutions. (Contrast this with Exercise 31.)

47. Suppose that

$$2\frac{d^2y}{dt^2} + c\frac{dy}{dt} + 5y = 0$$

is the equation of a damped oscillating motion. Find a value of c such that the motion is **a.** underdamped, **b.** critically damped, and **c.** overdamped. Using the case analysis of Example 5, explain why these terms are appropriate for each type of motion. (Answers to parts a. and c. will vary.)

48. The viscosity of hydraulic fluid in automobile shock absorbers will determine the value of c in an equation of damped oscillating motion (see Example 5 or Exercise 47). Discuss which of the three cases is appropriate for the design of automobile shock absorbers and why. (Shock absorbers are designed for cars to prevent or "smooth out" wheel oscillations caused by an uneven road surface.)

49. When suspended at one end of a spring, an object of mass 0.1 kg stretches the spring by 5 cm. Find the value of c such that the resulting motion is critically damped. (This value is called the *critical damping constant*. For a refresher on how to determine the spring constant, see Example 2 in Section 6.5.)

50. Suppose that a 5 kg object attached to a spring with spring constant $k = 13$ is pulled down 25 cm below equilibrium and released. Find and graph the displacement function if the surrounding medium offers resistance with a damping constant of $c = 2$ kg/s.

51. A 4 lb weight stretches a spring by 6 in., while the damping constant is $\frac{1}{4}$ slug/s. The weight is pulled down 9 in. below equilibrium and released with an upward velocity of 2 ft/s. Find the equation of motion, solve it for the displacement function, and graph your result. Use 1 slug ≈ 32 lb.

52. A 4 kg object stretches a spring by 12 cm. The object is then pushed upward from equilibrium by 20 cm and released. Find and graph the displacement function if the damping constant is $c = 3$ kg/s.

53. If we place the system of Exercise 52 in a high-viscosity fluid with a damping constant of 75 kg/s, the motion will become overdamped. Find and graph the displacement function in this case.

54. Show that if the mass-spring model of Example 5 is critically damped or overdamped, that is, if $c^2 - 4mk \geq 0$, then the oscillating object cannot pass through the equilibrium more than once. (**Hint:** Show that the equation $y(t) = 0$ cannot have two or more solutions.)

55. Prove that $y = c_1 \cosh(kx) + c_2 \sinh(kx)$ is a general solution of the equation in Exercise 17. Show that the above family of functions is the same as the one you obtained in Exercise 17. (**Hint:** Start by showing that both $y_1(x) = \sinh(kx)$ and $y_2(x) = \cosh(kx)$ satisfy the differential equation, and argue that they are also linearly independent. Compare this answer to the one from Exercise 16.)

56. Show that if $y(t)$ is the solution of a damped mass-spring model

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0,$$

then

$$\lim_{t\to\infty} y(t) = 0.$$

(**Hint:** Handle the critically, under-, and overdamped cases separately, using the fact that all constants m, c, and k are greater than 0 in the equation of motion. Note that the conclusion of this exercise is consistent with our everyday experience of damped oscillations "dying down" over time.)

57.* At an amusement park, a boat slides down a ramp and splashes into the water at a speed of 15 m/s. The resistance offered by the water is proportional to the boat's speed with a coefficient of 270 kg/s. If the combined mass of the boat and passengers is 300 kg, how long does it take for it to come to a complete stop? What distance will it travel in the water while slowing down? Consider the boat stopped if your model predicts a velocity less than 1 cm/s. (**Hint:** See Example 5.)

58.* An RLC circuit is a simple electric circuit with inductance L (in henries, H), resistance R (in ohms, Ω), and capacitance C (in farads, F). The differential equation describing an RLC circuit is as follows.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V$$

(Note that $I(t) = dq(t)/dt$. See Exercise 62 in Section 8.1 and Example 5 in Section 8.2.)

Suppose that in an RLC circuit a switch is open (i.e., there is no current), $V = 0$, and the capacitor has an initial charge of 3 coulombs. Then, at time $t = 0$, the switch is flipped closed. Find and graph the current $I(t)$ if the capacitance is $C = 10^{-2}$ F, the resistance is 1.5 Ω, and the inductance is 0.1 H. What happens to $I(t)$ as $t \to \infty$?

9.1 Exercises

1. Given the parametric equations $x = 5 + t$ and $y = \sqrt{t}/(t-2)$, construct a table of the points (x, y) that result from integer t-values from 0 to 6, and then sketch the curve.

2. Given the parametric equations $x = (\tan\theta)/2$ and $y = \cos^2\theta + 3$, construct a table of the points (x, y) that result from the values $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6$, and π. Using these points, sketch the graph of the equations.

3–9 A straightforward way of parametrizing the graph of a function $y = f(x)$ is with the equations $x = t$ and $y = f(t)$. Use this technique to construct parametric equations defining the graph of the given equation.

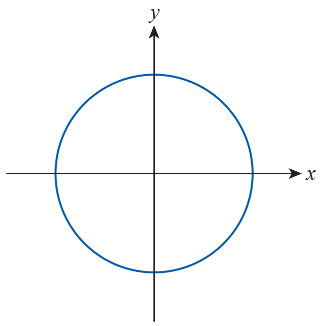
3. $y = -x^2 - 5$
4. $x^2 + \dfrac{y^2}{4} = 1$
5. $x = y^2 + 4$
6. $x = 4y - 6$
7. $y = |x-1|$
8. $x = 2(y-3)$
9. $y^2 = 1 - x^2$

10–23 Sketch the curve defined by the parametric equations by eliminating the parameter.

10. $x = 3(t+1), \quad y = 2t$
11. $x = \sqrt{t-2}, \quad y = 3t - 2$
12. $x = 1 + t, \quad y = \dfrac{t-3}{2}$
13. $x = |t+3|, \quad y = t - 5$
14. $x = \dfrac{t}{4}, \quad y = t^2$
15. $x = \dfrac{t}{t+2}, \quad y = \sqrt{t}$
16. $x = \sqrt{t+3}, \quad y = t + 3$
17. $x = \dfrac{2}{|t-3|}, \quad y = 2t - 1$
18. $x = \cos\theta, \quad y = 2\sin\theta$
19. $x = 3\sin\theta - 1, \quad y = \dfrac{\cos\theta}{2}$
20. $x = 1 - \sin\theta, \quad y = \sin\theta - 1$
21. $x = 2\cos\theta, \quad y = 3\cos\theta$
22. $x = 2\sin\theta + 2, \quad y = 2\cos\theta + 2$
23. $x = \sin\theta, \quad y = 4 - 3\cos\theta$

24–29 Match the parametrization with its graph (labeled A–F).

24. $x = 10\cos\theta, \quad y = \theta$
25. $x = 2 + 3t, \quad y = \dfrac{t^3 + 1}{4}$
26. $x = 1 + 4t, \quad y = 3 + 2t$
27. $x = t\cos t, \quad y = t\sin t$
28. $x = 4 - 2t^2, \quad y = t^3 - 9t$
29. $x = 4\sin\theta, \quad y = 4\cos\theta$

A.


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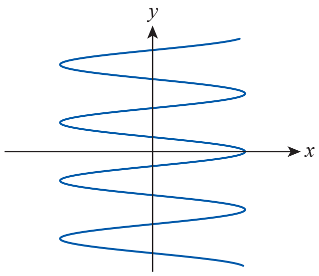

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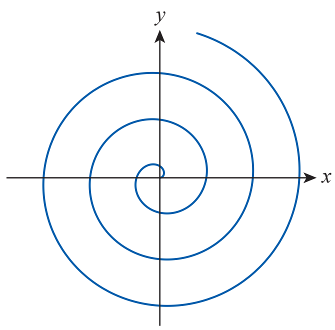

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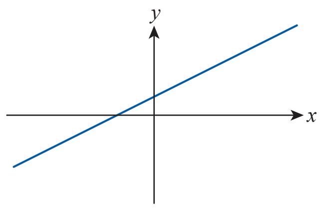

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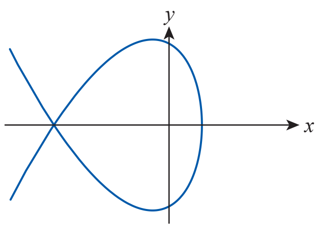
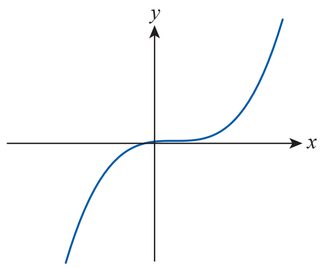


F.


30–33 Sketch the curve defined by the parametric equations and indicate the orientation of the curve using arrows.

30. $x = \dfrac{1}{\sqrt{t-1}} + 1, \quad y = \dfrac{1}{t-1}$

31. $x = (\ln t)^2, \quad y = \dfrac{1}{\sqrt{t}}, \quad 1 \le t \le 5$

32. $x = e^{-t/2}, \quad y = e^t, \quad t \ge 0$

33. $x = \sec^2 t, \quad y = \tan^2 t, \quad 0 \le t \le \dfrac{\pi}{3}$

34. Show that the curve defined by the parametric equations $x = r\cos(nt)$ and $y = r\sin(nt)$, $0 \le t \le 2\pi/n$, is a circle of radius *r*, centered at the origin.

35. Show that the curve defined by the parametric equations $x = a\cos(nt)$ and $y = b\sin(nt)$, $a > b$, $0 \le t \le 2\pi/n$, is an ellipse with a horizontal major axis centered at the origin with respective lengths of the major and minor axes being 2*a* and 2*b*. (**Hint:** Recall from precalculus that the equation of such an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.)

36. Taking advantage of Exercise 34, along with horizontal and vertical shifts, derive the general parametric form for a circle of radius *r*, centered at (h, k).

37. Using Exercise 35 along with horizontal and vertical shifts, derive the general parametric form for an ellipse with a horizontal major axis and axes of lengths 2*a* and 2*b*, centered at (h, k).

38–45 Find parametric equations to represent the graph described. (Answers will vary.)

38. Line, slope –2, passing through $(-5, -2)$

39. Line, passing through $(6, -3)$ and $(2, 3)$

40. Line segment connecting the points $(-2, -1)$ and $(3, 4)$

41. Line segment connecting the points $(-3, 1)$ and $(5, -5)$

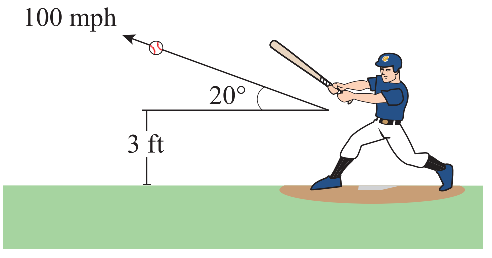
42. Circle, center $(7, -5)$, radius 4

43. Circle, center $(0, -2)$, radius 6

44. Ellipse, center $(5, -1)$, $a = 3$, $b = 2\sqrt{2}$, vertical major axis (**Hint:** See Exercises 35 and 37.)

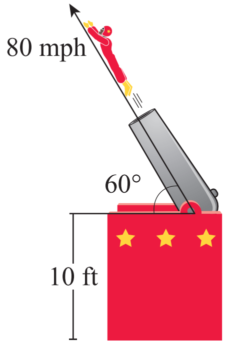
45. Ellipse, center $(0, 1)$, $a = 6$, $b = \sqrt{11}$, horizontal major axis (**Hint:** See Exercises 35 and 37.)

46. Suppose that a baseball is hit 3 feet above the ground, and it leaves the bat at a speed of 100 miles per hour at an angle of 20° from the horizontal. Construct parametric equations representing the path of the ball's flight, and sketch a graph of the ball's travel. (**Hint:** Supposing that the ball starts at the point $(0, 3)$ and treating time *t* as the parameter, express the ball's *x*- and *y*-coordinates as functions of *t*. Do not forget to decompose the initial velocity into horizontal and vertical components!)



47. Suppose the ball in Exercise 46 has been hit toward a 10-foot-high fence that is 400 feet from home plate. Will the ball clear the fence?

48. Suppose that a circus performer is shot from a cannon at a rate of 80 mph, at an angle of 60° from the horizontal. The cannon sits on a platform 10 feet above the ground.



a. Construct parametric equations representing the performer's path as he flies through the air.

b. Sketch a graph of his flight.

c. How high is the acrobat 1.5 seconds after leaving the cannon?

d. How far from the cannon should a landing net be placed, if it is placed at ground level?

e. At what time *t* will the performer land in the net?

f. If a 12-foot-high wall of flames is placed 70 feet from the cannon, will he clear it unharmed?

49. On his morning paper route, John throws a newspaper from his car window 3.5 ft from the ground. The paper has an initial velocity of 10 ft/s and is tossed at an angle of $10°$ from the horizontal.

a. Construct parametric equations modeling the path of the newspaper.

b. Sketch a graph of the paper's path.

50. François shoots a basketball at an angle of $48°$ from the horizontal. It leaves his hands 7 ft from the ground with a velocity of 21 ft/s.

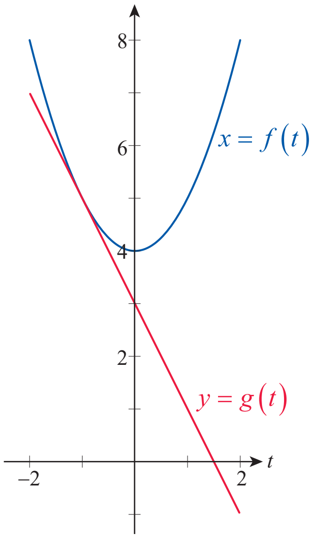
a. Construct parametric equations representing the path of the ball.

b. Sketch a graph of the basketball's flight.

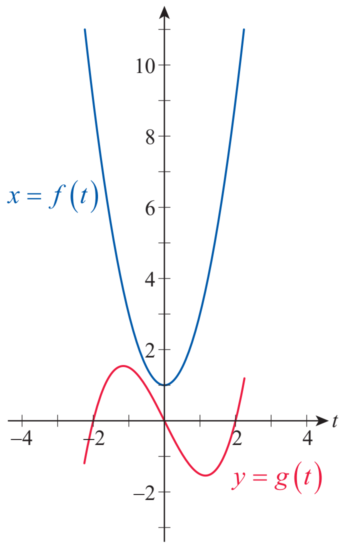
c. If the goal is 15 ft away and 10 ft high, will he make the shot?

51–54 Use the given graphs of $f(t)$ and $g(t)$ to make a rough sketch of the curve defined by the parametric equations $x = f(t)$ and $y = g(t)$. (**Hint:** Plotting a few points may help.)

51.



52.



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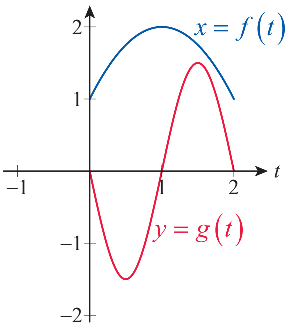

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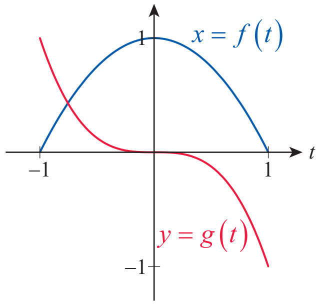


55. Verify that the three parametrizations below represent the same curve. Then come up with two parametrizations on your own. (Answers will vary.)

a. $x = t^2, \quad y = t^3, \quad -1 \le t \le 1$

b. $x = 1 - \cos^2 t, \quad y = \sin^3 t, \quad -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

c. $x = \sec^2\left(\dfrac{\pi}{4}t\right) - 1, \quad y = \tan^3\left(\dfrac{\pi}{4}t\right), \quad -1 \le t \le 1$

56–57 Find all intersection points of the given parametric curves.

56. $x = 3t - 2,\ y = 3t - 1;\quad x = 2u,\ y = 4u^2 - 4u + 1$

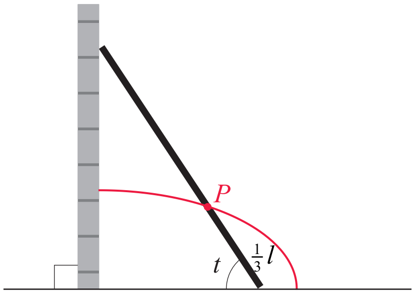
57. $x = t - 1,\ y = 3t^2 - 6t + 3;\quad x = \dfrac{u}{2},\ y = -\dfrac{u^2}{4} + 2u$

58. A wheel of radius 12 inches rolls along a flat surface in a straight line. There is a fixed point P that initially lies at the point $(0,0)$. Find parametric equations defining the cycloid traced out by P.

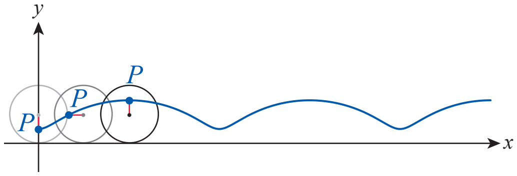
59. A ball is rolled on the floor in a straight line from one person to another. The ball has a radius of 3 cm and there is a fixed point P located on the ball. Let the person rolling the ball represent the origin. Find parametric equations defining the cycloid traced out by P.

60. Prove that the parametrizations $x_1 = \dfrac{1-t^2}{1+t^2}$, $y_1 = \dfrac{2t}{1+t^2}$, $-\infty < t < \infty$, and $x_2 = \cos t$, $y_2 = \sin t$, $-\pi < t < \pi$, represent the same curve. What is the curve?

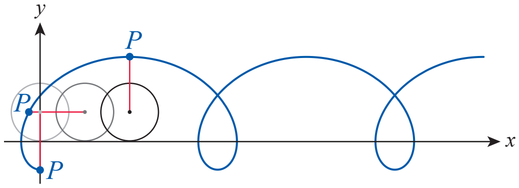
61. A ladder of length l is leaning against a wall, sliding slowly all the way down to a horizontal position. Suppose that there is a paint mark on the side of the ladder, exactly one-third of the way up from the bottom of the ladder (see point P in the figure). Assuming that the ladder started sliding from a vertical position, prove that the curve traced out by P during the slide is one-quarter of an ellipse. (**Hint:** Let t be the radian measure of the angle that the ladder makes with the horizontal, and use it as the parameter to determine the parametric equations of the curve.)



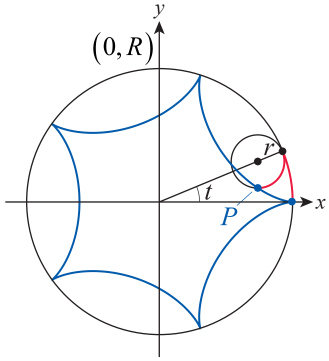
62. One way to generalize the cycloid is to consider the curve traced by a point P on a fixed radius (a "spoke") in the circle of Example 4. Generalize the argument of Example 4 to prove that if P is b units ($b < a$) from the center, then the parametric equations of the resulting trochoid (also called *curtate cycloid*) are $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$. (Note that the case of $a = b$ yields the equations we obtained in Example 4.)



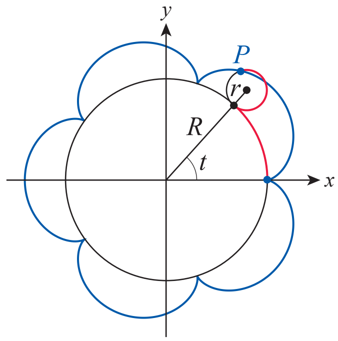
63. Repeat Exercise 62 for the case $a < b$ (imagine each spoke extending an appropriate length beyond the circumference of the circle; the resulting trochoid is called a *prolate cycloid*).



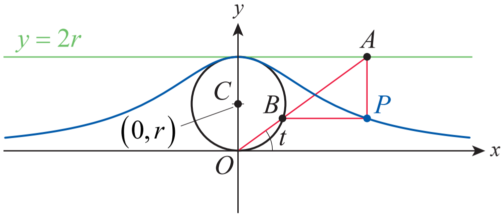
64.* Derive the parametrization of the hypocycloid seen in the Technology Note by using the angle t in the figure as a parameter. (**Hint:** Since the circle rolls without slipping, you may use the equality of the lengths of the two red arcs in the figure.)



65.* The *epicycloid* is the curve traced out by a fixed point P on a circle of radius r as it rolls without slipping on the outside of a larger circle with radius R. Using the technique of Exercise 62, derive the parametric equations of the epicycloid.



66.* The famous curve in the figure below is called the *witch of Agnesi* and is derived as follows. Suppose that a circle of radius r is centered at $(0, r)$ and the line $y = (\tan t)x$ intersects the horizontal line $y = 2r$ and the circle at the points A and B, respectively (B is not the origin). The curve is then traced out by the point P, which is the intersection of the horizontal segment through B and the vertical segment through A. Use t as a parameter to derive the parametric equations for the witch of Agnesi. (**Hint:** If O is the origin, and C is the center of the circle, start by examining the isosceles triangle OCB.)



67. Eliminate the parameter to find the Cartesian equation of the witch of Agnesi in Exercise 66.

Concept Check

68–71 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

68. If $x = f(t)$ and $y = g(t)$ are both quadratic functions, then the parametric curve defined by $x = f(t)$, $y = g(t)$ is a parabola or a parabolic arc.

69. The parametric equations $x = t$, $y = t^{2/3} - 2$ and $x = 8t$, $y = 4t^{2/3} - 2$ have the same graph.

70. The graph of $x = t^3, y = t^6$ is the prototypical parabola.

71. The graph of parametric equations can either be represented in the form $y = f(x)$ (i.e., y as a function of x), or in the form $x = g(y)$ (i.e., x as a function of y).

9.1 Technology Exercises

72–78 Use a graphing utility to sketch the given curve for various values of k and explore how the value of k affects the shape of your graph.

72. $x = 2t - k\sin t$, $y = 2 - k\cos t$ (trochoid)

73. $x = 2\cos t + k\cos\frac{2}{3}t$, $y = 2\sin t - k\sin\frac{2}{3}t$ (hypotrochoid)

74. $x = 2kt - 4t^3$, $y = 3t^4 - kt^2$ (swallowtail catastrophe curve)

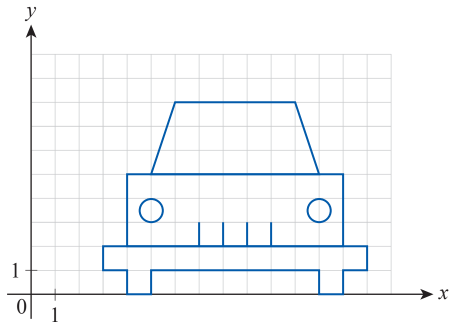
75. $x = \frac{3kt}{1+t^3}$, $y = \frac{3kt^2}{1+t^3}$ (folium of Descartes)

76. $x = \cos(t - \cos(kt))$, $y = \cos(kt)$

77. $x = 2t - \cos(kt)$, $y = t^2 - \sin(kt)$

78. $x = \frac{k^2\cos(kt)}{t^2+k^2}$, $y = \frac{k^2\sin(kt)}{t^2+k^2}$

79. Use technology and your parametrizing skills to display the following picture on your screen.



$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} 2\pi r\, ds \\ &= 2\pi \int_{-\pi/2}^{\pi/2} (R\cos t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi R \int_{-\pi/2}^{\pi/2} \cos t \sqrt{R^2 \sin^2 t + R^2 \cos^2 t}\, dt \\ &= 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos t\, dt \\ &= 2\pi R^2 \left[\sin t\right]_{-\pi/2}^{\pi/2} = 4\pi R^2 \end{aligned}$$

9.2 Exercises

1–4 Identify the curve as differentiable, smooth, or neither.

1. $x = t + 3, \quad y = 2|t-1|, \quad 0 \le t \le 2$
2. $x = t - 1, \quad y = 2t^2, \quad t \in \theta$
3. $x = -2t^2, \quad y = t^3 + 1, \quad -5 \le t \le 5$
4. $x = 3\cos\theta, \quad y = 2\sin\theta, \quad -\pi < \theta \le \pi$
5. Show that (t, t) is a smooth parametrization of the (smooth) curve $y = x$, while (t^3, t^3) is not.

6–14 Find the equations of any horizontal or vertical tangent lines to the curve.

6. $x = 5t - 2, \quad y = 6t^2 + 1$
7. $x = t^2 - t, \quad y = 1 + 2t^2$
8. $x = 2t^2 + 1, \quad y = (t-2)^2$
9. $x = t^3 + 1, \quad y = t$
10. $x = 3t^2, \quad y = \frac{1}{t+3}, \quad t > -3$
11. $x = \sqrt{t}, \quad y = \ln t, \quad t \ge 1$
12. $x = \sqrt{t}, \quad y = t^2 - 2$
13. $x = 2\cos t, \quad y = 5\sin t$
14. $x = \tan t, \quad y = \sec t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

15–24 Find the values of dy/dx and d^2y/dx^2 for the curve at the given point.

15. $x = 2t - 1, \quad y = 2t^2 + 1; \quad \left(0, \frac{3}{2}\right)$
16. $x = 2t^3 + 1, \quad y = (t+1)^2; \quad (3, 4)$
17. $x = \frac{1}{t+1}, \quad y = t^2 - 1; \quad (1, -1)$
18. $x = \sqrt{t}, \quad y = 2t + 1; \quad (1, 3)$
19. $x = te^t, \quad y = e^{-t}; \quad (0, 1)$
20. $x = \cos t, \quad y = \ln(\cos t); \quad \left(\frac{1}{2}, -\ln 2\right)$
21. $x = \sin t, \quad y = \sin 2t; \quad \left(\frac{\sqrt{2}}{2}, 1\right)$
22. $x = \frac{\sin^3 t}{3}, \quad y = \frac{\cos^3 t}{3}; \quad \left(\frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{12}\right)$
23. $x = \ln t, \quad y = 2\sqrt{t}; \quad (0, 2)$
24. $x = \cos 2t, \quad y = \sin 3t; \quad \left(\frac{1}{2}, 1\right)$

25–28 Find the value(s) of the parameter for any inflection point(s) of the given curve.

25. $x = t + 2, \quad y = t^3 - 10t^2$
26. $x = t^2(t^2 - 4), \quad y = 2t + 3$
27. $x = t^4 - t, \quad y = t^2$
28. $x = \sqrt{t}, \quad y = t^2 - 5t$

29–34 Find the area enclosed by the given curve.

29. $x = \sin t, \quad y = \sin\frac{t}{2}, \quad 0 \le t \le 2\pi$
30. $x = 2\cos t, \quad y = \sin 2t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$
31. $x = 2t\cos t, \quad y = 2t\sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$
32. $x = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t, \; y = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \; 0 \le t \le 2\pi$

33. $x = \sin t, \quad y = e^{-t/2}, \quad 0 \le t \le \pi$ and the y-axis

34. $x = t^4 - 4, \quad y = t^3 - 2t, \quad -\sqrt{2} \le t \le \sqrt{2}$

35. Use the parametric representation of the ellipse from Exercise 35 of Section 9.1 to arrive at its area formula of $A = \pi ab$.

36. Prove that the area A under one arch of the cycloid generated by a circle of radius a is three times the area of the circle, $A = 3\pi a^2$. (This result was first proved by the French mathematician Gilles de Roberval ca. 1630.)

37. Sometimes it may be necessary or convenient to divide a region between a curve and the y-axis into horizontal strips when approximating its area. Modify our discussion preceding Example 4 to arrive at the formula $\int_{y=g(a)}^{y=g(b)} x\,dy = \int_{t=a}^{t=b} f(t)g'(t)\,dt$.

38–39 Use Exercise 37 to determine the area between the given curve and the y-axis.

38. $x = 2t - 1, \quad y = t^2 - 2, \quad \frac{1}{2} \le t \le 3$

39. $x = 2\cos t, \quad y = 1 + \text{sint}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

40–47 Find the arc length of the given curve over the indicated interval.

40. $x = t^2, \quad y = t^3 - \frac{t}{3}, \quad 1 \le t \le 3$

41. $x = \sqrt{t}, \quad y = \frac{t^2}{8} + \frac{1}{4t}, \quad 1 \le t \le 9$

42. $x = e^t, \quad y = \frac{e^{2t}}{8} - t, \quad 0 \le t \le 2$

43. $x = \frac{1}{t}, \quad y = \frac{t}{4} + \frac{1}{3t^3}, \quad \frac{1}{2} \le t \le 1$

44. $x = \sin^3 t, \quad y = \cos^3 t, \quad 0 \le t \le 2\pi$

45. $x = \sqrt{2}e^t \sin t, \quad y = \sqrt{2}e^t \cos t, \quad 0 \le t \le \frac{\pi}{2}$

46. $x = t^3, \quad y = t\left(t^4 - \frac{9}{20}\right), \quad 0 \le t \le 1$

47. $x = -\ln t, \quad y = t + \frac{1}{4t}, \quad \frac{1}{2e} \le t \le 1$

48–57 Find the area of the surface generated by revolving the parametric curve about the indicated axis.

48. $x = t + 1, \quad y = \frac{t-1}{2}, \quad 1 \le t \le 5,$

a. about the x-axis **b.** about the y-axis

49. $x = 2t - 3, \quad y = 8 - 2t, \quad 2 \le t \le 4,$

a. about the x-axis **b.** about the y-axis

50. $x = 4t^2 - 4t + 4, \quad y = 2t - 1, \quad 1 \le t \le 2,$

about the x-axis

51. $x = t, \quad y = t^3 + \frac{1}{12t}, \quad 1 \le t \le 2,$

a. about the x-axis **b.** about the y-axis

52. $x = t^2, \quad y = 2t^3 - \frac{t}{6}, \quad 0 \le t \le 1,$

a. about the x-axis **b.** about the y-axis

53. $x = \ln t, \quad y = t + \frac{1}{4t}, \quad 1 \le t \le e,$

a. about the x-axis **b.** about the y-axis

54. $x = t^5 - \frac{9t}{20}, \quad y = t^3, \quad 0 \le t \le 2,$

about the x-axis

55. $x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \le t \le \pi,$

about the y-axis

56. $x = 2\cos t, \quad y = \sin t, \quad 0 \le t \le \pi,$

about the x-axis

57. $x = 2t + 1, \quad y = \cosh 2t, \quad 0 \le t \le 1,$

a. about the x-axis **b.** about the y-axis

58. Suppose one arch of the cycloid generated by a circle of radius 1 rolling along the x-axis is rotated about the x-axis. Find the area of the resulting surface.

59. Revisit Example 5 by using the more general parametrization found in Exercise 34 of Section 9.1. Do you obtain the same answer?

60. Evaluate the integral in Example 6 by writing it as

$$\int \frac{1+t^2}{\sqrt{1+t^2}}\,dt = \int \frac{1}{\sqrt{1+t^2}}\,dt + \int \frac{t^2}{\sqrt{1+t^2}}\,dt;$$

then show

$$\int \frac{t^2}{\sqrt{1+t^2}}\,dt = t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt,$$

and finally proceed to solve for $\int \sqrt{1+t^2}\,dt$.

61–64 Consider a particle moving along a curve in the *xy*-plane such that its coordinates at time *t* are $x = f(t)$, $y = g(t)$, $a \le t \le b$. Since the distance traveled by the particle at time *t* can be calculated from the arc length

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du,$$

by Part I of the Fundamental Theorem of Calculus (Section 5.3) we obtain

$$\text{speed} = \frac{ds}{dt} = \frac{d}{dt}\int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du$$
$$= \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

In Exercises 61–64, use the above formula to determine the speed of the particle traveling along the given curve at the specified time. (Distance is measured in meters, time in seconds.)

61. $x = t^2, \quad y = 2t + 1; \quad t = 2$

62. $x = \ln(t+1), \quad y = t^2 - 2; \quad t = 1$

63. $x = 3\cos 3t, \quad y = \sin 3t; \quad t = \frac{\pi}{2}$

64. $x = t^2, \quad y = \sqrt{t}; \quad t = 4$

65. Suppose that the position of a particle in the *xy*-coordinate system is given by $x = t^2 - 1, y = t^3 - 6t$, $t \ge 0$. When does the particle reach its minimum speed?

66. Using the discussion preceding Example 4, prove the following: If $x = f(t)$ and $y = g(t)$ define a parametric curve over $[a,b]$ such that *g* is continuous, *f* is continuously differentiable, and $y(x)$ is a continuous function of *x*, then

$$\int_{f(a)}^{f(b)} y\, dx = \int_a^b g(t) f'(t)\, dt.$$

67–70 Use the result of Exercise 66 to solve the exercise.

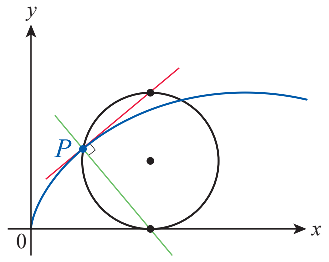
67. Find the volume of the solid generated by rotating the parametrically defined curve $x = 2t - 1, y = 1 - t^2$, $-1 \le t \le 1$, about the *x*-axis.

68. Repeat Exercise 67 for the parametric curve $x = \sin^3 t$, $y = \cos^3 t$, $-\pi/2 \le t \le \pi/2$.

69. Find the centroid of the region bounded by the parametric curve $x = 9 - t$, $y = \sqrt{t}$, $0 \le t \le 9$, and the coordinate axes.

70. Repeat Exercise 69 for the parametric curve $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le \pi/2$.

71. Prove the following property of cycloids: If *P* is the intersection point of the rolling circle and the cycloid, then the line tangent to the cycloid at *P* passes through the highest point of the circle, while the normal line intersects the circle at its lowest point.



72.* Prove that if the curve *C* defined by $x = f(t)$ and $y = g(t)$, $t \in [a,b]$, is differentiable with both f' and g' bounded (meaning there are constants K_1 and K_2 such that $|f'(t)| \le K_1$ and $|g'(t)| \le K_2$ for all $t \in [a,b]$), then *C* has finite length (such curves of finite length are called *rectifiable*). (**Hint:** As a first step, use the boundedness of the derivatives and the Mean Value Theorem to prove that there is a constant *M* such that whenever $P_1(f(t_1), g(t_1))$ and $P_2(f(t_2), g(t_2))$ are two points on *C*, $|f(t_2) - f(t_1)| \le M|t_2 - t_1|$, and $|g(t_2) - g(t_1)| \le M|t_2 - t_1|$. As a consequence, $|P_1P_2| \le 2M|t_2 - t_1|$. Now you can use a Riemann-sum argument to finish your proof.)

73. Use Exercise 72 to prove that if the parametric curve *C* is continuously differentiable on $[a,b]$ (i.e., both f' and g' have continuous derivatives on $[a,b]$), then *C* is rectifiable.

Concept Check

74–78 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. Every smooth parametric curve is differentiable.

75. Every continuous, differentiable parametric curve is smooth.

76. If *C* is smooth at (x_0, y_0), then for every parametrization such that $x_0 = f(t_0)$ and $y_0 = g(t_0)$, both f' and g' are continuous at t_0 and at least one of $f'(t_0)$ and $g'(t_0)$ is nonzero.

77. If the graph of a parametric curve is a continuously differentiable function $y = f(x)$, then the curve is smooth.

78. If a parametric curve is defined by $x = f(t)$ and $y = g(t)$, then $\dfrac{d^2y}{dx^2} = \dfrac{d^2y/dt^2}{d^2x/dt^2}$.

9.2 Technology Exercises

79. Use a computer algebra system to approximate the length of the ellipse parametrized by $x = 5\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$.

80–81 Use a computer algebra system to approximate the length of the curve with the given parametrization.

80. $\left(\sqrt{t}, \sin t\right), \quad 0 \le t \le 2\pi$

81. $\left(\ln(t+1), \sqrt{t}\right), \quad 0 \le t \le 9$

82–83 Use a computer algebra system to approximate the area of the surface obtained by rotating the given parametric curve about the **a.** *x*-axis and **b.** *y*-axis.

82. $\left(3t-2, \sqrt{1+\dfrac{1}{t}}\right), \quad 1 \le t \le 2$

83. $\left(4t+1, \sqrt{t^2+1}\right), \quad -\dfrac{1}{4} \le t \le 2$

9.3 Exercises

1–6 Plot the point given by the polar coordinates.

1. $\left(-1,\dfrac{5\pi}{4}\right)$
2. $\left(-5,\dfrac{3\pi}{2}\right)$
3. $\left(\dfrac{1}{4},-\dfrac{7\pi}{6}\right)$
4. $\left(\sqrt{3},-\dfrac{\pi}{3}\right)$
5. $\left(\dfrac{44}{9},-\pi\right)$
6. $\left(\dfrac{7}{\sqrt{2}},\dfrac{\pi}{2}\right)$

7–12 Convert the point from polar to Cartesian coordinates.

7. $\left(5,\dfrac{7\pi}{4}\right)$
8. $(0,2\pi)$
9. $\left(6.25,-\dfrac{3\pi}{4}\right)$
10. $\left(-2.25,\dfrac{\pi}{4}\right)$
11. $\left(3,-\dfrac{5\pi}{6}\right)$
12. $\left(-11,\dfrac{5\pi}{6}\right)$

13–18 Convert the point from Cartesian to polar coordinates.

13. $(-3,0)$
14. $(-6,\sqrt{3})$
15. $(12,-1)$
16. $(8,0)$
17. $(-\sqrt{3},9)$
18. $(-5,-5)$

19–30 Rewrite the rectangular equation in polar form.

19. $x^2+y^2=25$
20. $x^2+y^2=81$
21. $x=12$
22. $y=16$
23. $y=x$
24. $y=b$
25. $x=16a$
26. $x^2+y^2=a$
27. $x^2+y^2=4ax$
28. $x^2+y^2=4ay$
29. $y^2-4=4x$
30. $x^2+y^2=36a^2$

31–40 Rewrite the polar equation in rectangular form.

31. $r=5\cos\theta$
32. $r=8\sin\theta$
33. $r=7$
34. $\theta=\dfrac{\pi}{6}$
35. $18r=9\csc\theta$
36. $r=2\sec\theta$
37. $r^2=\sin 2\theta$
38. $r=\dfrac{2}{1-\cos\theta}$
39. $r=\dfrac{12}{4\sin\theta+7\cos\theta}$
40. $r=\dfrac{16}{4+4\sin\theta}$

41–46 Rewrite the polar equation in rectangular form; then sketch the graph.

41. $r=2$
42. $r=6$
43. $\theta=\dfrac{5\pi}{6}$
44. $\theta=\dfrac{\pi}{4}$
45. $r=7\sec\theta$
46. $r=2\csc\theta$

47–68 Sketch a graph of the given polar equation.

47. $r=4$
48. $r=5$
49. $\theta=\dfrac{4\pi}{3}$
50. $\theta=\dfrac{-\pi}{3}$
51. $r=6\cos\theta$
52. $r=2\sin\theta$
53. $r=3-3\sin\theta$
54. $r=6+5\cos\theta$
55. $r=7(1+\cos\theta)$
56. $r=2(1-2\sin\theta)$
57. $r=4-3\sin\theta$
58. $r=3+4\sin\theta$
59. $r=3\sin 3\theta$
60. $r=5\sin 3\theta$
61. $r=2\sin 2\theta$
62. $r=4\sin 2\theta$
63. $r=5\cos 5\theta$
64. $r=4\cos 5\theta$
65. $r=4\cos 4\theta$
66. $r=3\cos 4\theta$
67. $r^2=16\sin\theta$
68. $r^2=9\cos 2\theta$

69–72 Find all points of intersection of the given polar curves.

69. $r=\sin\theta,\quad r=\cos\theta$
70. $r=\sin 2\theta,\quad r=\cos\theta$
71. $r=1-\cos\theta,\quad r=1+\sin\theta$
72. $r^2=4\sin\theta,\quad r=1-\sin\theta$
73. For a fixed $a\in\theta$, explain in geometric terms how the graphs of $f(\theta)$ and $f(\theta-a)$ are related. (**Hint:** For guidance, recall the rectangular analogue.)
74. **a.** Describe the graph of $r=\sec(\theta-\pi/4)$.

    b. How are the graphs of $r=k\sec(\theta-\pi/4)$ related as k ranges over nonzero values? (Do not use graphing technology.)

9.3 Technology Exercises

75–81 Use a graphing utility to sketch the given curve. Whenever applicable, explore how different values of the parameter(s) affect the shape of the graph. Experiment with both integer and noninteger parameters.

75. $r = \cos k\theta$

76. $r = 1 - k_1 \sin k_2\theta$

77. $r = \dfrac{1 + k\sin\theta}{1 - k\sin\theta}$

78. $r = \theta\cos\theta, \quad -2\pi \le \theta \le 2\pi$ (Garfield curve)

79. $r = 1 + 2\sin\dfrac{\theta}{2}$ (nephroid of Freeth)

80. $r = k_1 + k_2\theta$

81. $r = 1 - k_1 \cos k_2\theta$

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$
$$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta \qquad \text{Note that } \sin(\theta/2) \geq 0 \text{ for } 0 \leq \theta \leq 2\pi.$$
$$= \left[-4\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$
$$= 4 + 4 = 8$$

9.4 Exercises

1–14 Find the slope of the line tangent to the given polar curve at the indicated point.

1. $r = 2\cos\theta;\quad \theta = \frac{\pi}{6}$

2. $r = 2\sin 2\theta;\quad \theta = \frac{\pi}{6}$

3. $r = a\theta;\quad \theta = \frac{\pi}{2}$

4. $r = a\sin\theta;\quad \theta = \frac{\pi}{6}$

5. $r = 5\sec^2\theta;\quad \theta = \frac{\pi}{4}$

6. $r = e^\theta;\quad \theta = \frac{\pi}{6}$

7. $r^2 = 4\csc\theta;\quad \theta = \frac{\pi}{6}$

8. $r^2 = \sin 2\theta;\quad \theta = \frac{\pi}{3}$

9. $r = \frac{1}{\theta};\quad \theta = \frac{2\pi}{3}$

10. $r = a(1 + \sin\theta);\quad \theta = \frac{2\pi}{3}$

11. $r = \ln\theta;\quad \theta = 1$

12. $r = \cos 4\theta;\quad \theta = \frac{\pi}{4}$

13. $r = \sin\frac{\theta}{4};\quad \theta = \pi$

14. $r = 3 - 2\sin\theta;\quad \theta = \frac{\pi}{4}$

15–18 Find all points where the given polar curve has a horizontal or vertical tangent line.

15. $r = 1 + \sin\theta$

16. $r = a\cos\theta$

17. $r = a(1 + \cos\theta)$

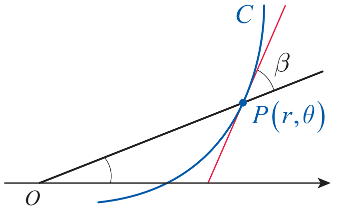
18. $r^2 = 4\cos 2\theta$

19–20 Notice that even in the case of polar curves, we still need to find dy/dx in order to determine tangents. The derivative $dr/d\theta$, while indirectly informing us about the tangent line at any given point, does not yield the slope of the tangent line. Exercises 19 and 20 shed some light on the relationship between $dr/d\theta$ and dy/dx.

19.* Suppose that the curve C is defined by the equation $r = f(\theta)$, P is a point on C, and there is a unique line tangent to C at P. Let β be the angle determined by the ray $\overrightarrow{OP}$ and the tangent at P. Prove that if $dr/d\theta$ is not equal to 0 at P, then

$$\tan\beta = \frac{r}{dr/d\theta}.$$

(**Hint:** In the formula we obtained for dy/dx in the text, divide the numerator and the denominator by $f'(\theta)\cos\theta$, and use the trigonometric identity $\tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha_2}$.)



20. Prove that if the graph of $r = f(\theta)$ passes through the pole and α is an angle such that $r = f(\alpha) = 0$, then the slope of the tangent to the graph at the pole is $\tan\alpha$; that is, the line $\theta = \alpha$ is tangent to the graph at the pole. (**Hint:** If $f'(\alpha) \neq 0$, use the formula we obtained for dy/dx in the text. Otherwise, examine $\lim_{\theta\to\alpha} \frac{\sin\theta \cdot f'(\theta)}{\cos\theta \cdot f'(\theta)}$.)

21–26 Notice that if the graph of $r=f(\theta)$ passes through the pole, we can use Exercise 20 to determine the polar equation of a tangent at the pole by solving the equation $f(\theta)=0$. Use this observation to find all lines tangent to the given curve at the pole.

21. $r=1-\cos\theta$

22. $r=\sin 3\theta$

23. $r^2=\cos 2\theta$

24. $r=\cos 4\theta$

25. $r=4\sin\theta$

26. $r=a\sin n\theta$

27–41 Find the area enclosed by the given curve.

27. $r=3\sin\theta$

28. $r=-2\cos\theta$

29. $r=1+\sin\theta$

30. $r=4-4\sin\theta$

31. $r=\dfrac{3}{2}-\sin\theta$

32. $r=3+2\cos\theta$

33. $r=3-\sin\theta$

34. $r=3(1+\cos\theta)$

35. $r=2+\cos\theta+\sin\theta$

36. $r=2\cos 3\theta$

37. $r=\sin 6\theta$

38. $r=6\sin\theta$

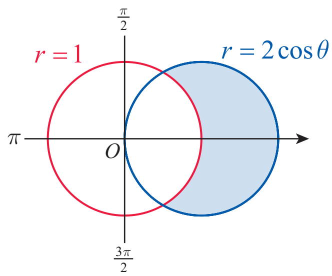
39. $r=4\sin 4\theta$

40. $r^2=2\sin 2\theta$

41. $r^2=4\cos 2\theta$

42–45 Find the area of the shaded region.

42.



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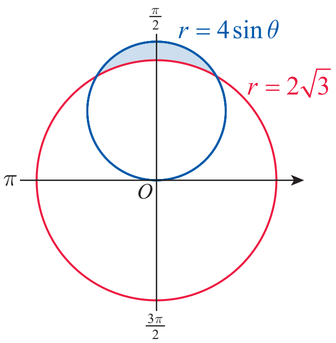

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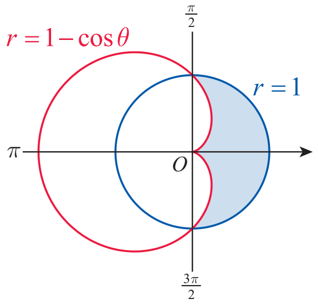

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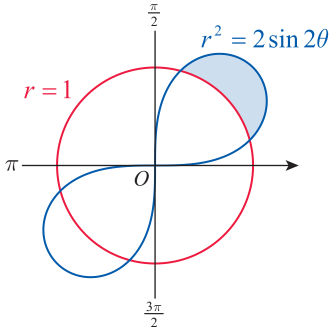


46–49 Find the area of the specified region.

46. The innermost loop of the spiral $r=3\theta$ bounded by the polar axis and $\theta=\pi/2$

47. The inner loop of the limaçon $1+2\sin\theta$

48. The region common to the circle $r=1$ and a petal of $r=2\sin 3\theta$

49. The region inside $r=3+2\sin\theta$ but outside $r=2$

50–57 Find the arc length of the given polar curve.

50. $r=3\sin\theta,\quad 0\le\theta\le\pi$

51. $r=3\theta,\quad 0\le\theta\le 4\pi$

52. $r=3e\theta,\quad 0\le\theta\le\pi$

53. $r=\csc\theta,\quad \dfrac{\pi}{3}\le\theta\le\dfrac{2\pi}{3}$

54.* $r=1-\sin\theta,\quad 0\le\theta\le 2\pi$

55. $r=\theta^2,\quad 0\le\theta\le 2\pi$

56.* $r=\sin^2\theta,\quad 0\le\theta\le\pi$

57.* $r=1+\sin\theta,\quad 0\le\theta\le\pi$

58. Use our derivation of the arc length formula for polar curves along with the surface area formula of Section 9.2 to arrive at the following formulas for the area of the surface generated by rotating the polar curve $r = f(\theta),\ a \le \theta \le b$, where f' is continuous on $[a,b]$ and the curve is traced out only once over the interval.

$$A = 2\pi\int_a^b r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

(rotation about the polar axis)

$$A = 2\pi\int_a^b r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

(rotation about $\theta = \pi/2$)

59–62 Use Exercise 58 to find the surface area of the solid generated by rotating the given curve as described.

59. $r = \sin\theta$, rotated about the polar axis

60. $r = \sin\theta$, rotated about $\theta = \pi/2$

61. The spiral $r = 3e\theta$, $0 \le \theta \le \pi$, rotated about the polar axis

62. $r = 2(1+\sin\theta),\ 0 \le \theta \le \pi/2,$ rotated about $\theta = \pi/2$

63. Show that the values of dy/dx at the tips of the petals of the curve in Example 2 are as claimed.

64.* Prove that if P is a point on the cardioid $r = a(1-\cos\theta)$, then the smaller of the two angles determined by $\overrightarrow{OP}$ and the tangent at P is one-half the angle determined by $\overrightarrow{OP}$ and the polar axis.

65. A point O inside a polar curve is called *equichordal* if every chord passing through O has the same length. (An obvious example is the center of a circle of radius r.) Prove that the pole is an equichordal point of the limaçon $r = a + \cos\theta$ (assume $a \ge 1$).

66. Find the polar equation of the circle whose diameter has the same length as the chords in the limaçon of Exercise 65, and prove that the areas of the limaçon and the circle are not equal. (As a consequence, we see that the existence of chords of equal length in every direction through a common point is insufficient to determine areas of regions.)

67. Suppose the polar curve $r = f(\theta),\ a \le \theta \le b$, has length L and encloses an area of A square units. Prove that for any constant k, the curve $r = kf(\theta)$, $a \le \theta \le b$, has length $|k|L$ and encloses an area of k^2A square units.

9.4 Technology Exercises

68–72 Use a graphing utility to sketch the given curve. Then use the integration capabilities of your technology to approximate its arc length.

68. $r = 2 + \cos 2\theta$

69. $r = \theta\cos\theta,\quad 0 \le \theta \le 3\pi$

70. $r = 3 - 2\sin 3\theta$

71. The inner loop of the limaçon $r = 1 + 2\sin\theta$

72. The 3-petaled rose $r = \sin 3\theta$

73–76 Use a graphing utility to approximate the surface area of the described solid of revolution. (**Hint:** See Exercise 58.)

73. $r = 3e\theta$, $0 \le \theta \le \pi$, rotated about the polar axis

74. $r^2 = \sin 2\theta$, rotated about the polar axis

75. The outer loop of the limaçon $r = 1 + 2\sin\theta$, rotated about $\theta = \pi/2$

76. $r = 4 - 4\cos 2\theta$, rotated about $\theta = \pi/2$

9.5 Exercises

1–6 Use the discriminant to determine whether the given equation represents an ellipse, a parabola, or a hyperbola.

1. $y^2 + 2y + 12x + 13 = 0$

2. $2x^2 + 12x - y^2 - 2y + 9 = 0$

3. $4x^2 + 3y^2 + 18y + 19 = 8x$

4. $-2x^2 - 8xy + 2y^2 + 2y + 5 = 0$

5. $3x^2 - 6xy + 3y^2 + 3x - 9 = 0$

6. $x^2 - xy + 4y^2 + 2x - 3y + 1 = 0$

7–21 Identify the type of conic section defined by the equation and match the equation with its graph (labeled A–O).

7. $\dfrac{(x-1)^2}{4} + \dfrac{y^2}{16} = 1$

8. $(x-2)^2 = 4y$

9. $x^2 - y^2 = 1$

10. $x^2 + \dfrac{(y-3)^2}{4} = 1$

11. $\dfrac{y^2}{4} - (x-1)^2 = 1$

12. $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{25} = 1$

13. $\dfrac{x^2}{9} - \dfrac{(y+2)^2}{4} = 1$

14. $(x+2)^2 = 3(y-1)$

15. $\dfrac{(x-1)^2}{4} + y^2 = 1$

16. $\dfrac{(x-3)^2}{4} - \dfrac{(y+1)^2}{9} = 1$

17. $y^2 = 4(x+1)$

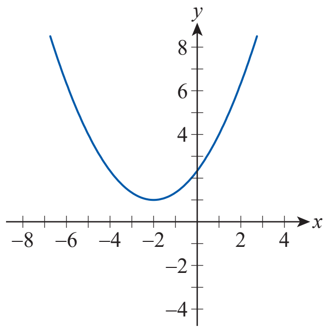
18. $(x-1)^2 = -(y-2)$

19. $(y-1)^2 = -2(x-2)$

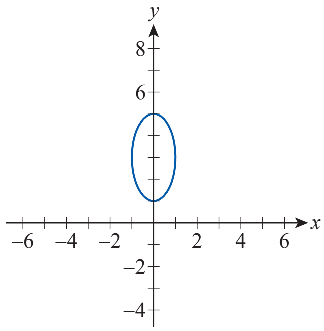
20. $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

21. $(y+2)^2 - \dfrac{(x-2)^2}{4} = 1$

A.



B.



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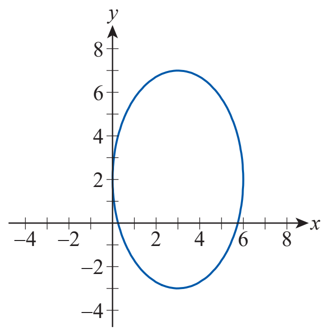

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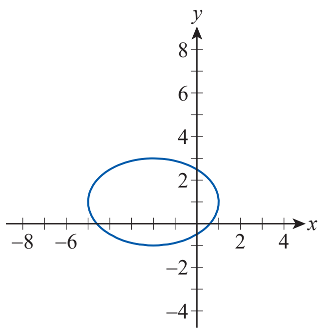

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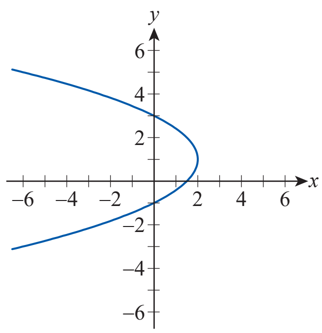

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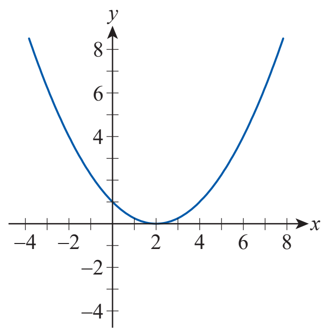

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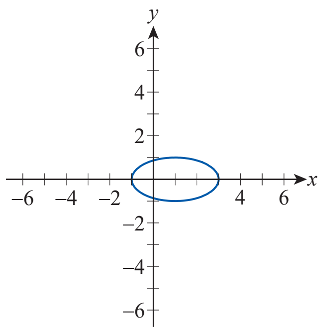

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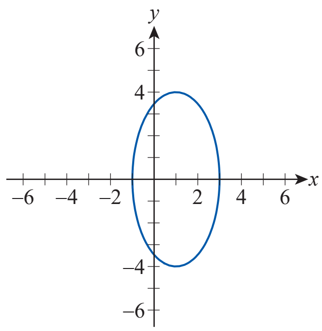

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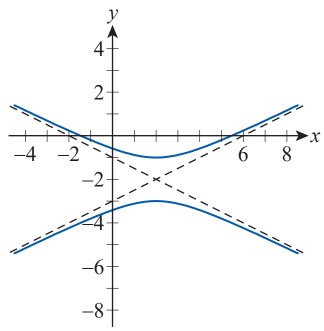

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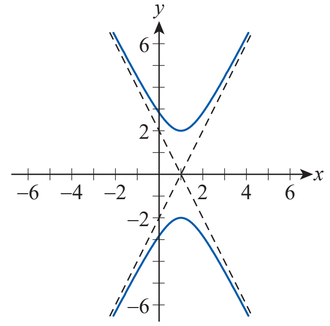

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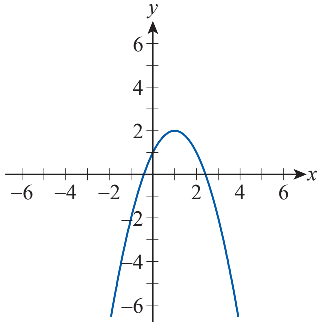

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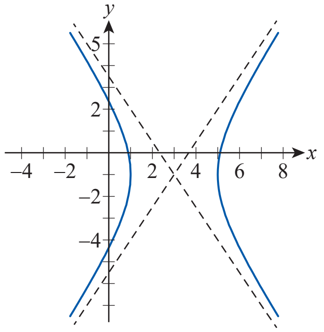

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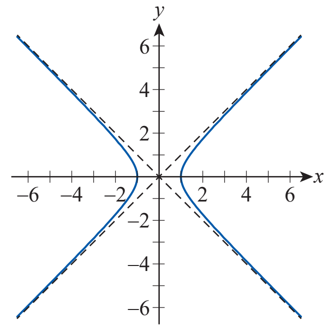

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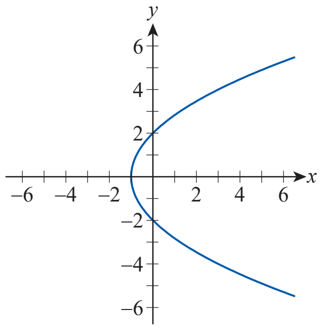

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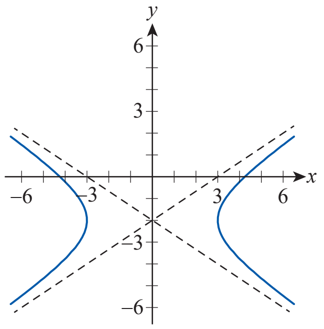


22–25 Find the eccentricity and the lengths of the major and minor axes of the ellipse.

22. $x^2 + 9y^2 = 36$

23. $5x^2 + 8y^2 = 40$

24. $20x^2 + 10y^2 = 40$

25. $\frac{1}{4}x^2 + \frac{1}{12}y^2 = \frac{1}{2}$

26–35 Graph the ellipse and determine the coordinates of the foci.

26. $\frac{(x-4)^2}{16} + \frac{(y-4)^2}{4} = 1$

27. $\frac{(x+2)^2}{16} + \frac{(y+1)^2}{9} = 1$

28. $9x^2 + 16y^2 + 18x - 64y = 71$

29. $9x^2 + 4y^2 - 36x - 24y + 36 = 0$

30. $16x^2 + y^2 + 160x - 6y = -393$

31. $25x^2 + 4y^2 - 100x + 8y + 4 = 0$

32. $4x^2 + 9y^2 + 40x + 90y + 289 = 0$

33. $16x^2 + y^2 - 64x + 6y + 57 = 0$

34. $4x^2 + y^2 + 4y = 0$

35. $9x^2 + 4y^2 + 108x - 32y = -352$

36–45 Graph the parabola and determine its focus and directrix.

36. $(x+1)^2 = 4(y-3)$ **37.** $(y-4)^2 = -2(x-1)$

38. $y^2 + 2y + 12x + 37 = 0$ **39.** $x^2 - 8y = 6x - 1$

40. $x^2 + 6x + 8y = -17$ **41.** $x^2 + 2x + 8y = 31$

42. $y^2 + 6y - 2x + 13 = 0$ **43.** $x^2 - 2x - 4y + 13 = 0$

44. $4y + 2x^2 = 4$ **45.** $2y^2 - 10x = 10$

46–55 Graph the hyperbola, using asymptotes as guides, and determine the coordinates of the foci.

46. $\frac{(x+3)^2}{4} - \frac{(y+1)^2}{9} = 1$

47. $4y^2 - x^2 - 24y + 2x = -19$

48. $x^2 - 9y^2 + 4x + 18y - 14 = 0$

49. $9x^2 - 25y^2 = 18x - 50y + 241$

50. $9x^2 - 16y^2 + 116 = 36x + 64y$

51. $\frac{(y-1)^2}{9} - (x+3)^2 = 1$

52. $9y^2 - 25x^2 - 36y - 100x = 289$

53. $9x^2 + 18x = 4y^2 + 27$

54. $9x^2 - 16y^2 - 36x + 32y - 124 = 0$

55. $x^2 - y^2 + 6x - 6y = 4$

56–73 Find the equation, in standard form, for the conic with the given properties or with the given graph.

56. Ellipse, center at $(-2,3)$, horizontal major axis of length 8, minor axis of length 4

57. Parabola, focus at $(-2,1)$, directrix is the x-axis

58. Ellipse, vertices at $(5,-1)$ and $(1,-1)$, minor axis of length 2

59. Hyperbola, foci at $(1,5)$ and $(1,-1)$, vertices at $(1,3)$ and $(1,1)$

60. Parabola, focus at $\left(-3,-\frac{3}{2}\right)$, directrix is the line $y=-\frac{1}{2}$

61. Hyperbola, foci at $(-1,3)$ and $(-1,-1)$, asymptotes given by $y=\pm(x+1)+1$

62. Parabola, vertex at $(-4,3)$, focus at $\left(-\frac{3}{2},3\right)$

63. Ellipse, foci at $(0,0)$ and $(6,0)$, $e=\frac{1}{2}$

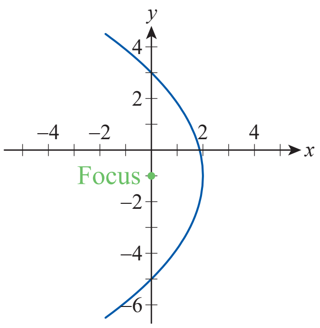
64. Hyperbola, asymptotes given by $y=\pm(2x+8)+3$, vertices at $(-6,3)$ and $(-2,3)$

65. Ellipse, vertices at $(-4,6)$ and $(-14,6)$, $e=\frac{2}{5}$

66. Hyperbola, foci at $(2,4)$ and $(-2,4)$, asymptotes given by $y=\pm 3x+4$

67. Parabola, symmetric with respect to the line $y=1$, directrix is the line $x=2$, and $p=-3$

68.



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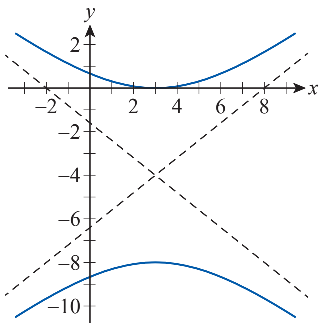

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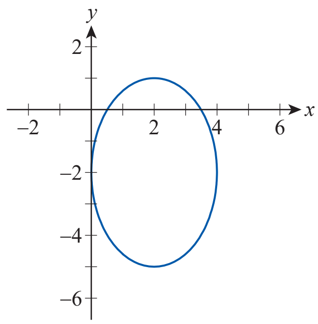

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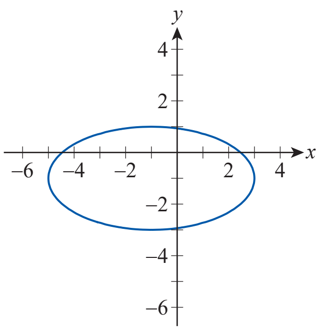

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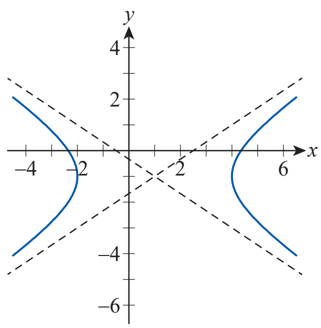

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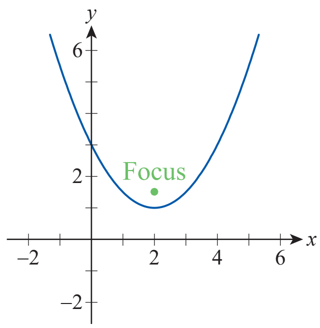
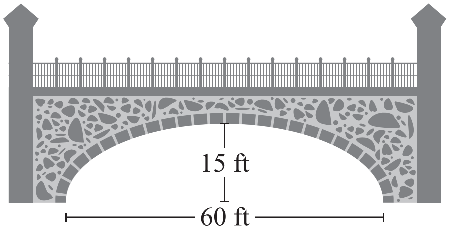


74. The orbit of Halley's Comet is an ellipse with the sun at one focus and an eccentricity of 0.967. Its closest approach to the sun is approximately 54,591,000 miles. What is the furthest Halley's Comet ever gets from the sun?

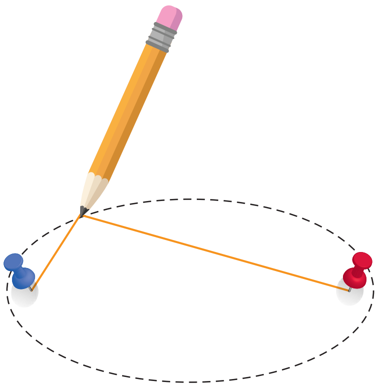
75. Pluto's closest approach to the sun is approximately 4.43×10^{9} kilometers, and its maximum distance from the sun is approximately 7.37×10^{9} kilometers. What is the eccentricity of Pluto's orbit?

76. The archway supporting a bridge over a river is in the shape of half an ellipse. The archway is 60 feet wide and is 15 feet tall at the middle. A large boat is 10 feet wide and 14 feet 9 inches tall. Is the boat capable of passing under the archway?



77. Use the information given in Example 2 to determine the length of the minor axis of the ellipse formed by Earth's orbit around the sun.

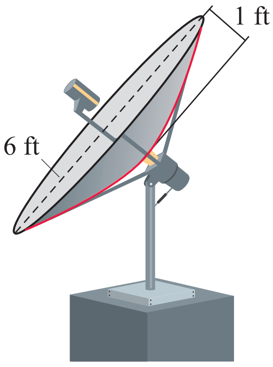
78. Since the sum of the distances from each of the two foci to any point on an ellipse is constant, we can draw an ellipse using the following method. Tack the ends of a length of string at two points (the foci) and, keeping the string taut by pulling outward with the tip of a pencil, trace around the foci to form an ellipse (the total length of the string remains constant). If you want to create an ellipse with a major axis of length 5 cm and a minor axis of length 3 cm, how long should your string be and how far apart should you place the tacks?



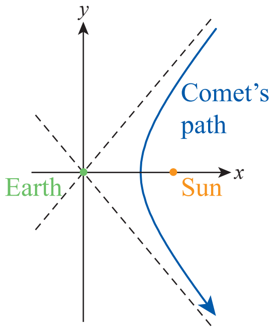
79. One design for a solar furnace is based on the paraboloid formed by rotating the parabola $x^2 = 8y$ around its axis of symmetry. The object to be heated in the furnace is then placed at the focus of the paraboloid (assume that x and y are in units of feet). How far from the vertex of the paraboloid is the hottest part of the furnace?

80. A certain satellite dish antenna is a paraboloid with a diameter of 6 feet and a depth of 1 foot. How far from the vertex of the dish should the receiver of the antenna be placed, given that the receiver should be located at the focus of the paraboloid?

81. A spotlight is made by placing a strong lightbulb inside a reflective paraboloid formed by rotating the parabola $x^2 = 6y$ around its axis of symmetry (assume that x and y are in units of inches). In order to have the brightest, most concentrated light beam, how far from the vertex should the bulb be placed?



82. As mentioned in this section, some comets trace one branch of a hyperbola through the solar system, with the sun at one focus. Suppose a comet is spotted that appears to be headed straight for Earth as shown in the figure. As the comet gets closer, however, it becomes apparent that it will pass between Earth, which lies at the center of the hyperbolic path of the comet, and the sun. In the end, the closest the comet comes to Earth is 60,000,000 miles. Using an estimate of 94,000,000 miles for the distance from Earth to the sun, and positioning Earth at the origin of a coordinate system, find the equation for the path of the comet.



83. Placing the foci at $(-c, 0)$ and $(c, 0)$ and introducing $d_1 + d_2 = 2a$, derive the standard form of the equation of an ellipse.

84. Denoting $|d_1 - d_2|$ by $2a$, use the approach suggested by Exercise 83 to derive the standard equation of a hyperbola.

85. Suppose two LORAN (LOng RAnge Navigation) radio transmitters are 26 miles apart. A ship at sea receives signals sent simultaneously from the two transmitters and is able to determine that the difference between the distances from the ship to each of the transmitters is 24 miles. By positioning the two transmitters on the y-axis, each 13 miles from the origin, find the equation of the hyperbola that describes the set of possible locations for the ship. (**Hint:** See Exercise 84.)

86.* Three high-sensitivity microphones are located in a forest preserve, with microphone A two miles due north of microphone B and microphone C two miles due east of microphone B. During an early morning thunderstorm, microphone A detects a thunderclap (and possible lightning strike) at 3:28:15 a.m. The same thunderclap is detected by microphone B at 3:28:19 a.m. and by microphone C at 3:28:25 a.m. Assuming that sound travels at 1100 feet per second, graphically approximate the source of the thunderclap. (**Hint:** Place microphone B at the origin, with A and C on the y- and x-axes, respectively. Then, by a repeated application of the method used in Exercise 85, construct two intersecting hyperbolas to locate the thunderclap.)

87–90 Find the $x'y'$-coordinates of the point for the given rotation angle θ.

87. $(8,6);\quad \theta = 30°$

88. $(-5,1);\quad \theta = \frac{\pi}{3}$

89. $\left(-\frac{1}{2},-\frac{1}{8}\right);\quad \theta = \frac{\pi}{4}$

90. $(-1,1);\quad \theta = \frac{\pi}{2}$

91–96 Use the discriminant to classify the conic section as an ellipse, parabola, or hyperbola. Then determine the appropriate angle θ by which to rotate the coordinate axes, and use that angle to convert the equation by eliminating the xy-term. Finally, sketch the graph of the conic section.

91. $xy - 4 = 0$

92. $x^2 + 2xy + y^2 - x + y = 0$

93. $7x^2 + 5\sqrt{3}xy + 2y^2 = 14$

94. $22x^2 + 6\sqrt{3}xy + 16y^2 - 49 = 276$

95. $2\sqrt{3}x^2 - 6xy + \sqrt{3}x - 9y = 0$

96. $34x^2 + 8\sqrt{3}xy + 42y^2 = 1380$

97–100 The given equation is that of a rotated conic. Match the equation with its graph (labeled A–D).

97. $3x^2 + 2xy + y^2 - 10 = 0$

98. $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$

99. $3x^2 + 8xy + 4y^2 - 7 = 0$

100. $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

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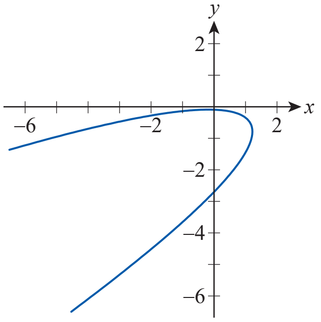

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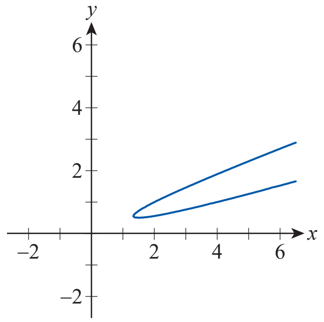

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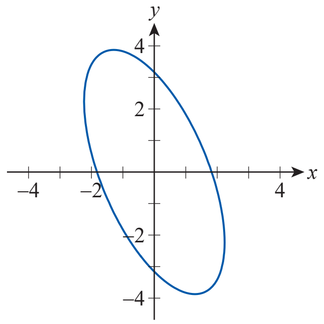

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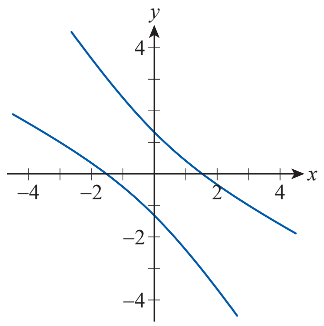

101. You have just used the rotation of axes to rotate the x- and y-axes until they were parallel to the axes of the conic. The resulting equation in the $x'y'$-plane is of the form

$$A'x'^2 + B'x'y' + E'y' + F' = 0,$$

where A', B', E', and F' are all nonzero. What is wrong with the resulting equation?

102. What must the angle of rotation θ be if the coefficients of x^2 and y^2 are equal and $B \neq 0$? Support your answer.

103. An expression involving the coefficients of the general form of a conic section is said to be *rotation invariant* if it has the same value under every possible rotation. Using the equation $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$,

a. show that the relationship $F = F'$ is true (so F is rotation invariant in this equation);

b. show that the relationship $A + C = A' + C'$ is true (so $A + C$ is rotation invariant in this equation);

c. show that the relationship $B^2 - 4AC = B'^2 - 4A'C'$ is true (so $B^2 - 4AC$ is rotation invariant in this equation).

104.* Show in general that the quantity $A + C$ and the discriminant $B^2 - 4AC$ are rotation invariant. (See Exercise 103.)

Concept Check

105–108 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

105. It is possible for a parabola to be tangent to its directrix.

106. The graph of $Ax^2 + Cy^2 + Dx + Ey = 0$ is a hyperbola if A and C have different signs and $D, E \neq 0$.

107. It is not possible for a line tangent to a hyperbola to have more than one point in common with the graph.

108. If the eccentricity of an ellipse is greater than 1, the ellipse is extremely narrow.

9.5 Technology Exercises

109–118 Use a graphing utility to sketch the given curve.

109. $15x^2 + 9y^2 + 150x - 36y = -276$

110. $5x^2 + 12y^2 - 20x + 144y + 392 = 0$

111. $x^2 - 6x + 12y + 21 = 0$

112. $x^2 - 5y^2 = 14x + 20y - 4$

113. $x^2 + 6xy + y^2 = 18$

114. $x^2 - 4xy + 3y^2 = 12$

115. $36x^2 - 19xy + 8y^2 = 72$

116. $72x^2 + 19xy + 4y^2 = 20$

117. $40x^2 + 20xy + 10y^2 + \left(2\sqrt{2} - 6\right)x - \left(4\sqrt{2} + 8\right)y = 90$

118. $72x^2 + 18xy - 9y^2 = 14$

119–120 Use a graphing utility to sketch the given curve. Explore how different values of the parameters k_1 and k_2 affect the graph. Experiment with both nonnegative and negative values. (Answers will vary.)

119. $k_1x^2 + k_2xy + 5y^2 - 6x + 7y + 15 = 0$

120. $k_1x^2 - 4xy + k_2y^2 + 2x + 3y - 1 = 0$

We will conclude this section with an example of rotation of conics in polar form. In general, the graph of an equation $r = f(\theta - \varphi)$ is the rotation of the graph of $r = f(\theta)$ by the angle θ counterclockwise. This makes rotation in polar coordinates particularly easy to handle.

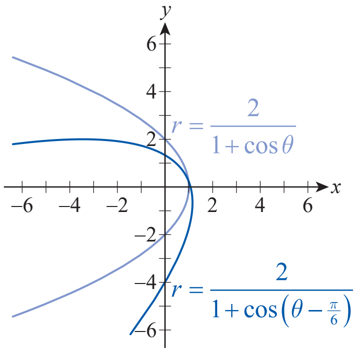

Figure 6

Example 5 Graphing a Rotated Conic Section in Polar Form

Sketch the graph of the conic section $r = \dfrac{2}{1+\cos\left(\theta - \dfrac{\pi}{6}\right)}$.

Solution

We constructed the equation $r = 2/(1+\cos\theta)$ in Example 2, so we know its graph is a parabola opening to the left with directrix $x = 2$.

The graph of $r = \dfrac{2}{1+\cos\left(\theta - \dfrac{\pi}{6}\right)}$ is the same shape rotated $\pi/6$ radians counterclockwise, as shown in Figure 6.

9.6 Exercises

1-6 Match the given polar equation with its graph (labeled A–F).

1. $r = \dfrac{3}{4-\cos\theta}$

2. $r = \dfrac{9}{6-2\sin\theta}$

3. $r = \dfrac{3}{3+4\sin\theta}$

4. $r = \dfrac{1}{2+2\cos\theta}$

5. $r = \dfrac{6}{1+3\sin\theta}$

6. $r = \dfrac{6}{1+3\cos\theta}$

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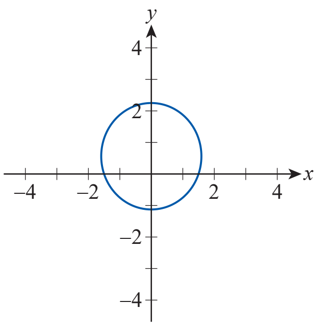

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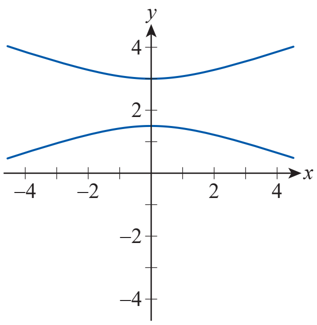

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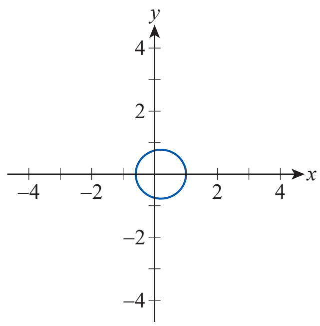

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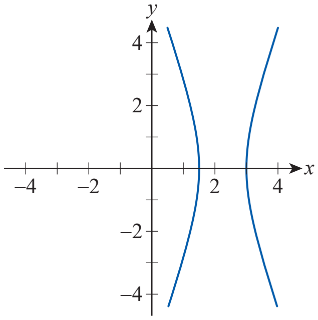

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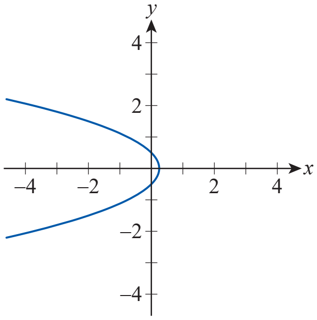

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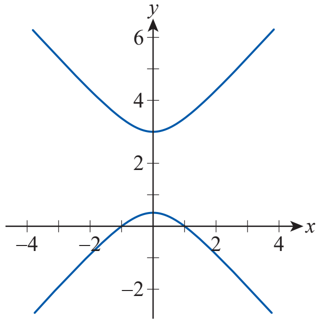


7–20 Identify the given conic section as an ellipse, parabola, or hyperbola and find the equation for its directrix.

7. $r=\dfrac{7}{1+6\sin\theta}$

8. $r=\dfrac{2}{1-\sin\theta}$

9. $r=\dfrac{3}{4-\cos\theta}$

10. $r=\dfrac{4}{2-2\cos\theta}$

11. $r=\dfrac{1}{1+3\cos\theta}$

12. $r=\dfrac{7}{3+2\sin\theta}$

13. $r=\dfrac{5}{2+\cos\theta}$

14. $r=\dfrac{3}{4-3\sin\theta}$

15. $r=\dfrac{6}{3-5\cos\theta}$

16. $r=\dfrac{8}{5-6\sin\theta}$

17. $r=\dfrac{3}{2+2\sin\theta}$

18. $r=\dfrac{-1}{3+4\cos\theta}$

19. $r=\dfrac{4}{6-7\cos\theta}$

20. $r=\dfrac{9}{5-4\sin\theta}$

21–26 Construct a polar equation for the conic section with the focus at the origin and the given eccentricity and directrix.

21. Parabola; eccentricity: $e=1$; directrix: $x=-2$

22. Hyperbola; eccentricity: $e=2$; directrix: $x=-3$

23. Hyperbola; eccentricity: $e=4$; directrix: $y=-\frac{3}{4}$

24. Parabola; eccentricity: $e=1$; directrix: $x=2$

25. Ellipse; eccentricity: $e=\frac{1}{4}$; directrix: $x=12$

26. Ellipse; eccentricity: $e=\frac{1}{2}$; directrix: $y=8$

27–36 Sketch the graph of the conic section.

27. $r=\dfrac{5}{1+3\cos\theta}$

28. $r=\dfrac{3}{2+\sin\theta}$

29. $r=\dfrac{4}{1-2\sin\theta}$

30. $r=\dfrac{6}{2-4\cos\theta}$

31. $r=\dfrac{9}{3-2\cos\theta}$

32. $r=\dfrac{5}{3+\sin\theta}$

33. $r=\dfrac{4}{1+2\cos\theta}$

34. $r=\dfrac{4}{2+2\sin\theta}$

35. $r=\dfrac{2}{1+\cos\left(\theta-\dfrac{\pi}{4}\right)}$

36. $r=\dfrac{4}{2+2\sin\left(\theta-\dfrac{\pi}{3}\right)}$

37. The planets of our solar system follow elliptical orbits with the sun located at one of the foci. If we assume that the sun is located at the pole and the major axes of these elliptical orbits lie along the polar axis, the orbits of the planets can be expressed by the polar equation

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

where e is the eccentricity. Verify the above equation.

38. Using the equation from Exercise 37, answer the following exercises.

a. Show that the shortest distance from the sun to a planet, called the *perihelion* distance, is $r=a(1-e)$.

b. Show that the longest distance from the sun to a planet, called the *aphelion* distance, is $r=a(1+e)$.

c. The distance from Uranus to the sun is approximately 2.74×10^9 km at perihelion and 3.00×10^9 km at aphelion. Find the eccentricity of Uranus' orbit.

d. The eccentricity of Neptune's orbit is 0.0113 and $a=4.495\times10^9$ km. Determine the perihelion and aphelion distances for Neptune.

39. Derive the polar form of the equation of a conic with vertical directrix $x=-d$ and focus at the origin.

40. Derive the polar form of the equation of a conic with horizontal directrix **a.** $y=d$, **b.** $y=-d$, and focus at the origin.

41. A chord through a focus of a conic section that is parallel to the directrix is called its *latus rectum* (from the Latin words "latus," meaning "side," and "rectum," meaning "straight"). Find the length of the latus rectum for the conic $r=ed/(1+e\cos\theta)$.

42. Find the polar coordinates of the vertices for the ellipse with polar equation $r=ed/(1+e\cos\theta)$, $0<e<1$.

43. Find the polar coordinates of the vertices for the hyperbola with polar equation $r=ed/(1+e\cos\theta)$, $e>1$.

44. Use Exercise 42 to find the rectangular equation of the ellipse $r=12/(5+\cos\theta)$.

45. Use Exercise 43 to find the rectangular equation of the hyperbola $r=12/(5+7\cos\theta)$.

9.6 Technology Exercises

46–55 Use a graphing utility to graph the conic section.

46. $r=\dfrac{-3}{4-9\cos\theta}$

47. $r=\dfrac{9}{-4+\frac{3}{2}\sin\theta}$

48. $r=\dfrac{-11}{3-\cos\theta}$

49. $r=\dfrac{2}{10+4\sin\theta}$

50. $r=\dfrac{3}{7+3\cos\theta}$

51. $r=\dfrac{2}{2+3\cos\left(\theta-\frac{\pi}{4}\right)}$

52. $r=\dfrac{-7}{5+3\sin\left(\theta-\frac{\pi}{6}\right)}$

53. $r=\dfrac{5}{-2-4\sin\left(\theta+\frac{2\pi}{3}\right)}$

54. $r=\dfrac{4}{-3-2\cos\left(\theta+\frac{\pi}{3}\right)}$

55. $r=\dfrac{1}{1+4\sin\left(\theta+\frac{\pi}{6}\right)}$

56. Use technology to sketch the conic section $r=ed/(1+e\cos\theta)$ for various values of d and e, $e>0$, and examine how these values affect the shape of the graph.

Squaring both sides, we obtain the equation $L^2 = 2 + L$, or $L^2 - L - 2 = 0$. One of the solutions of this equation is $L = 2$, and the other $(L = -1)$ is extraneous to this problem.

10.1 Exercises

1–4 List the first six terms of the given sequence.

1. $a_n = \dfrac{n}{n^2+1}$ **2.** $a_n = \left(-\dfrac{2}{5}\right)^n$ **3.** $a_n = \dfrac{3}{(n-1)!}$ **4.** $a_n = \dfrac{n(n+1)}{2}\cos(n\pi)$

5–8 Find the first six terms of the given recursively defined sequence.

5. $a_1 = 1, \quad a_n = 2a_{n-1} + 1$ for $n \geq 2$

6. $a_1 = 2, \quad a_2 = 3, \quad a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

7. $a_1 = 4, \quad a_{n+1} = (n+1)a_n$ for $n \geq 1$

8. $a_1 = 1, \quad a_2 = 1, \quad a_{n+1} = -2a_n + 3a_{n-1}$ for $n \geq 2$

9–12 Recognize the apparent pattern and find an explicit formula for the sequence. (Answers will vary.)

9. $\{2, 6, 12, 20, 30, 42, \ldots\}$

10. $\{2, 6, 18, 54, 162, 486, \ldots\}$

11. $\{9, 6, 1, -6, -15, -26, \ldots\}$

12. $\{2, 4, 7, 11, 16, 22, \ldots\}$

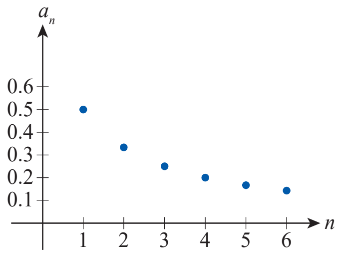
13. Notice that the sequences in Exercises 9–12 are actually not uniquely determined. For example, show that in each case the formula you obtained for a_n and the formula $b_n = a_n + (n-1)\cdots(n-j)$ define two sequences that match for the first j terms and then differ.

14–19 Match the sequence with its graph (labeled A–F).

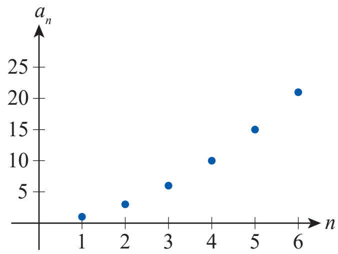
14. $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ **15.** $a_n = \dfrac{n(n+1)}{2}$ **16.** $a_n = (-2)^n$

17. $a_n = \cos(n\pi)$ **18.** $a_n = \dfrac{1}{n+1}$ **19.** $a_n = \dfrac{n}{n+1}$

A.



B.



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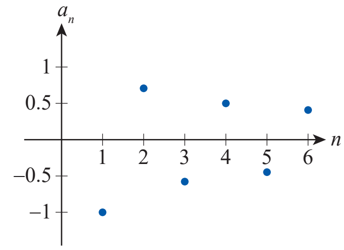
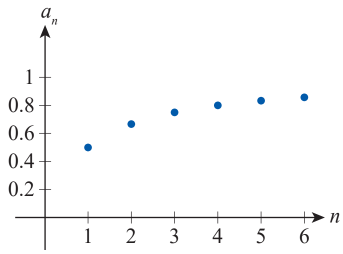
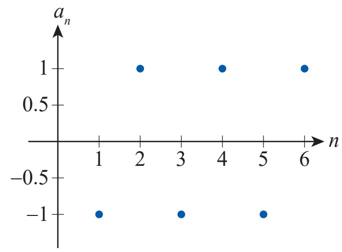


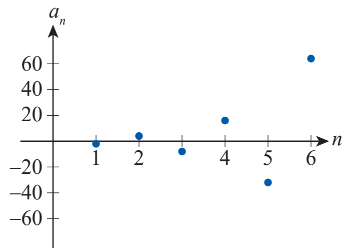
D.



E.



F.



20–23 Use the definition of the limit of a sequence to establish the given statement.

20. $\lim_{n\to\infty}\frac{1}{n^2}=0$

21. $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n=0$

22. $\lim_{n\to\infty}\frac{3n+1}{n+1}=3$

23. $\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n}\right)=0$

24–69 Find the limit of the sequence if it converges, or prove that the sequence diverges. (You can use any theorem from this section.)

24. $a_n=\frac{2n-3}{n}$

25. $a_n=\frac{3n+5}{6n-1}$

26. $a_n=\frac{4n^2+1}{2n-1}$

27. $a_n=\frac{2n-1}{4n^2+1}$

28. $a_n=\frac{n+1}{\sqrt{n}}$

29. $a_n=(-1)^n\frac{\sqrt{n}}{n+1}$

30. $a_n=\frac{n\sin n}{n^2+2}$

31. $a_n=\sqrt{n+1}-\sqrt{n}$

32. $a_n=\frac{n^4-5n^3+2n^2+1}{3n^4+n^2+2}$

33. $a_n=\frac{\pi^n}{n^2}$

34. $a_n=\frac{\pi^n}{2^n}$

35. $a_n=\frac{\pi^n}{4^n}$

36. $b_n=\frac{\ln n}{2n}$

37. $c_n=0.5+(-0.5)^n$

38. $a_n=4^n+\left(\frac{1}{4}\right)^n$

39. $a_n=\frac{\ln\frac{1}{\sqrt{n}}}{\sqrt{n}}$

40. $d_n=\frac{n3^n}{4^n}$

41. $h_n=\frac{n^4-4}{e^n}$

42. $a_n=\frac{n!}{4^n}$

43. $a_n=\ln\left(3n^2+2\right)-\ln\left(n^2+7\right)$

44. $a_n=(-1)^n\frac{\sqrt{n}}{\sqrt{n}+1}$

45. $p_n=2^{-1/n}$

46. $q_n=\sqrt{1+\frac{1}{n}}$

47. $a_n=\sin^{-1}\left(\frac{n^2+n+2}{2n^2+1}\right)$

48. $a_n=\left(\frac{n^2+1}{3n^2-2}\right)\left(3+\frac{1}{n}\right)$

49. $r_n=\frac{2}{(0.6)^n}$

50. $s_n=\frac{\ln 3n}{\ln 4n}$

51. $a_n=\left(\frac{1}{n}\right)^{1/n}$

52. $a_n=\frac{n^{1/n}}{\ln n}$

53. $t_n=\frac{n^n}{n!}$

54. $u_n=\tan^{-1}(\ln n)$

55. $a_n=\left(\frac{1}{\ln n}\right)^{1/n}$

56. $a_n=n\sin\frac{1}{n}$

57. $m_n=e^{-(\cos n)/n}$

58. $N_k=(-1)^k\frac{(\ln k)^2}{k}$

59. $a_n=\frac{2^n-1}{\pi^n}$

60. $K_n=\sqrt[n]{3n+1}$

61. $L_n=\left(\frac{1}{10}\right)^{-1/n}$

62. $S_k=\left(\frac{1}{k}\right)^{2/k}$

63. $T_k=\frac{\left(\frac{1}{2}\right)^k}{k^k-1}$

64. $a_n=\left(1+\frac{1}{n}\right)^n$

65. $a_n=\left(1+\frac{1}{n}\right)^{2n}$

66. $a_n=\left(1+\frac{1}{n^2}\right)^n$

67. $a_n=\left(1+\frac{3}{n}\right)^n$

68. $a_1=1,\quad a_{n+1}=\frac{1}{2}\left(a_n+\frac{5}{a_n}\right)$

69. $a_1=1,\quad a_n=\frac{3}{4}a_{n-1}+\frac{1}{a_{n-1}}$

70–73 Use the Squeeze Theorem to prove that the given sequence converges.

70. $\left\{\dfrac{\sin n}{n}\right\}_{n=1}^{\infty}$

71. $\left\{\dfrac{\cos^2 n}{3^n}\right\}_{n=1}^{\infty}$

72. $\left\{(-1)^n \dfrac{(\ln n)^2}{n^3}\right\}_{n=1}^{\infty}$

73. $\left\{2^{-n} \cos n\right\}_{n=1}^{\infty}$

74.* Prove the Bounded Monotonic Sequence Theorem. (**Hint:** Suppose first that $\{a_n\}$ is increasing and bounded above. Let L be the least upper bound of the set of values $\{a_n | n \in \mathbb{N}\}$ and fix an $\varepsilon > 0$. Since $L - \varepsilon$ is *not* an upper bound for $\{a_n | n \in \mathbb{N}\}$, there is an index N such that $a_N > L - \varepsilon$. Use monotonicity to finish the argument. Note that the decreasing case can be handled similarly, or by considering the sequence $\{-a_n\}$.)

75–80 Use the Bounded Monotonic Sequence Theorem to prove that the given sequence converges. In Exercises 75–78, find the limit.

75. $a_n = \dfrac{n^2}{n^2+1}$

76. $a_1 = \sqrt{6}, \quad a_{n+1} = \sqrt{6 + a_n}$

77. $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$

78. $a_1 = 0, \quad a_n = \dfrac{1}{2 - a_{n-1}}$

79. $a_n = \dfrac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)}$

80. $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3!} + \dfrac{1}{4!} + \cdots + \dfrac{1}{n!}$

81–82 The Fibonacci sequence has many interesting applications in combinatorics and the mathematics of computer algorithms. Fibonacci numbers also appear in nature, in the arrangements of leaves and flower petals and the geometry of some shells. We can derive the explicit formula for F_n, the n^{th} term of the Fibonacci sequence, as follows. First, notice that φ and ψ in Example 2 are the roots of the quadratic equation $x^2 - x - 1 = 0$ and, thus, we have $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$. Using, say, $\varphi^2 = \varphi + 1$ we obtain the following equations.

$$\varphi^3 = \varphi\varphi^2 = \varphi(\varphi+1) = \varphi^2 + \varphi = (\varphi+1) + \varphi = 2\varphi + 1$$

$$\varphi^4 = \varphi\varphi^3 = \varphi(2\varphi+1) = 2\varphi^2 + \varphi = 2(\varphi+1) + \varphi = 3\varphi + 2$$

Notice that the coefficients are Fibonacci numbers, and if we repeat the process for higher powers of φ, the Fibonacci numbers keep coming up. More precisely, using an induction argument one can show the following relationships.

$$\varphi^n = F_n\varphi + F_{n-1} \qquad (1)$$

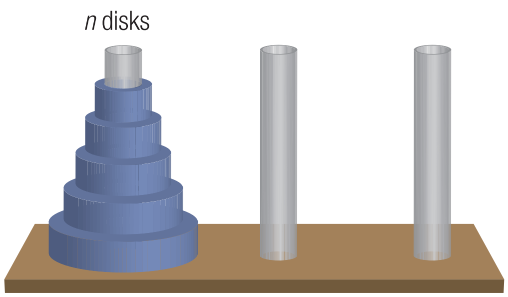
$$\psi^n = F_n\psi + F_{n-1} \qquad (2)$$

In Exercises 81–82, use this observation to derive the explicit formula for F_n.

81. Verify the explicit formula given in Example 2 for the n^{th} term of the Fibonacci sequence. (**Hint:** Subtract equation (2) from (1) above and solve for F_n.)

82.* Complete the induction argument referred to in the discussion preceding Exercise 81 to prove $\varphi^n = F_n\varphi + F_{n-1}$.

83–84 The popular Tower of Hanoi puzzle was invented by the French mathematician Édouard Lucas in 1883. It consists of three pegs, with n disks placed on the first peg in order of increasing size from top to bottom (see figure). The objective is to transfer all disks to the second peg so that they end up in the same order, according to the following two rules. Only one disk can be moved at a time, and no disk can be placed at any time on top of a smaller disk. In Exercises 83–84, you will find T_n, the number of moves necessary to solve the puzzle with n disks.



83. Find a recursive definition for the sequence $\{T_n\}$. (**Hint:** Use T_{n-1} steps to move the top $n - 1$ disks to the third peg, then place the largest disk on the second peg, and then repeat the previous moves to complete the tower on the second peg.)

84.* Prove that an explicit formula for $\{T_n\}$ is $T_n = 2^n - 1$.

85.* Prove that if $a > 1$, then $\lim\limits_{n\to\infty} a^n = \infty$.

86.* Prove that if $|a| < 1$, then $\lim\limits_{n\to\infty} a^n = 0$.

87.* Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ is convergent, then it is bounded.

88. Give an example of an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \not\to \infty$ and $a_n \not\to -\infty$.

89.* Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ is convergent, then it has a largest or a smallest term.

90. Suppose that $\lim_{n\to\infty} a_n = \infty$ and $\{b_n\}$ is a sequence such that there is an N with $b_n \geq a_n$ for all $n \geq N$. Prove that $\lim_{n\to\infty} b_n = \infty$.

91.* Prove that the limit of a convergent sequence is unique, that is, no convergent sequence can have two different limits. (**Hint:** Create an indirect argument by first assuming that $\{a_n\}$ converges to both L_1 and L_2, $L_1 \neq L_2$, and then using the definition of convergence.)

92.* We call $\{a_n\}$ a *null sequence* if $\lim_{n\to\infty} a_n = 0$. Prove that if $\{a_n\}$ is a null sequence and $\{b_n\}$ is bounded, then $\{a_n \cdot b_n\}$ is a null sequence.

93. Suppose that all terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are nonzero and $\lim_{n\to\infty} |a_n| = \infty$. Prove that $\{1/a_n\}$ is a null sequence.

94.* Prove that if $\lim_{n\to\infty} a_n = \infty$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n + b_n) = \infty$.

95.* If every term of a sequence is positive, it is called a *positive sequence*. Prove that if $\{a_n\}$ is a positive sequence and $\lim_{n\to\infty} (a_{n+1}/a_n) = L > 1$, then $\lim_{n\to\infty} a_n = \infty$.

96.* Prove that if $\{a_n\}$ is a positive sequence and $\lim_{n\to\infty} (a_{n+1}/a_n) = L < 1$, then $\{a_n\}$ is a null sequence.

97.* Prove that if $\lim_{n\to\infty} (a_{n+1} - a_n) = L > 0$, then $\lim_{n\to\infty} a_n = \infty$.

98.* Suppose that for two convergent sequences, $\lim_{n\to\infty} a_n = L_a$, $\lim_{n\to\infty} b_n = L_b$, and that $L_a < L_b$. Prove that there is a positive integer N such that $a_n < b_n$ for all $n \geq N$.

99.* Suppose that for two convergent sequences, $\lim_{n\to\infty} a_n = L_a$, $\lim_{n\to\infty} b_n = L_b$, and that $a_n < b_n$ for all n. Prove that $L_a \leq L_b$. Can we conclude that $L_a < L_b$?

100. Prove that for any irrational number $r \in \mathbb{R}$ there is a strictly increasing sequence $\{r_n\}$ of rational numbers so that $r_n \to r$.

101. Let $\{a_n\}$ be a recursively defined sequence as follows: $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \geq 2$. Prove that the sequence converges, find an explicit formula for its n^{th} term, and find its limit. (**Hint:** Start by finding the "gaps" between consecutive terms: $|a_{n+1} - a_n|$.)

102.* If $a > 0$, prove that the recursively defined sequence $a_1 = 1$, $a_{n+1} = (a_n^2 + a)/(2a_n)$ is convergent, and its limit is $\sqrt{a}$. (**Hint:** Think about Newton's method.)

Concept Check

103–114 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

103. If $\{a_n\}_{n=1}^{\infty}$ is convergent and its limit is L, then for any $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all but finitely many terms of $\{a_n\}_{n=1}^{\infty}$.

104. If for any $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$, then $\{a_n\}_{n=1}^{\infty}$ is convergent.

105. If $\{a_n\}_{n=1}^{\infty}$ is convergent, then so is $\{|a_n|\}_{n=1}^{\infty}$.

106. If $\{|a_n|\}_{n=1}^{\infty}$ is convergent, then so is $\{a_n\}_{n=1}^{\infty}$.

107. If the first n terms of a convergent sequence are altered, the resulting sequence still converges to the same limit.

108. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_n^2 = L^2$.

109. If $\lim_{n\to\infty} a_n^2 = L$, then $\lim_{n\to\infty} a_n = \sqrt{L}$.

110. If $\lim_{n\to\infty} a_n^3 = L$, then $\lim_{n\to\infty} a_n = \sqrt[3]{L}$.

111. If $\lim_{n\to\infty} a_n = L$, and $L > 1$, then $\lim_{n\to\infty} a_n^n = \infty$.

112. If both $\{a_n\}$ and $\{b_n\}$ diverge, then $\{a_n + b_n\}$ diverges.

113. If $\{a_n + b_n\}$ converges and $\{a_n\}$ converges, then $\{b_n\}$ also converges.

114. If $\{a_n b_n\}$ converges and $\{a_n\}$ diverges, then $\{b_n\}$ also diverges.

10.2 Exercises

1. Suppose that $a_n = 2n^2/(n^3-1)$ and consider the series $\sum_{n=2}^{\infty} a_n$. One of your classmates argues that the series converges, because the numerator of a_n has a lesser degree than the denominator, therefore $\lim_{n\to\infty} a_n = 0$, which makes $\{s_n\}$ convergent, and thus, $\sum_{n=2}^{\infty} a_n$ has a finite sum. Is this argument correct? Why or why not?

2–5 Find the first five terms of the sequence of partial sums $\{s_n\}$ for the given series.

2. $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$

3. $\sum_{n=0}^{\infty}(-1)^n$

4. $\sum_{n=0}^{\infty}\frac{3^n-1}{3^n}$

5. $\sum_{n=1}^{\infty}\sin\frac{n\pi}{3}$

6. Determine the index n so that the difference between the sum of the series $\sum_{n=1}^{\infty}(1/2^n)$ and the partial sum $\{s_n\}$ (the error) is less than 0.0001.

7. Write a short paragraph on the difference between the sequences $\{a_n\}$ and $\{s_n\}$ for the series in Example 5. (Mention convergence and limits.)

8–17 Determine whether the given geometric series converges. If so, find its sum.

8. $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots$

9. $5-1+\frac{1}{5}-\frac{1}{25}+\frac{1}{125}-\frac{1}{625}+\cdots$

10. $1-1.1+(1.1)^2-(1.1)^3+(1.1)^4-\cdots$

11. $\frac{4}{5}-1+\frac{5}{4}-\frac{25}{16}+\frac{125}{64}-\cdots$

12. $\sum_{n=0}^{\infty}7^{-n}$

13. $\sum_{n=0}^{\infty}(-0.7)^n$

14. $\sum_{n=0}^{\infty}3\cdot\left(\frac{11}{12}\right)^n$

15. $\sum_{n=0}^{\infty}2\cdot(-0.35)^n$

16. $\sum_{n=0}^{\infty}\frac{4}{5}\cdot\left(-\frac{5}{4}\right)^n$

17. $\sum_{n=0}^{\infty}\frac{5}{4}\cdot\left(-\frac{4}{5}\right)^n$

18–19 Find all values of x for which the geometric series converges.

18. $\sum_{n=1}^{\infty}2(1-3x)^{n-1}$

19. $\sum_{n=1}^{\infty}\frac{4}{(2x-5)^{n-1}}$

20–23 Recognize the repeating decimal as a geometric series and write the decimal as a ratio of two integers.

20. $0.\overline{5}$

21. $0.\overline{123}$

22. $0.5\overline{384}$

23. $3.37\overline{9}$

24–29 Find the sum of the series. (**Hint:** Use partial fraction decomposition wherever appropriate to express it as a telescoping series.)

24. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

25. $\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}$

26. $\sum_{n=1}^{\infty}\frac{2}{20n^2-5}$

27. $\sum_{n=1}^{\infty}\frac{3}{n^2+5n+6}$

28. $\sum_{n=1}^{\infty}\frac{1}{(4n+3)(4n-1)}$

29. $\sum_{n=1}^{\infty}\frac{2}{12n^2-3}$

30–49 Decide whether the given series converges. If so, find its sum.

30. $\sum_{n=0}^{\infty}\frac{3^{n-1}}{4^n}$

31. $\sum_{n=3}^{\infty}\frac{(-1)^{n-1}n}{n-2}$

32. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{\pi}{4}\right)^n$

33. $\sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{3}{n}\right)$

34. $\sum_{n=1}^{\infty}\frac{1}{3^{1/n}}$

35. $\sum_{n=1}^{\infty}\frac{2^n+2^{2n}}{5^n}$

36. $\sum_{n=0}^{\infty}\left(2\sqrt{3}\right)^{2-n}$

37. $\sum_{n=2}^{\infty}\frac{\sqrt{n+1}}{\ln n}$

38. $\sum_{n=0}^{\infty}\left(\frac{\pi-1}{e}\right)^n$

39. $\sum_{n=1}^{\infty}\left(3^{-n}-4^{-n}\right)$

40. $\sum_{n=1}^{\infty}\frac{n}{3n+2}$

41. $\sum_{n=1}^{\infty}\frac{3^n-1}{3^n}$

42. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{e}{\pi}\right)^n$

43. $\sum_{n=0}^{\infty}\frac{2\cdot 3^n+5\cdot 7^n}{11^n}$

44. $\sum_{n=0}^{\infty}\frac{2+3^n-5^n}{7^n}$

45. $\sum_{n=0}^{\infty}(-1)^n\frac{n}{n+2}$

46. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

47. $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

48. $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$

49. $\sum_{n=1}^{\infty}\frac{(-1)^n n}{2n-1}$

50. The series $\sum_{n=1}^{\infty}\left(1/2^n\right)$ converges and its sum is 1, as we have seen in Example 1. Examine what happens when we **a.** drop the first three terms of the series, and **b.** adjoin the three terms 3 + 2 + 1 to the series. Conclude that deleting or adjoining finitely many terms to a series may change its sum, but not the fact of convergence. Can you provide a rigorous proof of this more general statement?

51–52 Use the definition of series convergence to prove that the series is convergent.

51. $\sum_{n=1}^{\infty}\frac{1}{2^n}$

52. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

53. Use Example 5 to show that it is possible for $\sum_{n=1}^{\infty}(a_n-b_n)$ to converge if both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are divergent.

54. Give an example of two divergent series such that $\sum_{n=1}^{\infty}(a_n+b_n)$ is convergent.

55. Give an example of two divergent series such that $\sum_{n=1}^{\infty}(a_n+b_n)$ is divergent.

56. Give an example of two divergent series such that $\sum_{n=1}^{\infty}(a_n-b_n)$ is divergent.

57. Prove that the series $\sum_{n=0}^{\infty}(a+nd)$ $(a,\ d\in\mathbb{R})$ converges if and only if $a=d=0$. (Such series are called *arithmetic series*.)

58. In one of Zeno's famous motion paradoxes, Achilles races a tortoise, giving the tortoise a 100-meter head start. Even though we assume that Achilles is ten times faster, the statement is that Achilles will never actually catch the tortoise. The reasoning goes as follows. Soon after the start, Achilles will reach the starting point of the tortoise, but by that time, the tortoise will have advanced 10 meters. Achilles' job is to quickly cover that 10-meter distance, but during that time, the tortoise will have advanced, namely, a meter, and so on. Give a calculus-based solution to the paradox by proving that Achilles will actually catch the tortoise and find the total distance Achilles will have run when it happens.

59. By examining partial sums and using properties of logarithms, prove that $\sum_{n=3}^{\infty}\ln\frac{n}{n-2}$ diverges.

60. Prove that $\sum_{n=0}^{\infty}(1-x)^n$ is convergent if $0<x<2$, and find its sum.

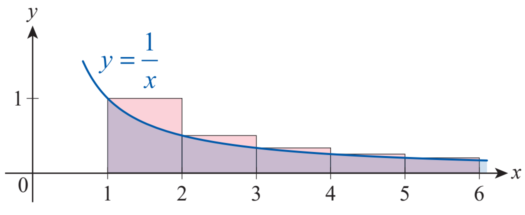
61. Prove the Sum Law for convergent series: $\sum(a_n+b_n)=\sum a_n+\sum b_n$. (**Hint:** Fix n, and write the statement for the n^{th} partial sums first; then take the limits and use the appropriate limit laws.)

62. Prove the Difference Law for convergent series: $\sum(a_n-b_n)=\sum a_n-\sum b_n$. (See the hint given in Exercise 61.)

63. Prove the Constant Multiple Law for convergent series: $\sum ka_n=k\sum a_n$. (See the hint given in Exercise 61.)

64. Suppose a large state injects a 2-billion-dollar stimulus package into its economy. Consumers and businesses in the state save approximately 30 percent of that money and respend 70 percent. Of that latter amount, approximately 70 percent is again spent, and so on. What is the total spending generated by the stimulus package? (**Hint:** Find the sum of the geometric series that models the process. In economics, this is called the *multiplier effect*.)

65. Use the figure below and the fact that $\int_1^{\infty}(1/x)\,dx=\infty$ to argue that the harmonic series diverges. (We will refine this idea in Section 10.3 and use it to "test" the convergence of various series.)



66. Prove that if the series $\sum a_n$ is convergent, then $\sum(1/a_n)$ is a divergent series.

67. Suppose that for the series $\sum a_n$ and $\sum b_n$ there exists a natural number N such that for all $n>N$, $a_n=b_n$. Prove that the series either both converge or both diverge.

68.* Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then it satisfies the so-called *Cauchy criterion for convergence*: For any $\varepsilon > 0$, there is a corresponding natural number N such that $\left| \sum_{n=n_1+1}^{n_2} a_n \right| < \varepsilon$ for all $n_1, n_2 > N$. (**Hint:** Choose an appropriately small ε-neighborhood around the sum of the series, noticing that $\left| \sum_{n=n_1+1}^{n_2} a_n \right| = \left| s_{n_2} - s_{n_1} \right|$.)

69.* Use Exercise 68 to prove that $\sum_{n=1}^{\infty} (1/n!)$ is convergent.

70. Prove that if $\sum_{n=1}^{\infty} a_n$ is a *positive series*, that is, $a_n > 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence $\{s_n\}_{n=1}^{\infty}$ of its partial sums is bounded.

71. Prove that $\sum_{n=1}^{\infty} \left(1/\sqrt{n}\right)$ diverges. (**Hint:** Prove that the n^{th} partial sum $s_n \ge \sqrt{n}$.)

72.* Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series such that $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \ge b_n$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} b_n$ is convergent. (**Hint:** Examine partial sums.)

73.* Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series such that $\sum_{n=1}^{\infty} a_n$ is divergent and $a_n \le b_n$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} b_n$ is divergent. (See the hint given in Exercise 72.)

74. Use Exercise 72 to prove that $\sum_{n=1}^{\infty} \left(1/n^2\right)$ is convergent. (**Hint:** Use the inequality below.)

$$\frac{1}{n^2} \le \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

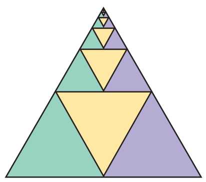
75. Use Exercise 73 and Example 6 to show that $\sum_{n=1}^{\infty} \left(1/n^p\right)$ diverges for all $0 < p < 1$.

76.* Two trains, 200 km apart, are on a collision course toward each other, each traveling at a rate of 50 km/h. A fly is zigzagging between the trains, flying at 75 km/h. Assuming constant rates and that the fly turns around in zero time, how much total distance will the fly be able to cover before being crushed to death by the trains upon their impending collision? (**Hint:** First, find the time required for the first "leg" of the flight, then, taking into consideration how much the original distance of 200 km between the trains has shrunk during this time, find the time required for the second "leg" of the fly's flight. Conclude that the time required for each leg is a constant times that required for the previous leg. Consequently, the fly's total time will be the sum of a geometric sequence; use this to find the total distance covered by the fly.)

77. Solve Exercise 76 "the easy way," without using an infinite series, that is, simply using the fact that the fly's total travel time equals the time necessary for the trains to reach each other. According to a well-authenticated story, when John von Neumann (born János Neumann), the great Hungarian American mathematician of the 20th century, was challenged with a version of this problem, he answered correctly within a few seconds. "Interesting," his challenger remarked, "most people try to solve this problem using infinite series." "Why," came von Neumann's reply, "that is how I did it!"

78–81 The following figures were generated by Rick Mabry as "pictorial proofs" for the convergence of various series. For example, dividing an equilateral triangle into four congruent parts and iterating the process, as below, provides illustration for the fact that $\sum_{n=1}^{\infty} \left(1/4^n\right) = \frac{1}{3}$ (assume that in the figure below, the area of the original triangle is 1; you can visually check what portion of the area is occupied by each color).

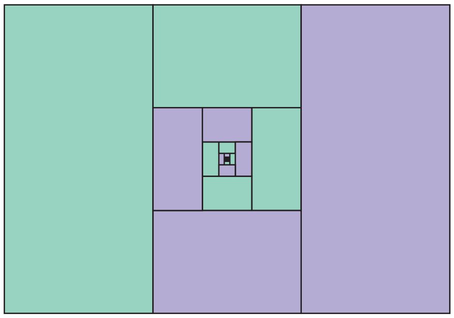
Source: www.lsus.edu/rick-mabry



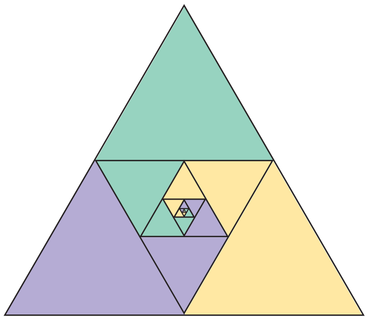
$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$

In Exercises 78–81, use the visual approach discussed above to identify the convergent series illustrated by the figure.

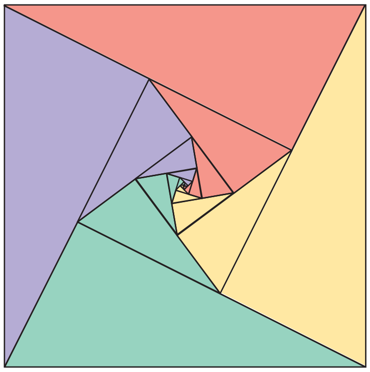
78.



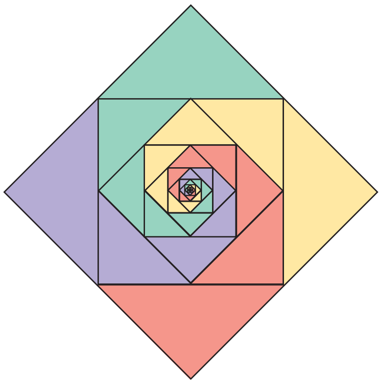
79.



80.



81.



10.2 Technology Exercises

82–85 Often we can readily show that a particular series is convergent, but finding the sum may be extremely challenging, if not impossible. For example, it is not difficult to prove that the series $\sum_{n=1}^{\infty}(1/n^p)$ is convergent if $p>1$ (see Exercise 74 and, for the full story, Example 2 in Section 10.3), but finding the sum often defies the best efforts of mathematicians. For example, letting $S(p)$ denote the sum of the above series for a particular $p>1$, we know that $S(2)=\pi^2/6$ (this surprising result was first proven by Euler), but formulas for the sums for odd p-values, such as $S(3)$, $S(5)$, etc. are still unknown.

In Exercises 82–85, use a graphing utility to verify the indicated sums.

82. $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

83. $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$

84. $\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$

85. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=\frac{1}{e}$

86. On the same screen, graph the function you obtained in Exercise 60 along with the partial sums $p(x)=\sum_{n=0}^{N}(1-x)^n$ on the interval $(0,2)$, for higher and higher N-values. What do you see?

Solution

Since

$$\int_n^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{n}\right) = \frac{1}{n},$$

we can make the following estimates.

a.
$$s_{10}+\frac{1}{11} \le s \le s_{10}+\frac{1}{10}$$
$$1.55+0.09 \le s \le 1.55+0.10$$
$$1.64 \le s \le 1.65$$

b.
$$s_{1000}+\frac{1}{1001} \le s \le s_{1000}+\frac{1}{1000}$$
$$1.643935+0.000999 \le s \le 1.643935+0.001$$
$$1.644934 \le s \le 1.644935$$

Since s lies within an interval of width 1×10^{-6}, we could approximate s by the average of the two values, 1.6449345, and know that the approximation is accurate to within $\frac{1}{2}\left(1\times10^{-6}\right) = 5\times10^{-7}$. (The exact value of s is $\pi^2/6$, as shown in Exercise 82 of Section 10.2)

Although the harmonic series diverges, we can still use an integral comparison to gain an appreciation for how slowly the sum grows. As we saw in proving the Integral Test, the relationship $\int_1^n f(x)\,dx \le s_{n-1}$ is valid for any series of positive but decreasing terms.

Example 4 Finding a Lower Bound for a Partial Sum

Find a lower bound for the sum of the first 1 million terms of the harmonic series.

Solution

Using $\int_1^n f(x)\,dx \le s_{n-1}$ with $n = 1{,}000{,}001$ gives us

$$\int_1^{1{,}000{,}001} (1/x)\,dx = \ln(1{,}000{,}001) \approx 13.816.$$

In fact, after a significant amount of computation, a computer algebra system can tell us that $s_{1{,}000{,}000} \approx 14.393$, so the lower bound given by the integral is a very quick and reasonable indication of the rate at which a divergent series of positive decreasing terms grows.

10.3 Exercises

1–32 Use the Integral Test to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2}{n+1}$ **2.** $\sum_{n=1}^{\infty} \frac{1}{3n-2}$ **3.** $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ **4.** $\sum_{n=1}^{\infty} \frac{2}{n^{5/4}}$

5. $\sum_{n=1}^{\infty} ne^{-n}$ **6.** $\sum_{n=4}^{\infty} \frac{6}{n^2-9}$ **7.** $\sum_{n=4}^{\infty} \frac{6}{(n-3)^2}$ **8.** $\sum_{n=1}^{\infty} \frac{n-1}{n^2-2n+0.75}$

9. $\sum_{n=1}^{\infty} \frac{2\ln n}{n}$ **10.** $\sum_{n=1}^{\infty} \frac{2\ln n}{n^2}$ **11.** $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$ **12.** $\sum_{n=2}^{\infty} \frac{2}{n\ln n}$

13. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}$

14. $\sum_{n=1}^{\infty} \frac{n}{(n^2+4)^2}$

15. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

16. $\sum_{n=3}^{\infty} \frac{1}{n^2-2n}$

17. $\sum_{n=1}^{\infty} \left(\frac{1}{n}+\frac{1}{2^n}\right)$

18. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+1}}$

19. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)^2}$

20. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{3n+1}}$

21. $\sum_{n=1}^{\infty} n^2e^{-n}$

22. $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$

23. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

24. $\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a \neq 0$

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

26. $\sum_{n=1}^{\infty} \frac{4n}{n^4+1}$

27. $\sum_{n=1}^{\infty} \frac{4n^3}{n^4+1}$

28. $\sum_{n=1}^{\infty} \left(2n^{-5/4}+\frac{1}{n^3}\right)$

29. $\sum_{n=1}^{\infty} \left(\frac{n}{(n^2+1)^3}+\frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)^2}$

31. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{4/5}}$

32. $\sum_{n=0}^{\infty} \left(2^{-n}+\left(\frac{e}{3}\right)^n\right)$

33–36 Explain why the Integral Test is not applicable to test the series for convergence.

33. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^3}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

35. $\sum_{n=0}^{\infty} \frac{\cos n}{2^n}$

36. $\sum_{n=0}^{\infty} e^{-\pi n}\cos(\pi n)$

37. Use the Integral Test to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

diverges when $p \leq 1$ and converges if $p > 1$. (These are called *logarithmic p-series*.)

38–43 Use our estimates immediately preceding Example 3 with the indicated number of terms to find an interval containing *s*, the sum of the series.

38. $\sum_{n=1}^{\infty} \frac{1}{n^4}$; four terms

39. $\sum_{n=1}^{\infty} \frac{1}{n^3}$; five terms

40. $\sum_{n=1}^{\infty} e^{-n}$; five terms

41. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$; six terms

42. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}(n+1)}$; five terms

43. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$; five terms

44–49 Find the smallest possible value of *n* to approximate the sum of the given series within the indicated error ε and provide the requested estimate.

44. $\sum_{n=1}^{\infty} \frac{1}{n^4}; \quad \varepsilon = 0.005$

45. $\sum_{n=1}^{\infty} \frac{1}{n^3}; \quad \varepsilon = 0.005$

46. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}; \quad \varepsilon = 0.01$

47. $\sum_{n=1}^{\infty} ne^{-n^2}; \quad \varepsilon = 5\times10^{-8}$

48. $\sum_{n=1}^{\infty} \frac{2}{(n+2)[\ln(n+2)]^3}; \quad \varepsilon = 0.01$

49. $\sum_{n=1}^{\infty} \frac{3}{1+n^2}; \quad \varepsilon = 0.05$

10.3 Technology Exercises

50–54 Use a graphing utility and remainder estimates to approximate the sum of the series with the given error ε. Find the smallest possible value of *n* you can use and give an approximation with the requested accuracy. (Answers may vary slightly. Recall that formulas for these sums are unavailable; see the discussion preceding Exercise 82 in Section 10.2.)

50. $\sum_{n=1}^{\infty} \frac{1}{n^3}; \quad \varepsilon = 10^{-8}$

51. $\sum_{n=1}^{\infty} \frac{1}{n^5}; \quad \varepsilon = 5\times10^{-9}$

52. $\sum_{n=1}^{\infty} \frac{1}{n^7}; \quad \varepsilon = 10^{-12}$

53. $\sum_{n=1}^{\infty} \frac{1}{n^9}; \quad \varepsilon = 10^{-12}$

54. Approximate π to eight decimal places using the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. What value of *n* did you use? (Answers will vary.)

10.4 Exercises

1–32 Use the Direct Comparison Test to determine whether the series converges or diverges. (Wherever applicable, combine the Direct Comparison Test with previous techniques, such as the Integral Test, *p*-series test, etc.)

1. $\sum_{n=1}^{\infty} \frac{2}{n^3+1}$

2. $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$

3. $\sum_{n=2}^{\infty} \frac{4n}{\sqrt{n^3-1}}$

4. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)}$

5. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+\sqrt{n}}$

6. $\sum_{n=1}^{\infty} \frac{2}{n^n}$

7. $\sum_{n=0}^{\infty} \frac{n^3+1}{n^4+1}$

8. $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

9. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+\sqrt{n}}$

10. $\sum_{n=0}^{\infty} \frac{1}{1+2^n}$

11. $\sum_{n=0}^{\infty} \frac{5n^2}{n^4+1}$

12. $\sum_{n=2}^{\infty} \frac{5n^2}{n^3-1}$

13. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{5n^3+2}}$

14. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

15. $\sum_{n=2}^{\infty} \frac{2}{\ln n}$

16. $\sum_{n=1}^{\infty} \frac{2}{n-\ln n}$

17. $\sum_{n=0}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+2}}$

18. $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$

19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1+\sqrt{n}}}$

20. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$

21. $\sum_{n=2}^{\infty} \frac{3n}{e^n \ln n}$

22. $\sum_{n=0}^{\infty} \sqrt{ne^{-n^2}}$

23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{ne^{\sqrt{n}}}}$

24. $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2}$

25. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3+1}$

26. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^4+2n+3}}$

27. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n^2-1}}$

28. $\sum_{n=0}^{\infty} \frac{2}{e^{2n}+n^{3/2}}$

29. $\sum_{n=2}^{\infty} \frac{\ln n}{n^{5/2}}$

30. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}-\frac{1}{2}}$

31. $\sum_{n=2}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}\ln n}$

32. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

33–50 Use the Limit Comparison Test to determine whether the series converges or diverges.

33. $\sum_{n=1}^{\infty} \frac{2n^2}{n^3+1}$

34. $\sum_{n=2}^{\infty} \frac{2n^2}{n^4-1}$

35. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}}$

36. $\sum_{n=1}^{\infty} \frac{n^4}{\sqrt{2n^9+n^5+3n^3+2}}$

37. $\sum_{n=5}^{\infty} \frac{2n+4}{n(n-2)(n-4)}$

38. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{2n}+\ln n}$

39. $\sum_{n=1}^{\infty} \frac{\sqrt{n}\ln(n+1)}{n^3}$

40. $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

41. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (**Hint:** Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.)

42. $\sum_{n=1}^{\infty} \frac{3^{1/n}-1}{3^{1/n}}$ (**Hint:** Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$.)

43. $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{\sqrt{n}\ln n}$

44. $\sum_{n=1}^{\infty} \frac{\sqrt{2+\frac{1}{n}}}{2n^2}$

45. $\sum_{n=1}^{\infty} \frac{ne^{-n^2}}{1+e^{-n}}$

46. $\sum_{n=1}^{\infty} \frac{n^2}{3n^{5/2}+1}$

47. $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n^2+3}}$

48. $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n^3+3}}$

49. $\sum_{n=0}^{\infty} \frac{n+3}{\sqrt{n^4+3}}$

50. $\sum_{n=0}^{\infty} \frac{n+3}{\sqrt{n^5+3}}$

51–70 Use any test covered so far in the text to determine whether the series converges or diverges.

51. $\sum_{n=0}^{\infty} \frac{1}{2^n+3}$

52. $\sum_{n=0}^{\infty} \frac{2^n}{2^n+3}$

53. $\sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{n+1}} - \frac{2}{\sqrt{n+2}}\right)$

54. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt[3]{n}}$

55. $\sum_{n=0}^{\infty} \frac{1}{2^{n^2}-3}$

56. $\sum_{n=1}^{\infty} \frac{n}{2^{-n}+3^{-n}}$

57. $\sum_{n=1}^{\infty} \frac{\ln n+1}{\sqrt{n}}$

58. $\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}+\cos n}$

59. $\sum_{n=1}^{\infty} \frac{1-\sin n}{n^{3/2}}$

60. $\sum_{n=1}^{\infty} \frac{n-3}{n^2\sqrt{n}}$

61. $\sum_{n=1}^{\infty} \frac{1}{\tan^{-1} n}$

62. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln(\ln n))}$

63. $\sum_{n=3}^{\infty} \frac{1}{\ln n - 1}$

64. $\sum_{n=3}^{\infty} \frac{\sqrt{n-2}}{n\sqrt{n^2-1}}$

65. $\sum_{n=1}^{\infty} \frac{n+3}{n3^n}$

66. $\sum_{n=1}^{\infty} \frac{2^{n-1}+2}{2^n}$

67. $\sum_{n=1}^{\infty} \frac{3^n+4^n}{4^n+5^n}$

68. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

69. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

70. $\sum_{n=0}^{\infty} \left(\frac{n}{n+2}\right)^n$

71. Prove part b. of the Limit Comparison Test. (**Hint:** Choose $K > 0$ and a natural number $N_0 \ge N$ for which $0 \le a_n/b_n \le K$ for all $n \ge N_0$ and apply the Direct Comparison Test.)

72. Prove part c. of the Limit Comparison Test. (**Hint:** One possibility is to notice that $\lim_{n\to\infty}(b_n/a_n) = 0$ and make use of the argument you gave in Exercise 71.)

73. Suppose that $\sum a_n$ is a positive series, $\{a_n\}$ is monotonically decreasing, and $\{na_n\}$ is convergent with $\lim_{n\to\infty} na_n \ne 0$. Prove that $\sum a_n$ diverges. (**Hint:** Use the Limit Comparison Test to compare $\sum a_n$ with an appropriate series. For the definition of a positive series, see Exercise 70 in Section 10.2.)

74–77 Use Exercise 73 to provide a quick proof of the divergence of the given series.

74. $\sum_{n=1}^{\infty} \frac{n^2}{3n^3+1}$

75. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$

76. $\sum_{n=2}^{\infty} \frac{2}{\ln n}$

77. $\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n}}$

78.* (The Cauchy Condensation Test) Suppose that $\sum_{n=1}^{\infty} a_n$ is a positive series and $\{a_n\}$ is monotonically decreasing. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. (**Hint:** Fix n and k, and let s_n be the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$, while letting c_k denote the k^{th} partial sum of the "condensed" series $\sum_{n=0}^{\infty} 2^n a_{2^n}$. Group the terms of $\sum_{n=1}^{\infty} a_n$ as

$$\sum_{n=1}^{\infty} a_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots,$$

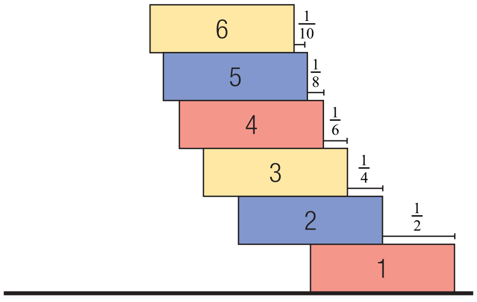
while $\sum_{n=0}^{\infty} 2^n a_{2^n}$ can be written as

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots.$$

Using the monotonicity of $\{a_n\}$, notice that if $n < 2^k$, then $s_n \le c_k$, while if $n > 2^k$, then $s_n \ge \frac{1}{2}c_k$. Use these observations to show that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded if and only if that of $\sum_{n=0}^{\infty} 2^n a_{2^n}$ is bounded. Now use Exercise 70 from Section 10.2 to finish your argument.)

79. Use Exercise 78 to give a new proof of the p-series test (see Example 2 of Section 10.3).

80.* Suppose we stack idealized building blocks (perfectly rectangular, perfectly homogeneous, perfectly smooth, perfectly level, etc.) with the indicated fraction of each block protruding to the right of the block above it, as shown in the figure below. If we continue building in this way, that is, by making sure exactly $1/(2n)$ of the n^{th} block protrudes to the right, how high can we build the structure before it topples? Assuming vertical sunrays (and an infinitely high sun), how far to the left will the shadow of the structure extend? (**Hint:** Starting from the bottom, consider the substructure consisting of the first n blocks and the location of its centroid relative to the right edge of the $(n+1)^{\text{th}}$ block.)



81.* Prove that a positive series $\sum a_n$ converges if and only if the series $\sum \frac{a_n}{a_n+1}$ converges. (**Hint:** Show that $a_n < 2a_n/(a_n+1)$ if n is large enough.)

Concept Check

82–91 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

82. If $\sum a_n$ and $\sum b_n$ are positive series, $\sum b_n$ diverges, and $a_n < b_n$ for all n, then $\sum a_n$ converges.

83. If $\sum a_n$ and $\sum b_n$ are convergent positive series, then $\sum a_n b_n$ is convergent.

84. If $\lim_{n\to\infty}(a_n/b_n) = 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

85. If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum b_n$ converges, then $\sum a_n$ converges.

86.* If $\sum a_n$ is a convergent positive series, then $\sum \ln(1+a_n)$ is convergent.

87. If $\sum a_n$ is a convergent positive series and $\sum b_n$ and $\sum c_n$ are positive series such that $b_n c_n = a_n$ for all n, then both $\sum b_n$ and $\sum c_n$ are convergent.

88. If $\sum a_n$ is a convergent positive series and $\sum b_n$ and $\sum c_n$ are positive series such that $b_n + c_n = a_n$ for all n, then both $\sum b_n$ and $\sum c_n$ are convergent.

89. If $p > 1$, then $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ diverges.

90.* The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

91.* The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges.

So as $n \to \infty$, a_{n+1}/a_n oscillates between values approaching 0 (for odd n) and values growing unboundedly large (for even n)—in other words, $\lim_{n\to\infty}(a_{n+1}/a_n)$ fails to exist in a fairly extreme fashion. But

$$\sqrt[n]{a_n} = \begin{cases} n^{1/n}/2 & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases}$$

and $n^{1/n} \to 1$ as $n \to \infty$, so

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \tfrac{1}{2} < 1$$

and the series converges by the Root Test.

10.5 Exercises

1–34 Use the Ratio Test to determine whether the series converges or diverges.

1. $\sum_{n=0}^{\infty} \frac{1}{n!}$
2. $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$
3. $\sum_{n=0}^{\infty} \frac{5^n}{n!}$
4. $\sum_{n=1}^{\infty} \frac{\pi^n}{n^\pi}$
5. $\sum_{n=0}^{\infty} \frac{3}{(2n)!}$
6. $\sum_{n=1}^{\infty} \frac{2^n}{n}$
7. $\sum_{n=1}^{\infty} \frac{n^5}{e^n}$
8. $\sum_{n=0}^{\infty} \frac{3}{(2n+1)!}$
9. $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$
10. $\sum_{n=2}^{\infty} \frac{2n}{(n-1)^2}$
11. $\sum_{n=0}^{\infty} \frac{3n+2}{5^{3n+2}}$
12. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$
13. $\sum_{n=1}^{\infty} \frac{(3n)!}{4^{n-1}5^n}$
14. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$
15. $\sum_{n=1}^{\infty} \frac{n}{5^n}$
16. $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{(n+1)!}$
17. $\sum_{n=1}^{\infty} \frac{4^n}{(n+4)n}$
18. $\sum_{n=1}^{\infty} \frac{n}{\pi^{2n+1}}$
19. $\sum_{n=1}^{\infty} \frac{n^2-4}{2^n}$
20. $\sum_{n=0}^{\infty} \frac{(n!)4^n}{(4n+1)!}$
21. $\sum_{n=2}^{\infty} \frac{2^n}{n(n^2-1)}$
22. $\sum_{n=1}^{\infty} \frac{n^2}{(\ln 2)^n}$
23. $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 3)^n}$
24. $\sum_{n=0}^{\infty} \frac{(3n-2)(3n+2)}{3^n}$
25. $\sum_{n=0}^{\infty} \frac{(n+3)3^n}{n!}$
26. $\sum_{n=1}^{\infty} \frac{n^2}{(\ln \pi)^n}$
27. $\sum_{n=1}^{\infty} \frac{2^{3n-2}}{2n^2+3n}$
28. $\sum_{n=1}^{\infty} \frac{(3n)!}{n^3}$
29. $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$
30. $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$
31. $\sum_{n=0}^{\infty} \frac{3^n}{2^n+1}$
32. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$
33. $\frac{2}{1\cdot 2\cdot 3} + \frac{4}{2\cdot 3\cdot 4} + \cdots + \frac{2^n}{n(n+1)(n+2)} + \cdots$
34. $\frac{1}{3} + \frac{1\cdot 5}{2\cdot 3\cdot 9} + \cdots + \frac{1\cdot 5\cdot\cdots\cdot(4n-3)}{3^n(n!)(2n-1)} + \cdots$

35–38 Verify that the Ratio Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

35. $\sum_{n=0}^{\infty} \frac{n^2+1}{(n+1)^2}$
36. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$
37. $\sum_{n=1}^{\infty} \frac{\arctan n}{n}$
38. $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n}(n+2)^2}$

39–62 Use the Root Test to determine whether the series converges or diverges.

39. $\sum_{n=1}^{\infty} \left(\frac{5}{n}\right)^n$
40. $\sum_{n=0}^{\infty} \frac{1}{(n+1)^n}$
41. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$
42. $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n^n}$
43. $\sum_{n=1}^{\infty} \left(\frac{3n}{n^2+3}\right)^n$
44. $\sum_{n=2}^{\infty} \frac{n^{2n}}{(\ln n)^n}$
45. $\sum_{n=1}^{\infty} \frac{n}{2^{2n}}$
46. $\sum_{n=2}^{\infty} \frac{e^{2n}}{(\ln n)^n}$

47. $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$ **48.** $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

49. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ **50.** $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

51. $\sum_{n=2}^{\infty} \frac{1}{(\ln(\ln n))^n}$ **52.** $\sum_{n=2}^{\infty} \left(\ln\frac{1}{n}\right)^{2n}$

53. $\sum_{n=1}^{\infty} n\left(\frac{4}{5}\right)^n$ **54.** $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

55. $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$ **56.** $\sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)^n$

57. $\sum_{n=2}^{\infty} \frac{n^{n/2}}{(\ln n)^n}$ **58.** $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$

59. $\sum_{n=1}^{\infty} \frac{(n+2)^n}{n^{2n}}$ **60.** $\sum_{n=1}^{\infty} \frac{2}{n^{n+2}}$

61. $\sum_{n=1}^{\infty} \sin^n \frac{1}{n^2}$ **62.** $\sum_{n=1}^{\infty} \frac{1}{\left(2+\frac{1}{n}\right)^{2n}}$

63–66 Verify that the Root Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

63. $\sum_{n=1}^{\infty} \left(\frac{n}{n+5}\right)^n$ **64.** $\sum_{n=1}^{\infty} \frac{(2\sqrt{n})^{2n}}{(4n+1)^n}$

65. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ **66.** $\sum_{n=1}^{\infty} \frac{2n}{(n+1)^3}$

67–70 Suppose that the series $\sum a_n$ satisfies the condition $\lim_{n\to\infty}(a_{n+1}/a_n) = \frac{1}{2}$. Decide whether the given series is convergent.

67. $\sum_{n=1}^{\infty} 3^n a_n$ **68.** $\sum_{n=1}^{\infty} n^2 a_n$

69. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n a_n$ **70.** $\sum_{n=1}^{\infty} a_n^2$

71. Prove that for all exponents p, the series $\sum(n^p/2^n)$ is convergent.

72. Prove that $\sum(p^n/n!)$ converges for all $p > 0$.

73. For what positive p-values does the series $\sum(p^n/n)$ converge?

Concept Check

74–79 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample. ($\sum a_n$ is a positive series in each problem.)

74. If $\sum a_n$ is convergent, then $\lim_{n\to\infty}(a_{n+1}/a_n) < 1$.

75. If $\sum a_n$ is divergent, then $\lim_{n\to\infty}\sqrt[n]{a_n} \geq 1$ or the limit doesn't exist.

76. If there is an $N > 0$ such that $\sqrt[n]{a_n} < 1$ for all $n \geq N$, then $\sum a_n$ is convergent.

77. The series $\sum_{n=1}^{\infty}(3^{n^2}/n!)$ is convergent.

78.* If $\sum a_n$ satisfies condition a. of the Ratio Test, then it satisfies condition a. of the Root Test.

79.* If $\sum a_n$ satisfies the condition a. of the Root Test, then it satisfies condition a. of the Ratio Test.

10.5 Technology Exercises

80. The following series, discovered by the great Indian mathematician Srinivasa Ramanujan, converges to $1/\pi$ with amazing speed.

$$\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(26390n+1103)}{(n!)^4\,396^{4n}} = \frac{1}{\pi}$$

a. Prove that the series is convergent.

b. Use Ramanujan's series with a graphing utility to approximate π, correct to 31 digits after the decimal. How many terms did you have to use to achieve this accuracy?

10.6 Exercises

1–24 Determine whether the alternating series converges and give a reason for your answer.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\sqrt{n}}$

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{3n-2}$

4. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2n-1}$

5. $\sum_{n=2}^{\infty}(-1)^{n}\frac{3}{n\ln n}$

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}-1}{4^n}$

7. $\sum_{n=0}^{\infty}(-1)^{n}\frac{n^2+1}{n^2+2}$

8. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2 2^n}$

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{\ln 2n}$

10. $\sum_{n=1}^{\infty}(-1)^{n}\frac{3^n}{n^3}$

11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n}$

12. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sqrt{n}}{n+2}$

13. $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{n+1}{5\sqrt{n}}\right)^n$

14. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{1}{n}$

15. $\sum_{n=0}^{\infty}(-1)^{n}\frac{\sqrt{n}+3}{\sqrt{n+3}}$

16. $\sum_{n=0}^{\infty}(-1)^{n}\frac{2^n}{n!}$

17. $\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{\sqrt[3]{n^2+1}}$

18. $\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{4}{5}\right)^n$

19. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\tan^{-1}n}$

20. $\sum_{n=1}^{\infty}(-1)^{n}\tan^{-1}\left(\frac{1}{n}\right)$

21. $\sum_{n=1}^{\infty}(-1)^{n}e^{-n}\sin n$

22. $\sum_{n=1}^{\infty}\left(\frac{-1}{1.1}\right)^n$

23. $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{1+\frac{1}{n}}$

24. $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^3}$

25–30 Approximate the sum of the alternating series, accurate to at least the indicated number of decimal places.

25. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$; accurate to 2 decimal places

26. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n3^n}$; accurate to 4 decimal places

27. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}$; accurate to 5 decimal places

28. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n n!}$; accurate to 6 decimal places

29. $\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{e^n}$; accurate to 3 decimal places

30. $\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{2n^4}$; accurate to 4 decimal places

31–58 Determine whether the given series converges absolutely, converges conditionally, or diverges. Use any convergence test discussed so far in this chapter.

31. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$

32. $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$

33. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^{1/n}}$

34. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{(2n)!}$

35. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sqrt[3]{n}}{\sqrt{n}}$

36. $\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{1}{n}$

37. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+4}$

38. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{5^n}{n!}$

39. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+4}$

40. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\sqrt{\ln n}}$

41. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{n^{1.1}}$

42. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{5n-1}$

43. $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$

44. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\left(\sqrt{n}+2\right)}$

45. $\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

46. $\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\frac{1}{n}$

47. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^4}{4^n}$

48. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\tan^{-1}n}{n^2}$

49. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

50. $\sum_{n=1}^{\infty}\left(\ln\frac{2}{n}\right)^n$

51. $\sum_{n=2}^{\infty}\left(-\frac{\ln n}{n}\right)^n$

52. $\sum_{n=1}^{\infty}\left(\frac{1-2n}{5n}\right)^n$

53. $\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{2^n}{n^2}\right)^n$

54. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^3}{e^n}$

55. $\sum_{n=2}^{\infty} \frac{(-\pi)^n}{\left(3+\frac{1}{n}\right)^n}$

56. $\sum_{n=1}^{\infty} (-1)^n \log\left(5+\frac{1}{n^2}\right)$

57. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{2}\right)^n$

58. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{2^n n!}$

59. Kate claims that the series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{9}+\cdots+\frac{1}{n}-\frac{1}{n^2}+\cdots$$

converges by Leibniz's Test, because it is an alternating series of the form $\sum (-1)^{n+1} a_n$ and $a_n \to 0$. Is she right? Why or why not?

60. Prove that the series consisting of the positive terms of the alternating harmonic series diverges to infinity.

61. Repeat Exercise 60 for the negative terms of the alternating harmonic series.

62.* Prove the first part of Riemann's Rearrangement Theorem; that is, the statement that the terms of a conditionally convergent alternating series can be rearranged so that the sum of the series is any preselected real number. (**Hint:** Let s be the desired, preselected sum, and start adding up the positive terms of the series until their sum first becomes greater than s. Then add enough negative terms until the resulting sum becomes less than n, and continue.)

63.* Prove the second and third parts of Riemann's Theorem; namely, that the terms of the series in Exercise 62 can be rearranged to diverge to ∞ or –∞, or to diverge in an oscillating manner.

64.* Prove the following so-called Polynomial Test for infinite series: If $p(x)$ and $q(x)$ are polynomials with degrees r and s, respectively, then the series $\sum \frac{p(n)}{q(n)}$ is convergent if and only if $s > r + 1$ (we are assuming $q(n) \neq 0$ for any value of the summation index n).

65. Prove that if the series $\sum a_n$ is absolutely convergent, then $\sum a_n^2$ is convergent. Then give an example to show that the statement is not true if $\sum a_n$ is conditionally convergent.

10.6 Technology Exercises

66–68 Use a graphing utility to solve the problem.

66. Find an approximation of e with an error no greater than 10^{-8}, knowing that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$. How many terms did you use?

67. Find an approximation of π with an error no greater than 10^{-3}, knowing that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. How many terms did you use?

68. Find an approximation of $\ln 2$ with an error no greater than 10^{-3}, knowing that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. How many terms did you use?

10.7 Exercises

1–32 Determine the radius and interval of convergence for the power series. Be sure to check for convergence at the endpoints.

1. $\sum_{n=0}^{\infty}(3x)^n$

2. $\sum_{n=0}^{\infty}3x^n$

3. $\sum_{n=1}^{\infty}\frac{(4x)^n}{n^4}$

4. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$

5. $\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$

6. $\sum_{n=0}^{\infty}\frac{(5x)^n}{2n!}$

7. $\sum_{n=0}^{\infty}\frac{(2n)!(5x)^n}{2n!}$

8. $\sum_{n=0}^{\infty}\frac{2n!(5x)^n}{(2n)!}$

9. $\sum_{n=1}^{\infty}\frac{(2x)^n}{(n-1)!}$

10. $\sum_{n=1}^{\infty}nx^n$

11. $\sum_{n=1}^{\infty}\frac{n^2(x+3)^n}{2^n}$

12. $\sum_{n=0}^{\infty}\frac{3x^n}{k^n}, \quad k\neq 0$

13. $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n n!$

14. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n3^n}$

15. $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n^3}$

16. $\sum_{n=0}^{\infty}\frac{(x+1)^n}{(n+2)3^n}$

17. $\sum_{n=0}^{\infty}\frac{(x-5)^n n!}{5^n}$

18. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(-1)^n n}$

19. $\sum_{n=0}^{\infty}\frac{(x+2)^n}{(n+2)2^n}$

20. $\sum_{n=0}^{\infty}\frac{(x-k)^n}{k^n}, \quad k>0$

21. $\sum_{n=4}^{\infty}\frac{(x-1)^n}{(n-2)(n-3)}$

22. $\sum_{n=1}^{\infty}\frac{(x+2)^n}{\sqrt{n}}$

23. $\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{(\ln n)^n}$

24. $\sum_{n=1}^{\infty}\frac{n^2(2x-3)^n}{n!}$

25. $\sum_{n=1}^{\infty}n3^n(x-1)^n$

26. $\sum_{n=2}^{\infty}\frac{x^n\ln n}{n!}$

27. $\sum_{n=2}^{\infty}\frac{x^{2n+5}}{\ln\sqrt{n}}$

28. $\sum_{n=0}^{\infty}x^{4^n}$

29. $\sum_{n=2}^{\infty}\frac{3^n(x+2)^n}{n\ln n}$

30. $\sum_{n=0}^{\infty}2^n(x+4)^{3n+1}$

31. $\sum_{n=0}^{\infty}\pi(x-3)^n$

32. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-3)^n$

33–40 Determine the interval of convergence for the given series and the limiting function of the series on that interval.

33. $\sum_{n=0}^{\infty}(3x-1)^n$

34. $\sum_{n=0}^{\infty}(2x+3)^n$

35. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^n$

36. $\sum_{n=0}^{\infty}2\left(\frac{x}{5}\right)^n$

37. $\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n$

38. $\sum_{n=0}^{\infty}\frac{(2x+3)^n}{3^n}$

39. $\sum_{n=0}^{\infty}\frac{(5x-6)^{2n}}{9^n}$

40. $\sum_{n=0}^{\infty}\frac{(2x-3)^n}{3^n}$

41–48 By "reversing" the process used to find the limiting function in Exercises 33–40, we can find the power series about 0 for certain functions. For example, if $f(x)=1/(1-3x)$, we can recognize it as the sum of the geometric series $\sum_{n=0}^{\infty}(3x)^n=\sum_{n=0}^{\infty}3^n x^n$. (This is valid only if x is in the interval of convergence, in this case, $-\frac{1}{3}<x<\frac{1}{3}$.)

Use this approach to find the power series representation of the given function. What is the radius of convergence? (**Hint:** In Exercises 47 and 48, use partial fractions.)

41. $f(x)=\frac{1}{1-2x}$

42. $f(x)=\frac{1}{1+4x}$

43. $f(x)=\frac{1}{1+x^2}$

44. $f(x)=\frac{1}{1-x^2}$

45. $f(x)=\frac{3}{3-4x}$

46. $f(x)=\frac{2}{3+8x^3}$

47. $f(x)=\frac{x+3}{1-x^2}$

48. $f(x)=\frac{2x}{1-4x^2}$

49. Find the power series representation of $f(x)=1/x$ centered at $a=1$. What is the radius of convergence? (**Hint:** Start by rewriting $1/x$ as $\frac{1}{1-(1-x)}$, and proceed along the lines of Exercises 41–48.)

50. Use series multiplication to find a second solution to Exercise 48. (**Hint:** Multiply the series expansions of $1/(1-2x)$ and $1/(1+2x)$, and then multiply the result by $2x$.)

51. Differentiating the result of Exercise 49, find the power series representation of $g(x)=1/x^2$ centered at $a=1$. What is the radius of convergence?

52. Use the result of Exercise 49 to find the power series representation about $a=1$ of $h(x)=\ln x$. What is the radius of convergence?

53. Use Exercise 52 to prove that $\ln 2=\sum_{n=1}^{\infty}\frac{1}{n2^n}$. Use this series to approximate $\ln 2$ to three decimal places. How many terms did you use? (**Hint:** Start with the series approximating $\ln\frac{1}{2}=-\ln 2$.)

54. Find the power series representation of $f(x)=1/x$ centered at $a=3$. What is the interval of convergence? (**Hint:** Start by rewriting $1/x$ as $\frac{1}{3-(3-x)}$ and proceed along the lines of Exercise 41–48.)

55. Find the second and third derivatives of the power series of Example 4. In both cases, express your answer as a power series and in closed form.

56. Find the closed form of the series $\sum_{n=1}^{\infty}\frac{(2x-3)^n}{2n}$ by first determining the closed form of the series $\sum_{n=0}^{\infty}(2x-3)^n$. What is the interval of convergence?

57. Find the closed form of the series $\sum_{n=2}^{\infty}9(n^2-n)(3x-1)^{n-2}$ by first determining the closed form of the series $\sum_{n=0}^{\infty}(3x-1)^n$. What is the interval of convergence? (**Hint:** Differentiate twice.)

58. Integrate the series you found in Exercise 43 to obtain a series expansion for $F(x)=\arctan x$ about $a=0$. Find the radius of convergence for the series you obtained.

59. Use differentiation and the result of Exercise 43 to find a series expansion of $g(x)=2x\big/\left(x^2+1\right)^2$. Find the radius of convergence for the series you obtained.

60. Even when differentiated, a convergent power series retains its radius of convergence. But as we have noted, the behavior at the interval endpoints can change—in particular, the original series may converge at an endpoint, while its derivative diverges. Verify this by examining the intervals of convergence of f and its first two derivatives for

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n^2+1}.$$

61. Show that for any natural number $k\in\mathbb{N}$, the power series expansion of $\frac{(k-1)!}{(1-x)^k}$ is

$$\sum_{n=0}^{\infty}(n+k-1)(n+k-2)\cdots(n+2)(n+1)x^n.$$

62.* In general, series of functions are not nearly as well behaved with regards to termwise differentiation and integration as power series are. As an illustration, examine the convergence set (the set of x-values for which the series converges) of $f(x)=\sum_{n=0}^{\infty}\frac{\sin(3^n x)}{2^n}$, and that of its first derivative. (**Note:** This is not a power series.)

63. Verify that $y=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ satisfies the differential equation $y'=2xy$. (**Hint:** Differentiate term by term.)

64. Find a power series solution of the differential equation, $y'=2y$ satisfying the initial condition $y(0)=1$. Then solve the equation by traditional means and conclude that the solutions are equal. (**Hint:** Starting with undetermined coefficients, write $y=\sum_{n=0}^{\infty}a_n x^n$, obtain the power series for both y' and $2y$, and finally equate terms.)

65. Find a power series solution of the differential equation, $y''+4y=0$ satisfying the initial conditions $y(0)=1$, $y'(0)=0$. Then solve the equation by traditional means and conclude that the solutions are equal. (See the hint given in Exercise 64.)

Concept Check

66–70 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

66. If the series $\sum a_n(x+1)^n$ converges at $x=-4$, then it must converge at $x=1$.

67. If the series $\sum a_n(x+1)^n$ converges at $x=-4$, then it must converge at $x=2$.

68. If the series $\sum a_n x^n$ converges at $x=1$, then $\sum na_n x^{n-1}$ must also converge at $x=1$.

69. If a series converges on an interval, we may differentiate it term by term to obtain the derivative of its sum.

70. If the interval of convergence of the series $\sum a_n x^n$ is $(-a,a)$ for an $a\in\mathbb{R}^+$, then the interval of convergence of $\sum a_n(x-a)^n$ is $(0,2a)$.

$\sin x$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$	$\lvert x\rvert<\infty$
$\cos x$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$	$\lvert x\rvert<\infty$
$\ln x$	$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots$	$0<x\le 2$

Table 1

10.8 Exercises

1. Find the Maclaurin polynomial of order 4 generated by $f(x)$ given that $f(0)=4$, $f'(0)=2$, $f''(0)=-1$, $f'''(0)=3$, and $f^{(4)}(0)=1$.

2. Find the Taylor polynomial of order 4 generated by $f(x)$ about $a=2$ given that $f(2)=1$, $f'(2)=-3$, $f''(2)=0$, $f'''(2)=5$, and $f^{(4)}(2)=-2$.

3–10 Determine the Taylor polynomials generated by the given function about the indicated point. Find the radius of convergence.

3. $f(x)=2x^4-x^3+5x^2+3x-7;\quad a=0$
4. $f(x)=e^{-x^2/2};\quad a=0$
5. $f(x)=e^x;\quad a=1$
6. $f(x)=\sin x;\quad a=\pi/2$
7. $f(x)=\frac{1}{x+4};\quad a=1$
8. $f(x)=\ln x;\quad a=1$
9. $f(x)=\frac{1}{x^2};\quad a=2$
10. $f(x)=\tanh^{-1}x;\quad a=0$

11–29 Use the Taylor series (or Maclaurin series if the center is not specified) we discussed so far in the text (see Table 1) to determine that of the given function. In each case, find the radius of convergence.

11. $f(x)=\cos(x^2)$
12. $f(x)=\cos^2 x=\frac{1+\cos 2x}{2}$
13. $f(x)=\sin x\cos x=\frac{\sin 2x}{2}$
14. $f(x)=x^3e^{-2x}$
15. $f(x)=\cosh x$
16. $f(x)=xe^x;\quad a=1$
17. $f(x)=\cos\sqrt{x}$
18. $f(x)=\frac{x+1}{x-1}$
19. $f(x)=(2x^3-x)e^x$
20. $f(x)=\sinh x$
21. $f(x)=\ln(1+x^2)$
22. $f(x)=\frac{x}{1+x^4}$
23. $f(x)=\frac{1}{x^2};\quad a=3$
24. $f(x)=\ln\frac{1+x}{1-x};\quad a=0$
25. $f(x)=\frac{2x+1}{x^2+x-6}$
26. $f(x)=\frac{1}{(x^2+1)^2}$
27. $f(x)=\frac{x^2}{1-x^2}$
28. $f(x)=\tan^{-1}x;\quad a=0$
29. $f(x)=x\cos(x^{3/2});\quad a=0$

30–35 Use the definition to find the first five nonzero terms of the Taylor series generated by the given function about the indicated point.

30. $f(x)=\tan x;\quad a=\pi/4$
31. $f(x)=\arctan x;\quad a=1$
32. $f(x)=\sec x;\quad a=0$

33. $f(x)=e^{\cos x}$; $a=0$

34. $f(x)=e^{\sin x}$; $a=0$

35. $f(x)=2^x$; $a=1$

36–47 Find the first five nonzero terms of the Maclaurin series generated by the indicated function by using operations on familiar series (try not to use the definition). In Exercises 40–42, try long division.

36. $f(x)=\sin x+\cos x$

37. $f(x)=\dfrac{e^x}{1-x}$

38. $f(x)=e^x\cos x$

39. $f(x)=e^{-2x}\sin x$

40. $f(x)=\dfrac{\cos x}{x+1}$

41. $f(x)=\tanh x$

42. $f(x)=\tan x$

43. $f(x)=\ln(1+x)\sin x$

44. $f(x)=x\sec 2x$

45. $f(x)=\ln(1-x)\cos x$

46. $f(x)=\cos(x^2+2x)$

47. $f(x)=(x-1)^2\cos \pi x$

48–57 Find the Taylor series approximation of the given function value or definite integral that guarantees the indicated accuracy. How many (nonzero) terms did you use? (**Hint:** See Example 8 and note the nonelementary integrals that we cannot evaluate exactly.)

48. $\ln 1.2$; error $\le 10^{-4}$ (**Hint:** Use the Taylor series expansion of $\ln(x+1)$.)

49. $\dfrac{1}{e}$; error $\le 10^{-4}$

50. $\sin 1$; error $\le 10^{-5}$

51. $\int_0^1 e^{-x^4}\,dx$; error ≤ 0.001

52. $\int_0^1 \cos(x^3)\,dx$; error $\le 10^{-4}$

53. $\int_0^1 \sin(x^2)\,dx$; error $\le 10^{-6}$

54. $\int_0^1 \dfrac{\sin x}{x}\,dx$; error $\le 10^{-6}$

55. $\int_0^1 \tan^{-1}(x^2)\,dx$; error ≤ 0.01

56. $\int_0^1 x^2\sin(x^2)\,dx$; error $\le 10^{-6}$

57. $\int_0^1 \dfrac{1}{\sqrt{x^4+1}}\,dx$; error ≤ 0.01

58–60 Prove that the Maclaurin series converges to the function by showing that the error $r_n(x)$ satisfies $\lim_{n\to\infty} r_n(x)=0$.

58. $e^{-x^2}=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{n!}$

59. $\cos x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{(2n)!}$

60. $\ln(x+1)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}x^n}{n}$, $|x|<1$

61–64 Use Taylor series to find the indicated limit.

61. $\lim_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}$

62. $\lim_{x\to 0}\dfrac{2x^3-1+\cos x}{x^5}$

63. $\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin(x^2)}{x}$

64. $\lim_{x\to 0}\dfrac{\ln(x+1)}{x\sin x}$

65. Use power series to show that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

66. Find the Taylor series expansion of the function $f(x)=3x^3-5x^2+2x-7$ about $a=2$. What can you tell from your answer?

67. Use the Maclaurin series expansion of $f(x)=1/(1-x)$ and differentiation to find $\sum_{n=1}^{\infty}(n/2^{n-1})$.

68.* Calculate the first few terms of the Maclaurin series expansion of $f(x)=\arcsin x$; then notice the apparent pattern and derive the general form of the series. What is the radius of convergence?

69.* Suppose that the function f has derivatives of all orders throughout an open interval I and there is an $M>0$ such that $\left|f^{(n)}(x)\right|\le M$ for all $n\in\mathbb{N}$ over I. Prove that if $a\in I$, then the Taylor series generated by $f(x)$ about a converges to f at any $x\in I$.

70. Use Exercise 68 to prove that if $f(x)=\sinh x$ or $f(x)=\cosh x$, then the Taylor series expansion of f about any $a\in I$ converges to f at every real number x.

71.* Find the value of the 162nd derivative of $f(x)=\tan^{-1}x$ at $x=0$. (**Hint:** Examine what happens to the Maclaurin series after repeated differentiation and substitution of $x=0$.) What is the 163rd derivative at 0?

72. Use the Maclaurin series expansion of $f(x)=(x+x^2)e^x$ to prove $\sum_{n=1}^{\infty}(n^2/n!)=2e$.

73.* Use an appropriate Maclaurin series expansion to prove $\sum_{n=1}^{\infty}(n^3/n!)=5e$. (**Hint:** See Exercise 72.)

10.9 Exercises

1–5 Use Euler's formula to express the complex number in the form $x + iy$.

1. $e^{i\pi/2}$

2. $e^{-3i\pi/4}$

3. $5e^{2+i\pi/3}$

4. e^{e^i}

5. $\dfrac{e^{2i} - e^{-2i}}{3i}$

6–10 Use Euler's formula to express the complex number in the form $e^x e^{iy} = re^{iy}$.

6. $\dfrac{1-i}{\sqrt{2}}$

7. $-2\left(1+\sqrt{3}i\right)$

8. $\dfrac{3+3i}{-\sqrt{3}-i}$

9. $(1+i)^4$

10. $\dfrac{3i}{2e^{4+i}}$

11. Use Euler's formula to verify the following formulas.

a. $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ **b.** $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

12. Use the Maclaurin series to verify the following formulas.

a. $\sinh i\theta = i\sin\theta$ **b.** $\cosh i\theta = \cos\theta$

13–21 Find the first six nonzero terms of the binomial series expansion of the indicated function.

13. $f(x) = \left(1+x^2\right)^5$

14. $f(x) = \dfrac{1}{\sqrt{1+x}}$

15. $f(x) = \sqrt{1-x}$

16. $f(x) = (1-x)^{3/2}$

17. $f(x) = \sqrt[3]{1+x^2}$

18. $f(x) = \left(1+\dfrac{x}{2}\right)^{-3}$

19. $f(x) = \left(1-2x^2\right)^{-2/3}$

20. $f(x) = \left(1-\dfrac{x}{3}\right)^{-3/2}$

21. $f(x) = \dfrac{5}{\sqrt[4]{1+2x}}$

22. Show that if $m \in \mathbb{N}$, the Maclaurin series of $(1+x)^m$ is finite, and use this observation to provide a "calculus-based" proof for the Binomial Theorem, which says $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k}B^k$, for any positive integer n and any two expressions A and B.

23. Use the binomial series of $f(x) = 1/\sqrt{1-x^2}$ to find the series expansion of $g(x) = \sin^{-1} x$ about 0. What is the radius of convergence? (**Hint:** Use the equality $\binom{-1/2}{n} = (-1)^n \binom{2n}{n} \Big/ 4^n$. Compare your result to that of Exercise 68 of Section 10.8.)

24–25 Use the method of Exercise 23 to find the Maclaurin series expansion of the indicated function along with the radius of convergence.

24. $h(x) = \cos^{-1} x$

25. $h(x) = \sinh^{-1} x$

26. By following the outline below, prove that the binomial series

$$\sum_{n=0}^{\infty} \binom{m}{n} x^n = \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n$$

converges to the function $(1+x)^m$ on $(-1,1)$.

a. Using the Ratio Test, it is straightforward to check that the above series converges for $|x| < 1$. Let us denote its sum by $f(x)$ and differentiate the series term by term to find the series expansion of $f'(x)$.

b. Multiply the series expansion of $f'(x)$ by $(1+x)$ and verify that the result is the series for $m \cdot f(x)$. (**Note:** Since $|x| < 1$, the series is absolutely convergent, so termwise differentiation and rearranging terms is possible without changing the sum.)

c. Conclude that $f(x)$ is a solution of the separable differential equation

$$\frac{y'}{y} = \frac{m}{1+x}.$$

d. Solve the above equation by conventional means to conclude that $f(x) = C(1+x)^m$ for some constant C.

e. Finally, use the initial condition $f(0) = 1$ to conclude that $C = 1$.

27–28 The period of a swinging pendulum of length L released from rest can be well approximated by simple harmonic motion and its period is approximately $T \approx 2\pi\sqrt{L/g}$, where g is the acceleration caused by gravity. However, this model is inaccurate for larger starting angles. In physics, it is shown that if the pendulum is released at angle θ_0 from vertical, the actual formula is

$$T = 4\sqrt{\frac{L}{g}}K(k),$$

where $k = \sin(\theta_0/2)$ and

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is a *complete elliptic integral of the first kind*.

In Exercises 27 and 28, use a Taylor series approximation to derive an estimate for the period of the swinging pendulum (keep in mind that we don't have a closed formula for this integral).

27.* Prove that if $|k| < 1$, the Maclaurin series expansion for $K(k)$ is

$$K(k) = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\frac{(1)(3)\cdots(2n-1)}{(2)(4)\cdots(2n)}\right]^2 k^{2n}.$$

(**Hint:** Substitute $k\sin\theta$ in the Maclaurin expansion of $1/\sqrt{1-x^2}$.)

28. Use the series from Exercise 27 to establish the following approximation for the period of the swinging pendulum, which is much more accurate for larger angles than the simple model we have seen previously.

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{\theta_0^2}{16}\right)$$

(**Hint:** In addition to the Maclaurin series, also use the approximation $\sin x \approx x$.)

29–30 Perhaps surprisingly, an explicit formula for the circumference of an ellipse is not known, for the problem leads to an integral that is (perhaps unsurprisingly) called the *complete elliptic integral of the second kind*. (By now you are probably expecting the fact that it cannot be evaluated in closed form, which is indeed the case.)

Use the Taylor series of this integral to approximate the circumference of an ellipse.

29.* The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\, d\theta.$$

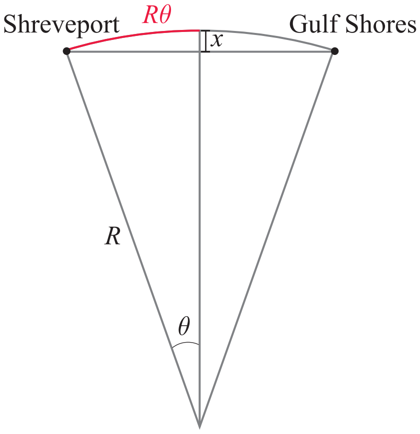
Prove that if $|k| < 1$, the Maclaurin series expansion for $E(k)$ is

$$E(k) = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\frac{(1)(3)\cdots(2n-1)}{(2)(4)\cdots(2n)}\right]^2 \frac{k^{2n}}{2n-1}.$$

(See the hint given in Exercise 27.)

30. Given that the circumference of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a<b$, is $C = 4bE(k)$ where $k=\sqrt{1-(a^2/b^2)}$, use the first four terms of the series expansion from Exercise 29 to approximate the circumference of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

31. The arc of a great circle of Earth connecting Shreveport, Louisiana, with Gulf Shores, Alabama, is approximately 430 miles long. Use Taylor series to estimate how much the arc will recede from its chord between these two cities. Use the first three nonzero terms of the series of $\cos\theta$. Approximate the radius of Earth with $R \approx 4000$ miles.



32. Use the Maclaurin series expansion of e^x to prove that e is irrational. (**Hint:** Expressing e as a power series, you can argue by contradiction as follows. Assuming p and q are positive integers with $e = p/q$ and multiplying the power series by $q!$, we obtain

$$p(q-1)! - 2q! - (3\cdot 4\cdot\cdots\cdot q) - (4\cdot 5\cdot\cdots\cdot q) - \cdots - q - 1$$
$$= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots$$

where the assumptions imply that the left-hand side must be an integer. Finally, argue that the expression on the right-hand side must be between 0 and 1, an apparent contradiction, finishing your proof.)

33. Recall from Section 8.2 that the differential equation obeyed by the current in an RL circuit is

$$L\frac{dI}{dt}+RI=V.$$

Solve this linear equation for $I(t)$ and use the series expansion of your solution to show that when t is small,

$$I(t)\approx\frac{V}{L}t.$$

34. The weight of an object at a height h above the surface of Earth is

$$W(h)=\frac{R^2W_0}{(R+h)^2},$$

where W_0 is the object's weight on Earth's surface. Use the first four nonzero terms of the Maclaurin expansion of $W(h)$ to approximate the height required for an object to lose 5% of its weight. (Use $R\approx 4000$ miles.)

35. According to Einstein's theory of special relativity, if a particle is moving with velocity v, the mass of the particle is given by

$$m(v)=\frac{m_r}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_r is the rest mass of the particle and c is the speed of light in a vacuum. Use the first three nonzero terms of the Maclaurin polynomial of $m(v)$ to estimate the speed necessary to increase the mass of the particle by 1%. (Use $c\approx 3\times 10^8$ m/s.)

36.* The fastest average qualifying speed at the Indy 500 race, 236.986 mph (approximately 106 m/s), was reached by Arie Luyendyk in 1996. Using the error term for the series of E_k found in Example 3, estimate the magnitude of the maximum error one makes when calculating the race car's kinetic energy at this speed using the classic Newtonian formula of $E_k\approx mv^2/2$. (**Hint:** The series of E_k can be rewritten as

$$E_k=mc^2\left[\frac{1}{2}\left(\frac{v^2}{c^2}\right)+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\frac{5}{16}\left(\frac{v^2}{c^2}\right)^3+\cdots\right].$$

Substituting $x=-v^2/c^2$, notice that the series in the brackets on the right-hand side is "almost" the binomial expansion of $1/\sqrt{1+x}$. Next, use Taylor's Theorem with $r_1(x)$, the remainder of order 1, note that $|v|\le 106$, and give an upper bound for $|r_1(v/c)|$.)

37. The phase speed at which a surface wave propagates on water of depth d is well approximated by the expression

$$s=\sqrt{\frac{\lambda g}{2\pi}\tanh\frac{2\pi d}{\lambda}},$$

where λ is the wavelength.

a. Explain why the following "rule of thumb" is valid: If the water is deeper than three times the wavelength, then $s\approx\sqrt{\lambda g/(2\pi)}$. (Note that this formula shows that the speed of propagation depends only on wavelength in deep water; in case of large wavelengths such as tidal waves, this speed can be enormous. In March 2011, just before the catastrophic Japan tsunami, s was about the same speed as that of a passenger jet!)

b. Use Taylor series to show that in shallow water, $s\approx\sqrt{gd}$. (In contrast to the previous case, the speed of propagation in shallow water depends only on water depth, rather than wavelength.)

38–41 Find a power series solution of the equation. In each case, use the initial conditions $y(0)=a$ and $y'(0)=b$.

38. $y''-2xy=0$

39. $y''+9y=0$

40. $y''=\dfrac{y'}{1-x}$

41. $(1-x^2)y''=4xy'+2y$

42. Explain why in our discussion preceding Example 5, we did not attempt to find a Fourier coefficient b_k for the index $k=0$.

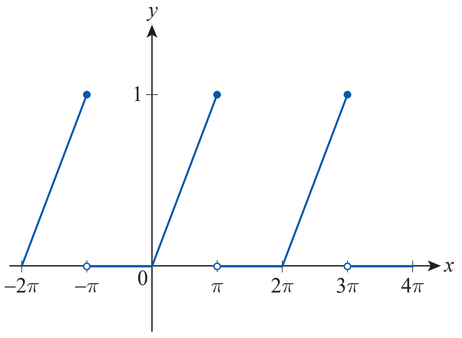
43. Verify the Fourier coefficients of Example 5.

a. $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}x\,dx=0$

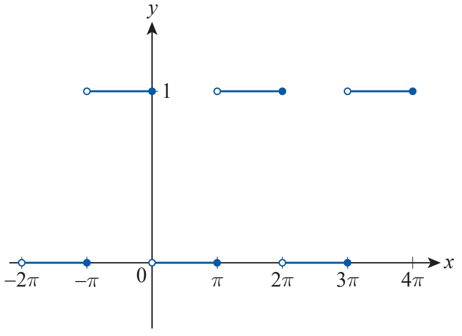
b. $a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}x\cos kx\,dx=0\quad(k\ge 1)$

c.
$$\begin{aligned}b_k&=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kx\,dx\\&=-\frac{2}{k}\cos k\pi+\frac{2}{k^2\pi}\sin k\pi\\&=(-1)^{k+1}\left(\frac{2}{k}\right)\quad(k\ge 1)\end{aligned}$$

44. Consider the function given by its graph below. After extending it to $\mathbb{R}$ in a 2π-periodic manner, find its Fourier coefficients.



45. Repeat Exercise 44 for the function graphed below.



46. Let $\hat{f}(x) = 3x^2$ be defined on the interval $[-\pi, \pi]$, and let $f(x)$ be the function obtained by extending $\hat{f}(x)$ to the entire real line in a 2π-periodic fashion. Find the Fourier series expansion of $f(x)$.

10.9 Technology Exercises

47–50 Use a graphing utility to graph the function along with its Maclaurin polynomial of order 5 on the same screen (use the polynomial of order 9 in Exercise 50). What is the largest interval on which the approximation is acceptable? (Answers will vary.)

47. $f(x) = \sqrt[5]{(2x-1)^7}$

48. $f(x) = \dfrac{x^2 - 1}{\sqrt[3]{(x+5)^2}}$

49. $f(x) = \dfrac{x+1}{\sqrt{1+x^2}}$

50. $f(x) = (x^3 - x)\sqrt[4]{x^4 + 1}$

Example 4 Finding the Distance between Points in Cartesian Three-Dimensional Space

Find the distance between the points $(-2,5,1)$ and $(3,0,-1)$.

Solution

Applying the previous formula, the distance is as follows.

$$\sqrt{(-2-3)^2+(5-0)^2+[1-(-1)]^2}=\sqrt{54}=3\sqrt{6}$$

Example 5 Using the Distance Formula to Construct an Equation of a Sphere

Construct an equation whose graph is the sphere of radius 2 centered at the point $(2,3,-1)$.

Solution

We seek an equation in x, y, and z that describes all those points (x,y,z) that are 2 units from $(2,3,-1)$. So using the distance formula, we need

$$\sqrt{(x-2)^2+(y-3)^2+[z-(-1)]^2}=2,$$

or, squaring both sides,

$$(x-2)^2+(y-3)^2+(z+1)^2=4.$$

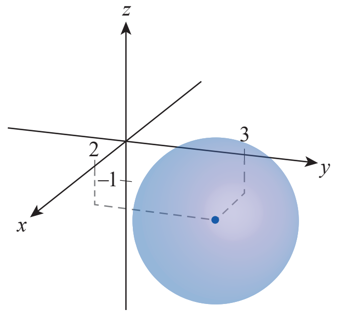


Figure 11
Sphere of Radius 2
Centered at $(2,3,-1)$

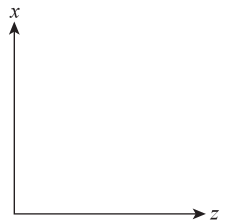
The graph of such a sphere is shown in Figure 11.

More generally, the graph of the equation $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ is a sphere of radius r centered at the point (a,b,c).

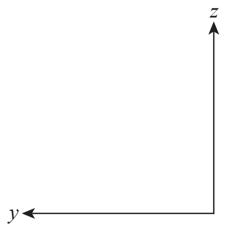
11.1 Exercises

1–2 Describe the orientation of the third axis if the resulting system is to be a right-handed system. (**Hint:** Feel free to use the words "out of the page" or "into the page.")

1.



2.



3–8 Plot the given point in the three-dimensional coordinate system.

3. $(1,0,0)$

4. $(-2,0,1)$

5. $\left(0,1,\frac{3}{2}\right)$

6. $(3,4,-1)$

7. $\left(1,-2,\frac{5}{2}\right)$

8. $\left(-2,-1,\frac{7}{4}\right)$

9–16 Explain what you know about the coordinates of point P if its location in the three-dimensional Cartesian system is as described below.

9. P is in the xy-plane

10. P is in the first octant

11. P is on the y-axis

12. P is in the yz-plane

13. P is on the negative x-axis

14. P is in the third quadrant of the xz-plane

15. P is in the plane $y=-1$

16. P is in the second quadrant of the plane $x=3$

17–20 Find the distance from the point to the indicated plane.

17. $(1,2,3)$; the xy-plane

18. $(2,-2,-5)$; the xz-plane

19. $(0,-7,-2)$; the yz-plane

20. $(1,-3,4)$; the plane $z=-2$

21–25 Find the coordinates of the projection of the given point.

21. The projection of the point $(5,3.2,-2)$ onto the xy-plane

22. The projection of the point $(-5,1,4)$ onto the xz-plane

23. The projection of the point $(3,0,0)$ onto the yz-plane

24. The projection of the point $(1,0,-7)$ onto the plane $z=-2$

25. The projection of the point $(1.3,\pi,-10)$ onto the plane $y=4.5$

26–35 Describe the set of points represented by the given equation(s).

26. $z=0$

27. $y=1$

28. $x=3-y$

29. $x+y=1,\quad z=0$

30. $x=y=z$

31. $xyz=0$

32. $y=0,\quad z=4x-2$

33. $z=2y-1,\quad x=5$

34. $z=|x|$

35. $z=[\![x]\!]$

36–41 Find the coordinates of the described point.

36. A point in the plane $y=-2$ whose projection onto the xz-plane is $(1,0,-5)$

37. A point on the line $x=y=z$, equidistant from the xy-plane and the plane $z=5$

38. A point on the x-axis, equidistant from the planes $z=x$ and $z=x-2$

39. A point P with an x-coordinate of -1, so that the line through P and the point $(1,-2,5)$ is parallel to one of the coordinate axes

40. The point on the sphere $x^2+y^2+6y+z^2-7=0$ that is closest to the plane $y=5$

41. The point on the sphere $x^2+y^2-2y+z^2-6z+6=0$ that is closest to the point $(2\sqrt{3},1,5)$

42–45 Match the equation to its graph (labeled A–D).

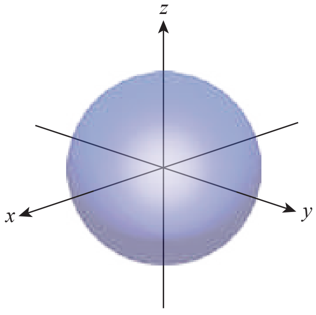
42. $z-x=1$

43. $x^2+y^2+z^2=10$

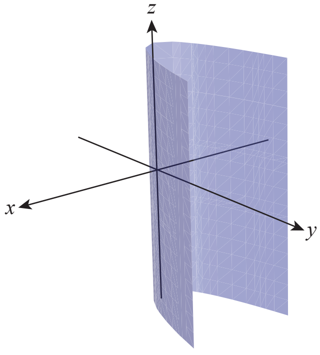
44. $x^2+z^2=5$

45. $y-x^2=0$

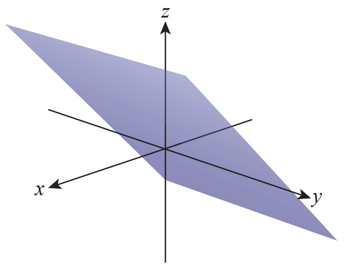
A.



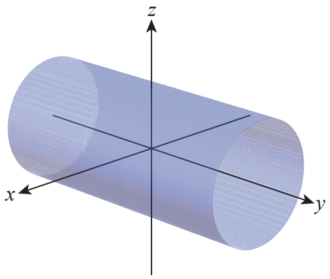
B.



C.



D.



46–59 Describe the set of points represented by the given relation(s). (**Hint:** A sketch may be helpful.)

46. $z^2\geq 1$

47. $x^2+y^2>1$

48. $x^2+y^2+z^2\leq 4$

49. $(x-3)^2+y^2+z^2=9$

50. $y^2+z^2<9$

51. $x^2+y=1$

52. $x^2+y^2+(z-3)^2=25,\quad z=0$

53. $x-z^2=2,\quad y=1$

54. $xz \geq 0, \quad y = 0$

55. $xyz = 0$

56. $x^2 - 4x + y^2 + z^2 + 6z + 12 = 0$

57. $x^2 + 4x + y^2 - 8y + z^2 - 10z = 0, \quad z \geq 5$

58. $x^2 - 6x + y^2 + z^2 + 4z \leq 3$

59. $x^2 + y^2 + 8y + z^2 - 2z + 8 = 0, \quad z = -2$

60–77 Find the equations or inequalities that define the indicated set.

60. The plane through $(2,0,-1)$ that is parallel to the xy-plane

61. The plane through $(1,0,0)$ that is perpendicular to the x-axis

62. The plane through $(-4,7,2)$ that is perpendicular to the y-axis

63. The plane through $\left(\frac{4}{5},1,e\right)$ that is perpendicular to the z-axis

64. The line through $(1,2,3)$ that is parallel to the z-axis

65. The line through $(0,-1,2)$ that is parallel to the x-axis

66. The line through $(-4,1,5)$ that is parallel to the y-axis

67. The sphere of radius 2, centered at the point $(0,2,0)$

68. The sphere of radius $\sqrt{5}$, centered at the point $(-1,3,5)$

69. The sphere centered at $(1,0,-2)$, passing through the point $(3,-3,4)$

70. The circle of radius 1 in the xy-plane, centered at $(1,0,0)$

71. The circle of radius 3 in the yz-plane, centered at $(0,-4,1)$

72. The circle of radius $\sqrt{2}$ in the plane $y = -1$, centered at $(-1,-1,1)$

73. The sphere centered at $(-1,4,-3)$ that is tangent to the yz-plane

74. The sphere with a diameter joining the points $(0,-2,5)$ and $(2,4,1)$

75. The intersection of the first octant and the sphere centered at $(3,2,-1)$, passing through the point $(5,3,1)$

76. A horizontal circular cylinder of inner radius $r = 4$ that is tangent to the xy-plane along the line $y = -1$

77. The set of points equidistant from $(-1,2,3)$ and $(7,2,3)$

78–81 Find the distance between the given pair of points.

78. $(0,4,2)$ and $(-1,1,0)$

79. $(-3,4,10)$ and $(5,-3,6)$

80. $\left(\frac{\sqrt{2}}{2},3,-1\right)$ and $\left(\sqrt{2},3,1\right)$

81. $\left(2,3\sqrt{3},-4\right)$ and $\left(-1,\sqrt{3},-2\right)$

82–85 A triangle is given by the coordinates of its vertices. For the triangle, select all that apply from the following list: **a.** isosceles, **b.** equilateral, **c.** scalene, **d.** right triangle.

82. $(0,6,2)$, $(3,4,1)$, and $(1,3,4)$

83. $(-1,2,7)$, $(1,1,3)$, and $(5,3,2)$

84. $(2,10,4)$, $(1,7,0)$, and $(-3,1,-1)$

85. $(2,1,1)$, $(4,3,1)$, and $(3,1,2)$

11.2 Exercises

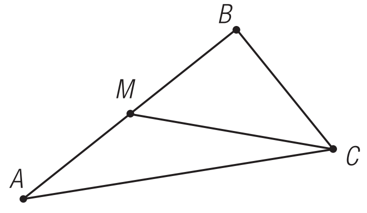
1–8 Quantities are given that arise in everyday life. Decide whether they are vectors or scalars.

1. The speed of your car
2. The cost of a telephone call
3. The velocity of your car
4. The mass of a golf cart
5. The weight of a gallon of milk
6. The displacement of a particle that moved from $(2,-1)$ to $(5,8)$ in the xy-system
7. The distance covered by a flight from Houston to San Diego
8. The restoring force exerted by a vertical spring when a mass is hung on it

9–14 Decide whether the following is a vector.

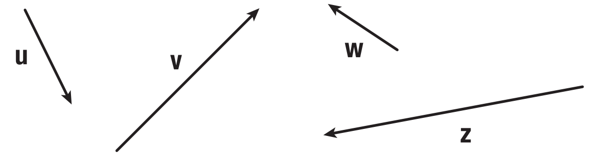
9. $(2,\pi)$
10. $(1,0,-4)$
11. $\langle 0,0\rangle$
12. $\left\langle 0,\frac{\pi}{2},\cos\frac{\pi}{2}\right\rangle$
13. $\langle 2,-1,3\rangle-\langle 1,5,10\rangle$
14. $\left|\langle 2,-1,3\rangle\right|$

15–20 Use the figure below to express the indicated sum or difference as a single vector. (M is the midpoint of $\overline{AB}$.)



15. $\overrightarrow{CA}+\overrightarrow{CB}$
16. $\overrightarrow{CA}-\overrightarrow{CB}$
17. $\overrightarrow{BC}-\overrightarrow{BM}$
18. $\overrightarrow{AB}+\overrightarrow{BM}+\overrightarrow{MC}$
19. $3\overrightarrow{AM}-\overrightarrow{AB}-\overrightarrow{AC}$
20. $2\overrightarrow{BM}+\overrightarrow{AC}$

21–24 Geometrically construct the indicated linear combination from the given vectors.



21. $\mathbf{u}-\mathbf{z}$
22. $\mathbf{v}+2\mathbf{w}$
23. $2\mathbf{u}+\mathbf{v}-\mathbf{w}$
24. $3\mathbf{w}+\frac{1}{2}\mathbf{z}-\mathbf{u}$

25–30 Find the linear combination in component form if the vectors **u** and **v** are given in component form as $\mathbf{u}=\langle 4,-1\rangle$ and $\mathbf{v}=\langle 5,8\rangle$.

25. $3\mathbf{u}$
26. $2\mathbf{u}+\mathbf{v}$
27. $4\mathbf{v}-\mathbf{u}$
28. $\frac{1}{4}\mathbf{u}+\frac{3}{4}\mathbf{v}$
29. $\mathbf{u}-\frac{1}{2}\mathbf{v}$
30. $\frac{2}{5}\mathbf{v}-\pi\mathbf{u}$

31–34 Find the coordinates of the endpoint of the indicated vector with the given initial point if $\mathbf{u}=\langle 3,-1,2\rangle$ and $\mathbf{v}=\langle 2,1,-1\rangle$.

31. $2\mathbf{u}$ with initial point $(4,0,-1)$
32. $-4\mathbf{v}$ with initial point $(2,5,-3)$
33. $\mathbf{u}-2\mathbf{v}$ with initial point $(3,7,-10)$
34. $5\mathbf{v}-3\mathbf{u}$ with initial point $(-3,6,11)$

35–36 Find a vector **v** that solves the vector equation.

35. $2\mathbf{v}+\langle 4,-8,2\rangle=\langle 2,2,2\rangle$
36. $\langle 4,-7,1\rangle-4\mathbf{v}=\langle 5,-1,0\rangle$

37–38 Use vectors to determine whether the given points are collinear (i.e., whether they fall on the same line).

37. $P(2,0,-1),\quad Q(-4,3,1),\quad R(8,-3,-3)$
38. $P\left(1,\frac{3}{2},-5\right),\quad Q(3,-1,-1),\quad R\left(9,-\frac{17}{2},13\right)$

39–41 Express the vector **w** as a linear combination of **u** and **v**. (**Hint:** Using undetermined coefficients, start with the vector equation $\mathbf{w}=a\mathbf{u}+b\mathbf{v}$, and solve the resulting linear system.)

39. $\mathbf{w}=\langle 1,2\rangle;\quad \mathbf{u}=\langle 2,3\rangle,\quad \mathbf{v}=\langle 0,1\rangle$
40. $\mathbf{w}=\langle 4,-7\rangle;\quad \mathbf{u}=\langle 1,5\rangle,\quad \mathbf{v}=\langle 2,1\rangle$

41. $\mathbf{w}=\langle -13,6\rangle;\quad \mathbf{u}=\langle -3,1\rangle,\quad \mathbf{v}=\langle 4,-8\rangle$

42–45 Find the magnitude of the given vector.

42. $\mathbf{u}=\langle -4,3\rangle$

43. $\mathbf{v}=\left\langle \frac{1}{5},\frac{\sqrt{3}}{5}\right\rangle$

44. $\mathbf{w}=\left\langle 3,\frac{9}{4}\right\rangle$

45. $\mathbf{z}=\left\langle -\frac{1}{2},-\sqrt{2}\right\rangle$

46–49 Find the indicated magnitude given that $\mathbf{u}=\langle -2,-5\rangle$ and $\mathbf{v}=\langle -1,3\rangle$.

46. $|\mathbf{u}+2\mathbf{v}|$

47. $|\mathbf{v}-4\mathbf{u}|$

48. $\left|\frac{\mathbf{u}-\mathbf{v}}{3}\right|$

49. $\left|-\frac{1}{10}\mathbf{u}+\frac{2}{5}\mathbf{v}\right|$

50–55 Determine whether the vectors $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ are equal.

50. $P_1(0,1),\quad Q_1(5,1),\quad P_2(-1,-2),\quad Q_2(4,-2)$

51. $P_1(-2,0),\quad Q_1(-2,-3),\quad P_2(1,-1),\quad Q_2(1,2)$

52. $P_1(0,0),\quad Q_1(0,\sqrt{2}),\quad P_2(1,1),\quad Q_2(2,2)$

53. $P_1(-5,0,2),\quad Q_1(1,1,-4),\quad P_2(0,-1,7),\quad Q_2(6,0,1)$

54. $P_1(0,0,0),\quad Q_1(3,5,-1),\quad P_2(4,1,-3),\quad Q_2(7,6,2)$

55. $P_1(2,1,3),\quad Q_1(-1,-4,4),\quad P_2(3,2,1),\quad Q_2(4,-1,-4)$

56–61 Find the component form and magnitude of the vector $\overrightarrow{PQ}$.

56. $P(4,0,-3),\quad Q(0,0,0)$

57. $P(2,9,-5),\quad Q(2,9,-5)$

58. $P(-1,3,4),\quad Q(-5,0,4)$

59. $P(-2,-1,3),\quad Q(1,-4,1)$

60. $P\left(-5,1,\frac{\sqrt{7}}{2}\right),\quad Q\left(-1,-\frac{1}{2},0\right)$

61. $P\left(\frac{1}{3},8,\frac{\sqrt{5}+1}{3}\right),\quad Q\left(-\frac{1}{3},5,\frac{1}{3}\right)$

62–67 Decide whether the points determine a parallelogram in three-dimensional Cartesian space.

62. $A(-1,2),\quad B(-2,-3),\quad C(6,-2),\quad D(7,3)$

63. $A(-1,4),\quad B(-3,1),\quad C(2,-5),\quad D\left(5,-\frac{1}{2}\right)$

64. $A(2,0,-3),\quad B(-4,1,0),\quad C(-1,2,7),\quad D(5,1,4)$

65. $A(-1,1,2),\quad B(5,0,-2),\quad C(9,-3,0),\quad D(3,-2,4)$

66. $A(0,1,0),\quad B(-3,0,4),\quad C(-1,1,4),\quad D(2,1,1)$

67. $A(-1,-2,-3),\quad B(4,1,1),\quad C(-5,-2,-4),\quad D(0,-1,0)$

68–71 Use the technique seen in Example 5 to find the coordinates of the indicated point.

68. The point one-third of the way from $P(12,-3,0)$ to $Q(0,6,-9)$

69. The point four-fifths of the way from $P(4,2,-5)$ to $Q\left(\frac{1}{4},2,-10\right)$

70. The point one percent of the way from $P(-3.8,-2.2,1.5)$ to $Q(2.4,-5.6,10)$

71. The point(s) on the line $\overleftrightarrow{PQ}$ with a distance from $Q(-3,1,7)$ equaling three times the distance from $P(1,5,3)$

72–77 For the given vector **v**, find the unit vector **u** pointing in the same direction. Express your answer in terms of the standard basis vectors.

72. $\mathbf{v}=\langle -8,6\rangle$

73. $\mathbf{v}=\langle 2,9\rangle$

74. $\mathbf{v}=\langle 2,0,-1\rangle$

75. $\mathbf{v}=\langle -4,-5,2\rangle$

76. $\mathbf{v}=\left\langle 3,-\frac{5}{4},\frac{13}{4}\right\rangle$

77. $\mathbf{v}=\left\langle 1,-\frac{3}{2},-\frac{1}{2}\right\rangle$

78. Find the unit vector **u** in $\mathbb{R}^2$ that makes a directed $2\pi/3$ radian angle with the positive x-axis. Express your answer as a linear combination of the standard basis vectors in $\mathbb{R}^2$.

79. Show that any vector **u** in $\mathbb{R}^2$ that can be written as $\mathbf{u}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$, for some $0\le\theta\le 2\pi$, is a unit vector.

80. Find an appropriate scalar a for the vector $\mathbf{v}=\langle 2,1,-1\rangle$ so that $a\mathbf{v}$ has a magnitude of 3.

81. Find an appropriate scalar a for the vector $\mathbf{w}=\langle -\sqrt{3},4,-5\rangle$ so that $a\mathbf{w}$ has a magnitude of 5.

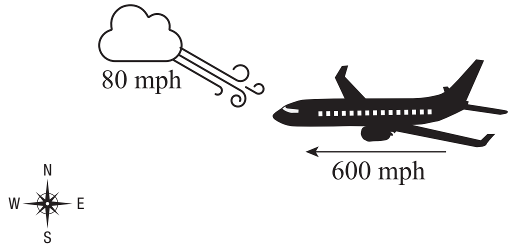
82. Find the vector **s** in $\mathbb{R}^2$ that makes a directed $7\pi/6$ radian angle with the positive x-axis and has a magnitude of 2. Express your answer in component form.

83. By first solving the linear system of Example 7 for $|\mathbf{T}_1|$ and $|\mathbf{T}_2|$, verify that $\mathbf{T}_1\approx\langle -4.55,2.12\rangle$ and $\mathbf{T}_2\approx\langle 4.55,7.88\rangle$.

84. Suppose that in Example 7 the ten-pound weight is suspended from two ropes that form angles of 75° and 51°, respectively, with the horizontal direction (refer to Figure 10). Find the tension forces $\mathbf{T}_1$ and $\mathbf{T}_2$ under these conditions.

85. A baseball bounces off a bat with an initial velocity that has a vertical component of 35.2 m/s and a horizontal component of 8 m/s. Ignoring air resistance, determine its velocity at time *t*, and sketch its position function. (Assume the initial height is 1 m. See Example 2.)

86. A jetliner flying at 600 mph due west encounters an 80 mph headwind that blows 30° south of east. If the captain wants to keep both his ground speed and direction, how much increase in speed will be needed and in what direction should he steer the plane?



87. A jetliner flying at 500 mph due southeast encounters a 60 mph tailwind that blows from the north. At the same time, a 20 mph updraft is affecting the flight. Find the actual ground speed of the jetliner under these conditions.

88. A guest in the restaurant section of a passenger train gets up from his table and cuts across toward the bar walking at a steady pace of 2 mph. If the train is moving at a constant eastward velocity of 80 mph, and the passenger is walking northeast (45° north of east), what is his resultant (ground) velocity?

89.* A plane is taking off with a flight plan that calls for a takeoff velocity of 250 mph in the direction of $\langle 1,4,2\rangle$. However, the plane experiences a tailwind, blowing at 15 mph in the direction $\langle 2,1,0\rangle$. Calculate the plane's actual direction and ground speed at takeoff. (Express your direction vector in a form where the first component is 1.)

90. Suppose the following three forces are acting simultaneously upon an object of mass 2 kilograms: $\mathbf{F}_1 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{F}_2 = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, and $\mathbf{F}_3 = 7\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ (units are in newtons). Find a unit vector $\mathbf{u}$ pointing in the direction of acceleration as well as the magnitude of the acceleration of the object. (**Hint:** Use Newton's Second Law of Motion.)

91. Use the component definition of scalar multiplication to verify the five scalar multiplication properties of vectors, as listed immediately preceding Example 3.

92. Use the component definition of vector addition to verify the four vector addition properties listed immediately preceding Example 3.

93.* Suppose that for some scalars *a*, *b* and *c*, $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{0}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors in $\mathbb{R}^3$, and $\mathbf{0}$ is the zero vector. Prove that $a = b = c = 0$.

94–98 Generalize the rules of this section to answer the following problems for vectors in higher dimensions.

94. Find $|\mathbf{u}|$ for $\mathbf{u} = \langle 1,-4,2,3\rangle$.

95. Find $3\mathbf{u} - 2\mathbf{v}$ for $\mathbf{u} = \langle 1,-4,2,3\rangle$ and $\mathbf{v} = \langle 2,-5,0,-1\rangle$.

96. Find $|2\mathbf{u} + \mathbf{v}|$ for $\mathbf{u} = \langle 3,1,\frac{1}{2},5\rangle$ and $\mathbf{v} = \langle -2,0,1,-4\rangle$.

97. Find $\mathbf{u} - \dfrac{\mathbf{v}}{3}$ for $\mathbf{u} = \langle 2,0,-4,6,-8,0\rangle$ and $\mathbf{v} = \langle 0,-3,9,-12,1,6\rangle$.

98. Find $\left|\dfrac{\mathbf{u}}{2} + 3\mathbf{v}\right|$ for $\mathbf{u} = \langle 4,-2,0,8,0,-6\rangle$ and $\mathbf{v} = \langle \frac{2}{3},-2,1,0,-\frac{1}{3},1\rangle$.

11.3 Exercises

1–10 Find the dot product of **u** and **v**.

1. $\mathbf{u}=\left\langle 2,\frac{1}{3}\right\rangle,\quad \mathbf{v}=\left\langle \frac{5}{2},-3\right\rangle$

2. $\mathbf{u}=\langle 3,-1\rangle,\quad \mathbf{v}=\langle 0,0\rangle$

3. $\mathbf{u}=\langle 2,0,1\rangle,\quad \mathbf{v}=\langle 1,1,-3\rangle$

4. $\mathbf{u}=\left\langle \frac{4}{3},-1,\frac{2}{5}\right\rangle,\quad \mathbf{v}=\left\langle -1,6,-\frac{5}{3}\right\rangle$

5. $\mathbf{u}=\langle 4s,-2,2s\rangle,\quad \mathbf{v}=\langle 3,5,-6\rangle$

6. $\mathbf{u}=\langle -5s,s,2\rangle,\quad \mathbf{v}=\langle 2t,4t,-1\rangle$

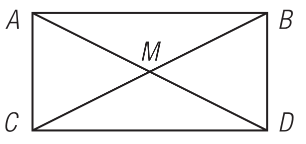
7. $\mathbf{u}=\mathbf{i}+3\mathbf{j}-2\mathbf{k},\quad \mathbf{v}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$

8. $\mathbf{u}=\frac{1}{2}\mathbf{i}+\mathbf{j}+\frac{5}{3}\mathbf{k},\quad \mathbf{v}=8\mathbf{i}-5\mathbf{j}-9\mathbf{k}$

9. $|\mathbf{u}|=4,\quad |\mathbf{v}|=3\sqrt{2},$ their angle is $45°$

10. $|\mathbf{u}|=2.5,\quad |\mathbf{v}|=5,$ their angle is $2\pi/3$

11–13 Suppose that the length and width of the rectangle below are $2\sqrt{3}$ and 2 units, respectively. Find the indicated dot product.



11. $\overrightarrow{CM}\cdot\overrightarrow{MD}$

12. $\overrightarrow{CA}\cdot\overrightarrow{MD}$

13. $\overrightarrow{CA}\cdot\left(\overrightarrow{CM}+\overrightarrow{MD}\right)$

14–21 Find the angle between the given vectors.

14. $\mathbf{u}=\langle 2,-\sqrt{5}\rangle,\quad \mathbf{v}=\langle -4,3\rangle$

15. $\mathbf{u}=\langle -1,1\rangle,\quad \mathbf{v}=\langle 3,3\rangle$

16. $\mathbf{u}=\mathbf{i}-2\mathbf{j},\quad \mathbf{v}=3\mathbf{i}+\mathbf{j}$

17. $\mathbf{u}=(\cos 32°)\mathbf{i}+(\sin 32°)\mathbf{j},$
$\mathbf{v}=-(\cos 87°)\mathbf{i}+(\sin 87°)\mathbf{j}$

18. $\mathbf{u}=\langle 4,-2,1\rangle,\quad \mathbf{v}=\langle 2,-1,5\rangle$

19. $\mathbf{u}=\langle -1,1,2\rangle,\quad \mathbf{v}=\langle 3,2,0\rangle$

20. $\mathbf{u}=\mathbf{i}+2\mathbf{j}-2\mathbf{k},\quad \mathbf{v}=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

21. $\mathbf{u}=\frac{1}{2}\mathbf{i}-\mathbf{j}+\mathbf{k},\quad \mathbf{v}=\mathbf{i}-\frac{3}{2}\mathbf{j}-\sqrt{3}\mathbf{k}$

22–26 Prove the indicated property of the dot product. (**Hint:** Start by representing vectors **u**, **v**, and **w** in component form.)

22. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

23. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

24. $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

25. $\mathbf{0}\cdot\mathbf{u}=0$

26. $a(\mathbf{u}\cdot\mathbf{v})=(a\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(a\mathbf{v}),$ a is a scalar

27. Prove that for any positive c,

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}=\frac{\mathbf{u}\cdot(c\mathbf{v})}{|c\mathbf{v}|}.$$

(**Hint:** Use the properties of the dot product.)

28. Prove the equation

$$|\mathbf{u}-\mathbf{v}|^2=|\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}||\mathbf{v}|\cos\theta$$

for the two cases $\theta=0$ and $\theta=\pi$. (**Hint:** Use the fact that $\mathbf{v}=c\mathbf{u}$ for some constant c in these two cases, with c positive if $\theta=0$ and c negative if $\theta=\pi$.)

29–40 Determine whether the given vectors are parallel, orthogonal, or neither.

29. $\langle 2,6\rangle$ and $\langle -1,-3\rangle$

30. $\langle 6,4\rangle$ and $\langle -2,3\rangle$

31. $\langle 2,-3\rangle$ and $\left\langle -\frac{1}{2},\frac{1}{3}\right\rangle$

32. $\langle s,2t\rangle$ and $\left\langle -3t,\frac{3}{2}s\right\rangle$

33. $\langle 1,0,-7\rangle$ and $\left\langle 0,-\frac{5}{3},0\right\rangle$

34. $\langle -s,5s,2s\rangle$ and $\left\langle \frac{3}{2}s,\frac{-15}{2}s,-3s\right\rangle$

35. $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}-\mathbf{i}$

36. $\mathbf{i}+\mathbf{j}$ and $-\frac{3}{7}\mathbf{k}$

37. $\frac{1}{2}\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+\frac{3}{2}\mathbf{j}-\mathbf{k}$

38. $-\mathbf{i}+\frac{2}{3}\mathbf{j}-4\mathbf{k}$ and $5\mathbf{i}+2\mathbf{j}-\frac{5}{4}\mathbf{k}$

39. $\langle \cos\theta,\sin\theta\rangle$ and $\langle -\sin\theta,\cos\theta\rangle$

40. $\langle \cos(90°-\theta),-\cos\theta,1\rangle$ and $\langle -\sin\theta,\cos\theta,1\rangle$

41–44 Use the vector method to decide which of the following are true of ΔABC: **a.** acute, **b.** obtuse, **c.** isosceles, **d.** equilateral, **e.** right triangle. (**Hint:** Determine interior angles.)

41. $A(1,2,5),\ \ B(-1,2,-4),\ \ C(8,8,5)$

42. $A(-4,5,6),\ \ B(-2,4,13),\ \ C(-1,-3,4)$

43. $A(2,-1,1),\ \ B(6,2,0),\ \ C(3,1,5)$

44. $A(3,-4,1),\ \ B(1,-3,4),\ \ C(0,-6,2)$

45. Find the angles determined by the two diagonals of the quadrilateral with vertices $\left(1,2,\frac{1}{3}\right)$, $(0,-1,0)$, $\left(3,1,-\frac{4}{3}\right)$, and $(2,-3,-2)$.

46–49 Determine the value of the parameter so that the vectors are orthogonal.

46. $\left\langle s,\frac{1}{2}s,2\right\rangle$ and $\langle -5,2s,3\rangle$

47. $\langle 2,s,-3s\rangle$ and $\langle s,1,1\rangle$

48. $\langle t,-2,t\rangle$ and $\langle t,t,t^2\rangle$

49. $\langle 4,s,-2\rangle$ and $\langle 4,5t,8\rangle$

50. Find a vector of length $\sqrt{3}$ in $\mathbb{R}^3$ that is orthogonal to both $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$.

51. Use vectors to show that a parallelogram is a rhombus if and only if its diagonals are perpendicular.

52. Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

53. Thales' Theorem states that any point on a circle determines a right triangle with the endpoints of any diameter not containing the point. Use vectors to prove Thales' Theorem. (**Hint:** It is enough to prove the statement for the unit circle centered at the origin.)

54. Find an equation of the line that contains the point (a,b) and is perpendicular to the vector $\mathbf{n}=\langle n_1,n_2\rangle$ ($\mathbf{n}$ is called a normal vector to the line). (**Hint:** Notice that for any point (x,y) on the line, the vector $\langle x-a,y-b\rangle$ must be perpendicular to $\mathbf{n}$, so their dot product is 0.)

55. Use Exercise 54 to derive the point-slope form of the equation of a line of slope m through the point (a,b). (**Hint:** Notice that $\mathbf{n}=\langle -m,1\rangle$ is a normal vector.)

56. Use normal vectors to determine the angle between the lines with slopes $m=1$ and $m=3$, respectively. (**Hint:** Note that the angle between the lines is the same as that between their respective normal vectors. See the hint given in Exercise 55.)

57–58 Find a unit vector that is normal (or perpendicular) to the given line.

57. $x-3y=7$

58. $y=\frac{1}{2}x-5$

59. Generalize Exercise 54 to find an equation of the plane in $\mathbb{R}^3$ that contains the point (a,b,c) and is perpendicular to $\mathbf{n}=\langle n_1,n_2,n_3\rangle$. (The vector $\mathbf{n}$ is called a normal vector to the plane. We will elaborate on this approach in Section 11.5.)

60. Use Exercise 59 to find the equation of the plane through $(2,-4,1)$ that is perpendicular to $\mathbf{n}=\langle 5,-3,1\rangle$.

61–62 Find a normal vector to the given plane.

61. $x-4y-2z=7$

62. $2x+y-5z=1$

63. Use the results of Exercises 61 and 62 to find the angle between the planes $x-4y-2z=7$ and $2x+y-5z=1$. (**Hint:** The angle between two planes is the same as that between their respective normal vectors.)

64–67 Find the direction angles of the given vector.

64. $\langle 1,0,-1\rangle$

65. $\langle 2,-1,4\rangle$

66. $\left\langle -\frac{1}{3},2,-\frac{2\sqrt{3}}{3}\right\rangle$

67. $\left\langle \frac{1}{2},-4,-\frac{5}{3}\right\rangle$

68. Suppose the first and second direction angles of a vector in the first octant are $\pi/6$ and $\pi/3$. Find the third direction angle.

69. Prove the *Cauchy-Schwarz Inequality*: For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$|\mathbf{u}\cdot\mathbf{v}|\le|\mathbf{u}||\mathbf{v}|.$$

Under what conditions does equality hold? (Note that on the left-hand side, $|\cdot|$ means the absolute value of a scalar, while the right-hand side is the product of the magnitudes of the vectors.)

70. Use the Cauchy-Schwarz Inequality (Exercise 69) to prove the famous *Triangle Inequality*: For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$|\mathbf{u}+\mathbf{v}|\le|\mathbf{u}|+|\mathbf{v}|.$$

Under what conditions does equality hold? (Note that this inequality states that the sum of the lengths of two vectors never exceeds the sum of their individual lengths. **Hint:** Estimate $|\mathbf{u}+\mathbf{v}|^2$ by writing $|\mathbf{u}+\mathbf{v}|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$. Then use the properties of the dot product and the Cauchy-Schwarz Inequality.)

71. What can you say about $\mathbf{v}$ if $|\mathbf{u}|=2$, $|\mathbf{v}|=4$, and $|\mathbf{u}\cdot\mathbf{v}|$ is maximum?

72. Use Exercise 70 to prove the following.

$$\big||\mathbf{u}|-|\mathbf{v}|\big|\le|\mathbf{u}-\mathbf{v}|$$

(This is often called the "left-hand part of the Triangle Inequality," or the "Reverse Triangle Inequality.")

73. Prove the following so-called *Parallelogram Law*. Can you give a reason for its name?

$$|\mathbf{u}+\mathbf{v}|^2+|\mathbf{u}-\mathbf{v}|^2=2|\mathbf{u}|^2+2|\mathbf{v}|^2.$$

74–83 Decompose **u** into a sum of two vectors, one parallel to **v** and one perpendicular to **v**.

74. $\mathbf{u}=\langle 3,-1\rangle,\quad \mathbf{v}=\langle 1,7\rangle$

75. $\mathbf{u}=\left\langle 2,\frac{1}{3}\right\rangle,\quad \mathbf{v}=\left\langle \frac{5}{2},-3\right\rangle$

76. $\mathbf{u}=\langle 2,0,1\rangle,\quad \mathbf{v}=\langle 1,1,-3\rangle$

77. $\mathbf{u}=\left\langle \frac{4}{3},-1,\frac{2}{5}\right\rangle,\quad \mathbf{v}=\left\langle -1,6,-\frac{5}{3}\right\rangle$

78. $\mathbf{u}=\langle 4s,-2,2s\rangle,\quad \mathbf{v}=\langle 3,5,-6\rangle$

79.* $\mathbf{u}=\langle -5s,s,2\rangle,\quad \mathbf{v}=\langle 2t,4t,-1\rangle$

80. $\mathbf{u}=\mathbf{i}+3\mathbf{j}-2\mathbf{k},\quad \mathbf{v}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$

81. $\mathbf{u}=\frac{1}{2}\mathbf{i}+\mathbf{j}+\frac{5}{3}\mathbf{k},\quad \mathbf{v}=8\mathbf{i}-5\mathbf{j}-9\mathbf{k}$

82. $|\mathbf{u}|=4,\quad |\mathbf{v}|=3\sqrt{2},$ their angle is $45°$

83. $|\mathbf{u}|=2.5,\quad |\mathbf{v}|=5,$ their angle is $2\pi/3$

84. For $\mathbf{v}=\langle 2,-1\rangle$, find a vector **u** such that

a. $|\text{proj}_{\mathbf{v}}\mathbf{u}|=\frac{5}{2}$, **b.** $|\text{proj}_{\mathbf{u}}\mathbf{v}|=\frac{5}{2}$.

(Answers will vary.)

85. For $\mathbf{v}=\langle -3,0,1\rangle$, find a vector **u** such that $|\text{proj}_{\mathbf{v}}\mathbf{u}|=\frac{5}{2}$.

86.* Prove that the distance d from a point $Q(x_0,y_0)$ to a line $ax+by=c$ is

$$d=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}.$$

(**Hint:** Revisit Exercises 59 and 60 to identify a normal vector of the line, then pick a point $P(x,y)$ on the line, and consider $\text{proj}_{\mathbf{n}}\overrightarrow{PQ}$.)

87.* Prove that the distance d between parallel lines $ax+by=c_1$ and $ax+by=c_2$ is

$$d=\frac{|c_2-c_1|}{\sqrt{a^2+b^2}}.$$

88.* Generalize your solution to Exercise 86 to three dimensions to show that the distance d from a point $Q(x_0,y_0,z_0)$ to a plane $ax+by+cz=f$ is

$$d=\frac{|ax_0+by_0+cz_0-f|}{\sqrt{a^2+b^2+c^2}}.$$

89.* Generalize Exercise 87 to show that the distance d between parallel planes $ax+by+cz=f_1$ and $ax+by+cz=f_2$ is

$$d=\frac{|f_2-f_1|}{\sqrt{a^2+b^2+c^2}}.$$

90–93 Use the formulas from Exercises 86–89 to find the indicated distance.

90. The distance between the point $(1,2)$ and the line $x-y=4$

91. The distance between the lines $2x+3y=2$ and $2x+3y=-5$

92. The distance between the point $(1,2,3)$ and the plane $x-3y+z=2$

93. The distance between the planes $x-2y+5z=1$ and $x-2y+5z=7$

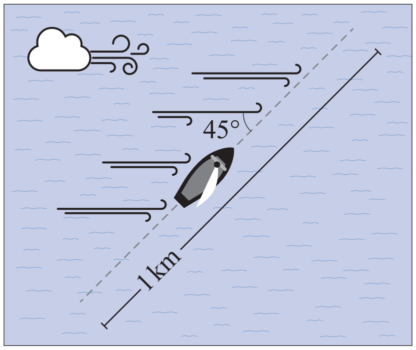
94–95 Find the work done by the force **F** as it moves an object from *P* to *Q*. (Suppose **F** is measured in pounds and a unit distance is 1 foot.)

94. $\mathbf{F}=-2\mathbf{i}-\mathbf{j}+5\mathbf{k}$ from $P(1,2,3)$ to $Q(4,-7,6)$

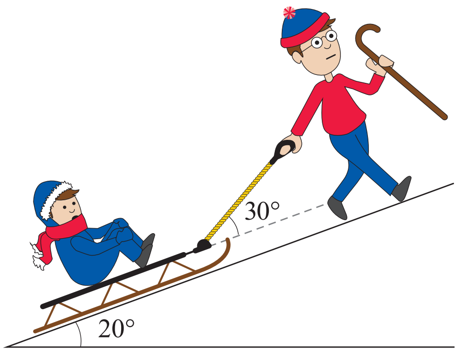
95. $\mathbf{F}=\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ from $P(-7,1,-4)$ to $Q(-3,-6,2)$

96. In order to close a curtain, a hotel guest pulls a rod at an angle of 45° with a constant force of 10 pounds. Find the work done if the curtain moves 5 feet to its closed position.

97. A sailboat is propelled a distance of 1 km by a wind that makes a 45° angle with the boat's direction of travel. Find the work done by the wind if its force is 2000 newtons.



98. A father is pulling his little son in a sled up a 20° slope that is 20 meters long. The rope makes a 30° angle with the surface of the slope and the combined weight of the child and the sled is 200 newtons. (Ignore friction and any acceleration of motion.)



a. Find the work done by the father on the sled.

b. Find the force of tension in the rope.

99.* Repeat Exercise 98 under the assumption that the coefficient of friction is $\mu = 0.13$.

100. Use vectors to find the angle between the diagonal of a cube and

a. one of its edges,

b. the diagonal of one of the faces.

101.* Prove that the points $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$ determine a regular tetrahedron and find the angle any edge makes with the face that doesn't contain it.

102. Use two-dimensional unit vectors to prove the following well-known formula from trigonometry:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(**Hint:** Considering two unit vectors, represent them as in Exercise 17, and interpret their dot product.)

103. A company manufactures three different products, producing n_j of each during a given production cycle ($j = 1, 2, 3$). Accordingly, the total production of a given cycle can be arranged into a three-dimensional production vector: $\mathbf{p} = \langle p_1, p_2, p_3 \rangle$. If the selling price of the j^{th} product is s_j dollars each, the price vector can similarly be defined as $\mathbf{s} = \langle s_1, s_2, s_3 \rangle$. Interpret the dot product $\mathbf{p} \cdot \mathbf{s}$ for a given production cycle.

Concept Check

104–120 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

104. $\langle 1,3,0 \rangle \cdot \langle -2,1 \rangle = 1$

105. $|\mathbf{u}|\mathbf{v} = \mathbf{v}|\mathbf{u}|$

106. $|\mathbf{u}|\mathbf{v} = \mathbf{u}|\mathbf{v}|$

107. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

108. $|\mathbf{u}| \cdot (\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (|\mathbf{u}|\mathbf{w})$

109. $|\mathbf{u}|(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (|\mathbf{u}|\mathbf{w})$

110. If $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w} = 0$, then $(\mathbf{u} \cdot \mathbf{v}) = -\mathbf{w}$.

111. $(\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{x}) = [(\mathbf{w} \cdot \mathbf{x})\mathbf{u}] \cdot \mathbf{v} = \mathbf{u} \cdot [(\mathbf{w} \cdot \mathbf{x})\mathbf{v}]$

112. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

113. $|\mathbf{u}|(\mathbf{v} + \mathbf{w}) = |\mathbf{u}|\mathbf{v} + |\mathbf{u}|\mathbf{w}$

114. $\mathbf{u} \cdot \mathbf{v} < 0$ if and only if the angle between **u** and **v** is obtuse.

115. $(\text{proj}_{\mathbf{v}}\mathbf{u} - \mathbf{u}) \cdot \text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{0}$

116. If **u** is orthogonal to both **v** and **w**, then **u** is orthogonal to **v** + **w**.

117. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\mathbf{u}|$, then **u** and **v** are parallel.

118. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = 0$, then **u** and **v** are orthogonal.

119. If $|\mathbf{u}| < |\mathbf{v}|$, then $\mathbf{u} \cdot \mathbf{w} < \mathbf{v} \cdot \mathbf{w}$.

120. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\text{proj}_{\mathbf{u}}\mathbf{v}|$, then $|\mathbf{u}| = |\mathbf{v}|$.

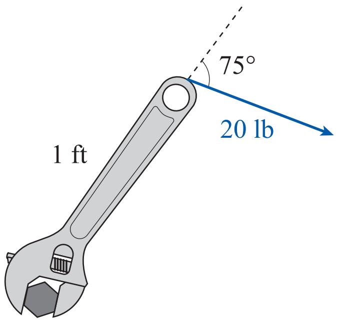


Figure 9

Example 7 Using a Cross Product to Find the Torque Generated by a Wrench

A bolt is tightened by a wrench supplying 20 pounds of force at an angle of $\theta = 75°$ (see Figure 9). The length of the wrench is 1 foot. How much torque is applied to the bolt at the pivot point?

Solution

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta = (1)(20)\sin 75° \approx 19.32 \text{ lb-ft}$$

Note that the direction of the torque vector is into the page.

11.4 Exercises

1–6 Use the determinant formula to find the cross product.

1. $\langle 2,-5,1\rangle \times \langle 1,1,-1\rangle$
2. $\langle -3,0,1\rangle \times \langle 2,1,4\rangle$
3. $\langle 5,-5,2\rangle \times \langle 3,1,-1\rangle$
4. $\left\langle \frac{1}{2},2,-3\right\rangle \times \left\langle 1,\frac{3}{2},-1\right\rangle$
5. $\left\langle \frac{1}{3},3,0\right\rangle \times \left\langle 0,-\frac{5}{3},-\frac{1}{4}\right\rangle$
6. $\langle 0.2,-0.8,1.25\rangle \times \langle -2,8,-12.5\rangle$

7–13 Use the determinant formula to prove the indicated property of the cross product. (Assume **u**, **v**, and **w** represent vectors in $\mathbb{R}^3$, while *a* and *b* represent scalars.)

7. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
8. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
9. $(a\mathbf{u})\times(b\mathbf{v}) = (ab)(\mathbf{u}\times\mathbf{v})$
10. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$
11. $\mathbf{0}\times\mathbf{u} = \mathbf{0}$
12. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$
13. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
14. Prove that the cross products of the standard unit vectors obey the following vector equations.
    a. $\mathbf{i}\times\mathbf{j} = \mathbf{k}$
    b. $\mathbf{j}\times\mathbf{k} = \mathbf{i}$
    c. $\mathbf{k}\times\mathbf{i} = \mathbf{j}$

15–18 Notice that using the properties of the cross product and results of Exercise 14, we can evaluate cross products as illustrated by the following example.

$$\begin{aligned}(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\times(2\mathbf{i}-4\mathbf{j}+5\mathbf{k}) &= 2(\mathbf{i}\times\mathbf{i})-4(\mathbf{i}\times\mathbf{j})+5(\mathbf{i}\times\mathbf{k})+6(\mathbf{j}\times\mathbf{i})-12(\mathbf{j}\times\mathbf{j})\\ &\quad+15(\mathbf{j}\times\mathbf{k})+4(\mathbf{k}\times\mathbf{i})-8(\mathbf{k}\times\mathbf{j})+10(\mathbf{k}\times\mathbf{k})\\ &= -4(\mathbf{i}\times\mathbf{j})-5(\mathbf{k}\times\mathbf{i})-6(\mathbf{i}\times\mathbf{j})+15(\mathbf{j}\times\mathbf{k})+4(\mathbf{k}\times\mathbf{i})+8(\mathbf{j}\times\mathbf{k})\\ &= -10(\mathbf{i}\times\mathbf{j})-(\mathbf{k}\times\mathbf{i})+23(\mathbf{j}\times\mathbf{k})\\ &= -10\mathbf{k}-\mathbf{j}+23\mathbf{i}\\ &= 23\mathbf{i}-\mathbf{j}-10\mathbf{k}\end{aligned}$$

Note that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$.

Use the above method to evaluate the cross product $\mathbf{u}\times\mathbf{v}$.

15. $\mathbf{u} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}, \quad \mathbf{v} = -\mathbf{i}-\mathbf{j}+2\mathbf{k}$
16. $\mathbf{u} = \frac{1}{3}\mathbf{i}+\frac{5}{2}\mathbf{j}-\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i}-9\mathbf{j}-12\mathbf{k}$
17. $\mathbf{u} = \langle 4,-2,1\rangle, \quad \mathbf{v} = \langle 2,5,-6\rangle$
18. $\mathbf{u} = \langle -5,1,2\rangle, \quad \mathbf{v} = \langle 3,4,-1\rangle$

19–22 Find both unit vectors perpendicular to **u** and **v**.

19. $\mathbf{u} = \langle 1,0,-1 \rangle, \quad \mathbf{v} = \langle 2,-2,1 \rangle$

20. $\mathbf{u} = \langle 3,1,0 \rangle, \quad \mathbf{v} = \langle -1,0,2 \rangle$

21. $\mathbf{u} = \mathbf{j} - 3\mathbf{k}, \quad \mathbf{v} = -2\mathbf{i} + \mathbf{j}$

22. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

23–26 Construct a vector normal to the plane containing the indicated points.

23. $P(0,0,0), \quad Q(2,-1,1), \quad R(3,3,4)$

24. $P(-5,0,4), \quad Q(2,2,2), \quad R(6,-1,3)$

25. $P\left(\frac{1}{2},-1,-1\right), \quad Q\left(\frac{5}{2},3,\frac{7}{2}\right), \quad R\left(2,\frac{1}{2},0\right)$

26. $P\left(-\frac{4}{3},-\frac{7}{6},\frac{1}{4}\right), \quad Q\left(-\frac{1}{3},-\frac{5}{6},\frac{3}{4}\right), R\left(\frac{5}{6},-\frac{1}{6},1\right)$

27. Prove that $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$ if and only if the points A, B, and C are collinear (i.e., they lie on the same line).

28–31 Use Exercise 27 to check whether the given points are collinear.

28. $P(3,1,1), \quad Q(2,-1,0), \quad R(1,4,-1)$

29. $P(3,1,1), \quad Q(2,-1,0), \quad R(5,5,3)$

30. $P\left(2,0,-\frac{1}{2}\right), \quad Q\left(\frac{1}{2},-1,3\right), \quad R\left(\frac{3}{2},-1,-1\right)$

31. $P\left(0,\frac{1}{3},1\right), \quad Q\left(\frac{1}{3},1,2\right), \quad R\left(-1,-\frac{5}{3},-2\right)$

32–35 Find the area of the parallelogram spanned by the given vectors.

32. $\langle 1,4 \rangle, \quad \langle -3,1 \rangle$

33. $\langle -3,-2 \rangle, \quad \left\langle 1,-\frac{5}{2} \right\rangle$

34. $\langle 7,3 \rangle, \quad \langle 6,-10 \rangle$

35. $\langle 6,9 \rangle, \quad \langle -9,-6 \rangle$

36–39 Find the area of the triangle with the given vertices.

36. $A(0,0,0), \quad B(1,2,3), \quad C(-3,-2,-1)$

37. $A(1,1,1), \quad B(4,-2,5), \quad C(-3,1,-1)$

38. $A(-4,1,2), \quad B(-1,3,5), \quad C(-3,0,-5)$

39. $A\left(\frac{1}{2},-1,\frac{5}{2}\right), \quad B\left(\frac{3}{2},-2,\frac{3}{2}\right), \quad C\left(-\frac{3}{2},-3,\frac{7}{2}\right)$

40. Suppose that the vertices of a triangle ABC in the xy-plane have coordinates $(x_A, y_A, 0)$, $(x_B, y_B, 0)$, and $(x_C, y_C, 0)$, respectively. Prove that the area of ΔABC is half the absolute value of the following determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{vmatrix}$$

41–46 Suppose $\mathbf{u} = \langle 1,3,-2 \rangle$, $\mathbf{v} = \langle 4,-1,1 \rangle$, and $\mathbf{w} = \langle -2,2,-1 \rangle$. If possible, evaluate each of the following expressions.

41. $\mathbf{u} + (\mathbf{v} \times \mathbf{w})$

42. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w}$

43. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

44. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

45. $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

46. $(\mathbf{u} \times \mathbf{v}) + \mathbf{w}$

47. Describe the conditions **u** or **v** have to satisfy in order for both their dot product and cross product to be zero, that is, for $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

48. Prove the following determinant formula for the triple scalar product.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

49–52 Find the volume of the parallelepiped spanned by the indicated vectors.

49. $\mathbf{u} = \langle 2,-1,3 \rangle, \quad \mathbf{v} = \langle 3,0,4 \rangle, \quad \mathbf{w} = \langle -1,1,-2 \rangle$

50. $\mathbf{u} = \langle 1,2,3 \rangle, \quad \mathbf{v} = \langle 2,-1,4 \rangle, \quad \mathbf{w} = \langle -1,0,3 \rangle$

51. $\mathbf{u} = \langle 1,1,3 \rangle, \quad \mathbf{v} = \langle 1,3,1 \rangle, \quad \mathbf{w} = \langle 3,1,1 \rangle$

52. $\mathbf{u} = \langle 1,0,-3 \rangle, \quad \mathbf{v} = \langle 0,-5,2 \rangle, \quad \mathbf{w} = \langle 3,1,1 \rangle$

53. In light of Exercises 49–52, state a condition in terms of the triple scalar product for three vectors to be coplanar (that is, to lie on the same plane).

54–55 Use the condition you found in Exercise 53 to determine whether the vectors are coplanar.

54. $\mathbf{u} = \langle 2,-1,3 \rangle, \quad \mathbf{v} = \langle -1,2,3 \rangle, \quad \mathbf{w} = \langle 3,2,-1 \rangle$

55. $\mathbf{u} = \langle 1,5,1 \rangle, \quad \mathbf{v} = \langle -2,-4,0 \rangle, \quad \mathbf{w} = \langle -3,-15,-3 \rangle$

56–57 Use Exercises 53–55 to determine whether the given points are coplanar.

56. $A(0,0,0), \quad B(1,3,4), \quad C(-1,-2,-3), \quad D(1,1,1)$

57. $A(1,5,0),\quad B(2,4,-1),\quad C(0,3,0),\quad D(4,2,-3)$

58. Use the cross product to prove the following well-known formula from trigonometry.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

(**Hint:** See the hint given in Exercise 102 of Section 11.3. Turn the unit vectors into three-dimensional vectors as in Example 4, and interpret their cross product.)

59. A bolt is tightened by an 18-pound force at the end of a 10-inch wrench at an angle of $\theta = 60°$. What is the magnitude of the torque applied to the bolt at the pivot point?

60. Repeat Exercise 59 if the length of the wrench is 16 inches and a rotating force of 15 pounds is applied at $\theta = 45°$.

61. The force **F** exerted by the uniform magnetic field with induction vector **B** on a wire carrying current **I** obeys the vector equation

$$\mathbf{F}=l\,\mathbf{I}\times\mathbf{B},$$

where l is the length of the wire. (The standard SI unit for **B** is the tesla (T). If, in addition, we measure **I** in amperes (A), the above equation will return **F** in newtons (N).) Find the magnitude of the force experienced by an 8 cm wire in a uniform magnetic field of **B** = 2 T if it carries a current of 0.3 A and the angle between the wire and **B** is $\theta = 30°$.

62. The force **F** experienced by a charged particle q moving at velocity $\mathbf{v}\ \mathrm{m/s}$ in the uniform magnetic field **B** obeys the vector equation

$$\mathbf{F}=q\,\mathbf{v}\times\mathbf{B},$$

where (just like in Exercise 61) we obtain the force in newtons (N) if we measure **B** in teslas (T), q in coulombs (C), and **v** in meters per second $(\mathrm{m/s})$. Find the magnitude of the force experienced by an electron moving in a uniform magnetic field of **B** = 0.001 T at $\mathbf{v}=1.2\cdot10^{6}\ \mathrm{m/s}$ if the velocity vector and **B** form a 25° angle. Note that the magnitude of the charge carried by an electron, or the *elementary charge*, is $e\approx1.6\cdot10^{-19}$ C.

63. Use an appropriate property of the cross product (see Exercises 7–13) to verify the following vector equation.

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=\mathbf{0}$$

64. Prove that if vectors **s**, **u**, **v**, and **w** are coplanar (i.e., contained by the same plane), then

$$(\mathbf{s}\times\mathbf{u})\times(\mathbf{v}\times\mathbf{w})=\mathbf{0}.$$

65. Let A, B, C, and D be four distinct points in three-dimensional Cartesian space. Prove that $\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\times\overrightarrow{AD}$ is parallel to the plane containing A, B, and C.

66. Find the equation of the plane through the points $(-5,0,4)$, $(2,2,2)$, and $(6,-1,3)$. (**Hint:** See Exercise 60 of Section 11.3 and Exercise 24 of this section.)

67. Repeat Exercise 66 for the plane through $\left(\frac{1}{2},-1,-1\right)$, $\left(\frac{5}{2},3,\frac{7}{2}\right)$, and $\left(2,\frac{1}{2},0\right)$.

68.* (Section 11.3 Exercise 86 revisited) Prove that the distance d from a point Q to a line containing the points R and S is

$$d=\frac{\left|\overrightarrow{RS}\times\overrightarrow{RQ}\right|}{\left|\overrightarrow{RS}\right|}.$$

69.* (Section 11.3 Exercise 88 revisited) Prove that the distance d from a point Q to a plane containing the points R, S, and T is

$$d=\frac{\left|\left(\overrightarrow{RS}\times\overrightarrow{RT}\right)\cdot\overrightarrow{RQ}\right|}{\left|\overrightarrow{RS}\times\overrightarrow{RT}\right|}.$$

70–73. Use Exercises 68–69 to find a second solution for each of Exercises 90–93 of Section 11.3.

74. For vectors **u**, **v**, and **w** in three-dimensional Cartesian space, prove *Lagrange's identity*.

$$|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

75.* A rectangular tetrahedron is one with a vertex such that any pair of incident edges form right angles. (Such is obtained by "chopping a corner off" a cube.) Denoting the areas of the faces containing a right angle by a, b, and c, respectively, and that of the fourth face by d, prove the following "three-dimensional generalization" of the Pythagorean Theorem.

$$a^2+b^2+c^2=d^2$$

(**Hint:** Place the tetrahedron appropriately into the three-dimensional Cartesian system, and use the techniques of this section to find the areas of its faces.)

76. Suppose that the nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ satisfy

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=\mathbf{0}.$$

Prove that $\mathbf{v}$ and $\mathbf{w}$ are parallel, or both are perpendicular to $\mathbf{u}$.

77. Prove: $(\mathbf{s}\times\mathbf{u})\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}\mathbf{s}\cdot\mathbf{v} & \mathbf{u}\cdot\mathbf{v}\\ \mathbf{s}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{w}\end{vmatrix}.$

78–81 Determine whether the given expression is a vector, a scalar, or nonsense.

78. $\mathbf{u}\times(|\mathbf{u}|\,\mathbf{v})$

79. $(\mathbf{u}\cdot\mathbf{v})\times|\mathbf{v}|$

80. $(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|$

81. $|\mathbf{v}|(\mathbf{u}\times\mathbf{v})$

Concept Check

82–94 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

82. $\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{u}$

83. $\mathbf{u}\times\mathbf{v}=(-\mathbf{u})\times(-\mathbf{v})$

84. $(-\mathbf{u})\times\mathbf{u}=\mathbf{0}$

85. $|\mathbf{u}|\times\mathbf{v}=\mathbf{v}\times|\mathbf{u}|$

86. If $(\mathbf{u}\times\mathbf{v})+\mathbf{w}=\mathbf{0}$, then $(\mathbf{u}\times\mathbf{v})=-\mathbf{w}$.

87. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$

88. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$

89. $\mathbf{u}\times\mathbf{u}=|\mathbf{u}|^2$

90. If $|\mathbf{u}|<|\mathbf{v}|$ and $\mathbf{w}\neq\mathbf{0}$, then $\mathbf{u}\times\mathbf{w}<\mathbf{v}\times\mathbf{w}$.

91. $(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v})=2(\mathbf{u}\times\mathbf{v})$

92. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=0$

93. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=0$

94. If $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$ and $\mathbf{u}\neq\mathbf{0}$, then $\mathbf{v}=\mathbf{w}$.

11.4 Technology Exercises

95–96. Use a computer algebra system or programmable calculator to write a program that decides whether four points (given by their coordinates) in three-dimensional space lie in the same plane. Use your program to check your answers for Exercises 56–57.

11.5 Exercises

1–3 Determine which of the given points lie on the given line L.

1. L is the line $\mathbf{r}(t) = \langle 2+3t, -1+t, 4+2t \rangle$; $A(5,0,6)$, $B(0,0,2)$, $C(-1,-2,2)$

2. L is the line through the point $(-1,0,5)$ and parallel to $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $A(1,-6,3)$, $B(-3,6,3)$, $C\left(-\frac{1}{2}, -\frac{3}{2}, \frac{11}{2}\right)$

3. L is the line through the points $(-6,-2,1)$ and $(-4,1,3)$; $A(-2,4,5)$, $B(4,13,11)$, $C(0,7,2)$

4–11 Find the vector form and the parametric equations of the given line.

4. The line through the point $(2,5,-3)$ and parallel to $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

5. The line through the point $(1,2,3)$ and parallel to $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

6. The line through the point $\left(14, 0, -\frac{13}{4}\right)$ and parallel to $\left\langle -1, \frac{1}{2}, \frac{1}{3} \right\rangle$

7. The line through the point $(0,-5,8)$ and parallel to $\langle -4,-15,-19 \rangle$

8. The line through the points $(0,-1,0)$ and $(4,1,1)$

9. The line through the point $(-9,11,-6)$ and parallel to $4\mathbf{i} + 3\mathbf{k}$

10. The line through the points $(3,-5,1)$ and $(0,2,7)$

11. The line through the points $(16,21,28)$ and $(-10.5,32,12)$

12–19. By changing the initial point and the tracing rate, find alternative parametrizations for each of the lines in Exercises 4–11.

20–24 Find parametric equations for the line segment between the two given points. (**Hint:** First determine a direction vector and be sure to appropriately restrict the possible values of the parameter t.)

20. The line segment between $(-1,0,1)$ and $(3,2,0)$

21. The line segment between $(1,2,3)$ and $(3,2,1)$

22. The line segment between $(-2,4,7)$ and $(-1,5,8)$

23. The line segment between $\left(-3, \frac{2}{3}, -11\right)$ and $(5,-2,-7)$

24. The line segment between $\left(-\frac{1}{2}, 1, \frac{2}{3}\right)$ and $\left(\frac{4}{3}, -6, \frac{3}{2}\right)$

25–32. Find the intersection points of the lines in Exercises 4–11 with the coordinate planes.

33–37 If possible, determine the point of intersection of the given pair of lines.

33. $\mathbf{r}(t) = \langle 1+t, -2+5t, 1 \rangle$ and $\mathbf{s}(u) = \langle 4+3u, 2u, 2+u \rangle$

34. $\mathbf{r}(t) = \langle 4+2t, -1-t, 2+3t \rangle$ and $\mathbf{s}(u) = \left\langle \frac{1}{2}+7u, \frac{1}{2}-3u, \frac{3}{2}+u \right\rangle$

35. $\mathbf{r}(t) = \langle 3+t, 3t, 1+2t \rangle$ and $\mathbf{s}(u) = \langle 1+u, -1-2u, -2+u \rangle$

36. $\mathbf{r}(t) = \left\langle 3+2t, 3+2t, \frac{3}{2}+\frac{1}{2}t \right\rangle$ and $\mathbf{s}(u) = \left\langle 5-2u, 6u-19, \frac{43}{2}-7u \right\rangle$

37. $\mathbf{r}(t) = \langle 1-t, 2t-1, 4t \rangle$ and $\mathbf{s}(u) = \langle 2+u, 5+2u, 2 \rangle$

38–42 Find a scalar equation for the plane containing the given point and having the indicated normal vector. Then describe the plane as a two-parameter set of points in $\mathbb{R}^3$.

38. The plane through the origin, with normal vector $\mathbf{n} = \langle -1,3,1 \rangle$

39. The plane through the point $(2,0,9)$, with normal vector $\mathbf{n} = \langle 5,-4,-1 \rangle$

40. The plane through the point $(7,2,4)$, with normal vector $\mathbf{n} = \langle -1,2,-5 \rangle$

41. The plane through the point $(1,2,3)$, with normal vector $\mathbf{n} = \left\langle \frac{1}{2}, \frac{3}{4}, \frac{5}{6} \right\rangle$

42. The plane through the point $(-2,3,5)$, with normal vector $\mathbf{n} = \left\langle 7, -\frac{2}{5}, -\frac{5}{2} \right\rangle$

43–47 Find an equation for the plane containing the given points.

43. $A(2,0,0)$, $B(0,1,0)$, $C(0,0,-3)$

44. $A(2,0,-1)$, $B(3,-4,4)$, $C(-1,5,2)$

45. $A(1,2,0)$, $B(3,0,4)$, $C(0,5,6)$

46. $A\left(-3, \frac{1}{2}, 1\right)$, $B\left(-1, 0, \frac{5}{2}\right)$, $C(-4,-1,2)$

47. $A\left(4, 0, -\frac{3}{5}\right)$, $B(1,0,1)$, $C(2,-1,-5)$

48–51 Find the parametric equations of the line as described.

48. The line through the point $(-5,7,8)$ that is perpendicular to the plane $2x - y + 3z = 4$

49. The line through $(-3,-1,4)$ that is parallel to the line $\mathbf{r}(t) = \langle 1+t,-2+4t,1-3t\rangle$

50. The line through the origin that is perpendicular to the vectors $\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - \mathbf{k}$

51. The line through the point $(-1,-1,-5)$ that forms a right angle with the line $\mathbf{r}(t) = \langle 1+t,1+2t,1-t\rangle$

52–56. Solving each of the parametric equations $x = x_0 + at,\quad y = y_0 + bt,$ and $z = z_0 + ct$ for t and equating the results yields the so-called **symmetric equations** for a line.

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Use this method to find the symmetric equations for the lines in Exercises 4–8.

57–65 Find the equation of the plane as described.

57. The plane through the point $(-2,1,5)$, parallel to the plane $3x - 2y + 5z = 1$

58. The plane through the point $(-5,0,1)$, perpendicular to the vector $\langle 1,-1,7\rangle$

59. The plane through the point $(6,-1,2)$, perpendicular to the line $\mathbf{r}(t) = \langle 1-t,3+4t,2+t\rangle$

60. The plane that contains the z-axis and makes an angle of $\pi/6$ with the positive x-axis

61. The plane through the point $(3,5,-2)$ that contains the line $x = 2 - t, y = 4 + 2t, z = t - 3$

62. The plane containing the lines $\mathbf{r}(t) = \langle 1+t,-2+5t,1\rangle$ and $\mathbf{s}(u) = \langle 4+3u,2u,2+u\rangle$

63. The plane containing the lines $\mathbf{r}(t) = \langle 4+2t,-1-t,2+3t\rangle$ and $\mathbf{s}(u) = \left\langle \frac{1}{2}+7u,\frac{1}{2}-3u,\frac{3}{2}+u\right\rangle$

64. The plane through the points $(4,-1,1)$ and $(1,0,-2)$ that is perpendicular to the plane $2x - y + z = 9$

65. The plane that contains the line $\mathbf{r}(t) = \langle t-2,4+t,3-2t\rangle$ and the intersection of the planes $x - y + 2z = 0$ and $2x + 2y - z = 1$

66–69 Find the point of intersection between the given line and plane.

66. The line $\mathbf{r}(t) = \langle t-1,4t,t\rangle$ and the plane $3x - y + 4z = 3$

67. The line $\mathbf{r}(t) = \langle 2+5t,3t-2,4+t\rangle$ and the plane $x + 2y - 6z = 2$

68. The line $\mathbf{r}(t) = \left\langle 8+t,4+2t,3+\frac{t}{3}\right\rangle$ and the plane $x + 2y - 2z + 3 = 0$

69. The line $\mathbf{r}(t) = \left\langle 1+\frac{t}{9},1-\frac{t}{12},1+\frac{t}{4}\right\rangle$ and the plane $5y + z = 15 + x$

70–73 Find the parametric equations for the line formed by the intersection of the two given planes.

70. $x - 2y + z = 4,\quad$ the xz-plane

71. $x + 3y - z = 2,\quad 2x - y + z = 1$

72. $x - 2y - z = 3,\quad -x + y + 2z = 0$

73. $3x + y - 2z + 1 = 0,\quad x + y - 4z = 5$

74. Find an equation for the set of points that are equidistant from the points $(2,-1,6)$ and $(-4,5,2)$.

75. Find the equation of the plane having x-, y-, and z-intercepts of $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, respectively.

76. Find three planes that intersect along the line $\mathbf{r}(t) = \langle t-1,4t,t\rangle$.

77–80 Find the shortest distance between the point and the plane.

77. $A(0,0,0);\quad 4x - y + 5z = 1$

78. $A(-2,-1,7);\quad x - 3y + 2z = 0$

79. $A(-3,1,-1);\quad 2x - 2y + z = 3$

80. $A(1,2,3);\quad x + 2y + 3z + 4 = 0$

81–84 Find the shortest distance d between the given skew lines.

81. $\mathbf{r}(t) = \langle 1,1+t,t\rangle$ and $\mathbf{s}(u) = \langle -2u,1-u,2-2u\rangle$

82. $\mathbf{r}(t) = \langle 1+4t,0,-2t\rangle$ and $\mathbf{s}(u) = \langle 4u-4,2u,u\rangle$

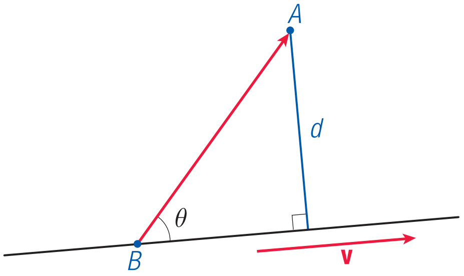
83. $\mathbf{r}(t) = \langle 2t,t-1,-1\rangle$ and $\mathbf{s}(u) = \langle u-1,2u,2u-1\rangle$

84. $\mathbf{r}(t) = \langle 1+2t,2-9t,1-3t\rangle$ and $\mathbf{s}(u) = \langle 2u,16u-7,2u\rangle$

85–88 We can find the shortest distance in three-dimensional space from a point A to a line containing a point B by the following argument. Denoting the direction vector of the line by **v**, and the angle formed by **v** and the vector $\overrightarrow{BA}$ by θ, note that the distance d we are seeking is $\left|\overrightarrow{BA}\right| \cdot \sin\theta$ (see figure below). Recalling that $\left|\mathbf{v} \times \overrightarrow{BA}\right| = |\mathbf{v}| \cdot \left|\overrightarrow{BA}\right| \cdot \sin\theta$, d can be expressed as

$$d = \frac{\left|\mathbf{v} \times \overrightarrow{BA}\right|}{|\mathbf{v}|}.$$

(Note that this is an improvement over Exercise 86 of Section 11.3 where both the point and the line were lying in the xy-plane. However, it is worth comparing the above formula with that of Exercise 68 of Section 11.4.)



In Exercises 85–88, use the above formula to find the shortest distance between the point and the line.

85. $A(2,-1,1)$; $\mathbf{r}(t) = \langle 3t, 1-t, 1+t \rangle$

86. $A(-2,1,0)$; $\mathbf{r}(t) = \langle 1+t, t, 2-t \rangle$

87. $A(1,0,0)$; $\mathbf{r}(t) = \langle 1-2t, 1-t, 1+3t \rangle$

88. $A\left(\frac{1}{2}, 5, -4\right)$; $\mathbf{r}(t) = \left\langle \frac{3}{2} - t, 6+t, -3-2t \right\rangle$

Concept Check

89–116 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

89. If the corresponding coefficients of x, y, and z in the equations of two planes are equal, then the planes are parallel. (Here we assume that both equations are in the standard form $ax + by + cz = d$.)

90. If two planes are parallel, then the corresponding coefficients of x, y, and z (as in Exercise 89) in their equations are equal.

91. If the corresponding coefficients in the equations of two planes (as in Exercise 89) are negative reciprocals, then the planes are perpendicular.

92. If two lines are parallel to a common line, then they are parallel.

93. If two lines are parallel to a common plane, then they are parallel.

94. If two lines are perpendicular to a common plane, then they are parallel.

95. If two lines are perpendicular to a common line, then they are parallel.

96. If two planes are perpendicular to a common plane, then they are parallel.

97. If two planes are perpendicular to a common line, then they are parallel.

98. If two planes are parallel to a common line, then they are parallel.

99. If two planes are parallel to a common plane, then they are parallel.

100. If two lines do not intersect, then they are parallel.

101. If two planes do not intersect, then they are parallel.

102. If a line and a plane do not intersect, then they are parallel.

103. The intersection of a line and a plane is empty or a single point.

104. The intersection of two distinct planes is empty or it is a line.

105. If plane P is parallel to the xz-plane, then $\langle 1,0,-1 \rangle$ is normal to P.

106. If plane P is parallel to the xz-plane, then $\langle 0,-1,0 \rangle$ is normal to P.

107. If plane P is parallel to the xz-plane, then the coefficient of y in its equation is 0.

108. If plane P is parallel to the xz-plane, then the coefficients of both x and z in its equation are 0.

109. The plane $y = 0$ contains the y-axis.

110. The plane $2z - x = 3$ contains the y-axis.

111. The plane $2z - x = 0$ contains the y-axis.

112. The plane with equation $ax + by + cz = d$ goes through the origin if and only if $d = 0$.

113. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors to the planes P_1 and P_2, respectively, then $\mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector for their line of intersection.

114. If L_1 and L_2 are skew lines, then there is a pair of parallel planes containing them.

115. The vector equation $\mathbf{r}(t) = \langle 3+t, 5-2t, 1-4t \rangle$ and the parametric equations $x = 5 - \frac{t}{2}$, $y = 1 + t$, and $z = -7 + 2t$ describe the same line.

116. The equations $4(1-x) + 2(y-7) - 14(z-1) = 0$ and $2x - y + 7z = 2$ describe the same plane.

11.5 Technology Exercises

117–121. Use a computer algebra system or programmable calculator to write a program that returns the equation of a plane through three given points in three-dimensional space. Use your program to check your answers for Exercises 43–47.

122–130. Use a computer algebra system or programmable calculator to write a program that decides whether two lines intersect in three-dimensional space. If they do, the program should return the coordinates of the intersection point, otherwise the message "skew lines" should appear, along with the shortest distance between them. Use your program to revisit Exercises 33–37 and 81–84.

```
In[1]:= ContourPlot3D[(y^2) / 9 - (z^2) / 4 == x / 5,
          {x, -5, 5}, {y, -4, 6}, {z, -3, 7}]
```

Out[1]=

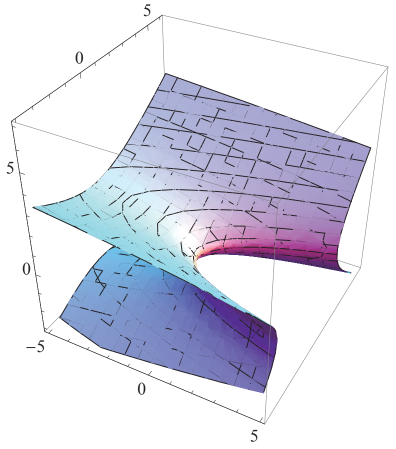


Figure 10

```
In[2]:= ContourPlot3D[(y^2) / 9 - (z^2) / 4 == x / 5,
          {x, -5, 5}, {y, -4, 6}, {z, -3, 7},
          ContourStyle → Opacity[0.5], Mesh → None]
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Out[2]=

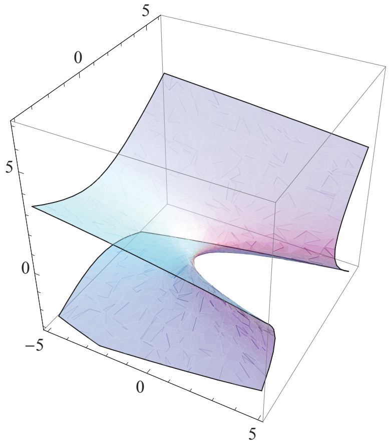


Figure 11

11.6 Exercises

1–8 Identify the surface defined by the equation and match it to the appropriate graph (labeled A–H).

1. $x^2 = 2(1 - y)$

2. $y^2 - 4y + z^2 = 4$

3. $\dfrac{y^2}{5} - \dfrac{z^2}{8} = \dfrac{x}{6}$

4. $\dfrac{x^2}{9} + \dfrac{z^2}{4} = \dfrac{y^2}{25}$

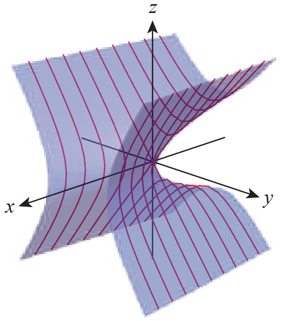
5. $\dfrac{x^2}{16} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

6. $6y^2 + 4z^2 - 3x^2 = 12$

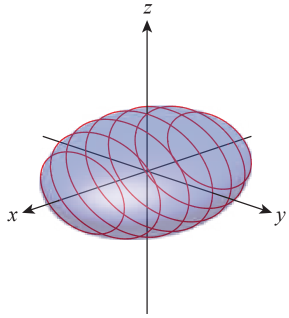
7. $5x^2 - 8y^2 + 32y - 10z^2 + 20z = 82$

8. $15x^2 + 30x - 20y + 12z^2 + 55 = 0$

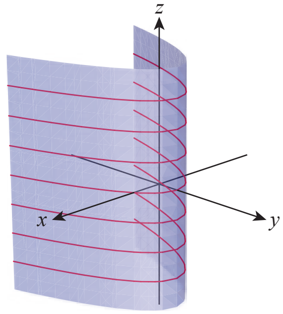
A.



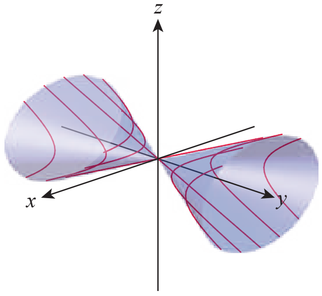
B.



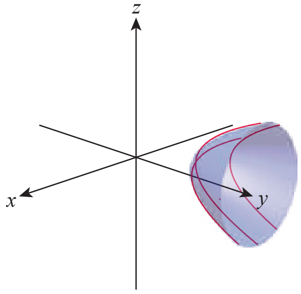
C.



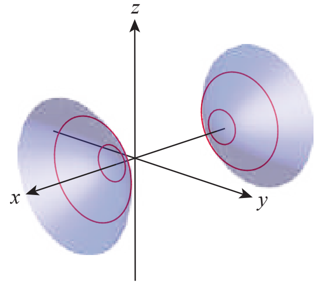
D.



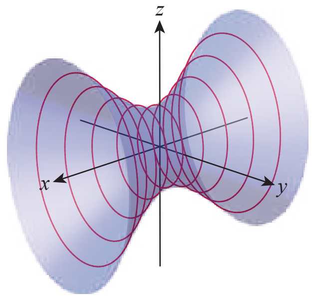
E.



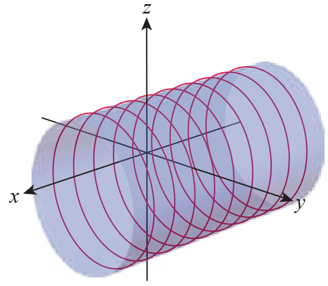
F.



G.



H.



9–16 Sketch the surface by hand. Use cross-sections to help you with your sketch.

9. $x^2+(z-1)^2=1$

10. $y^2+\frac{(z-1)^2}{4}=1$

11. $y^2-\frac{(z-1)^2}{4}=1$

12. $y^2-\frac{(z-1)}{4}=0$

13. $x^2-6x+2y+11=0$

14. $x^2-6x+2y^2+8=0$

15. $\cos\frac{z}{2}-y=1$

16. $\ln x-z=0$

17–18 Find the lengths of the major and minor axes as well as the foci of the indicated cross-section of the surface $z=\frac{x^2}{2}+y^2$.

17. $z=4$

18. $z=8$

19–20 The intersection of the given plane and the surface from Exercises 17–18 is a parabola. Find the coordinates of its vertex and focus.

19. $x=1$

20. $y=1$

21–34 Identify and sketch the quadric surface by hand. Use cross-sections to help you with your sketch.

21. $\frac{x^2}{9}+\frac{y^2}{16}+\frac{(z-2)^2}{4}=1$

22. $\frac{(y+1)^2}{9}+\frac{z^2}{4}=x$

23. $x^2+\frac{9y^2}{16}-\frac{9z^2}{25}-9=0$

24. $3z^2+2y^2=\frac{3x^2}{2}$

25. $3z^2 - 2y^2 = 1 + \frac{3x^2}{2}$

26. $3z^2 + 2y = \frac{3x^2}{2}$

27. $3z^2 + 2y^2 = \frac{3x}{2}$

28. $3z^2 + 2y^2 = 1 - \frac{3x^2}{2}$

29. $2x^2 + 2y^2 = z$

30. $2x^2 + 2y^2 = z^2$

31. $x^2 + 2x - y^2 + z^2 = 0$

32. $x^2 + 2x + 2y^2 + 3z^2 = 8y$

33. $x^2 + 2x - 2y^2 - 3z = 8y + 7$

34. $-x^2 + 2x + 2y^2 - 3z^2 = 8y + 2$

35. Consider the quadric surface of Exercise 29, $2x^2 + 2y^2 = z$. Find the intersection of this surface with the xz-plane. Explain why it is called a "generating curve" of the surface.

36. Find another generating curve for the surface in Exercise 35, and argue that the generating curve of a surface of revolution is not unique. (**Hint:** Consider the intersection of the surface with another coordinate plane.)

37–38 Find the indicated generating curve for the given surface.

37. $x^2 + y^2 = 5z$, the generating curve that lies in the yz-plane

38. $x^2 + z^2 = 1 - 3y^2$, the generating curve that lies in the xy-plane

39–46 Find an equation for the surface that results from rotating the curve about the indicated axis. (**Hint:** See Exercises 35–38.)

39. $x^2 = 2y$; about the y-axis

40. $x = 2y$; about the y-axis

41. $y = \sqrt{z-2}$; about the z-axis

42. $y = \sqrt{1-x^2}$; about the x-axis

43. $x = \frac{a}{c}\sqrt{z^2 + c^2}$; about the z-axis

44. $2xz = 3$; about the x-axis

45. $2xz = 3$; about the z-axis

46. $z = \frac{e^y}{5}$; about the y-axis

47. Assuming that Earth is a perfect ellipsoid with equatorial and polar radii of 6378 and 6357 kilometers, respectively, find the equation of this ellipsoid assuming it is centered at the origin and the axis of rotation is the z-axis.

48. What do you know about a, b, and c if the horizontal cross-sections of the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are circles?

49. Prove that all horizontal cross-sections of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ have the same eccentricity. Can you make a similar statement about vertical cross-sections? (**Hint:** For a refresher, see the definition of eccentricity in Section 9.5).

50. Find the equation of the set of points in three-dimensional space which are equidistant from the point $(0,0,1)$ and the plane $z = -1$.

Concept Check

51–55 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

51. A sphere is not an ellipsoid.

52. A vertical paraboloid whose horizontal cross-sections are circles is not an elliptic paraboloid.

53. If $a = b$ in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ of a hyperboloid, then its horizontal cross-sections are circles.

54. A quadric surface that is a surface of revolution has a unique generating curve.

55. A quadric surface that is a surface of revolution has a unique axis of rotation.

11.6 Technology Exercises

56–63. Use a graphing utility to sketch the cylindrical surfaces of Exercises 9–16.

64–77. Use a graphing utility to sketch the quadric surfaces of Exercises 21–34.

Example 7 Using Integration to Find Velocity and Position Vector Functions

A ball is shot by a slingshot into the air with an initial velocity vector $\mathbf{v}(0) = \langle 0, 10, 64 \rangle$, measured in ft/s. Determine its velocity $\mathbf{v}$ and its position $\mathbf{r}$ as functions of time t. Find the positions of the ball at 2 seconds and 4 seconds.

Solution

We begin with the fact that the acceleration of the ball is given by the vector $\mathbf{a}(t) = \langle 0, 0, -32 \rangle$, reflecting the fact that Earth's gravity is pulling it (and every other object) in the negative z-direction at a rate of 32 ft/s^2. So the ball's velocity vector is the indefinite integral of $\mathbf{a}$.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 0, 0, -32t \rangle + \mathbf{C}$$

Since we are given its velocity at time $t = 0$,

$$\langle 0, 10, 64 \rangle = \mathbf{v}(0) = \langle 0, 0, -32 \cdot 0 \rangle + \mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \langle 0, 10, 64 \rangle,$$

so $\mathbf{v}(t) = \langle 0, 10, -32t + 64 \rangle$.

If we let $\mathbf{r}$ denote the ball's position vector, then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \langle 0, 10t, -16t^2 + 64t \rangle + \mathbf{C}.$$

We can locate the origin of the coordinate system wherever convenient, so we may as well specify that the ball's position at time $t = 0$ is the origin, meaning $\mathbf{C} = \langle 0, 0, 0 \rangle$. We previously determined that the ball reaches its maximum height when $t = 2$ and lands on the ground when $t = 4$, and its position at these times is given by the following definite integrals of its velocity.

$$\text{Position at } t = 2\text{: } \int_0^2 \mathbf{v}(t)\,dt = \langle 0, 20, 64 \rangle$$

$$\text{Position at } t = 4\text{: } \int_0^4 \mathbf{v}(t)\,dt = \langle 0, 40, 0 \rangle$$

In words, the ball has traveled 20 feet horizontally when it reaches its maximum height of 64 feet, and has traveled 40 feet horizontally when it lands.

12.1 Exercises

1–4 Find the domain of the vector function. If possible, evaluate the vector function at the indicated points.

1. $\mathbf{r}(t) = \frac{1}{t^2}\mathbf{i} + 2t\,\mathbf{j} - t\,\mathbf{k}$; **a.** $t = 2$ **b.** $t = -5$

2. $\mathbf{r}(t) = 3t\,\mathbf{i} - e^t\,\mathbf{j} - \sqrt{t-1}\,\mathbf{k}$; **a.** $t = 2$ **b.** $t = -5$

3. $\mathbf{r}(t) = \frac{1}{\sqrt{9-t^2}}\mathbf{i} - t^3\,\mathbf{j} + \ln t\,\mathbf{k}$; **a.** $t = -4$ **b.** $t = 1$

4. $\mathbf{r}(t) - \mathbf{s}(t)$, where $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} - 5t^2\,\mathbf{k}$, $s(t) = e^{-t}\,\mathbf{i} + t^2\,\mathbf{j}$; **a.** $t = -1$ **b.** $t = 1$

5. If $\mathbf{r}(t) = \mathbf{i} - t\,\mathbf{j}$ and $\mathbf{s}(t) = t\,\mathbf{i} + 3\,\mathbf{j} + t^3\,\mathbf{k}$, find a formula for $\mathbf{u}(t) = \mathbf{r}(t) \cdot \mathbf{s}(t)$. Is it a space curve?

6. Repeat Exercise 5 for $\mathbf{v}(t) = \mathbf{r}(t) \times \mathbf{s}(t)$.

7–14 Match the vector function with its graph (labeled A–H).

7. $\mathbf{r}(t)=\left\langle \frac{1}{2}t\cos t, \frac{1}{2}t\sin t, \frac{t}{2}\right\rangle;\quad t\in[0,6\pi]$

8. $\mathbf{r}(t)=\left\langle 5\sin^2 t, \cos^2 t, t\right\rangle;\quad t\in[0,4\pi]$

9. $\mathbf{r}(t)=\langle \ln t, \sin t, \cos t\rangle;\quad t\in(0,6\pi)$

10. $\mathbf{r}(t)=\left\langle 3\cos(t^2), 3\sin(t^2), 3\sqrt{t}\right\rangle;\quad t\in[0,4\pi]$

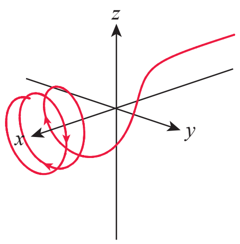
11. $\mathbf{r}(t)=\left\langle \cos\sqrt{t}, \sin\sqrt{t}, \frac{t}{50}\right\rangle;\quad t\in[0,36\pi^2]$

12. $\mathbf{r}(t)=\left\langle 2\sin t, 4\cos t, \frac{t}{4}\right\rangle;\quad t\in[0,6\pi]$

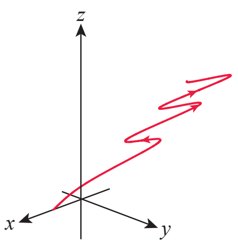
13. $\mathbf{r}(t)=\langle 1+t^2, 2t, 2t\rangle;\quad t\in[-6,25]$

14. $\mathbf{r}(t)=\left\langle \cos t, \sqrt{t}, \sqrt{t}\right\rangle;\quad t\in[0,6\pi]$

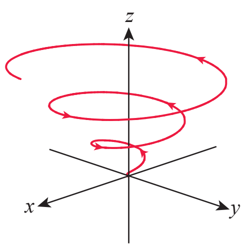
A.



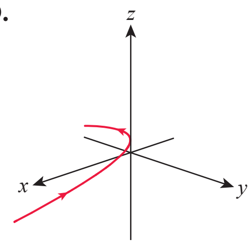
B.



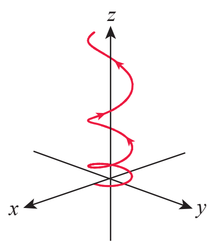
C.



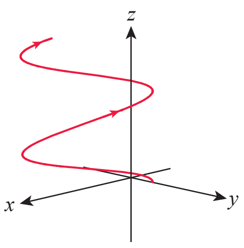
D.



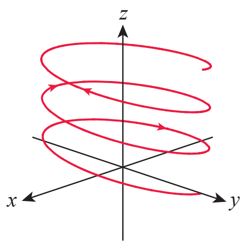
E.



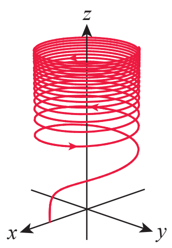
F.



G.



H.



15–22 Sketch the space curve by hand.
(**Hint:** See Exercises 7–14.)

15. $\mathbf{r}(t)=\left\langle \sin t, \cos t, \sqrt{t}\right\rangle;\quad t\in[0,4\pi]$

16. $\mathbf{r}(t)=\langle 2t\cos t, 3t\sin t, t\rangle;\quad t\in[0,6\pi]$

17. $\mathbf{r}(t)=\langle t, t, 3\sin t\rangle;\quad t\in[-2\pi,2\pi]$

18. $\mathbf{r}(t)=\left\langle 2\cos(t^2), 4\sin(t^2), 2\sqrt{t}\right\rangle;\quad t\in\left[0,2\sqrt{3\pi}\right]$

19. $\mathbf{r}(t)=\langle \cos t, \sin t, \ln t\rangle;\quad t\in\left[\frac{1}{e^3},6\pi\right]$

20. $\mathbf{r}(t)=\left\langle \frac{t}{3}, \frac{3\cos 2t}{t}, \frac{3\sin 2t}{t}\right\rangle;\quad t\in(0,4\pi)$

21. $\mathbf{r}(t)=\left\langle \frac{t}{2}, 4\sin t, 2\cos t\right\rangle;\quad t\in[0,4\pi]$

22. $\mathbf{r}(t)=\langle 4\cos t, 4\sin t, \cos 10t\rangle;\quad t\in[0,2\pi]$

23–29 Describe the intersection of the surfaces as a vector function. (Use the suggested parameter.)

23. The elliptic cylinder $2x^2+3y^2=6$ and the plane $3x+2z=2$ $(x=\sqrt{3}\cos t)$

24. The paraboloid $x^2+y^2=z$ and the plane $x+z=2$ $(z=t)$

25. The cylinder $x^2+y^2=1$ and the hyperbolic paraboloid $2x^2-y^2=z$ $(x=\cos t)$

26. The cylinder $x^2+y^2=9$ and the surface $y=z/x$ $(x=3\cos t)$

27. The elliptic paraboloid $2x^2+y^2=2z$ and the parabolic cylinder $y^2=x$ $(y=t)$

28. The cone $x^2+y^2=z^2$ and the plane $2z=y+4$ $(y=t)$

29. The semiellipsoid $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{8}=1,\ z\ge 0$ and the parabolic cylinder $2x=y^2$ $(y=t)$

30–35 Determine whether the indicated limit exists. If so, find it.

30. $\lim_{t\to -1}\left\langle t^2-2t, \sqrt{t+5}, \frac{1}{t}\right\rangle$

31. $\lim_{t\to 1}\left\langle e^{-3t}, \sqrt{t-2}, \cos 2t\right\rangle$

32. $\lim_{t\to 3}\left\langle \frac{t+2}{t-3}, \ln t, \cot \pi t\right\rangle$

33. $\lim_{t\to 0}\left\langle \ln\left(t^2+1\right), |t|, 2^t\right\rangle$

34. $\lim_{t\to 0}\left\langle \sqrt{1-\cos t}, e^{\tan t}, \frac{2\sin t}{t}\right\rangle$

35. $\lim_{t\to 0}\left\langle \frac{2t}{t+1}, \ln\left(t+1\right), \sin\frac{\pi}{t}\right\rangle$

36–39 Find any discontinuities of the given vector function.

36. $\mathbf{r}(t)=e^t\,\mathbf{i}+\frac{2}{t+1}\mathbf{j}+3t^2\,\mathbf{k}$

37. $\mathbf{r}(t)=2t^2\,\mathbf{i}-5|t|\,\mathbf{j}+\cos t\,\mathbf{k}$

38. $\mathbf{r}(t)=\mathbf{i}+\frac{t}{t^2+1}\mathbf{j}-\cot t\,\mathbf{k}$

39. $\mathbf{r}(t)=\left(t^3-1\right)\mathbf{i}-\sin\frac{\pi}{t}\mathbf{j}-\sqrt{t^2+2}\,\mathbf{k}$

40. Prove: If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector functions so that both have limits at $t=t_0$, then the limit of their dot product is the dot product of their limits, that is,

$$\lim_{t\to t_0}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\lim_{t\to t_0}\mathbf{u}(t)\cdot\lim_{t\to t_0}\mathbf{v}(t).$$

41. Prove: If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector functions so that both have limits at $t=t_0$, then the limit of their cross product is the cross product of their limits, that is,

$$\lim_{t\to t_0}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\lim_{t\to t_0}\mathbf{u}(t)\times\lim_{t\to t_0}\mathbf{v}(t).$$

(**Hint:** Use the determinant rule for determining cross products.)

42. Prove that the differentiability of a vector function implies its continuity, that is, if $\mathbf{u}(t)$ is differentiable at $t=t_0$, then $\mathbf{u}(t)$ is continuous at $t=t_0$.

43. Give a rigorous proof of the fact that the vector function $\mathbf{r}(t)=\left\langle f(t), g(t), h(t)\right\rangle$ is continuous if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous.

44. Prove that if the vector function $\mathbf{r}(t)$ is continuous at $t=t_0$, then the scalar function $|\mathbf{r}(t)|$ is also continuous at $t=t_0$.

45. Prove that the converse of Exercise 44 is false by finding a vector function $\mathbf{r}(t)$ with a discontinuity at $t=t_0$ so that $|\mathbf{r}(t)|$ is continuous at $t=t_0$.

46–51 Find $\mathbf{r}'(t)$.

46. $\mathbf{r}(t)=t^2\,\mathbf{i}+2\sqrt{t}\,\mathbf{j}-t\,\mathbf{k}$

47. $\mathbf{r}(t)=\left(2-t^3\right)\mathbf{i}-\pi^2\,\mathbf{j}+\frac{t^5}{5}\mathbf{k}$

48. $\mathbf{r}(t)=\sin t\,\mathbf{i}+e^t\,\mathbf{j}-\tan t\,\mathbf{k}$

49. $\mathbf{r}(t)=\ln t\,\mathbf{i}-\csc t\,\mathbf{j}+\sqrt{4-t^2}\,\mathbf{k}$

50. $\mathbf{r}(t)=\left\langle \frac{1}{\sqrt[3]{t^2}}, \arctan 2t, \sin^3 t\right\rangle$

51. $\mathbf{r}(t)=\left\langle \cos\left(t^2+1\right), \frac{t+1}{t-1}, 3\arcsin t\right\rangle$

52–55 Find a unit vector that is tangent to the graph of the vector function at the specified value of t.

52. $\mathbf{r}(t)=t\,\mathbf{i}-2t^2\,\mathbf{j}+2\sqrt{t}\,\mathbf{k}; \quad t=1$

53. $\mathbf{r}(t)=2\sin t\,\mathbf{i}-e^t\,\mathbf{j}+8\sqrt{4+t}\,\mathbf{k}; \quad t=0$

54. $\mathbf{s}(t)=\left\langle \arctan t, -\cos^2 t, \sqrt{3t}\right\rangle; \quad t=0$

55. $\mathbf{u}(t)=\left\langle 4t, 3\sin t, 3\cos t\right\rangle; \quad t=0$

56–59 Find the vector form of an equation for the line tangent to the curve at the specified value of t.

56. $\mathbf{r}(t)=2t\,\mathbf{i}+\left(t^2-4\right)\mathbf{j}+\sqrt{t+1}\,\mathbf{k}; \quad t=3$

57. $\mathbf{r}(t)=e^{3t}\mathbf{i}-e^{2t}\mathbf{j}+e^t\,\mathbf{k}; \quad t=0$

58. $\mathbf{s}(t)=\left\langle t, \sin 2t, \cos 2t\right\rangle; \quad t=0$

59. $\mathbf{u}(t)=\left\langle \arcsin t, \arccos t, \ln t\right\rangle; \quad t=\frac{1}{2}$

60–64 Prove the indicated differentiation rule, assuming that **u** and **v** are differentiable vector functions, **C** is a constant vector, *c* is a scalar, and *f* is a differentiable scalar function.

60. Constant Vector Rule: $\frac{d}{dt}\mathbf{C}=\mathbf{0}$

61. Scalar Multiple Rules: $\frac{d}{dt}\left[c\,\mathbf{u}(t)\right]=c\,\mathbf{u}'(t)$ and $\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$

62. Sum/Difference Rules: $\frac{d}{dt}\left[\mathbf{u}(t)\pm\mathbf{v}(t)\right]=\mathbf{u}'(t)\pm\mathbf{v}'(t)$

63. Dot Product Rule: $\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

64. Chain Rule: $\frac{d}{dt}\left[\mathbf{u}(f(t))\right]=f'(t)\mathbf{u}'(f(t))$

65. Prove the following differentiation rule for the triple scalar product of vector functions: If $\mathbf{u}(t)$, $\mathbf{v}(t)$, and $\mathbf{w}(t)$ are differentiable, then

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w})=\frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w}+\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w}+\mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}.$$

66. Prove that if a point moves along a sphere, then its velocity vector is tangential to the sphere. (**Hint:** See Example 6.)

67. Prove that if a point moves along a curve in $\mathbb{R}^3$ with constant speed, then its velocity and acceleration vectors are orthogonal. (**Hint:** See Exercise 66.)

68. (A converse of Example 6) Assume $\mathbf{r}(t)$ is a differentiable vector function satisfying $\mathbf{r}'(t)\cdot\mathbf{r}(t)=0$ for all *t*. Show that $|\mathbf{r}(t)|$ is constant, that is, the graph of $\mathbf{r}(t)$ lies on a sphere centered at the origin.

69–74 Find the indefinite integral.

69. $\int\left\langle 3t^2,t^3-t,-\sqrt{t}\right\rangle dt$

70. $\int\left(\frac{t}{t^2+1}\mathbf{i}-\frac{1}{t^2+1}\mathbf{j}+\frac{t}{t^2-1}\mathbf{k}\right)dt$

71. $\int\left(t\,\mathbf{i}+3\,\mathbf{j}-4t^3\,\mathbf{k}\right)dt$

72. $\int\left(\cos t\,\mathbf{i}-2\sin t\,\mathbf{j}-\sec^2 t\,\mathbf{k}\right)dt$

73. $\int\left(2\,\mathbf{i}-\frac{1}{t}\mathbf{j}+t^{3/2}\,\mathbf{k}\right)dt$

74. $\int\left\langle\frac{1}{t^2},\ln t,-e^{-t}\right\rangle dt$

75–80 Evaluate the definite integral.

75. $\int_0^3\left[(2-t)\mathbf{i}-4\,\mathbf{j}+t^2\,\mathbf{k}\right]dt$

76. $\int_0^1\left[2t^4\,\mathbf{i}+t\,\mathbf{j}-\left(t^2-2\right)\mathbf{k}\right]dt$

77. $\int_{-1}^1\left\langle\sqrt[3]{t},t,t^4\right\rangle dt$

78. $\int_0^{\pi}\left(\sin t\,\mathbf{i}+t\sin t\,\mathbf{j}-\mathbf{k}\right)dt$

79. $\int_1^e\left\langle 2e^t,-\ln t,\frac{1}{t}\right\rangle dt$

80. $\int_0^3\left\langle\sqrt{t+1},\frac{6t}{t^2+1},\frac{-1}{(t+1)(t-4)}\right\rangle dt$

81. A projectile is launched from the ground with an initial speed of 78.48 m/s at an angle of elevation of 30° from horizontal. After determining the vector functions $\mathbf{v}(t)$ and $\mathbf{r}(t)$, as in Example 7, find the maximum altitude reached by the projectile as well as its range. (Suppose that the launch takes place in the positive *x*-direction. Use $g\approx 9.81\text{ m/s}^2$ and ignore air resistance.)

82. Use Exercise 81 to determine the effect on the maximum altitude and range of the projectile if we double its initial velocity.

83. A particle is moving in $\mathbb{R}^3$ so that its acceleration function is $\mathbf{a}(t)=\langle 2t,1,0\rangle$. Find the velocity and position functions of the particle if it starts at the point $\mathbf{r}(0)=\langle -5,0,2\rangle$ with initial velocity $\mathbf{v}(0)=\langle 3,1,-1\rangle$.

84.* Prove that the force acting on a mass moving along a circle of radius *R* with constant angular speed ω is always pointing toward the center of the circle. (Such a force is called a *center-seeking* or *centripetal* force.) (**Hint:** Parametrize the path of the object and differentiate twice to find its acceleration, then use Newton's Second Law of Motion.)

85.* The plane curve $\mathbf{r}(t)=\left\langle ae^{bt}\cos t,ae^{bt}\sin t\right\rangle$ is called a *logarithmic spiral* or *Bernoulli spiral*. One of its intriguing properties is that for any fixed point $P=\mathbf{r}(t_0)$, the corresponding radial and tangent lines form a constant angle φ. Prove this fact, and find the angle φ.

86.* The *angular momentum* (with respect to the origin) of a mass m that is moving along a space curve $\mathbf{r}(t)$, is defined as

$$\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$$

Use Newton's Second Law to demonstrate that $\boldsymbol{\tau}$, the net external torque acting on m, is equal to the derivative of $\mathbf{L}(t)$; that is,

$$\boldsymbol{\tau} = \mathbf{L}'(t).$$

(**Hint:** Use the fact that torque is the cross product of the displacement vector and the force vector, that is, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.)

12.1 Technology Exercises

87–92. Use the integration capabilities of a graphing utility to verify your answers to Exercises 69–74.

93–98. Use the integration capabilities of a graphing utility to verify your answers to Exercises 75–80.

99–104. Use the `Limit` command of a graphing utility to verify your answers to Exercises 30–35.

105–112. Use a graphing utility to graph the curves of Exercises 15–22.

113–119. Use a graphing utility to display the curves of Exercises 23–29 as intersections of the given surfaces.

12.2 Exercises

1–10 Find the arc length of the curve over the given interval.

1. $\mathbf{r}(t)=\langle 4\cos t, 4\sin t, 2t\rangle$; $[0,6\pi]$
2. $\mathbf{r}(t)=\langle t, 2\sin t, 2\cos t\rangle$; $[0,4\pi]$
3. $\mathbf{r}(t)=\langle 2-5t, 1+3t, 4-\sqrt{2}t\rangle$; $[0,3]$
4. $\mathbf{r}(t)=\langle e^t, 2e^t\cos t, 2e^t\sin t\rangle$; $[0,1]$
5. $\mathbf{r}(t)=\langle 2t, t^2, \ln t\rangle$; $[1,e]$
6. $\mathbf{r}(t)=\left\langle \frac{t^3}{3}, t^2, 2t\right\rangle$; $[0,2]$
7. $\mathbf{r}(t)=\langle t^3, \sqrt{6}t^2, 4t\rangle$; $[0,5]$
8. $\mathbf{r}(t)=\langle t^2, 2t\sin t, 2t\cos t\rangle$; $[0,1]$
9. $\mathbf{r}(t)=\left\langle \frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t, -\ln(\cos t)\right\rangle$; $\left[0,\frac{\pi}{3}\right]$
10. $\mathbf{r}(t)=\langle t-\tanh t, \operatorname{sech} t\rangle$; $[0,\ln 3]$ (This curve is called a *tractrix*.)

11–16 Reparametrize the given curve with respect to arc length.

11. The line $\mathbf{r}(t)=\langle 1+2t, 3-5t, 4+4t\rangle$
12. The circle $\mathbf{r}(t)=\langle 0, 4\cos t, 4\sin t\rangle$
13. The helix $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 3t\rangle$
14. The curve $\mathbf{r}(t)=\langle t, \cosh t, \sinh t\rangle$
15. The helix $\mathbf{r}(t)=\langle bt, a\cos\omega t, a\sin\omega t\rangle$
16. The curve $\mathbf{r}(t)=\langle e^t, e^t\cos t, e^t\sin t\rangle$

17–24 Find the unit tangent vector for the given curve.

17. $\mathbf{r}(t)=\langle t+1, t^3, -t^2\rangle$
18. $\mathbf{r}(t)=\langle 2t, \cos t, \sin t\rangle$
19. $\mathbf{r}(t)=\langle e^t, e^t\cos t, e^t\sin t\rangle$
20. $\mathbf{r}(t)=\left\langle \frac{1}{3}t^3, \frac{1}{t}, \sqrt{2}t\right\rangle$
21. $\mathbf{r}(t)=\langle t\cos t-\sin t, t, t\sin t+\cos t\rangle$
22. $\mathbf{r}(t)=\langle bt, a\cos\omega t, a\sin\omega t\rangle$
23. $\mathbf{r}(t)=\langle e^t, e^{-t}, \sqrt{2}t\rangle$
24. $\mathbf{r}(t)=\left\langle t, \sqrt{2-2t^2}, t\right\rangle$

25. Find an arc length parametrization of the straight line $y=mx+b$.

26. A circle of radius 5 is located in the plane $x=3$, centered at $(3,2,1)$. Find an arc length parametrization for this circle.

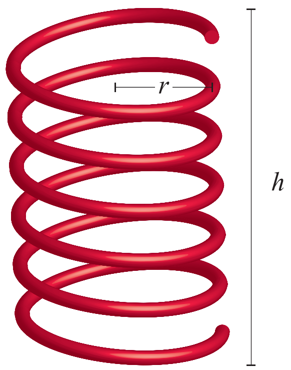
27. The following are all parametrizations of the same helix. Which one is the arc length parametrization?
   a. $\mathbf{r}(t)=\left\langle \cos t, \sin t, \frac{t}{3}\right\rangle$
   b. $\mathbf{r}(t)=\langle \cos 3t, \sin 3t, t\rangle$
   c. $\mathbf{r}(t)=\left\langle \cos\frac{3t}{\sqrt{10}}, \sin\frac{3t}{\sqrt{10}}, \frac{t}{\sqrt{10}}\right\rangle$
   d. $\mathbf{r}(t)=\left\langle \cos\sqrt{10}t, \sin\sqrt{10}t, \frac{\sqrt{10}t}{3}\right\rangle$

28. Suppose a bug starts crawling at $(0,0,0)$ along the curve $\mathbf{r}(t)=\left\langle 3t, 2t^2, 4\sqrt{2/3}\,t^{3/2}\right\rangle$. After crawling exactly 5 unit lengths, it runs into a spider web. Find the coordinates of the point where the curve pierces the spider web.

29. Recall from Section 9.1 the parametrization of the first full arch of the cycloid: $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$, $\theta\in[0,2\pi]$. Find the arc length parametrization of this curve.

30. One version of the Bernoulli spiral can be parametrized as $\mathbf{r}(t)=\langle e^t\cos t, e^t\sin t\rangle$ (see Exercise 85 of Section 12.1). Find the arc length parametrization of this curve. (**Hint:** For the lower limit of integration in calculating $s(t)$, use $-\infty$.)

31.* Suppose a spring has radius r and it reaches height h while making n full revolutions. Find a formula for the length of the wire used in manufacturing this spring.



32. Suppose you calculate the arc length parametrization for the curve $\mathbf{r}(t)=\langle f(t), g(t), h(t)\rangle$, while your classmate does the same, but she starts out with the parametrization $\mathbf{r}(t)=\langle f(t^3), g(t^3), h(t^3)\rangle$. Do you obtain equivalent answers? Explain.

33. Use the vector function given in Example 5, $\mathbf{r}(t)=\langle (v_0\cos\theta)t+x_0, -\frac{1}{2}gt^2+(v_0\sin\theta)t+y_0\rangle$, to find a formula for the range of the projectile, assuming it was launched at ground level. (The range of the projectile is the distance between its launching and landing points.)

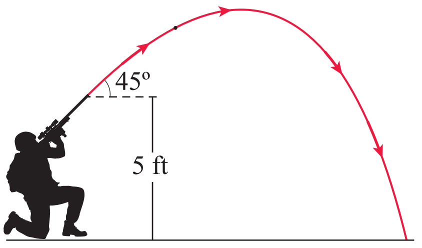
34. Use the formulas found in Example 5 and Exercise 33 to revisit Exercise 81 of Section 12.1.

35. Use Exercise 33 to find the angle θ which corresponds to the maximum range for the projectile.

36.* Repeat Exercise 35 to find the angle θ that corresponds to the maximum downhill range if the projectile is launched on a downhill terrain that drops at an angle of δ from horizontal. (θ is still the angle of elevation measured from horizontal.)

37.* Repeat Exercise 36 to find the angle θ that corresponds to the maximum uphill range if the projectile is launched on an uphill terrain with an angle of elevation of φ from horizontal.

38. A pellet is shot from an air rifle with a muzzle velocity of 1200 ft/s, leaving the rifle 5 ft above ground level and at a 45° angle of elevation. Assuming the surrounding terrain is flat and level, how far does the pellet travel, and with what speed does it hit the ground? (As usual, ignore air resistance.)



39. A projectile is launched from a 2 m high platform with an initial speed of 30 m/s, in a direction 60° upward from horizontal. Ignore air resistance.

a. Find a vector function that models the projectile's path.

b. Find the maximum height attained by the projectile, its range, and the speed of impact. (**Hint:** Suppose the projectile was launched from the point $(0,2)$.)

40. Answer the questions of Exercise 39, assuming that the launch took place on the moon. (Use $g/6$ for the acceleration caused by gravity near the moon's surface.)

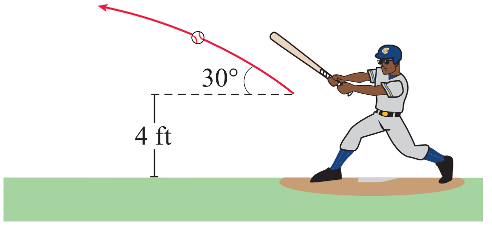
41. The exit velocity of a baseball (its velocity as it leaves the bat) is 128 feet per second, in the direction of 30° above horizontal. If it was hit 4 feet above ground level, find

a. a vector function that models the path of the baseball,

b. the maximum height attained by the baseball,

c. the horizontal distance traveled by the ball and its speed of impact.

(Ignore air resistance.)



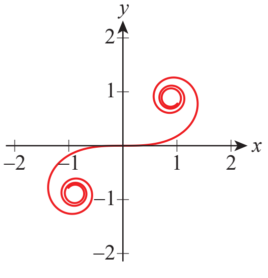
42. A golf ball is hit with an initial speed of 140 feet per second, 50° upward from horizontal, toward a hole 158 yards away (when measured horizontally). If the elevation of the hole is 121 feet higher than that of the starting point of the golf ball, will the ball land in the hole? (**Hint:** Use a vector function to examine the trajectory of the golf ball. Ignore air resistance.)

43.* A projectile is launched from a 6-foot platform with an initial speed of 200 feet per second and at a firing angle of x degrees above horizontal. Find the value of x that will result in a range of 1000 feet. (Ignore all retarding forces but gravity. Express your answer in degrees.)

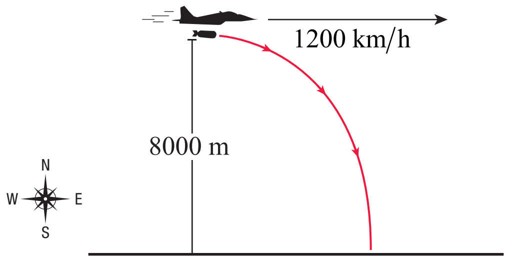
44. The curve pictured below is a *Cornu spiral* over the interval $[-2\pi, 2\pi]$ (also known as *Euler's spiral*, though initially discovered by Johann Bernoulli). It is defined by

$$\mathbf{r}(t)=\left\langle \int_0^t \cos\frac{u^2}{2}\,du, \int_0^t \sin\frac{u^2}{2}\,du \right\rangle.$$

Find the arc length of the Cornu spiral over the interval $[0,b]$.



45. A bomber plane is flying eastward at a speed of 1200 kilometers per hour when it releases a bomb at an altitude of 8000 meters. Ignoring air resistance, find a vector function modeling the path of the bomb, the horizontal distance traveled by the bomb, and its speed of impact. (**Hint:** Suppose the bomb was released at the point $(0, 8000)$, and that the x-axis points to the east.)

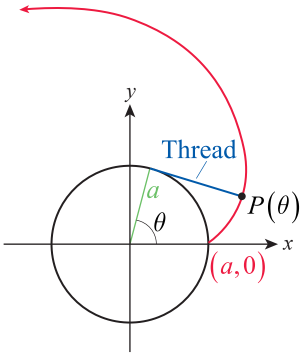


46. Suppose a certain type of medieval cannon, at a 10-degree angle of elevation, is able to fire to a distance of 2400 feet. Find the initial speed of the cannonball. What maximum range can be achieved with this type of cannon? (**Hint:** To simplify matters, suppose that the cannon ball was shot from ground level. Ignore air resistance. See Exercises 33 and 35.)

47.* When air resistance is taken into consideration, assuming it is proportional to the projectile's velocity, then the vector function $\mathbf{r}(t)$ for the projectile's motion (see Example 5) satisfies the differential equation $\frac{d^2}{dt^2}\mathbf{r}(t) = -C\frac{d}{dt}\mathbf{r}(t) - g\,\mathbf{j}$, where C is the drag coefficient. Assuming that the projectile is launched from the origin (i.e., $x_0 = y_0 = 0$), find $\mathbf{r}(t)$ under these conditions.

48. Suppose that the path of a moving point is a straight line. Prove that in this case, $\mathbf{T}'(t) = \mathbf{0}$.

49.* If we unwind a thread from a fixed circular spool of radius a, starting at the point $(a, 0)$ and keeping the thread taut in the xy-plane throughout the process, the curve traced out by the endpoint of the thread is called the *involute* of the circle. (**Note:** In the figure below, the point $P(\theta)$ denotes the endpoint of the thread at the instant when the radius to the point of tangency makes an angle of θ with the positive x-axis.)



a. Use the figure to derive the following parametrization of the involute of a circle of radius a.

$$\mathbf{r}(\theta) = a\langle \cos\theta + \theta\sin\theta, \sin\theta - \theta\cos\theta \rangle,\ \theta > 0$$

b. Reparametrize the involute with respect to arc length.

Concept Check

50–54 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

50. The circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ has the property that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular for all t.

51. The helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ has the property that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular for all t.

52. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $f(t)$, $g(t)$, and $h(t)$ are linear polynomials, then $\mathbf{T}(t)$ is constant.

53. When the launching speed of a projectile is doubled, its range doubles. (Suppose it is launched at angle α, $0 < \alpha < \pi/2$, upward from horizontal.)

54. For a space curve $\mathbf{r}(t)$, we have $|\mathbf{r}(t)|' = |\mathbf{r}'(t)|$.

Normal Component	$a_N = \dfrac{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert}{\lvert \mathbf{r}' \rvert}$
Third Derivative	$\mathbf{r}''' = \left[s''' - \kappa^2 (s')^3 \right]\mathbf{T} + \left[3\kappa s' s'' + \kappa' (s')^2 \right]\mathbf{N} + \kappa\tau (s')^3 \mathbf{B}$
Curvature	$\kappa = \left\lvert \dfrac{d\mathbf{T}}{ds} \right\rvert = \dfrac{\lvert \mathbf{T}' \rvert}{\lvert \mathbf{r}' \rvert} = \dfrac{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert}{\lvert \mathbf{r}' \rvert^3}$
Torsion	$\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert^2}$
Frenet-Serret Formulas	$\dfrac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$, $\dfrac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}$, and $\dfrac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$

Table 1

12.3 Exercises

1–10 Find the unit tangent, normal, and binormal vectors for the given curve.

1. $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$
2. $\mathbf{r}(t) = \langle \sin t, \cos t, \sqrt{3}t + 1 \rangle$
3. $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$
4. $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$
5. $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$
6. $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$
7. $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$
8. $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$
9. $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$
10. $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$

11–20 Use the results of Exercises 1–10 to determine equations for the osculating, normal, and rectifying planes associated with the curve at the indicated point.

11. $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$; $t = 0$
12. $\mathbf{r}(t) = \langle \sin t, \cos t, \sqrt{3}t + 1 \rangle$; $t = \frac{\pi}{2}$
13. $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$; $t = 0$
14. $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$; $t = 1$
15. $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$; $t = \frac{1}{2}$
16. $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$; $t = 2$
17. $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$; $t = \pi$
18. $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$; $t = 0$
19. $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$; $t = \frac{\pi}{2}$
20. $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$; $t = \frac{\pi}{4}$

21–28 Use Exercises 3–10 to find a parametric description of the surface.

21. The ribbon of width $\frac{1}{2}$ centered on $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$
22. The ribbon of width 2 centered on $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$
23. The circular tube of radius 1 centered on $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$
24. The circular tube of radius $\frac{1}{3}$ centered on $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$
25. The elliptical tube of major axis 0.6 (in the normal direction) and minor axis of 0.4 centered on $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$
26. The ribbon of width $\frac{1}{4}$ centered on $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$
27. The hypocycloid $x = \cos t + 2\cos(t/2)$ and $y = \sin t - 2\sin(t/2)$, $-2\pi < t \le 2\pi$ wrapped around $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$
28. The circular tube of radius $\frac{1}{8}$ centered on $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$

29. Show how the fact that the unit tangent vector function $\mathbf{T}(t)$ corresponding to a smooth curve $\mathbf{r}(t)$ has constant length implies that $\mathbf{T}\cdot\mathbf{T}'=0$ for all t.

30. Show that if $\mathbf{r}(t)$ is a plane curve, then $\mathbf{N}(t)$ always points toward its "concave side," that is, "in the direction the curve bends."

31–33 Use your reparametrization results from Exercises 11, 13, and 14 of Section 12.2 to calculate the curvature and torsion functions for these curves.

31. $\mathbf{r}(t)=\langle 1+2t, 3-5t, 4+4t\rangle$

32. $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 3t\rangle$

33. $\mathbf{r}(t)=\langle t, \cosh t, \sinh t\rangle$

34. Find the curvature of the ellipse

$$\mathbf{r}(t)=\langle 4\cos t, 3\sin t, 0\rangle$$

at $t=\pi/2$. What about $t=0$?

35. Prove that the curvature of the helix $\mathbf{r}(t)=\langle a\cos t, a\sin t, bt\rangle$ is obtained by the following formula.

$$\kappa=\frac{a}{a^2+b^2}$$

What can you say if $b=0$?

36. Show that the torsion for the helix of Exercise 35 is $\tau=b/(a^2+b^2)$.

37. Generalize Exercises 35 and 36 to obtain formulas for the curvature and torsion functions of the general helix $\mathbf{r}(t)=\langle a\cos\omega t, a\sin\omega t, bt\rangle$.

38. Prove that if a curve $\mathbf{r}(t)$ lies in a plane, its torsion function is identically zero.

39–48 Calculate the curvature and torsion functions for the given curve. (**Hint:** Use the relevant formulas from the summary table.)

39. $\mathbf{r}(t)=\langle t, 2\sin t, 2\cos t\rangle$

40. $\mathbf{r}(t)=\langle t, 3t+2, 3t-1\rangle$

41. $\mathbf{r}(t)=\left\langle 1, t, \frac{1}{t}\right\rangle$

42. $\mathbf{r}(t)=\langle 1, t, e^t\rangle$

43. $\mathbf{r}(t)=\langle 2\cos 3t, 2\sin 3t, 1\rangle$

44. $\mathbf{r}(t)=\langle 1, t, t^2\rangle$

45. $\mathbf{r}(t)=\langle \sin 3t, \cos 3t, 4t\rangle$

46. $\mathbf{r}(t)=\langle 2e^t\sin t, 2e^t\cos t, 1\rangle$

47. $\mathbf{r}(t)=\langle e^t, e^t\cos t, e^t\sin t\rangle$

48. $\mathbf{r}(t)=\langle t\cos t+\sin t, 0, \cos t-t\sin t\rangle$

49. Prove that for the Bernoulli spiral of Exercise 85 in Section 12.1, the arc length $s(t)$ and curvature $\kappa(t)$ are inversely proportional. (**Note:** This is commonly interpreted as the arc length $s(t)$ and the radius of curvature being directly proportional; for the radius of curvature see the discussions preceding Exercises 60 and 74.)

50. If $f(x)$ is at least twice differentiable, prove that the curvature function of the plane curve $y=f(x)$ can be determined as

$$\kappa(x)=\frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}.$$

What can you conclude about the curvature of a plane curve at any of its inflection points? (**Hint:** Use $t=x$ as a parameter.)

51–53 Use Exercise 50 to evaluate the curvature of the plane curve at the indicated point.

51. $y=x^3;\quad x=1$

52. $y=\sin x;\quad x=\frac{\pi}{4}$

53. $y=\ln|\cos x|;\quad x=\frac{\pi}{6}$

54–56 Use Exercise 50 to find the point(s) of maximum curvature for the curve.

54. $f(x)=\frac{x^2}{2}$

55. $f(x)=\cos x$

56. $f(x)=e^x$

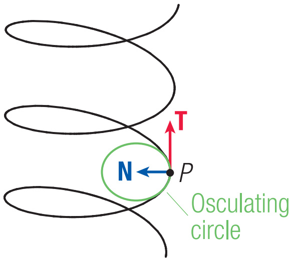
57. Generalize Exercise 50 for a plane curve $\mathbf{r}(t)=\langle x(t), y(t)\rangle$ to obtain the formula

$$\kappa(t)=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left([x'(t)]^2+[y'(t)]^2\right)^{3/2}}.$$

58. Use implicit differentiation along with Exercise 50 to find the curvature of $(6-x)y^2=2x^3$ at the point $(2,2)$. (Recall from Exercise 27 of Section 3.5 that this curve is a *cissoid*.)

59. Find the curvature of the cycloid $x=t-\sin t$, $y=1-\cos t$ at $t=\pi$.

60–68 Suppose the curvature of a curve $\mathbf{r}(t)$ at the point $P=\mathbf{r}(t_0)$ is nonzero. The **osculating circle** of the curve at P is the circle of radius $1/\kappa(t_0)$ that has the same tangent at P as $\mathbf{r}(t)$ does, and whose center lies in the direction of $\mathbf{N}(t_0)$ from P. (You can think of this as the "tangent circle." Note that it lies in the osculating plane, being tangent to $\mathbf{r}(t)$ so that both the circle's tangent line and curvature at P are the same as those of $\mathbf{r}(t)$. The osculating circle is rightfully called the "best-fitting circle" at P.)



Find the osculating circle of the graph of the equation at the indicated point.

60. $y=x^2;\quad x=0$

61. $y=x^2-1;\quad x=-\frac{1}{2}$

62. $y=\sqrt{x};\quad x=1$

63. $y=\cos x;\quad x=0$

64. $y=\cos x;\quad x=\frac{\pi}{4}$

65. $y=\frac{1}{x};\quad x=1$

66. $y=e^x;\quad x=0$

67. $y=x^3-x;\quad x=1$

68. $xy+2x+y=2;\quad x=1$

69–72 Parametrize the osculating circle of the curve at the indicated point. (**Hint:** For space curves, remember that the osculating circle lies in the plane spanned by **T** and **N**.)

69. $\mathbf{r}(t)=\langle t-\sin t, 1-\cos t\rangle;\quad t=\pi$
(See Exercise 59.)

70. $\mathbf{r}(t)=\langle \sin t, \cos t, 2t\rangle;\quad t=\pi$

71. $\mathbf{r}(t)=\left\langle t^2, t, \frac{t^3}{3}\right\rangle;\quad t=0$

72. $\mathbf{r}(t)=\left\langle 2\cos t, \frac{4}{3}\sin t, t\right\rangle;\quad t=0$

73.* Suppose that $f(x)$ is at least twice differentiable on an interval containing a, with nonzero first and second derivatives at $x=a$. Let $C(c_1,c_2)$ be the center of the osculating circle of the plane curve $y=f(x)$ at $(a,f(a))$. Prove that c_1 and c_2 can be determined as follows.

$$c_1=a-\frac{f'(a)\left(1+\left[f'(a)\right]^2\right)}{f''(a)}$$

$$c_2=f(a)+\frac{1+\left[f'(a)\right]^2}{f''(a)}$$

74–75 The center of the osculating circle of the curve $\mathbf{r}(t)$ at P is called the curve's **center of curvature** at P, while the radius of the osculating circle is the **radius of curvature** of $\mathbf{r}(t)$ at P. The locus of all centers of curvature is called the **evolute** of $\mathbf{r}(t)$. Exercises 74 and 75 will use this concept.

74. Show that the evolute of the parabola $y=x^2$ can be parametrized as $\mathbf{r}(t)=\left\langle -4t^3, 3t^2+\frac{1}{2}\right\rangle$. (**Hint:** Use Exercise 73.)

75. Find the evolute for the curve $y=x^3$ $(x>0)$.

76–87 Find the tangential and normal components of acceleration for the given position function.

76. $\mathbf{r}(t)=\langle \sin t, \cos t, 2t\rangle$

77. $\mathbf{r}(t)=\langle t, 2\sin t, 2\cos t\rangle$

78. $\mathbf{r}(t)=\langle 2t, t^2, 0\rangle$

79. $\mathbf{r}(t)=\langle 2t, t^2, t\rangle$

80. $\mathbf{r}(t)=\langle 2t, t^2, \ln t\rangle$

81. $\mathbf{r}(t)=\langle t-\sin t, 1-\cos t, 0\rangle$

82. $\mathbf{r}(t)=\left\langle \frac{1}{t}, \sqrt{2}t, \frac{1}{3}t^3\right\rangle$

83. $\mathbf{r}(t)=\left\langle 2t, t^2, \frac{t^3}{3}\right\rangle$

84. $\mathbf{r}(t)=\left\langle \frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t, -\ln(\cos t)\right\rangle$

85. $\mathbf{r}(t)=\langle e^t, e^{-t}, \sqrt{2}t\rangle$

86.* $\mathbf{r}(t)=\langle bt, a\cos\omega t, a\sin\omega t\rangle$

87.* $\mathbf{r}(t)=\langle e^t, e^t\cos t, e^t\sin t\rangle$

88. According to Newton's Second Law of Motion, the magnitude of the friction force that keeps a car from skidding out of a curve is $\mathbf{F}(t) = ma_N(t)$, where m is the mass of the car, and $a_N(t)$ is the normal component of acceleration. Find the minimum friction force needed to keep a 1500 kg car from skidding as it navigates a curve of radius 15 m, at a constant speed of 43 km/h. (Note that such a normal force, also called centripetal or center-seeking force, is needed to keep the car on a circular path.)

89. Generalize your solution to Exercise 88 to obtain a formula for the centripetal force acting on an object of mass m that moves along a circular path of radius r at a constant speed of v.

90. Suppose an object is moving along the space curve $\mathbf{r}(t)$ when at time $t = t_0$ its velocity vector is $\mathbf{v}(t_0) = \langle 32, -10, 15 \rangle$ and its acceleration is $\mathbf{a}(t_0) = \langle 2, 1, -3 \rangle$. Is the object's speed increasing or decreasing at this instant?

91. When navigating a curve, a driver should minimize the normal component of acceleration, which, if too big, could cause the car to skid. According to common advice, a driver should slow down before entering a curve, and then gently accelerate once in the curve. Use your knowledge of the acceleration vector to explain why this is sound advice. (Mention changes in the tangential and normal components of the acceleration vector.)

92.* The radius of the "Singapore Flyer," which was the world's tallest Ferris wheel from 2008 to 2014, is 75 meters. Suppose the wheel is rotating at an angular speed of 0.02 radians per second, which is increasing at a rate of 1.33×10^{-4} radians per second. Find the tangential and normal components, as well as the magnitude of acceleration of the riders, in a capsule that is at the very top of the wheel at this instant.

93. Prove the second Frenet-Serret formula,

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

(**Hint:** Noting that $d\mathbf{T}/ds = \kappa\mathbf{N}$ and $d\mathbf{B}/ds = -\tau\mathbf{N}$, use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.)

94. Prove that the tangential component of acceleration can be computed by the formula $a_T = (\mathbf{r}' \cdot \mathbf{r}'')/|\mathbf{r}'|$. (**Hint:** Making a sketch is helpful.)

95. Prove that $a_N = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$. (See the hint given in Exercise 94.)

96. Prove the following formula.

$$\mathbf{r}''' = \left[s''' - \kappa^2 (s')^3\right]\mathbf{T} + \left[3\kappa s' s'' + \kappa' (s')^2\right]\mathbf{N} + \kappa\tau (s')^3 \mathbf{B}$$

97. Prove the formula $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$. (**Hint:** Use Exercise 96.)

Concept Check

98–106 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

98. $\mathbf{T} \times \mathbf{B} = \mathbf{0}$

99. $(\mathbf{T} \times \mathbf{N}) \cdot \mathbf{B} = 0$

100. $\left|\dfrac{d(\mathbf{T} \cdot \mathbf{T})}{ds}\right| = \kappa^2$

101. The radius of the osculating circle for $y = \sin x$ at the origin is 1.

102. The acceleration vector of a particle moving on a curve $\mathbf{r}(t)$ is always in the osculating plane.

103. If we double the speed of a car in a curve, the force required to keep it from skidding is also doubled.

104. If a car is moving in a curve, then its acceleration is perpendicular to the direction of motion.

105. If an object is moving along a smooth curve that is not a straight line, then a normal force is acting on the object.

106. If the acceleration of a moving object is nonzero, and not a multiple of **N**, then its speed is changing.

12.3 Technology Exercises

107–109 Use a graphing utility to plot the curves in Exercises 79, 83, and 87. Then find and graph their respective curvature and torsion functions. Interpret these graphs in terms of your three-dimensional plot.

107. $\mathbf{r}(t) = \langle 2t, t^2, t \rangle$

108. $\mathbf{r}(t) = \left\langle 2t, t^2, \dfrac{t^3}{3} \right\rangle$

109. $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$

110. Write a program for a computer algebra system or programmable calculator that returns the parametric form of the osculating circle of a given space curve at a specified point. Use the program to check your answers for Exercises 69–72.

12.4 Exercises

1–6 Describe in terms of **u** and $\mathbf{u}_{\perp}$ the velocity and acceleration vectors of a particle with the given position function.

1. $\mathbf{r}(t)=\langle 2t\cos 2t, 2t\sin 2t\rangle$
2. $\mathbf{r}(t)=\langle (t^2+1)\cos 5t, (t^2+1)\sin 5t\rangle$
3. $\mathbf{r}(t)=\langle \sqrt{t}\cos(4t+1), \sqrt{t}\sin(4t+1)\rangle$
4. $\mathbf{r}(t)=\left\langle e^t\cos\frac{t}{2}, e^t\sin\frac{t}{2}\right\rangle$
5. $\mathbf{r}(t)=\langle 2\sin 2t\cos(t^3), 2\sin 2t\sin(t^3)\rangle$
6. $\mathbf{r}(t)=\langle a(\cos t-1)\cos bt, a(\cos t-1)\sin bt\rangle$
7. Prove that if a satellite or planet is moving in a circular orbit, then its speed is constant. (**Hint:** Use the fact that $\mathbf{C}_1$ is constant, an observation made in the proof of Kepler's First Law.)
8. Prove that in order for a moon or satellite to stay in a circular orbit of radius R around a planet of mass M, the required orbital speed is $v=\sqrt{GM/R}$. (**Hint:** Recall that v is constant by Exercise 7, thus the magnitude of acceleration is $a=v^2/R$. Use this and Newton's Second Law to finish the proof. Alternatively, use the last equation in our proof of Kepler's Third Law, noting that $a=R$.)
9. A satellite is in a circular orbit 219.2 kilometers above Earth's surface. Use Exercise 8 to find its orbital speed. Express your answer in kilometers per hour. (Approximate the radius of Earth by 6371 kilometers and its mass by 5.9736×10^{24} kilograms. Recall that $G\approx 6.6738\times 10^{-11}\ \text{Nm}^2/\text{kg}^2$.)
10. Given that the perihelion of Earth's orbit is approximately 147,098,290 km, with its *aphelion* (the distance farthest from the sun) being 152,098,232 km, and using its period of T = 365.256 days, estimate the mass of the sun. (**Hint:** Find the length of the orbit's semimajor axis first, then use the last equation in our derivation of Kepler's Third Law. For the value of G, see Exercise 9.)
11. The period of one revolution of the moon around Earth is approximately 27.3217 days, its *perigee* (distance from Earth upon closest approach) is approximately 361,400 km, while its *apogee* (its greatest distance from Earth) is about 405,000 km. Use these data to estimate the mass of Earth.
12. Use the data given in Exercise 11, along with Kepler's Third Law, to estimate the necessary height above Earth's surface for a *geostationary satellite*. (A geostationary satellite is one that is in a near-circular orbit over the equator, orbiting in the direction of Earth's rotation with a period of 24 hours, thus appearing stationary for an observer on the ground. **Hint:** While you may compute the height directly, an easier approach suggested by the problem is to compare the satellite's orbit with that of the moon and use Kepler's Third Law.)
13. Use Kepler's Third Law, along with Earth's orbital data given in Exercise 10, and Mars' period of 686.971 (Earth) days to estimate the semimajor axis of Mars' orbit.
14. The length of the semimajor axis of Neptune's orbit is 30.0476 *astronomical units* (1 astronomical unit, abbreviated AU, is equal to 149,597,870.700 km, which is approximately the mean distance between Earth and the sun). Estimate the period of Neptune in Earth years. (For orbital data on Earth, see Exercise 10.)
15. Prove that the ratio of the perihelion and aphelion of a planet is equal to the inverse ratio of its speeds at the perihelion and aphelion positions. (**Hint:** Use Kepler's Second Law, namely, the equation that says $dA/dt=\frac{1}{2}r^2\theta'=C_1/2$ is constant.)
16. Show that the minimum distance of a moon or satellite from the planet it is orbiting (the perigee) is $r_p=a(1-e)$, while the maximum distance (the apogee) is $r_a=a(1+e)$, where e is the eccentricity of the orbit. (**Hint:** In order to express ed in terms of a and b in the polar equation of the planet, see the proof of Kepler's Third Law in the text.)
17. Use Exercise 16 along with the orbital data of Earth given in Exercise 10 to find the eccentricity of Earth's orbit, then write a polar equation of the orbit (with the sun at the origin).
18. Find the aphelion of the orbit of Halley's comet and write a polar equation for the orbit given that $e\approx 0.967$ and $a\approx 17.94$ astronomical units (AU). (See Exercise 16. Use the unit AU in your answer.)
19. Find how much time passes between two consecutive visits of Halley's comet to the solar system. (See Exercises 14 and 18.)

20. Define $r_0 = |\mathbf{r}(0)|$ and $v_0 = |\mathbf{v}(0)|$, and show that with this notation,

$$A(t) = \frac{r_0 v_0}{2} t.$$

(**Hint:** Note that $v_0 = r_0 \theta'(0)$ since $r'(0) = 0$.)

21. Modify your proof of Exercise 20 to show that the equality $v = \dfrac{2\pi ab}{rT}$ holds at perigee or apogee for an orbiting planet or satellite.

22. If v_p and v_a denote a planet's speeds at perigee and apogee, respectively, prove that

$$v_p(1-e) = v_a(1+e).$$

(**Hint:** As in Exercise 20, note that $v = r\theta'$ at perigee and apogee, and that $r^2\theta' = C_1$ is a constant. Use this latter equation for both perigee and apogee positions, along with Exercise 16.)

23. Use Exercise 21 and Earth's orbital data given in Exercise 10 to find the speeds of Earth at perihelion and aphelion, respectively. Express your answer in kilometers per second, and then convert it to miles per hour.

24. Repeat Exercise 23 for Mars, if its perihelion and aphelion are 2.0662×10^8 kilometers and 2.4923×10^8 kilometers, respectively, with its period being 1.88079 years.

25. Use Exercises 11 and 21 to find the moon's speed at perigee and apogee, respectively. Express your answer in kilometers per second, then convert it to miles per hour.

26. Repeat Exercise 25 for Jupiter's moon Europa, if given that the eccentricity of its orbit is 0.0101, the length of its semimajor axis is 671,100 kilometers, and its period is 3.5512 days.

27. Repeat Exercise 26 for Halley's comet. Express your answer in the following units: astronomical units per day, kilometers per second, and miles per hour. (For orbital data, see Exercises 18 and 19.)

28.* Suppose a moon or planet is orbiting another planet (or star) of mass M. As before, let r_p denote the perigee or perihelion, while v_p is the speed at perigee or perihelion, as applicable. Prove that the orbit can be classified as an ellipse, a parabola, or a hyperbola according to the values of r_p and v_p as follows:

if $r_p v_p^2 = GM$, the orbit is a circle;

if $GM < r_p v_p^2 < 2GM$, the orbit is an ellipse;

if $r_p v_p^2 = 2GM$, the orbit is a parabola;

if $r_p v_p^2 > 2GM$, the orbit is a hyperbola.

(The last two orbit types are called *open*; such orbits are exhibited by comets entering the solar system once and then leaving it forever. **Hint:** Referring to the proof of Kepler's First Law in the text, since $e = \dfrac{C_2}{GM}$, if you show that $\dfrac{C_2}{GM} = \dfrac{r_p v_p^2}{GM} - 1$, the conclusion will follow. To that end, it will suffice to show that $C_2 = r_p v_p^2 - GM$. From the proof of Kepler's First Law, convince yourself that $r_p(GM + C_2) = C_1^2$ and note from Exercise 22 that $C_1 = r_p^2\theta'$, which will finish the proof. Finally, we note that in the case of $C_1 = 0$, the moon or planet falls along a straight line into the star or planet it is orbiting.)

29. A novice astronaut is in an elliptical orbit around Earth. In an attempt to slow down to better behold the beautiful view, she plans to apply reverse thrust to decrease (tangential) speed. However, Mission Control advises her to check her calculations, for this may actually cause the craft to go around Earth faster. Who is correct and why? Use Kepler's Third Law to provide an explanation.

30.* Suppose the astronaut of Exercise 29 erroneously applied reverse thrust and ended up in a circular orbit of radius $R = 6600$ km, instead of a planned new orbit with perigee of $r_p = 6600$ km and apogee $r_a = 10{,}000$ km. At a time determined by Mission Control, she is instructed to apply forward thrust to enter into the desired elliptical orbit. If her thrusters lend the spacecraft an acceleration of $0.0205\ \text{km/s}^2$, how long does she have to burn them in order to accomplish this task? (**Hint:** By Exercise 8, the speed in the circular orbit is $v = \sqrt{GM/R}$. Next, note that the speed at perigee in the new orbit will be $v_p = C_1/r_p$. By determining the eccentricity of the planned orbit and recalling from the proof of Kepler's Third Law that $ed = C_1^2/GM = a(1-e^2)$, you can determine C_1, and then v_p. For data on Earth, see Exercises 9 and 10.)

13.1 Exercises

1–4 Evaluate the given multivariable function at the indicated points.

1. $f(x,y)=xy-y^3x^2;\quad (0,0),\quad (2,1)$
2. $f(u,v)=(2u+1)(v-u^2);\quad (3,-1),\quad (1,1)$
3. $f(x,y,z)=\dfrac{xz}{2y^2+z^4};\quad (5,0,-1),\quad (-4,2,1)$
4. $f(t,u,v,w)=\dfrac{(4t-v)^w}{(u^2+3)};\quad (1,1,3,1),\quad (2,-1,7,12)$

5–12 Determine the domain and range of the given function and evaluate the function at the indicated point.

5. $f(x,y)=xy+yx^3;\quad (3,-1)$
6. $f(x,y)=2y^2\left(x+\dfrac{y}{2}\right);\quad (0,2)$
7. $f(x,y)=\sqrt{x^2y};\quad (2,9)$
8. $f(x,y)=\dfrac{\sqrt{36-4x^2-9y^2}}{6};\quad (2,0)$
9. $f(x,y)=\ln|xy|;\quad (e,-1)$
10. $f(x,y)=\sqrt{1-x^2y^2};\quad \left(-2,\dfrac{1}{4}\right)$
11. $f(x,y,z)=\arctan(xz-y^2);\quad (5,3,2)$
12. $f(x,y,z)=\dfrac{ze^{x/y}}{\sqrt{x-z}};\quad (0,-1,-2)$

13–20 Describe in words the graph of the function. (**Hint:** It is helpful to review quadric surfaces from Section 11.6.)

13. $f(x,y)=3x+y-2$
14. $f(x,y)=\dfrac{\sqrt{144-9x^2-16y^2}}{6}$
15. $f(x,y)=\dfrac{x^2}{4}+\dfrac{(y-1)^2}{9}$
16. $f(x,y)=\dfrac{\sqrt{9x^2+4y^2-36}}{6}$
17. $f(x,y)=\sqrt{x^2+2y^2}$
18. $f(x,y)=\sqrt{x^2+2y^2+1}$
19. $f(x,y)=2x^2+3y^2$
20. $f(x,y)=\dfrac{3x^2-6y^2}{4}$

21–26 A region R and a point P in the Cartesian plane $\mathbb{R}^2$ are given. Classify the point as an interior point of R, a boundary point, or neither.

21. $R=\{(x,y)\mid y>|x|\};\quad P(1,1)$
22. $R=\{(x,y)\mid x^2+y^2\le 1\};\quad P\left(\dfrac{1}{2},\dfrac{1}{2}\right)$
23. $R=\{(x,y)\mid 1<x^2+y^2<2\};\quad P(0,1)$
24. $R=\{(x,y)\mid y\le x^2\};\quad P(1,2)$
25. $R=\{(x,y)\mid |x+2|+|x-3|<y\};\quad P(1.5,5)$
26. $R=\{(x,y)\mid y-|2-x|\ge 0\};\quad P(3,0)$

27–34 Classify the given subset R of $\mathbb{R}^2$ as open, closed, or neither.

27. $R=\{(x,y)\mid (x-2)^2+y^2\le 4\}$
28. $R=\{(x,y)\mid x^2+2y^2\ge 3\}$
29. $R=\{(x,y)\mid 0<x^2+y^2<9\}$
30. $R=\{(x,y)\mid y\ne 2x-3\}$
31. $R=\{(x,y)\mid xy\ne 0\}$
32. $R=\{(x,y)\mid |x|+|y|\le 1\}$
33. $R=\{(x,y)\mid \sqrt{x}+\sqrt{y}<1\}$
34. $R=\{(x,y)\mid x>0 \text{ or } y\ge 0\}$

35–42 The graphs of the given equations are quadric surfaces as seen in Exercises 27–34 of Section 11.6. Express each as a function of two variables other than the pair x and y. (Note that the graph of the resulting function may not be the entire surface. Can you see why?)

35. $3z^2+2y^2=\dfrac{3x}{2}$
36. $3z^2+2y^2=1-\dfrac{3x^2}{2}$
37. $2x^2+2y^2=z$
38. $2x^2+2y^2=z^2$
39. $x^2+2x-y^2+z^2=0$
40. $x^2+2x+2y^2+3z^2=8y$
41. $x^2+2x-2y^2-3z=8y+7$
42. $-x^2+2x+2y^2-3z^2=8y+2$

43–48 Match the function with its graph (labeled A–F).

43. $f(x,y)=\dfrac{(x+y)^2}{2}$

44. $f(x,y)=\cos|xy|$

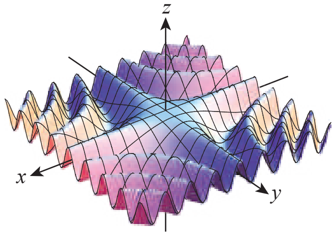
45. $f(x,y)=\dfrac{3}{4x^2+3y^2+1}$

46. $f(x,y)=|x-1|+|y|$

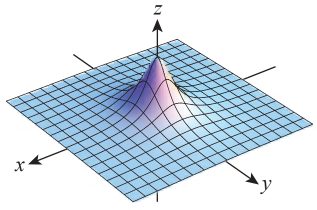
47. $f(x,y)=\sin(x+2y)$

48. $f(x,y)=\cos(x^2+y^2)$

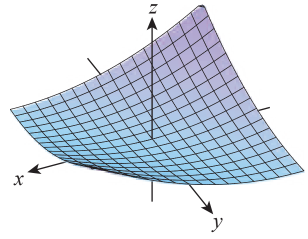
A.



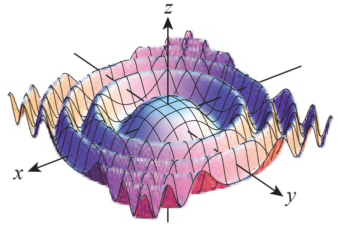
B.



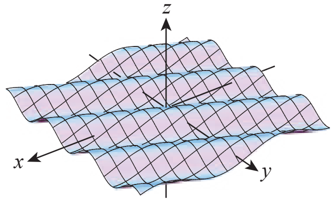
C.



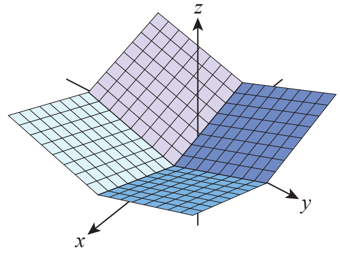
D.



E.

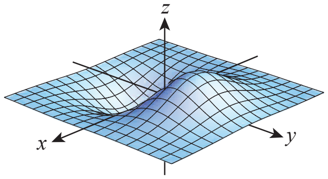


F.



49–54 Match the graph with its contour map (labeled A–F).

49.



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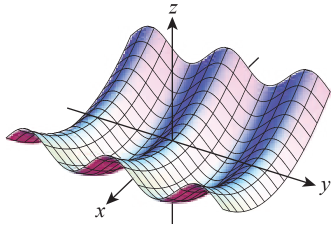

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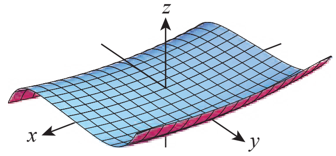

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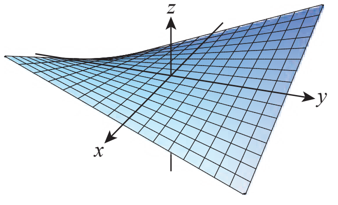

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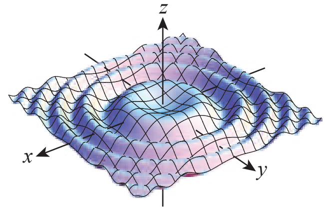

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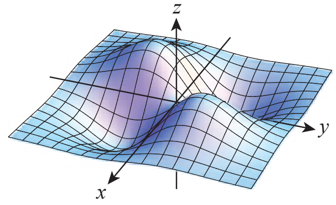
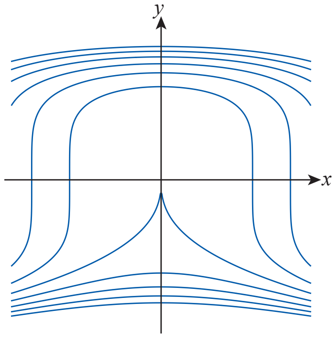
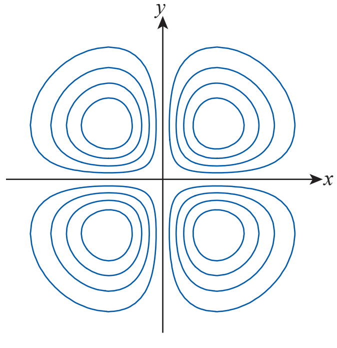


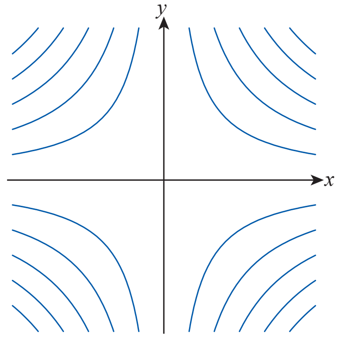
A.



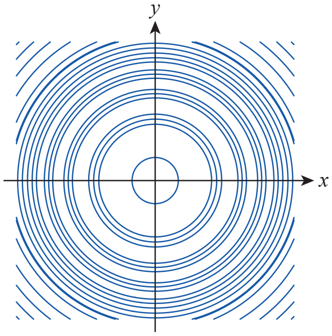
B.



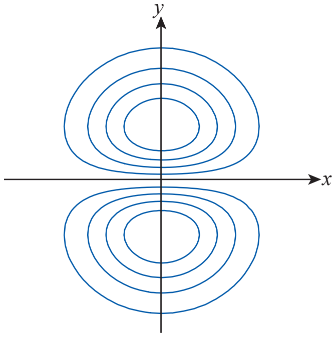
C.



D.



E.



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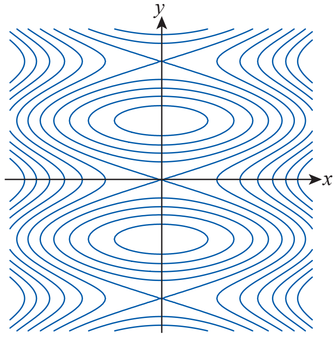


55–60 Sketch a rough graph of the given function by hand. Then sketch its contour map by selecting a few representative contours.

55. $f(x,y) = x - 3y + 1$

56. $f(x,y) = \frac{x^2}{9} + \frac{y^2}{4}$

57. $f(x,y) = y^2 - x$

58. $f(x,y) = \ln(x^2 + y^2)$

59. $f(x,y) = \frac{y}{x^2 + y^2 + 1}$

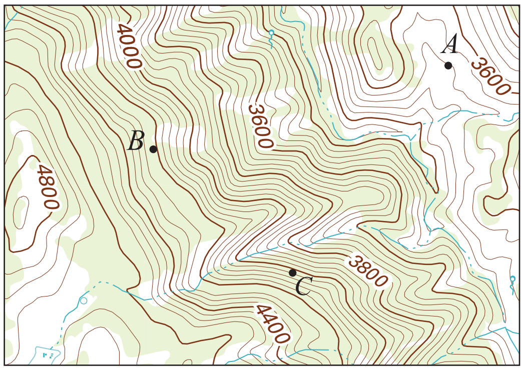
60. $f(x,y) = \sin\sqrt{x^2 + y^2}$

61–62 Describe the level surfaces of the given three-variable function.

61. $f(x,y,z) = x^2 + 2y^2 + 3z^2$

62. $f(x,y,z) = x^2 + 2y^2 - 3z^2$

63. The figure below shows a portion of a topographical map of an area near Julian, California. Examine the map and answer the following.



Source: www.usgs.gov

a. Estimate the direction of the steepest slope from point *A*.

b. Find a possible "steepest path" from point *A* to point *B*.

c. Estimate the elevation of point *C*.

d. Find a point *D* where the northern direction is uphill, while it is downhill to the southwest.

64.* Prove that if S is a finite subset (i.e., a set consisting of finitely many points) of $\mathbb{R}^2$, then $\mathbb{R}^2 - S$ is an open subset of $\mathbb{R}^2$. (**Hint:** Pick an arbitrary point P in $\mathbb{R}^2 - S$ and prove that it is an interior point.)

65.* Prove that the open interval $(0,1)$, when viewed as a subset of $\mathbb{R}^2$, is neither open nor closed.

13.1 Technology Exercises

66–71. Use a computer algebra system to generate the graphs and contour maps of the functions in Exercises 55–60.

13.2 Exercises

1–4 Use the limit laws to find the indicated limits, assuming that $\lim_{(x,y)\to(a,b)} f(x,y) = 4$ and $\lim_{(x,y)\to(a,b)} g(x,y) = -1$.

1. $\lim_{(x,y)\to(a,b)} \left[f(x,y) - 2g(x,y)\right]$

2. $\lim_{(x,y)\to(a,b)} \left[5f(x,y)g(x,y)\right]$

3. $\lim_{(x,y)\to(a,b)} \left[\frac{4g(x,y)}{f(x,y)} + f(x,y)\right]$

4. $\lim_{(x,y)\to(a,b)} \frac{3f(x,y)+g(x,y)}{g(x,y)}$

5–22 Determine whether the indicated limit exists. If so, find it.

5. $\lim_{(x,y)\to(0,0)} \frac{x\sqrt{y} - y^{3/2}}{\sqrt{x}+\sqrt{y}}$

6. $\lim_{(x,y)\to(1,2)} \frac{\sqrt{x} - \sqrt{y-1}}{x^2 - xy + x}$

7. $\lim_{(x,y)\to(0,0)} \frac{(x+1)\left(\sqrt{y+5}-\sqrt{5}\right)}{xy+y}$

8. $\lim_{(x,y)\to(0,0)} \left(2x - y^2\right)$

9. $\lim_{(x,y)\to(2,-1)} \frac{x+2y}{y+3x^2}$

10. $\lim_{(x,y)\to(1,4)} \left(2xy^2 + 5x^4\sqrt{y}\right)$

11. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

12. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

13. $\lim_{(x,y)\to(0,0)} \frac{(1-\cos x)y}{x^4+y^4}$

14. $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^3-y^3}$

15. $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$

16. $\lim_{(x,y)\to(1/2,2)} \arctan\left(\frac{xy}{y-1}\right)$

17. $\lim_{(x,y)\to(0,0)} \frac{2\sqrt{x}y^3}{x^3+y^4}$

18. $\lim_{(x,y)\to(1,0)} \frac{e^{xy}+e^{-\sqrt{y}}}{2xy+1}$

19. $\lim_{(x,y)\to(0,0)} \sin\frac{2}{x^2+y^2}$

20. $\lim_{(x,y)\to(0,0)} \frac{2\sin\left(x^2+y^2\right)}{x^2+y^2}$

21. $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-3y^2+4z^2}{x^2+3y^2+4z^2}$

22. $\lim_{(x,y,z)\to(0,0,0)} \frac{5xy^4}{x^4+y^4+z^4}$

23. Find the domains of the functions in Exercises 5 and 6.

24. With reference to the function of Example 1,

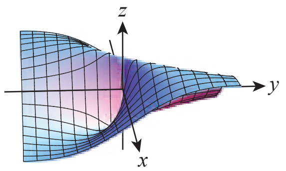
$$f(x,y) = \frac{x^2-y^2}{x^2+y^2},$$

show that if the origin is being approached in the xy-plane along the lines $y = mx$ for various values of m, all limiting values between -1 and 1 are possible.

25. Recall the following function introduced in the conclusion of Example 3:

$$g(x,y) = \frac{5xy}{x^2+y^2}.$$

Show that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. (**Hint:** See Exercise 24.)



26. Show that the limit of the function

$$f(x,y) = \frac{x^4y^4}{\left(x^2+y^4\right)^3}$$

does not exist at $(0,0)$. (**Hint:** In addition to the line $y = x$, consider the limit along the curve $y^2 = x$. Explain why it would not be helpful to restrict the paths of approach to straight lines through the origin.)

27. Show that the function

$$g(x,y) = \frac{2x^{3/2}y}{x^3+y^2}$$

has a limiting value of 0 when the origin is approached in the xy-plane along any parabola $y = cx^2$, but any limiting value between -1 and 1 can be achieved by considering the curves $y = cx^{3/2}$.

28. Prove directly that the limit in Example 3 is 0. (**Hint:** First argue that if $x = 0$, the function is 0 on its domain, and in the case of $x \neq 0$, divide the numerator and denominator by x^2 to obtain an expression whose limit is 0 as $y \to 0$.)

29. Use the Squeeze Theorem to give a new proof for Exercise 28. (**Hint:** Notice that $0 \le \left|\frac{x^2}{x^2+y^2}\right| \le 1$, and multiply this inequality by $|5y|$; then apply the Squeeze Theorem.)

30. Prove or disprove: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = 0$.

31. Prove that for $h(x,y) = 5xy/\sqrt{x^2+y^2}$, we have $\lim_{(x,y)\to(0,0)} h(x,y) = 0$, and contrast this with $g(x,y)$ of Exercise 25. (**Hint:** Rewrite $h(x,y)$ using polar coordinates, so that $x = r\cos\theta$ and $y = r\sin\theta$ and see what happens as $r \to 0$.)

32. The following function looks algebraically similar to $f(x,y)$ of Example 1.

$$k(x,y) = \frac{x^3 - y^3}{x^2 + y^2}$$

Show that $\lim_{(x,y)\to(0,0)} k(x,y) = 0$. (See the hint given in Exercise 31.)

33. Consider the polar coordinate approach of Exercise 31 and use it to find a new proof of the fact that the limit in Example 3 is 0.

34.* Find three different proofs of the fact that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \ln(x+1)}{x^2+y^2} = 0.$$

(**Hint:** Use a direct proof, then the Squeeze Theorem, and finally a polar coordinate approach.)

35.* Use an ε-δ argument to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{2(x^2+y^2)} = 0.$$

36–39 Prove that the indicated limit exists, and find its value.

36. $\lim_{(x,y)\to(0,0)} \frac{4y^2 \sin x}{x^2+4y^2}$

37. $\lim_{(x,y)\to(0,0)} \frac{x^4 - 3x^2 - 9y^2}{x^2+3y^2}$

38. $\lim_{(x,y)\to(0,0)} \frac{4xy - \sqrt{4x^2+4y^2}}{\sqrt{x^2+y^2}}$

39. $\lim_{(x,y,z)\to(0,0,0)} \frac{2y^5}{x^4+y^4+z^4}$

40. Is there a value of a so that $\lim_{(x,y)\to(a,a-1)} \frac{x}{y-x+1}$ exists? Explain.

41. For $\alpha, \beta \ge 0$, prove that $\lim_{(x,y)\to(0,0)} \frac{x^\alpha y^\beta}{x^2+y^2} = 0$ if $\alpha + \beta > 2$; otherwise the limit does not exist. Can you generalize this result? Use it to revisit Exercises 28, 30, and 35.

42. Use an ε-δ argument to show that $\lim_{(x,y)\to(a,b)} (x+y) = a+b$.

43. Suppose that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $L > 0$. Prove that there exists a δ-neighborhood of (a,b) such that $f(x,y)$ is positive for every point (x,y) in that neighborhood.

44–53 Determine where the given function is continuous.

44. $f(x,y) = \frac{2}{1+x^2+y^2}$

45. $g(x,y) = \frac{3y^2}{(x^2+3)(y^2+3)}$

46. $h(x,y) = \cot^{-1}\left(\frac{2y}{\sqrt{x+y}}\right)$

47. $r(x,y) = \sqrt{e^{xy}}$

48. $s(x,y) = 5\sqrt{x}y^2 + \frac{3}{\ln(x^2y^2)}$

49. $k(x,y) = e^{\arctan(2x^2y)}$

50. $q(x,y) = \frac{x^2}{\sqrt{4-x^2-y^2}}$

51. $m(x,y,z) = \ln(3z - x^2 - 2y^2)$

52. $n(x,y,z) = \sqrt{z^2 - x^2 - 2y^2}$

53. $p(x,y,z)$

$$= \begin{cases} \dfrac{1-\cos\sqrt{9-x^2-y^2-z^2}}{\sqrt{9-x^2-y^2-z^2}} & \text{if } x^2+y^2+z^2 < 9 \\ 0 & \text{if } x^2+y^2+z^2 = 9 \end{cases}$$

54–59 Find any discontinuities of the given function and classify them as removable or nonremovable.

54. $f(x,y) = \frac{(x-1)y^2}{(x-1)^2+y^2}$

55. $g(x,y) = \frac{xy}{x^2+y^2}$

56. $h(x,y) = \frac{x}{x^2+y^2}$

57. $p(x,y) = \ln\left(\frac{1}{x^2+y^2}\right)$

58. $m(x,y) = \frac{x}{\sqrt{x^2+(y-2)^2}}$

59. $k(x,y,z) = \frac{x^2-2y^2+z^2}{x^2+y^2+z^2}$

Concept Check

60–65 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. If $\lim_{(0,y)\to(0,0)} f(x,y) = L$ and $\lim_{(x,0)\to(0,0)} f(x,y) = L$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

61. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and its value is L, then $f(a,b) = L$.

62. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and its value is L, then $\lim_{x\to a} f(x,b) = L$.

63. If $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and the limiting value of $f(x,y)$ along the line $y = x$ is $f(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

64. If $f(x,y) < 0$ in some ε-neighborhood $B_\varepsilon(0,0)$ of the origin and $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then $\lim_{(x,y)\to(0,0)} f(x,y) < 0$.

65. If $f(0,0) = L$ and $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

Proof

For convenience, we will prove the theorem in the two-variable case.

If f is differentiable at (a,b), then by definition the increment Δf can be written in such a way that

$$\begin{aligned}\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[f(a+\Delta x,b+\Delta y)-f(a,b)\right] &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\right]\\ &= f_x(a,b)\lim_{\Delta x\to 0}\Delta x+f_y(a,b)\lim_{\Delta y\to 0}\Delta y\\ &\quad+\lim_{\Delta x\to 0}\varepsilon_1\lim_{\Delta x\to 0}\Delta x+\lim_{\Delta y\to 0}\varepsilon_2\lim_{\Delta y\to 0}\Delta y\\ &= 0.\end{aligned}$$

That is, $f(a+\Delta x,b+\Delta y)\to f(a,b)$ as $\Delta x\to 0$ and $\Delta y\to 0$.

By implication, then, the function in Example 9 does not meet the conditions of the Increment Theorem of Differentiability at $(0,0)$. In Exercise 103, you will show that this is indeed the case.

13.3 Exercises

1–4 Determine z_x and z_y at the indicated point; then find equations for the corresponding tangent lines that are parallel to, respectively, the xz-plane and the yz-plane.

1. $z=x+3xy^3$; $(2,1)$
2. $z=x^4y-4xy^3$; $(2,-1)$
3. $z=x(2-xy)^2$; $(-1,0)$
4. $z=(xy^2+3)(x-3y)$; $(3,1)$

5–28 Find all first-order partial derivatives of the given function.

5. $f(x,y)=x^3+y^3$
6. $g(x,y)=xy^2-2xy$
7. $h(x,y)=5x^3y+y^4$
8. $r(x,y)=x^4-2x^2y^2+3y^6$
9. $V(r,h)=\dfrac{\pi r^2h}{3}$
10. $s(x,y)=(x^2+y^4)^5$
11. $k(x,y)=xy^2+\sqrt{2xy}$
12. $l(x,y)=(4xy-3)(x^2+1)$
13. $m(x,y)=(xy-3x^2)^4$
14. $n(x,y)=x(1-\sqrt{x}y)(y^2+2)$
15. $p(x,y)=\dfrac{x^2y}{x+y}$
16. $F(x,y)=\sqrt{1-x^2-y^2}$
17. $G(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$
18. $H(x,y)=e^{x^2y}$
19. $I(x,y)=\sin\dfrac{x}{y}$
20. $J(x,y)=y\ln\sqrt{x^2+y^2}$
21. $K(x,y)=\dfrac{2y^2}{x}-\dfrac{x^2}{2y}$
22. $L(x,y)=e^x\cos(xy)$
23. $M(x,y)=\arctan\sqrt{xy}$
24. $f(x,y,z)=e^xy^2\sin z$
25. $g(x,y,z)=x^{y/z}$
26. $A(a,b,c)=2(ab+ac+bc)$
27. $h(x,y,z)=x\cos(y+z^2)$
28. $w(r,s,t)=(r^2+2s^2+3t^2)^{3/2}$

29–34 Find a vector equation for L, the line tangent to the surface $z^2-4x^2-5y^2=0$ of Example 3 at the given point and parallel to the indicated coordinate plane.

29. $(1,1,3)$; the xz-plane
30. $(1,1,-3)$; the yz-plane
31. $\left(2,\dfrac{3}{\sqrt{5}},5\right)$; the xz-plane
32. $\left(2,\dfrac{3}{\sqrt{5}},-5\right)$; the yz-plane

33. $(-1,1,3)$; the xz-plane

34. $(-1,1,-3)$; the yz-plane

35–37 Use implicit differentiation to determine z_x and z_y.

35. $z^2 + 2xy - yz = 0$

36. $zx - (z+x)^2 = 3y$

37. $xz + \ln z - x^2 y = 0$

38. Treating y and z as independent variables, determine x_y and x_z from Exercise 36.

39. Find y_x and y_z by implicitly differentiating $y^2 - xy - z\ln y = 5$.

40. Recall the Ideal Gas Law, the equation relating the pressure P, volume V, and temperature T of an ideal gas: $PV = nRT$ (Section 3.4, Exercise 97). Assuming that n (i.e., the amount of gas in moles) is constant, differentiate implicitly to find the following partial derivatives and explain their physical meaning.

a. $\dfrac{\partial P}{\partial T}$ **b.** $\dfrac{\partial P}{\partial V}$ **c.** $\dfrac{\partial V}{\partial T}$

41–48 Verify the equality of the mixed partials f_{xy} and f_{yx}.

41. $f(x,y) = x^3 y - 2y^2 + 5xy^4$

42. $f(x,y) = (2y - x)^4$

43. $f(x,y) = (x^4 + y^4)^8$

44. $f(x,y) = xy^4 + 2y^{2/3}$

45. $f(x,y) = x^2 y^2 e^{xy}$

46. $f(x,y) = \ln(x^2 + y^2)$

47. $f(x,y) = e^{\sqrt{x^2+y^2}}$

48. $f(x,y) = 2y\cos(3x + 4y)$

49–54 Verify that the third-order mixed partials g_{xyz}, g_{yzx}, and g_{zxy} are equal.

49. $g(x,y,z) = x^3 + 3yz^2 - xy^2 + 3z^3$

50. $g(x,y,z) = 2x(y - 3z)^3$

51. $g(x,y,z) = x^3 y^3 z^3$

52. $g(x,y,z) = \sin(xyz)$

53. $g(x,y,z) = e^{xy}\cos z$

54. $g(x,y,z) = \dfrac{y}{x^2 + z^2}$

55. Find the partial derivative f_{xyy} of the function

$$f(x,y) = x\left(\frac{y^3}{3} + \ln y\right) + y(\cos x + x\sin x) + x\sec^2 x$$

by judiciously choosing the order of differentiation. (**Hint:** See Example 7.)

56–63 Use the most convenient order of differentiation to find the indicated partial derivative, as in Exercise 55.

56. g_{xzyz}; $g(x,y,z) = \ln(x^2 y^2) + z\cos(y^2) + z^2 x^3 y^4$

57. h_{rstu}; $h(r,s,t,u) = s\left(u^2 e^r + \dfrac{tr^2}{u}\right) + re^{rs}\sin t + t(s^2 + t\ln u)$

58. f_{xy}; $f(x,y) = xe^{y^2} + \dfrac{\ln x}{\sqrt{x}}$

59. f_{yxyx}; $f(x,y) = \sin(x^2 - y)$

60. f_{xyz}; $f(x,y,z) = z\left(xy^2 + \sqrt{x}\ln x\right) + x^2\sin y$

61. f_{xyz}; $f(x,y,z) = \dfrac{x^2 y}{2}(1 + 3z^2) + \cos y - \cos x$

62. f_{utst}; $f(r,s,t,u) = ru(t^2\sin s + 1) + \dfrac{e^{(t^2+1)}}{r}$

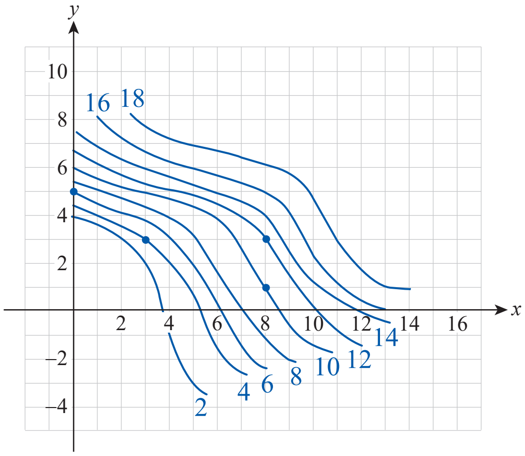
63. f_{xyzw}; $f(w,x,y,z) = \sqrt{w^2 + x^2 + y^2 + z^2}$

64–65 Use the definition of the partial derivative to find $\partial f/\partial x$ and $\partial f/\partial y$ for the given function.

64. $f(x,y) = x^2 y$ **65.** $f(x,y) = \dfrac{\sqrt{y}}{x}$

66. Use the contour map below to estimate the values of f_x and f_y at the indicated points. (Answers will vary.)

a. $(0,5)$ **b.** $(3,3)$ **c.** $(8,1)$ **d.** $(8,3)$



67–70 If possible, find a function that has the indicated partial derivatives. If such a function doesn't exist, explain why.

67. $f_x(x,y)=2xy, \quad f_y(x,y)=x^2-\cos y$

68. $f_x(x,y)=\dfrac{1}{\sqrt{x}}+\ln\left(y^2+1\right), \quad f_y(x,y)=\dfrac{2xy}{y^2+1}$

69. $f_x(x,y)=x-y^2, \quad f_y(x,y)=\sqrt{x}-2y$

70. $f_x(x,y)=y\sin x+\dfrac{1}{2\sqrt{x}}, \quad f_y(x,y)=\cos x+3y^2$

71. Recall the lens equation from Exercise 44 of Section 3.8:

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f},$$

where o is the object distance, i is the image distance, and f is the focal length of the lens. Find the partial derivative $\partial i/\partial o$ and explain its physical meaning.

72. Explain why there is a difference in sign between the first two formulas for $\Delta\rho$ in Example 4.

73. Recall that the kinetic energy E of an object of mass m moving at speed v is found from the formula $E=\frac{1}{2}mv^2$. Suppose that a 1 kg mass is moving at a speed of 5 m/s, but due to inaccuracies of the measuring devices and human error, its mass was recorded as 1002 grams and its speed was clocked at 495 cm/s. Estimate the relative error this causes in the value of E. (**Hint:** See Example 4.)

74. Consider the lens equation of Exercise 71, and suppose that a lens that is thought to be a 50 mm lens has an actual focal length of 52.56 mm. If, in addition, the actual object distance of 2.03 m is erroneously measured to be exactly 2 m, estimate the relative error all of this causes in the value of i.

75–80 Show that the given function satisfies the wave equation of Example 8.

75. $u(x,t)=\sin(x+ct)+e^{x-ct}$

76. $u(x,t)=\cosh(x+ct)$

77. $u(x,t)=\sin(\omega x)\cos(\omega ct)$

78. $u(x,t)=(x+ct)^4+(x-ct)^4$

79. $u(x,t)=\cos(x+ct)+\dfrac{1}{x-ct}$

80. $u(x,t)=\ln(x+ct)+\sqrt{x-ct}$

81–86 Verify that f satisfies the two-dimensional form of Laplace's equation: $f_{xx}+f_{yy}=0$.

81. $f(x,y)=e^x\cos y$

82. $f(x,y)=\sinh y\cos x$

83. $f(x,y)=x^2-y^2$

84. $f(x,y)=e^x\sin y+e^y\cos x$

85. $f(x,y)=\arctan\dfrac{x}{y}$

86. $f(x,y)=\ln\left(x^2+y^2\right)^2$

87–89 Verify that the three-variable function satisfies Laplace's equation: $f_{xx}+f_{yy}+f_{zz}=0$.

87. $f(x,y,z)=kxyz$

88. $f(x,y,z)=\sin(3x)e^{2y-\sqrt{5}z}$

89. $f(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$

90–92 The temperature $u(x,t)$ of an insulated rod that is aligned with the x-axis satisfies the heat equation $u_t=\kappa u_{xx}$, where κ is a constant, called *thermal diffusivity*. Show that the functions in Exercises 90–92 satisfy the heat equation.

90. $u(x,t)=e^{-\kappa t}\sin x$ **91.** $u(x,t)=e^{-4\kappa t}\sin 2x$

92. $u(x,t)=5e^{-\kappa n^2 t}\cos(nx)$

93. Suppose that the temperature of an insulated planar surface is $T(x,y,t)=e^{-5t}\sin x\cos 2y$. Find the rates of change of temperature in the x- and y-directions, respectively, at the point $(\pi/4,\pi/4,0)$.

94. Show that $T(x,y,t)$ of Exercise 93 satisfies the two-dimensional heat equation $u_t=\kappa\left(u_{xx}+u_{yy}\right)$ with $\kappa=1$.

95. Generalizing Exercises 92 and 94, show that all functions of the form

$$u(x,y,t)=e^{-\kappa\left(m^2+n^2\right)t}\sin(mx)\cos(ny)$$

satisfy the two-dimensional heat equation ($m, n\in\mathbb{R}$).

96. Suppose a guitar string, originally aligned with the positive x-axis, is plucked and the function $g(x,t)$ describes the displacement of the point $(x,0)$ as a function of t. What can you say about a point $(x_0,0)$ at time $t=t_0$ if both partial derivatives $g_x(x_0,t_0)$ and $g_t(x_0,t_0)$ are positive? What if, in addition, $g_{xx}(x_0,t_0)$ and $g_{tt}(x_0,t_0)$ are both negative?

97.* Recall from trigonometry that the area of a circular sector of radius r and central angle α is $A = \frac{1}{2}r^2\alpha$. Denoting the circumference of the sector by C, prove that the area can also be expressed as $A = \frac{1}{2}Cr - r^2$; then determine $\partial A/\partial r$ from both of the given area formulas and explain why your answers are not equivalent.

98. Consider the general cubic polynomial in the variables x and y:

$$P(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J$$

Find conditions on the coefficients to ensure that $P(x,y)$ satisfies Laplace's equation (such functions are called *harmonic*).

99.* If we invest P dollars and take inflation and taxes into consideration, the future value of our investment in n years is

$$A = P\left[\frac{1+r(1-T)}{1+I}\right]^n \text{ dollars,}$$

where I and T are the inflation and tax rates, respectively, and r is the annual interest rate. Suppose we invest \$15,000 for 10 years at a rate of 12%. Use the partial derivatives $\partial A/\partial I$ and $\partial A/\partial T$ to decide whether it is inflation or the tax rate that affects the investment more drastically.

100.* Prove that the first partial derivatives of a harmonic function are themselves harmonic, if they have continuous second partial derivatives (see Exercise 98 for the definition of harmonic functions).

101.* Let R denote the net resistance of two resistors R_1 and R_2 in a parallel circuit, which satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a. Generalize the above equation to show that the net resistance of n resistors in a parallel circuit is as follows.

$$R = \frac{\prod_{i=1}^{n} R_i}{\sum_{i=1}^{n} \prod_{j \neq i} R_j}$$

b. For a fixed index k, find a formula for $\partial R/\partial R_k$. (**Hint:** It is helpful to look at the cases $n = 2$ and $n = 3$ before generalizing.)

102. Prove that if $u(x,t)$ can be written in the form $u(x,t) = u_1(x+ct) + u_2(x-ct)$ for some one-variable functions u_1 and u_2 that are at least twice differentiable, then $u(x,t)$ is a solution of the wave equation of Example 8.

103. Show that the function in Example 9 does not meet the conditions of the Increment Theorem of Differentiability at $(0,0)$. (**Hint:** Show, for example, that f_x is not continuous at the origin by looking at f_x at nearby points on the x-axis.)

104–107 Decide whether the given function is differentiable at the origin. Give a reason for your answer.

104. $f(x,y) = y^2e^x - x^2y$

105. $f(x,y) = \sqrt{x^2 + 2y^2}$

106. $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$

107. $f(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

108–113 Show that f is not differentiable at the origin, even though both $f_x(0,0)$ and $f_y(0,0)$ exist. (In Exercises 112 and 113, generalize to the three-variable case.)

108. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

109. $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

110. $f(x,y) = \begin{cases} \dfrac{-4x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

111. $f(x,y) = \begin{cases} 1 & \text{if } y^2 < x < 2y^2 \\ -1 & \text{otherwise} \end{cases}$

112. $f(x,y,z) = \begin{cases} 2 & \text{if } xyz = 0 \\ -2 & \text{if } xyz \neq 0 \end{cases}$

(**Hint:** See Example 9.)

113. $f(x,y,z) = \begin{cases} 1 & \text{if } x^2 + y^2 < z < 2x^2 + 2y^2 \\ -1 & \text{otherwise} \end{cases}$

(**Hint:** See Exercise 111.)

114.* Consider the given piecewise-defined function.

$$F(x,y)=\begin{cases}\dfrac{xy^2}{2(x^2+y^2)} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

a. In contrast to some of the previous exercises, show that F is continuous at $(0,0)$. (**Hint:** See, for example, Exercise 31 or 41 of Section 13.2.)

b. Prove that both partial derivatives of F are equal to 0 at $(0,0)$. (**Hint:** Use the definition of partial derivatives.)

c. Prove that F is not differentiable at $(0,0)$. (**Hint:** As noted after Example 9 in the text, the derivative of F at $(0,0)$ must take into account *all* the different limiting approaches to $(0,0)$; however, compare the common value of the two partial derivatives of F at $(0,0)$ with $\lim\limits_{h\to 0}\dfrac{F(h,h)-F(0,0)}{h}$.)

115–118 For certain functions, it is fairly straightforward to demonstrate differentiability by using the definition. For example, let $f(x,y)=x+y^2$ and note the following.

$$\begin{aligned}\Delta f(x,y)&=f(x+\Delta x,y+\Delta y)-f(x,y)\\&=x+\Delta x+(y+\Delta y)^2-(x+y^2)\\&=x+\Delta x+y^2+2y\Delta y+(\Delta y)^2-x-y^2\\&=\Delta x+2y\Delta y+(\Delta y)^2\\&=f_x(x,y)\Delta x+f_y(x,y)\Delta y+0\cdot\Delta x+\Delta y\cdot\Delta y\end{aligned}$$

Letting $\varepsilon_1=0$ and $\varepsilon_2=\Delta y$, we see that ε_1 and ε_2 approach 0 as both Δx and Δy approach 0, as needed. In Exercises 115–118, mimic this process of using the definition to prove that the function is differentiable.

115. $f(x,y)=x^2-2y$ **116.** $g(x,y)=xy^2$

117. $h(x,y)=2(x^2+y^2)$

118. $k(x,y)=x^3-4x+3y$

119. Find all points (x,y) where the function $f(x,y)=|x-y|$ is differentiable.

120. Repeat Exercise 119 for the function $g(x,y)=\sqrt{x^2+y^2}$.

$$z_x = -\frac{F_x}{F_z} = -\frac{2xy - z^3 - yz}{-3xz^2 - xy} = \frac{2xy - z^3 - yz}{3xz^2 + xy}$$

$$z_y = -\frac{F_y}{F_z} = -\frac{x^2 - xz}{-3xz^2 - xy} = \frac{x^2 - xz}{3xz^2 + xy}$$

13.4 Exercises

1. Obtain the result of Example 1 by first expressing y explicitly as a function of x.

2–5 Use a tree diagram to apply the Chain Rule to express the indicated derivative of the given function.

2. $\dfrac{df}{dt}$; $f = f(x(t), y(t))$

3. $\dfrac{dg}{dt}$; $g = g(x(t), y(t), z(t))$

4. $\dfrac{\partial h}{\partial u}$; $h = h(x(u,v), y(u,v), z(u,v))$

5. $\dfrac{\partial k}{\partial z}$; $k = k(u(x,y,z), v(x,y,z))$

6–17 Determine dy/dx, given y as a function of $u(x)$ and $v(x)$. In Exercises 6–13, check your answer by expressing y explicitly as a function of x and differentiating. In Exercises 14–17, generalize to three intermediate variables.

6. $y = u^2 v$; $u = 3x + 1$, $v = x^4$

7. $y = uv^2 - \cos u$; $u = 2x^2$, $v = \sqrt{x}$

8. $y = u^3 v - \sin u$; $u = 2x$, $v = x^2$

9. $y = \ln \dfrac{u}{\sqrt{v}}$; $u = \sin x$, $v = \cos^2 x$

10. $y = \sqrt{u^2 + v^2}$; $u = e^x \cos x$, $v = e^x \sin x$

11. $y = \dfrac{1}{u} + \dfrac{1}{v}$; $u = \csc^2 x$, $v = \cot x$

12. $y = u \arctan v$; $u = e^x$, $v = \tan x$

13. $y = \arcsin \dfrac{u}{v}$; $u = 2x$, $v = x^3$

14. $y = uv + uw + vw$; $u = 2x$, $v = x + 2$, $w = x^2 + 2$

15. $y = u \sin^2(vw)$; $u = 2x$, $v = x^2$, $w = \dfrac{1}{x}$

16. $y = uvw$; $u = e^x$, $v = 3x$, $w = x^3$

17. $y = uve^w$; $u = 5x - 2$, $v = \sin x$, $w = \ln x$

18. Determine the value of $f'(7\pi/6)$ for the function in Example 2.

19–21 After determining the rate of change with respect to t of the function $f(x,y)$ along the indicated parametric curve, find $f'(t)$ at the given point.

19. $f(x,y) = x^2 y$ along the curve $x = 10\cos t$, $y = t$, at $t = \pi/2$

20. $f(x,y) = (x+y)^2$ along the curve $x = t\cos t$, $y = t\sin t$, at $t = \pi$

21. $f(x,y) = xe^{-y}$ along the curve $x = 4 - 2t^2$, $y = t^3 - 9t$, at $t = 0$

22. Find the rate of change of the function $f(x,y,z) = x^2 y + \sin z$ along the helix $x = \cos 2t$, $y = \sin 2t$, $z = t/2$. Express your answer in terms of the variable t.

23–33 Use the Chain Rule to determine the partial derivatives z_x and z_y. (Answers may be left in terms of the intermediate and independent variables.)

23. $z = u^2 + v^2$; $u = x + 2y$, $v = y - x$

24. $z = u \sin v$; $u = x^2 + y$, $v = x - y^2$

25. $z = v^3 - 2u^2 v$; $u = \cos y$, $v = \sin x$

26. $z = v^2 - 2uv$; $u = x \sin y$, $v = y \cos x$

27. $z = 2v^4 - 3u\sqrt{v}$; $u = e^x$, $v = e^y$

28. $z = \cos(u + v^2)$; $u = y - \dfrac{x^2}{2}$, $v = 3x - 2y$

29. $z = u^2 v - uw^3$; $u = x + y$, $v = x^2 - y$, $w = 2xy$

30. $z = u(v+w)^2 - v^2; \quad u = 3x, \quad v = 5y^2, \quad w = x - 2y$

31. $z = v^2 e^{2u+3w}; \quad u = x - y, \quad v = 2y - x, \quad w = 3xy$

32. $z = \ln(u^2 + v^2 + w^2); \quad u = 2x - y, \quad v = xy, \quad w = e^y$

33. $z = v\sin(uw^2); \quad u = x + y^3, \quad v = 3y - x, \quad w = x^2 y$

34. Find w_ρ, w_θ, and w_ϕ if $w = 2x + y^2 z$ and $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\theta$. (Answers may be left in terms of the intermediate and independent variables.)

35. Prove that for $z = f(x(u,v), y(u,v))$ of Example 5,

$$z_{vv} = 2z_x + 4v^2 z_{xx} + 4uvz_{xy} + u^2 z_{yy}.$$

36. Assume (as in Example 5) that $z = f(x,y)$ has continuous second-order partial derivatives and that $x = u^2 v$ and $y = u + v^2$. Find z_{uu} and z_{vv}.

37. Suppose $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Prove the following.

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

(**Hint:** Start by determining f_r and f_θ.)

38. Suppose $z = f(x,y)$ is as in Exercise 37. Prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial f}{\partial r}.$$

(This is called the *Laplacian* of f. Can you see why?)

39–47 Find dy/dx, where y is given implicitly by the given equation.

39. $x^2 - xy + y^2 = \frac{1}{4}$

40. $(4-x)y^2 = 2x^3$

41. $(x^2+4)y = 8$

42. $\frac{y^2}{4}(8-x) = x^3$

43. $x^{2/3} + y^{2/3} = 8$

44. $(2x^2+y^2)^2 - 4x^2 y = 0$

45. $x^2 + y^2 = (x^2 + y^2 - 3x)^2$

46. $(x^2+y^2)^2 = 9xy$

47. $\frac{y}{x^2+y^2} = 3 + x^2$

48–55 Find z_x and z_y, where z is defined implicitly by the given equation.

48. $x^2 + y^2 + z^2 = 1$

49. $xyz = e^{x+y+z}$

50. $xy - 3y^3 - 4xz^2 = 1$

51. $x^2 z^3 + xy = \sin(yz)$

52. $x\sin y + y^2 z - e^{xyz} = 1$

53. $e^x \sin y - 2z^2 + \frac{yz^2}{2} = 2$

54. $x^2 y + z^2 + y\ln z = 4$

55. $\ln(x^2+y^2+z^2) = 5 - xyz$

56. If $F(x,y) = 0$ implicitly defines y as a function of x and both F and y are twice-differentiable, show that

$$y''(x) = -\frac{F_{xx}(F_y)^2 - 2F_x F_y F_{xy} + F_{yy}(F_x)^2}{(F_y)^3}.$$

57. Use a tree diagram to write out the Chain Rule for the first partial derivatives f_x and f_y of $f(t,u,v,w)$, where $t = t(x,y)$, $u = u(x,y)$, $v = v(x,y)$, and $w = w(x,y)$.

58. If $f(x,y)$ is differentiable, where $x(u,v) = u + v$ and $y(u,v) = u - v$, prove the following.

$$\left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

59. Suppose $F(x,y,z)$ is differentiable, has nonzero first partial derivatives, and that $F(x,y,z) = 0$ defines each variable as a function of the other two variables (i.e., $x = x(y,z)$, $y = y(x,z)$, and $z = z(x,y)$). Prove the following equation.

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x} = -1$$

60–63 A function f is said to be *homogeneous of degree n* if

$$f(tx,ty) = t^n f(x,y)$$

for all n, $t \in \mathbb{R}$. In these exercises, you will work with homogeneous functions. To begin, show that the given function in Exercises 60 and 61 is homogeneous and state the degree of homogeneity.

60. $f(x,y) = 5x^2y^2 - 3x^3 y$

61. $f(x,y) = \frac{xy^2 + 2x^2 y}{\sqrt{x^4 + y^4}}$

62. Let $f(x,y)$ be homogeneous of degree n. Prove the following formula.

$$x\cdot\frac{\partial f(x,y)}{\partial x}+y\cdot\frac{\partial f(x,y)}{\partial y}=nf(x,y)$$

(**Hint:** Consider the equation of homogeneity; differentiate both sides with respect to t, and let $t = 1$.)

63. Let $f(x,y)$ be homogeneous, as in Exercise 62. Prove the following formulas.

$$\frac{\partial f(tx,ty)}{\partial x}=t^{n-1}\cdot\frac{\partial f(x,y)}{\partial x}\quad\text{and}$$
$$\frac{\partial f(tx,ty)}{\partial y}=t^{n-1}\cdot\frac{\partial f(x,y)}{\partial y}$$

64. An ice "cube" in the form of a rectangular prism with a square base is melting so that the edge of the base is shrinking at 0.5 mm/min while the height is decreasing at 0.75 mm/min. Determine the rate of change of its volume and surface area when the edge of the base is 20 mm and the height is 30 mm.

65. Consider a circular sector of radius r and central angle θ. Suppose that θ is increasing at a rate of 0.1 radians per minute, while r is decreasing at a rate of 0.2 inches per minute. Find the rate of change of the area at the instant when $\theta = 1$ radian and $r = 15$ inches.

66. Suppose the height of a right circular cylinder is increasing at 1 millimeter per second. Determine the rate of change of the radius of the cylinder if the instantaneous rate of change of its volume is 0 when the radius is 50 millimeters and the height is 100 millimeters.

67. Consider a sand cone such as one formed by a child pouring sand out of a bucket. Assume that its height is growing at a rate of 0.1 inches per second, while its radius at 0.05 inches per second, at the instant when its height is 4 inches and its radius is 6 inches. Find the rate of change of the volume of the sand cone at this instant.

68. Find the rate of change of the lateral surface area of the sand cone at the instant described in Exercise 67.

69. Suppose that at a certain moment during takeoff, a plane's speed is 100 m/s, its acceleration 3 m/s^2, while its mass of 63,350 kg is decreasing at a rate of 1.15 kg/s due to fuel consumption. Find the rate of change of the plane's kinetic energy at this instant.

70. Suppose the temperature of two moles of an ideal gas in a 50-liter (L) container is 323 kelvins (K) and increasing at a rate of 0.2 K/s, when at the same time, the volume of the container is increasing at a rate of 0.05 L/s. Find the rate of change of pressure at this instant. (For a refresher on the Ideal Gas Law, see Exercise 97 in Section 3.4.)

71.* Consider the insulated plane of Exercise 93 of Section 13.3, with temperature measured in degrees Celsius and time in minutes. Suppose a point is moving along the line $y=-x+\pi/2$, in the southeastern direction, at a speed of $\sqrt{2}/2$ unit lengths per minute. Supposing that it is at the point $(\pi/4,\pi/4)$ at $t = 5$ seconds, find the rate of temperature change from the moving point's perspective at that instant. (**Hint:** Determine the rates dx/dt, dy/dt, and use the Chain Rule.)

Solution

We begin with the gradient.

$$\nabla T(x,y,z) = -100\frac{\langle 4x, 2y, z\rangle}{\left(1+2x^2+y^2+0.5z^2\right)^2}$$

The direction of the greatest decrease in rate of change of T from the point $(1,3,2)$ is then

$$-\nabla T(1,3,2) = 100\frac{\langle 4,6,2\rangle}{(1+2+9+2)^2} = \frac{100}{14^2}\langle 4,6,2\rangle \approx \langle 2.04, 3.06, 1.02\rangle,$$

and the rate of decrease in that direction is approximately $-3.82\ °\text{C/m}$.

13.5 Exercises

1–6 Find the gradient of the function at the indicated point.

1. $f(x,y) = 5x^3 - 2y^2;\quad (1,2)$
2. $g(x,y) = 2xy + \dfrac{\sqrt{y}}{x};\quad (1,4)$
3. $h(x,y) = e^{x^2+y^2};\quad (-1,1)$
4. $k(x,y) = \dfrac{y}{\sqrt{x^2+y^2}};\quad (-4,3)$
5. $F(x,y,z) = x\sqrt{\dfrac{y^3}{z}};\quad (-2,3,3)$
6. $G(x,y,z) = \sqrt{2}x\sin\left(\pi y + z^2\right);\quad \left(3,\dfrac{1}{4},0\right)$

7–12 Use the definition to find the directional derivative of *f* at the given point in the direction of the given vector. Then compute the directional derivative with the help of the gradient vector to check your answer.

7. $f(x,y) = 2x - y;\quad (0,0);\quad \mathbf{v} = \langle 3,4\rangle$
8. $f(x,y) = x^2y;\quad (1,-1);\quad \mathbf{v} = \langle 1,2\rangle$
9. $f(x,y) = \dfrac{x}{y};\quad (1,2);\quad \mathbf{v} = \langle 2,2\rangle$
10. $f(x,y) = x\sqrt{y};\quad (4,1);\quad \mathbf{v} = \left\langle 1,\sqrt{3}\right\rangle$
11. $f(x,y,z) = 2xy - z^2;\quad (1,1,1);\quad \mathbf{v} = \langle 1,2,3\rangle$
12. $f(x,y,z) = xyz;\quad (5,0,-2);\quad \mathbf{v} = \left\langle 2,4,\sqrt{5}\right\rangle$
13. Try to evaluate the limit in Example 1 by using the following parametrizations.

    a. $\langle t,-t\rangle$ **b.** $\langle 2t,-2t\rangle$

    What do you find?

14–29 Find the derivative of the function at the given point in the indicated direction. (Note that sometimes the direction is conveniently specified in terms of a direction angle, the one determined by a direction vector and the positive *x*-axis. In this case, the corresponding unit vector is $\mathbf{u} = \langle \cos\alpha, \sin\alpha\rangle$.)

14. $f(x,y) = x^2 - 4y^2;\quad (1,2);\quad \mathbf{v} = \langle 3,4\rangle$
15. $f(x,y) = 3x^2 - 2xy + y^2;\quad (0,1);\quad \mathbf{v} = \langle 1,1\rangle$
16. $f(x,y) = 2x^3y^2;\quad (1,2);\quad \mathbf{v} = \langle -2,1\rangle$
17. $f(x,y) = \dfrac{x^3}{3} + x^2y - 3xy^2 + y^3;\ (-1,-1);$
    $\mathbf{v} = \langle -4,-3\rangle$
18. $f(x,y) = ye^x;\quad (1,0);$
    in the direction toward $(-1,4)$
19. $f(x,y) = 4e^x\cos x;\quad (0,0);\quad \alpha = 60°$
20. $f(x,y) = \arcsin(xy);\quad (0,1);$
    in the direction toward $(3,4)$
21. $f(x,y) = 2\cos x\sin y;\quad \left(-\dfrac{\pi}{6},\dfrac{5\pi}{6}\right);\quad \mathbf{v} = \langle 5,12\rangle$
22. $f(x,y) = \dfrac{2x}{\sqrt{x^2+y^2}};\quad (0,1);\quad \mathbf{v} = \langle 0.7,2.4\rangle$
23. $f(x,y) = \ln\left(x^2+y^2\right);\quad (1,0);\quad \alpha = 30°$
24. $f(x,y,z) = xy - yz + xz;\quad (1,-3,-2);\quad \mathbf{v} = \langle 2,1,-2\rangle$
25. $f(x,y,z) = e^{-\left(x^2+y^2+z^2\right)};\quad (2,1,1);\quad \mathbf{v} = \langle -6,2,3\rangle$
26. $f(x,y,z) = \sqrt{x^2+y^2+z^2};\quad (1,1,1);\quad \mathbf{v} = \langle 1,1,1\rangle$
27. $f(x,y,z) = x\arctan\dfrac{y}{z};\quad (1,1,1);\quad \mathbf{v} = \langle 2,-2,-2\rangle$

28. $f(x,y,z)=\ln(x+y+z)$; $(e,1,-1)$; in the direction toward $(e-1,1,1)$

29. $f(x,y,z)=ze^{xy}$; $(2,0,1)$; in the direction toward $(-1,4,1)$

30–33 Find the direction and value of the greatest rate of increase for the function at the given point.

30. $f(x,y)=2x^2-5xy+y^2$; $(2,1)$

31. $f(x,y)=x^2e^{2xy}$; $(1,0)$

32. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$; $(12,5)$

33. $f(x,y,z)=\dfrac{x-y}{z+2}$; $(4,2,0)$

34–37 Find the direction and value of the greatest rate of decrease for the function at the given point.

34. $f(x,y)=\dfrac{x^2+y^2}{x^2-y^2}$; $(-1,2)$

35. $f(x,y)=\cos(\pi xy)$; $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$

36. $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$; $(2,1,-2)$

37. $f(x,y,z)=xe^{-yz}$; $(-1,3,0)$

38–39 Find the direction of no change for the function at the given point.

38. $f(x,y)=x^2y+2x^3-3y^2$; $(-1,1)$

39. $f(x,y)=\dfrac{x-y}{x^2+y^2}$; $(1,-1)$

40. If possible, find a direction angle θ for which the rate of change at the point $(1,-1)$ of the function in Exercise 39 is **a.** 0, **b.** 1.

41–48 Find an equation for the line tangent to the graph of the given equation at the indicated point.

41. $x^2-xy+y^2=7$; $(1,3)$

42. $(6-x)y^2=2x^3$; $(2,2)$

43. $(x^2+4)y=10$; $(-1,2)$

44. $x^{2/3}+y^{2/3}=13$; $(8,27)$

45. $(2x^2+y^2)^2-9x^2y=0$; $(1,1)$

46. $x^2+y^2=(x^2+y^2-5x)^2$; $(4,3)$

47. $(x^2+y^2)^2=4xy$; $(1,1)$

48. $\dfrac{y}{x^2+y^2}=\dfrac{3}{4}+x^2$; $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$

49–52 Generalize Example 4 to three variables to obtain the equation of the tangent plane to the surface at the given point.

49. $x^2+3y^2+4z^2=11$; $(2,1,-1)$

50. $y^3z^2+5xz^2+2xy=32$; $(2,-1,2)$

51. $x^2-2y^2=z$; $(3,2,1)$

52. $3x-2y^2=3z^2$; $(7,3,1)$

53. Prove the linearity of the gradient, that is, the Sum/Difference Law and the Constant Multiple Law: If f and g are differentiable functions and k is a real number, then $\nabla(f\pm g)=\nabla f\pm\nabla g$, and $\nabla(kf)=k\nabla f$.

54. Prove the Product Law for gradients, that is, if f and g are differentiable functions, then $\nabla(fg)=f\nabla g+g\nabla f$.

55. Prove the Quotient Law for gradients, that is, if f and g are differentiable functions and $g\neq 0$, then $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$.

56–59 Use the properties of the gradient to determine ∇f.

56. $f(x,y)=\dfrac{2xy^2+yx^3}{x^2+2xy}$

57. $f(x,y)=\dfrac{2x^3-\sqrt{y}}{x(x+y)}$

58. $f(x,y)=\dfrac{xy}{x^3-y^3}$

59. $f(x,y,z)=\dfrac{xyz}{x^2+y^2+z^2}$

60–65 Find a function with the given gradient. If it is not possible, explain why. (Answers will vary.)

60. $\nabla f=\langle 1,2\rangle$

61. $\nabla f=\langle 2y,2x\rangle$

62. $\nabla f=\langle 3y,x\rangle$

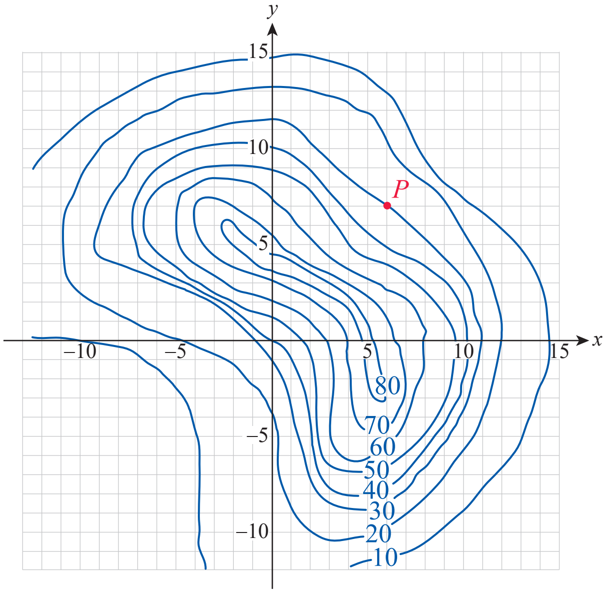
63. $\nabla f=\langle y,2x^2\rangle$

64. $\nabla f=\langle 2y,2x,2z\rangle$

65. $\nabla f=\langle 6xyz^2,3x^2z^2,6x^2yz\rangle$

66. Prove that the rate of change of the function $f(x,y)=\sqrt{x^2+y^2}$ is greatest along rays emanating from the origin.

67. Use the contour map below to estimate the directional derivative $D_{\mathbf{u}}f$ at the point $P(6,7)$ in the direction of **a.** $\mathbf{u}=\langle 1,2\rangle$, **b.** $\mathbf{u}=\langle -1,-1\rangle$. Then draw a possible path of steepest ascent starting at *P*. (Answers will vary.)



68. Sarah is standing at an intersection on a mountain trail where, according to the trail markings, her route continues to the southeast. Her current position can be modeled by the point $P(400,200,3560)$ on the graph of $f(x,y)=4000-0.002x^2-0.003y^2$ (units in feet).

a. What will be the angle of elevation (or depression) of her route immediately after leaving the intersection?

b. What is the direction and angle of steepest ascent from her current position? (Assume that the northern direction coincides with the positive *y*-axis.)

69. Suppose that the temperature of a metal plate is given by

$$T(x,y)=\frac{150}{\sqrt{x^2+y^2+1}}.$$

Find the rate of change of temperature at the point $(8,4)$ in the direction toward the point $(7,2)$.

70. Suppose that the temperature around the origin in three-dimensional space is given by $T(x,y,z)=300e^{-\left(x^2+2y^2+3z^2\right)}$.

a. Find the rate of change of temperature at the point $(2,1,0)$ in the direction toward the point $(4,0,2)$.

b. Find the direction at $(2,1,0)$ in which the rate of decrease is greatest.

c. Find the rate of greatest decrease at the point $(2,1,0)$.

71.* Consider a path on the contour map of a differentiable two-variable function $f(x,y)$ that follows the gradient at each point. Such is a possible path of a heat-seeking object, if *f* were a temperature function, or a path of steepest ascent on a geographical map (see Exercise 67). If such a path is parametrized as $\langle x(t),y(t)\rangle$, prove that

$$\frac{y'(t)}{x'(t)}=\frac{f_y}{f_x}.$$

72. Use Exercise 71 to find the path of a heat-seeking object if it starts at the point $(5,25)$ on a plane whose temperature is given by the function $T(x,y)=500-x^2-3y^2$. (**Hint:** Notice that this is a separable initial value problem. See Section 8.1.)

73. Use Exercise 71 to find the equation in the *xy*-plane of the steepest path from the point $(400,200)$ (this is the projection of the spot where Sarah is standing) for the function in Exercise 68. (See the hint given in Exercise 72.)

74.* Consider the piecewise–defined function of Exercise 114 of Section 13.3.

$$F(x,y)=\begin{cases}\dfrac{xy^2}{2\left(x^2+y^2\right)} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Prove that all directional derivatives of *F* exist at $(0,0)$, but *F* is not differentiable at $(0,0)$. (**Hint:** To prove the existence of the directional derivatives, use the definition.)

75.* Suppose $f(x,y)$ is defined and differentiable on an open region *R*, and $\nabla f(x,y)=\mathbf{0}$. Prove that $f(x,y)$ is constant on *R*.

Example 7 Linearizing a Function of Four Variables

Find the linear approximation of the following function at the given point.

$$f(w,x,y,z)=xy^2-\frac{3w}{z}+\sin(x^3yz);\quad (-2,1,0,3)$$

Solution

The linearization of f at $(-2,1,0,3)$ is given by the formula

$$\begin{aligned}L(w,x,y,z)&=f(-2,1,0,3)+f_w(-2,1,0,3)(w+2)+f_x(-2,1,0,3)(x-1)\\&\quad+f_y(-2,1,0,3)(y-0)+f_z(-2,1,0,3)(z-3),\end{aligned}$$

so we calculate the following values of the function and its four partial derivatives.

$$\begin{aligned}f(-2,1,0,3)&=(1)(0)^2-\frac{3(-2)}{3}+\sin\left((1)^3(0)(3)\right)=2\\f_w(-2,1,0,3)&=\left[-\frac{3}{z}\right]_{(-2,1,0,3)}=-1\\f_x(-2,1,0,3)&=\left[y^2+(3x^2yz)\cos(x^3yz)\right]_{(-2,1,0,3)}=0\\f_y(-2,1,0,3)&=\left[2xy+(x^3z)\cos(x^3yz)\right]_{(-2,1,0,3)}=3\\f_z(-2,1,0,3)&=\left[\frac{3w}{z^2}+(x^3y)\cos(x^3yz)\right]_{(-2,1,0,3)}=-\frac{2}{3}\end{aligned}$$

Hence,

$$\begin{aligned}L(w,x,y,z)&=2-(w+2)+3y-\frac{2}{3}(z-3)\\&=-w+3y-\frac{2}{3}z+2.\end{aligned}$$

13.6 Exercises

1–8 Determine the tangent plane and normal line at the indicated point of the given surface.

1. $z-x^2+2y^2=0;\quad (1,1,-1)$
2. $z-xy+y^3x^2=0;\quad (2,-1,2)$
3. $z^2+(3x+1)(y-x^2)=0;\quad (1,-3,4)$
4. $z^2-3x^2-y^2=15;\quad (2,-3,6)$
5. $(xy+2)^2+3(y+1)^2-\sqrt{z}=5;\quad (2,-2,4)$
6. $xyz=18;\quad (-3,2,-3)$
7. $x^2z-x^2y^2=3;\quad (1,-3,12)$
8. $z=\sin(2xy)-5;\quad \left(\frac{\sqrt{\pi}}{2},\frac{\sqrt{\pi}}{2},-4\right)$

9–16 Determine the tangent plane at the indicated point of the given surface.

9. $z=xy^2-x^3y;\quad (1,2,2)$
10. $z=\tan(xy)+2;\quad \left(1,\frac{\pi}{4},3\right)$
11. $z=\frac{y}{\sqrt{x}};\quad (9,3,1)$
12. $z=\ln(x^2+2y^2);\quad (1,1,\ln 3)$
13. $z=\frac{1}{x^2}+y^2;\quad (1,3,10)$
14. $z=2\sin x\cos y;\quad \left(\frac{\pi}{4},\frac{\pi}{4},1\right)$
15. $z=\frac{1}{\sqrt{x^2+y^2}};\quad \left(3,4,\frac{1}{5}\right)$
16. $z=e^{y/x};\quad (3,6,e^2)$

17. Given a differentiable function $f(x,y)$, use your knowledge of partial derivatives to show that the vectors $\mathbf{u} = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\mathbf{v} = \langle 0, 1, f_y(x_0, y_0)\rangle$ are direction vectors for the plane tangent to the surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$. Then use the cross product to find a normal vector $\mathbf{n}$ and arrive at the following equation of the tangent plane that we derived immediately preceding Example 2.

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

18. Assuming that $F(x,y,z)$ is differentiable and $F(x,y,z)=0$ defines a function $z=f(x,y)$ implicitly, use the formulas $f_x = -F_x/F_z$ and $f_y = -F_y/F_z$ to provide yet another derivation of the equation of the tangent plane described in Exercise 17.

19–22 Find parametric equations for the line tangent to the intersection of the two surfaces at the indicated point.

19. $z = \dfrac{x^2+y^2}{5}$; $z = 8-3x^2-y^2$; $(1,2,1)$

20. $3x^2+y^2-z^2=2$; $x^2+y^2+z^2=14$; $(2,1,3)$

21. $x^2+5y^2+2z^2=11$; $xyz=2$; $(-2,1,-1)$

22. $4x^3+y^3+5x^2y-xy=z^2$; $4x^2-y^2-z=0$; $(1,1,3)$

23. Find a parametrization $\mathbf{r}(t)$ of the curve C of Example 3; then determine $\mathbf{r}'$ at the point $(1,1,1)$, and use it as a direction vector to obtain parametric equations for the tangent line to C at $(1,1,1)$.

24. Show that, in general, the tangent to curve C at the point (x,y,z) of the intersection of the surfaces in Example 3 points in the direction of $\langle -10y, 6x, -4xy\rangle$. (**Hint:** Determine the cross product of ∇F and ∇G.)

25. Find all points on the surface $8x^2+4y^2+z^2=4$ where the tangent plane is normal to the vector $\langle 4,2,1\rangle$. (Notice that this surface is an ellipsoid. How many such points do you expect?)

26. Repeat Exercise 25 for the hyperboloid $x^2+3y^2-2z^2=6$ and the vector $\langle 1,3,2\rangle$.

27–34 Find **a.** the differential df at an arbitrary point (x,y) and **b.** the linearization of f at the indicated point.

27. $f(x,y)=x^3+y-(y+2)^2$; $(1,3)$

28. $f(x,y)=xy^3-x^2y^2+x^3y$; $(-2,1)$

29. $f(x,y)=\sqrt{x^2+y^2}$; $(\sqrt{3},-1)$

30. $f(x,y)=x\cos y+y\cos x$; $(\pi,0)$

31. $f(x,y)=\dfrac{e^{x^2-y^2}}{2}$; $(1,1)$

32. $f(x,y)=\arctan\dfrac{x}{y}$; $(\sqrt{3},1)$

33. $f(x,y)=2(\sqrt{x}-\sqrt{y})$; $(1,4)$

34. $f(x,y)=e^{3x}\cos 2y$; $(0,\pi/2)$

35–39 Find the linear approximation of the function at the indicated point.

35. $f(w,x,y,z)=2x^2(wy-z^3)+\dfrac{4wx}{yz}$; $(3,-1,1,2)$

36. $f(x,y,z)=\dfrac{xyz}{x^2+y^2+z^2}$; $(2,1,-1)$

37. $f(w,x,y,z)=e^{\sqrt{w^2+x^2+y^2+z^2}}$; $(-2,2,0,-1)$

38. $f(w,x,y,z)=\arctan(wxyz)$; $(-1,1,1,-1)$

39. $f(x,y,z)=e^x\sqrt{\ln(y-z)}$; $(1,e,0)$

40. Using the linear approximation of the function $f(x,y)=\sqrt{x^2-y^2}$ at the point $(5,4)$, find an approximation for the number $\sqrt{(4.9)^2-(4.1)^2}$. Compare your approximation with the value returned by your calculator.

41–46 Find an appropriate function and mimic the process from Exercise 40 to approximate the given number. Compare your approximation with the value returned by your calculator. (In Exercises 45–46, generalize to three variables.)

41. $\dfrac{(1.98)^3}{(3.01)^2-1}$

42. $\sqrt{\dfrac{16.5}{24.2}}$

43. $\dfrac{1}{\sqrt{(3.8)^2+(3.1)^2}}$

44. $\dfrac{6.16}{2.9}$

45. $\sqrt{(8.98)(3.01)(2.9)}$

46. $\dfrac{\ln(2.8)}{(2.15)(4.95)}$

47. Suppose the legs of a right triangle are known to within 3% relative error and within 4% relative error, respectively. Find an upper bound on the relative error with which the triangle's area is known.

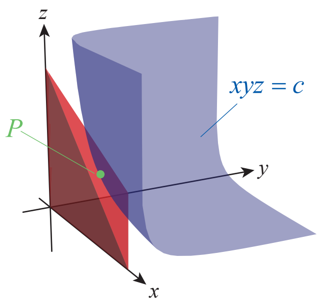
48. Suppose the length and width of the base of a rectangular pyramid were measured within a relative error of 1%, while its height is known to within 3% relative error. What is the greatest possible relative error this causes in the volume of the pyramid?

49. Repeat Exercise 48 for a circular cone if the radius of the base is measured within a relative error of 1%, while its height is known to within 3% relative error.

50. The electrical power used by a resistor is given by the equation $P = V^2/R$, where R is its resistance and V is the voltage. Suppose R and V are known to within 2% and 1% relative error, respectively. Find an upper bound on the relative error with which the power is known.

51. As we have seen earlier (see, for example, Exercise 17 in Section 3.7), the speed of impact of a body falling from a height h in the absence of air resistance is $v = \sqrt{2hg}$. Find the maximum relative error if one uses $g \approx 10 \text{ m/s}$ (instead of its "true value" of 9.81 m/s), and if h is only known to within 3% of its true value.

52. Consider the first-octant tetrahedron formed by the coordinate planes and the plane tangent to the surface $xyz = c$ at a fixed point P on the surface. Find the volume of this tetrahedron and show that its value is independent of the choice of P.



53. Show that every plane tangent to the elliptic cone $x^2 + y^2 - z^2 = 0$ passes through the origin.

54. Suppose you are working on a mathematical model that involves positive numbers less than 1000 and that your calculator rounds all your data to two decimal places. Use linear approximation to estimate the maximum error this might cause when multiplying three numbers.

55. Consider the two moles of ideal gas in the 50-liter container in Exercise 70 of Section 13.4. Use linear approximation to estimate the change in pressure if the temperature increases from 323 K to 328 K and the volume increases to 51 liters.

56.* Suppose four resistors are connected in parallel, with the following resistances: $R_1 = 12$ ohms (Ω), $R_2 = 16\ \Omega$, $R_3 = 9\ \Omega$, and $R_4 = 6\ \Omega$. If the actual resistances might differ by up to 2% from the above specifications, use linear approximation to estimate the maximum relative error in the calculated value of the net resistance. (See Exercise 101 of Section 13.3.)

57. Recall that the standard equation of the ellipsoid centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Show that the tangent plane to its surface at the point (x_0, y_0, z_0) has equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$.

58. Find the equation of the tangent plane to the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ at (x_0, y_0, z_0). (See Exercise 57.)

Concept Check

59–62 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

59. If both $f_x(a,b)$ and $f_y(a,b)$ exist, then a tangent plane to the graph of $f(x,y)$ exists at the point (a,b).

60. If a tangent plane to the graph of $f(x,y)$ exists at the point (a,b), then every directional derivative of f exists at (a,b).

61. If $f(x,y)$ is differentiable at (a,b), then $\nabla f(a,b)$ is contained in the tangent plane and points in the direction of greatest increase of f.

62. If (x_0, y_0, z_0) is a point on the level surface $F(x,y,z) = k$ of a differentiable function $F(x,y,z)$, then $\nabla F(x_0, y_0, z_0)$ is normal to the plane tangent to that level surface at (x_0, y_0, z_0).

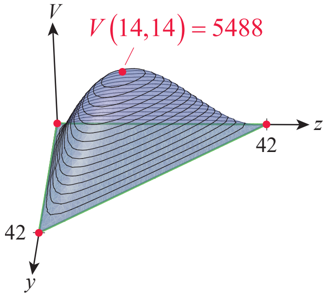


Figure 8

We leave it to the reader (Exercise 32) to show that $V(y,z)=0$ on each edge of the boundary D and also to show that V has a relative (and hence absolute) maximum value at the point $(14,14)$. Thus, the dimensions that maximize volume are a length of 28 inches with a square cross-section of width 14 inches, and the maximum possible volume is 5488 cubic inches. Figure 8 shows a graph of the function V over its triangular region of definition.

13.7 Exercises

1–12 Find any extrema of the given function. Classify critical points without using the Second Partial Derivative Test.

1. $f(x,y)=x^2+4x+y^2-6y+10$
2. $f(x,y)=x^2-12x+y^2+8y$
3. $f(x,y)=10x-x^2-4y-y^2$
4. $f(x,y)=-x^2-y^2-4y+1$
5. $f(x,y)=x^2-8x-y^2-2y+15$
6. $f(x,y)=y^2-x^2-6x$
7. $f(x,y)=y^4-4y^2+x^2+5$
8. $f(x,y)=x^4-2x^2+y^2$
9. $f(x,y)=\dfrac{1}{2+x^4+y^4}$
10. $f(x,y)=\dfrac{1}{x^2+y^2-4y+5}$
11. $f(x,y)=\ln\left(x^2+y^2+2x+2\right)$
12. $f(x,y)=\dfrac{1}{x^2-4x+y^2-2y+6}$

13–30 Use the Second Partial Derivative Test (if necessary) to classify the critical points of the given function. If the test fails, classify the critical point by other means. Identify absolute extrema wherever appropriate.

13. $f(x,y)=x^2+6x+6y^3-8y$
14. $f(x,y)=2x^2-4x+y^2+6y-1$
15. $f(x,y)=y^4-2y^2+4x^2$
16. $f(x,y)=6x^2y-x^2-3y^2$
17. $f(x,y)=x^3+y^3-3x^2-2y^2$
18. $f(x,y)=x^3+2xy^2-y^2$
19. $f(x,y)=3xy-x^3-y^3$
20. $f(x,y)=2xy+\dfrac{4}{x}+\dfrac{1}{y}$
21. $f(x,y)=3-\sqrt[3]{x^2+y^2}$
22. $f(x,y)=\dfrac{x^2+2y^2}{2e^x}$
23. $f(x,y)=2xy-x^4-y^4$
24. $f(x,y)=ye^{x^2-y^2}$
25. $f(x,y)=2xe^{2x-y^2}$
26. $f(x,y)=x\ln(x+y)$
27. $f(x,y)=\left(x^2-y^2\right)e^x$
28. $f(x,y)=e^x-xe^y$
29. $f(x,y)=x\cos y$

30.* $f(x,y)=\sin x+\cos(x+y)$

31. Find and classify all extrema of the function $f(x,y)=\sqrt{x^2+y^2}$.
32. Show that $V(y,z)=0$ on each edge of the boundary of D in Example 6, and demonstrate that V has a relative (and hence absolute) maximum value at the point $(14,14)$.

33–42 Find the absolute extrema of the function on the given region.

33. $f(x,y)=2x+4y-3;$ $D=\{(x,y)\mid -1\le x\le 1, 0\le y\le 2\}$
34. $f(x,y)=3x-\dfrac{y}{2}+1;$ $D=\{(x,y)\mid -2\le x\le 3, -1\le y\le 5\}$

35. $f(x,y)=2xy-x-y+4$; D: The triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$

36. $f(x,y)=4xy-2x-y+1$; D: The triangle with vertices $(0,0)$, $(4,0)$, and $(0,2)$

37. $f(x,y)=(x-2y)^2$; D: The triangle with vertices $(0,0)$, $(12,0)$, and $(0,3)$

38. $f(x,y)=1-\sqrt{x^2+y^2}$; $D=\{(x,y)\,|\,x^2+y^2\le 1\}$

39. $f(x,y)=x^2-2xy+y$; $D=\{(x,y)\,|\,-2\le x\le 2, -1\le y\le 1\}$

40. $f(x,y)=x^2-2xy+y$; $D=\{(x,y)\,|\,0\le x\le 2, x^2\le y\le 4\}$

41. $f(x,y)=\dfrac{xy}{(x^2+1)(y^2+1)}$; $D=\{(x,y)\,|\,0\le x\le 2, 0\le y\le 2\}$

42. $f(x,y)=\dfrac{xy}{(x^2+1)(y^2+1)}$; $D=\{(x,y)\,|\,0\le x\le 2, 0\le y\le \sqrt{4-x^2}\}$

43. Find the absolute extrema of the function $g(x,y)=(x^2+2y^2)e^{-(x^2+y^2)}$ on the square $S=\{(x,y)\,|\,|x|\le 2, |y|\le 2\}$.

44–49 Show that the Second Partial Derivative Test fails for the given function and classify any critical points by other means.

44. $f(x,y)=x^2y^2$

45. $g(x,y)=\dfrac{1}{x^2+y^2}$

46. $h(x,y)=(x-1)^3+(y+2)^3$

47. $F(u,v)=(u+1)^{2/3}+(v-1)^{2/3}$

48. $R(s,t)=s^3+t^3-3t^2-2$

49. $k(x,y)=x^3-2x^2y$

50. Demonstrate that even though the function $f(x,y)=4+2x-x^2$ has infinitely many critical points, the Second Partial Derivative Test fails to classify any of them. Are those points extrema, and if so, what kind?

51. Determine m and b such that the sum of the squares of vertical distances from the line $y=mx+b$ to the points $(0,3)$, $(1,1)$, and $(4,8)$ is minimal.

52. Repeat Exercise 51 for the points $(1,0)$, $(2,5)$, and $(6,9)$.

53. Find a line that minimizes the sum of squares of the horizontal distances between the points in Exercise 51 and the line.

54. Find a line that minimizes the sum of squares of the horizontal distances between the points in Exercise 52 and the line.

55. By generalizing Exercise 51 to n points, this exercise will prove the formulas used in the least-squares method of curve fitting (see Section 1.5). Given n data points (x_1,y_1), (x_2,y_2), ..., (x_n,y_n) in the plane, let $y=mx+b$ be the line that minimizes the sum of the squares of the vertical distances from the line to the points.

a. Using the notation $S(m,b)=\sum_{i=1}^{n}\left[y_i-f(x_i)\right]^2$, show that in order for $S(m,b)$ to be minimal, m and b have to satisfy

$$\left(\sum_{i=1}^{n}x_i\right)m+nb=\sum_{i=1}^{n}y_i$$

and

$$\left(\sum_{i=1}^{n}x_i^2\right)m+\left(\sum_{i=1}^{n}x_i\right)b=\sum_{i=1}^{n}x_iy_i.$$

b. Solve the above system to derive the formulas introduced in Section 1.5 for m and b.

56. Find the minimum distance from the origin to the surface $xyz^2=2$.

57. Repeat Exercise 56 for the surface $x^3y^2z=1$.

58. Repeat Exercise 56 for the surface $z^2-xy^2=3$.

59. Use the methods of this section to find the minimum distance between the point $(-3,-4,1)$ and the plane $4x+y-2z=3$.

60. Find the dimensions of a rectangular prism with a fixed surface area of 6 square units and maximum volume.

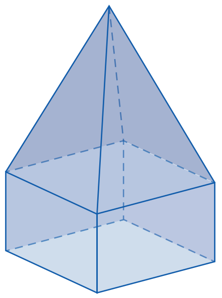
61. Find the dimensions of a rectangular prism with a fixed volume V and minimum surface area.

62. Repeat Exercise 61 for an open box that has no lid.

63. Suppose we want to paint the inside of a rectangular box of volume V and that the paint used on the sides costs \$2 per square unit, the paint for the top is \$3 per square unit, while the sealant used to paint the bottom is \$5 per square unit. What are the dimensions of the box that is the most cost-effective to paint under these conditions?

64. A rectangular box is placed in the three-dimensional coordinate system with one vertex at the origin and the three edges containing it lying along the positive coordinate axes. If the vertex opposite the origin lies in the plane $4x + y + 2z = 9$, what is the greatest possible volume for such a box?

65.* A right rectangular pyramid with a square base is sitting atop a right rectangular prism with a congruent base to form the solid seen in the given figure. Find the side length of the base, and the respective heights of the pyramid and the prism that minimize the lateral surface area of the solid, if its volume is $\frac{100}{3}$ cubic units.



66. Find the dimensions of the rectangular box of maximum volume inscribed in a hemisphere of radius R. (**Hint:** First argue that one of the box's faces should lie in the base plane of the hemisphere.)

67.* Find the volume of the largest box inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

68. If the sum of three positive numbers is 300, what is the greatest possible value for their product?

69. Again, as in Exercise 68, assume the sum of three positive numbers is 300. Find the minimum value for the sum of their squares.

70. A 30 in. piece of wire is cut into three pieces which are then bent into squares. If possible, determine how to **a.** minimize, and **b.** maximize the sum of the areas of the three squares.

71. Suppose $a, b, c > 0$ are given and consider the tetrahedron formed by the three coordinate planes and an arbitrary plane containing the point (a, b, c). Find the minimum volume for such a tetrahedron.

72.* Find the coordinates of the point Q so that the sum of the squares of the distances between Q and the given points $(x_1, y_1), \ldots, (x_n, y_n)$ is minimal.

73. Show that the function $f(x, y) = 3x^4 - 8x^2y + 4y^2$ has a relative minimum along every line $y = mx$ through the origin, but $(0, 0)$ is *not* a relative minimum for f. (**Hint:** Examine the behavior of f along the parabola $y = x^2$.)

Concept Check

74–78 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. If $f(x, y)$ has a relative minimum at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

75. If $f_x(a, b) = f_y(a, b) = 0$, then $f(x, y)$ has a relative extremum at (a, b).

76. If $f(x, y)$ has a relative extremum at an interior point (a, b) of its domain, and if f_x and f_y both exist at (a, b), then $f_x(a, b) = 0 = f_y(a, b)$.

77. If $f(x, y)$ has exactly two relative maxima, then it must have a relative minimum also.

78. If $f(x, y)$ has an extremum at (a, b) along every straight line $y = mx$, then (a, b) is an extremum for f.

13.7 Technology Exercises

79. With the help of a graphing utility, create an example of a two-variable function that has two maxima, but no relative minima. Sketch the graph of your example. (To start off, you may want to review, for example, Exercise 7, 15, or 23 of this section. Answers will vary.)

80–85 Use a graphing utility to graph the function of the indicated exercise and graphically reinforce your conclusions made in the referenced exercise.

80. Exercise 7

81. Exercise 15

82. Exercise 21

83. Exercise 29

84. Exercise 30

85. Exercise 43

13.8 Exercises

1. Find the absolute extreme values of the function $f(x,y)=2y^2-x^2$ on the circle $x^2+y^2=4$.

2. Find the absolute extreme values of $f(x,y)=xy$ on the ellipse $\frac{x^2}{4}+\frac{y^2}{8}=1$.

3–18 Find any extreme values of the function subject to the given constraint.

3. $f(x,y)=x^2+y^2$; constraint: $xy=2$

4. $f(x,y)=x^2+y^2$; constraint: $2x+y=5$

5. $f(x,y)=x^2-3xy+y^2$; constraint: $x^2+y^2=2$

6. $f(x,y)=x^2+3xy+y^2$; constraint: $y-x=2$

7. $f(x,y)=\frac{x^2+y^2}{2}$; constraint: $x^2+2y^2=4$

8. $f(x,y)=x^2+y^2$; constraint: $2x^2+3y^2=8$

9. $f(x,y)=x^2-y^2$; constraint: $2x+y=6$

10. $f(x,y)=x^2-y^2$; constraint: $y^2=4x$

11. $f(x,y)=2xy$; constraint: $\frac{x^2}{9}+\frac{y^2}{4}=1$

12. $f(x,y)=\sqrt{3x^2+y^2-2}$; constraint: $x^2+2y^2=14$

13. $f(x,y,z)=2x-3y+z$; constraint: $6z=3x^2+y^2$

14. $f(x,y,z)=x^2+y^2+z^2$; constraint: $2x-4y+z=14$

15. $f(x,y,z)=x+2y+8z$; constraint: $x^2+9y^2+16z^2=144$

16. $f(x,y,z)=xyz$; constraint: $x^2+y^2+z^2=4$

17. $f(x,y,z)=x^2-2y^2+z^2$; constraint: $2x^2+y^2+3z^2=18$

18. $f(x,y,z)=x+2y+3z$; constraint: $x^2+y^2+z^2=1$

19–26 Use Lagrange multipliers to find the coordinates of the point on the given curve that is closest to the point *P*.

19. $y=2x-3$; $P(2,2)$

20. $y=1-3x$; $P(-3,0)$

21. $y=x^2$; $P(3,0)$

22. $y=x^2+x$; $P(2,-2)$

23. $y=x^2+x$; $P(2,0)$

24. $y=2x-x^2$; $P(1,0)$

25. $x^2-2y^3=1$; $P(0,0)$

26. $(x^2+4)y=10$; $P(0,0)$

27–34 Use Lagrange multipliers to find the coordinates of the point on the given surface that is closest to the point *P*.

27. $3x-2y+\sqrt{3}z+16=0$; $P(0,0,0)$

28. $z=3x-2y+3$; $P(2,1,0)$

29. $2z+x-y=1$; $P(3,-2,-1)$

30. $z^2=x^2+y^2$; $P\left(1,0,\frac{1}{2}\right)$

31. $z=x^2+2y^2$; $P(0,0,1)$

32. $xyz=2$; $P(0,0,0)$

33. $x=yz+1$; $P(2,0,0)$

34. $x^2=yz+1$; $P(0,0,0)$

35. Generalize Exercise 27 by showing that the location on the plane $ax+by+cz=d$ closest to the origin is the following point.
$$\left(\frac{ad}{a^2+b^2+c^2},\frac{bd}{a^2+b^2+c^2},\frac{cd}{a^2+b^2+c^2}\right)$$

36. Suppose we want to cut a rectangular beam from a log that has a circular cross-section of radius $\sqrt{2}$ feet. Use Lagrange multipliers to find the dimensions of the beam's cross-section if it is to have maximum area. (**Hint:** Place the cross-section in the *xy*-system with its center at the origin, and maximize the area of the beam's cross-section subject to the constraint $x^2+y^2=2$.)

37. Suppose that when manufacturing three different products, a company is able to make a profit of \$4 on each unit of the first product, while the profits on the second and third types of products are \$8 and \$6 per unit, respectively. Let *x*, *y*, and *z* denote thousands of units produced from each product.
    - **a.** Find the profit function $P(x,y,z)$.
    - **b.** Assuming that the manufacturing process is under the constraint $x^2+2y^2+z^2\leq 756$, find the maximum profit for the company under these conditions.

38–48 We will revisit some exercises from Section 13.7. Use Lagrange multipliers to provide a second solution for each.

38. Find the dimensions of a rectangular prism with a fixed surface area of 6 square units and maximum volume.

39. Find the dimensions of a rectangular prism with a fixed volume V and minimum surface area.

40. Repeat Exercise 39 for an open box that has no lid.

41. Suppose we want to paint the inside of a rectangular box of volume V and that the paint used on the sides costs \$2 per square unit, the paint for the top is \$3 per square unit, while the sealant used to paint the bottom is \$5 per square unit. What are the dimensions of the box that is the most cost-effective to paint under these conditions?

42. A rectangular box is placed in the three-dimensional coordinate system with one vertex at the origin and the three edges containing it lying along the positive coordinate axes. If the opposite vertex lies in the plane $4x + y + 2z = 9$, what is the greatest possible volume for such a box?

43. Find the dimensions of the rectangular box of maximum volume inscribed in a hemisphere of radius R. (**Hint:** First argue that one of the box's faces should lie in the base plane of the hemisphere.)

44. If the sum of three positive numbers is 300, what is the greatest possible value for their product?

45. Find the minimum distance from the origin to the surface $xyz^2 = 2$.

46. Repeat Exercise 45 for the surface $x^3y^2z = 1$.

47. Repeat Exercise 45 for the surface $z^2 - xy^2 = 3$.

48.* Find the volume of the largest box inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

49. Use Lagrange multipliers to prove that among all triangles inscribed in a circle of radius R, the equilateral triangle has the largest area. (**Hint:** Draw the three radii connecting the center of the circle with the vertices of the triangle, and mark the angles formed by these radii α, β, and γ. Then use the formula for the area of a triangle with sides a, b and included angle θ, $A = \frac{1}{2}ab\sin\theta$.)

50–53 Find the absolute extreme value(s) of the function subject to the given constraints.

50. $f(x, y, z) = x + 2y + 2z$;
constraints: $x^2 + z^2 = 4$ and $2x + y + z = 1$

51. $f(x, y, z) = x^2 + y^2 + z^2$;
constraints: $x + y = 1$ and $2x - y + 3z = 2$

52. $f(x, y, z) = xyz$;
constraints: $x - y + z = 4$ and $x + y + z = 6$

53. $f(x, y, z) = 2x + y - 2z$;
constraints: $x^2 + y^2 = 4$ and $x - y + 2z = 1$

54. Supposing that $x, y, z > 0$, maximize the function $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$, and use your result to prove the famous inequality between geometric and arithmetic means for three positive numbers

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3},$$

with equality occurring precisely when $x = y = z$.

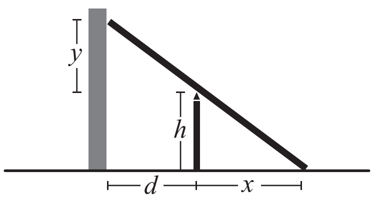
55.* Generalize Exercise 54 for n positive numbers.

56. The graph of the equation $x^2 + xy + y^2 = 4$ is a rotated ellipse. Use Lagrange multipliers to find its points closest to and farthest from the origin.

57. Find the points closest to and farthest from the origin on the intersection of the surfaces $z = 2x^2 + 2y^2$ and $z = 20 - x - y$.

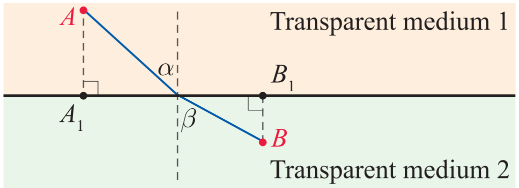
58. Find the highest and lowest points on the curve of intersection of the two surfaces $x^2 + y^2 + z^2 = 9$ and $x - 2y + z = 2$.

59. Suppose a fence of height h is located d units from a wall. Find the minimum length of a ladder that is able to reach the wall over the fence. (**Hint:** Minimize $f(x, y) = (x + d)^2 + (y + h)^2$ subject to the constraint $\frac{h}{x} = \frac{y}{d}$.)



60. Using Lagrange's method, show that the critical point of the function $f(x, y) = x^2y$ subject to the constraint $y - x = 0$ does not yield a local extremum. Explain.

61.* When a light ray travels from a transparent medium into another transparent medium, it "bends" or "refracts" as shown in the figure below. Using the fact that the traveling of light waves is governed by the principle of making the "best time" between A and B, and supposing that the speed of light in the two different media is v_A and v_B, respectively, use Lagrange's method to derive the equality $\frac{\sin\alpha}{\sin\beta} = \frac{v_A}{v_B}$. (This is called Snell's Law of refraction. **Hint:** Minimize the travel time of light subject to the constraint that the distance between A_1 and B_1 is constant.)



13.8 Technology Exercises

62–64 Use a computer algebra system or a programmable calculator to write a program that accepts a three-variable function along with one or two constraints, and uses Lagrange's method to return the critical points. Use it to check the answers you obtained for the following exercises.

62. Exercise 18

63. Exercise 50

64. Exercise 53

14.1 Exercises

1. Suppose $f(x,y) = x^2$ is defined on the square $R = [0,4] \times [0,4]$. Estimate $\iint_R f(x,y)\,dA$ using the Riemann sum approximation corresponding to $n = m = 4$, with each sample point (x_{ij}^*, y_{ij}^*) chosen to be the center point of the respective subsquare, $1 \le i, j \le 4$.

2. Use Fubini's Theorem to evaluate the integral of Exercise 1 and compare its true value to your estimate.

3. Evaluate the iterated integral and verify that your answer is equal to that obtained in Example 3.
$$\int_1^{5/2} \int_{1/2}^{2} \left(15 - x^2 - 2y^2\right) dy\,dx$$

4–15 Evaluate the iterated integral.

4. $\int_0^1 \int_0^4 (3x - 2y)\,dx\,dy$

5. $\int_0^3 \int_1^2 xy^2\,dy\,dx$

6. $\int_2^4 \int_{-1}^3 (5x + xy - 3y)\,dx\,dy$

7. $\int_{-2}^2 \int_0^1 (x^3 y - 9)\,dy\,dx$

8. $\int_{-1}^0 \int_0^3 (xy^4 - x^4 y)\,dx\,dy$

9. $\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \sin y \cos x\,dx\,dy$

10. $\int_1^2 \int_0^1 y^2 e^{-x}\,dy\,dx$

11. $\int_{-3/2}^0 \int_0^{\pi} (x^2 y - \cos x)\,dx\,dy$

12. $\int_1^2 \int_1^2 \left(\frac{x}{y} - \frac{y}{x}\right) dx\,dy$

13. $\int_{1/2}^2 \int_1^e \frac{1}{x^2 y}\,dy\,dx$

14. $\int_2^3 \int_0^1 \frac{x}{\sqrt{1-y^2}}\,dy\,dx$

15. $\int_0^1 \int_1^e \frac{1}{x\sqrt{y}}\,dx\,dy$

16–19 Suppose $\iint_{R_1} f(x,y)\,dA = 4$, $\iint_{R_1} g(x,y)\,dA = 5$, $\iint_{R_2} g(x,y)\,dA = -1$, and $\iint_{R_1 \cup R_2} f(x,y)\,dA = 12$ on the rectangular regions $R_1 = [-1,3] \times [2,4]$ and $R_2 = [-5,5] \times [-2,2]$. Use the properties of double integrals to evaluate the integral.

16. $\iint_{R_1} [2f(x,y) + 3]\,dA$

17. $\iint_{R_1} [4f(x,y) - g(x,y)]\,dA$

18. $\iint_{R_1 \cup R_2} [3f(x,y) + 5g(x,y)]\,dA$

19. $\iint_{R_2} [3f(x,y) - 2g(x,y)]\,dA$

20–22 Evaluate the given integral on the rectangular region $R = [-1,3] \times [0,4]$. (**Hint:** Recall that $[\![x]\!]$ denotes the greatest integer less than or equal to x.)

20. $\iint_R \left([\![y]\!] - [\![x]\!]\right) dA$

21. $\iint_R [\![x]\!] \cdot [\![y]\!]\,dA$

22. $\iint_R x \cdot [\![y]\!]\,dA$

23–34 Evaluate the iterated integral.

23. $\int_0^1 \int_{x^2}^{x} (x^2 + y^2)\,dy\,dx$

24. $\int_0^{1/2} \int_0^y (x + y)^2\,dx\,dy$

25. $\int_0^2 \int_{x/2}^{3-x} xy\,dy\,dx$

26. $\int_0^3 \int_{y/3}^{7-2y} x^2 y\,dx\,dy$

27. $\int_1^e \int_0^{\sqrt{9-x^2}} \frac{2y}{x}\,dy\,dx$

28. $\int_0^3 \int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{9-x^2}}\,dy\,dx$

29. $\int_0^1 \int_{y^2}^{y} e^y\,dx\,dy$

30. $\int_0^1 \int_y^{\sqrt{y}} (x^3 - x)\,dx\,dy$

31. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2y\,dy\,dx$

32. $\int_0^6 \int_{y/2}^{5-y/3} 2x\,dx\,dy$

33. $\int_0^4 \int_1^{e^x} \frac{1}{y}\,dy\,dx$

34. $\int_0^{\pi/2} \int_0^{\sin x} \frac{1}{\sqrt{1-y^2}}\,dy\,dx$

35–49 Rework the indicated exercise by reversing the order of integration and verify that the answer does not change. (**Hint:** It is helpful, sometimes even necessary, to sketch the region of integration before reversing the order.)

35. Exercise 4
36. Exercise 5
37. Exercise 6
38. Exercise 23
39. Exercise 24
40. Exercise 25
41. Exercise 26
42. Exercise 27
43. Exercise 28
44. Exercise 29
45. Exercise 30
46. Exercise 31
47. Exercise 32
48. Exercise 33
49. Exercise 34

50–52 Use the symmetry of the graph of f over R to evaluate the double integral. (Do not use repeated integration.)

50. $\iint_R x\,dA;\; R = \{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4\}$

51. $\iint_R \sin y\,dA;\; R = \left\{(x,y) \mid -1 \le x \le 5, -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

52. $\iint_R (x^3 + 1)\,dA;\; R = \{(x,y) \mid -3 \le x \le 3, -1 \le y \le 7\}$

53–54 Set up, but do not evaluate, the integral of $f(x, y)$ as an iterated integral over the triangle with the given vertices. Choose the most convenient order of integration. (Answers may vary.)

53. $\iint_R f(x,y)\,dA$; R is the triangle with vertices $(1,0)$, $(3,2)$, and $(4,0)$.

54. $\iint_R f(x,y)\,dA$; R is the triangle with vertices $(0,0)$, $(2,4)$, and $(4,3)$.

55. Evaluate $\iint_R 2x\,dA$ over the region R bounded by $y = \cos x$ and $y = \sin x$, $0 \le x \le \pi/4$.

56. Let R be the region between the circle $x^2 + y^2 = 9$ and the ellipse $4x^2 + 9y^2 = 36$. By integrating $f(x,y) = 1$ over R, find the area of the region.

57. Evaluate $\iint_R y\,dA$ over the region R bounded by $y + x = 2$ and $y = x^2 - 2x$.

58. Evaluate $\iint_R x^2\,dA$ over the region R bounded by $y^2 = 5 - x$ and $y^2 = 4x$.

59. Evaluate $\iint_R xy\,dA$ over the region R bounded by $y = x^3 - 3x$ and $4y = x^3$, $x \ge 0$.

60. Evaluate $\iint_R \sqrt{1 - x^2}\,dA$, where R is the first quadrant of the disk $x^2 + y^2 = 1$.

61. Evaluate $\iint_R y\,dA$, where R is the circular sector bounded by $y = 0$, $y = \sqrt{3}x$, and $x^2 + y^2 = 4$.

62. Evaluate $\iint_R xe^y\,dA$, where R is the first quadrant region bounded by $y = \ln x$ and $x = e$.

63–67 The iterated integral represents the (signed) volume of a well-known solid. Use a formula from geometry to evaluate the integral. (**Hint:** For Exercises 65 and 66, see Exercise 83 of Section 6.1 for the formula for the volume of a generalized cone.)

63. $\int_0^4 \int_1^3 2\,dy\,dx$

64. $\int_{-1}^2 \int_{-3/2}^1 (-3)\,dx\,dy$

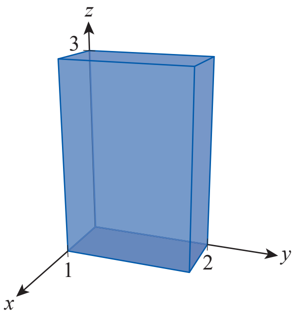
65. $\int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx$

66. $\int_0^1 \int_0^{3-3y} \left(\frac{5}{6}x + \frac{5}{2}y - \frac{5}{2}\right) dx\,dy$

67. $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{9 - x^2 - y^2}\,dx\,dy$

68–70 Set up and evaluate an appropriate double integral to determine the volume of the solid shown in the figure. Use the order of integration of your choice.

68.



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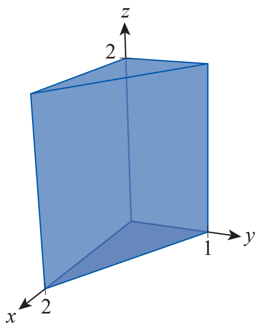

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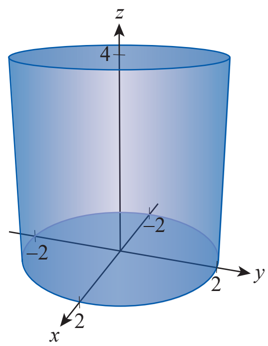


71–74 Evaluate the integral by reversing the order of integration.

71. $\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos(x^2)\,dx\,dy$

72. $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

73. $\int_0^1 \int_y^{\sqrt{y}} \frac{e^x}{x}\,dx\,dy$

74. $\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)\,dy\,dx$

75. Suppose a thin plate has variable density given by a continuous two-variable function $\rho(x,y)$. Interpret the double integral $\iint_R \rho(x,y)\,dA$.

76. Use the Domination Property to show that if $f(x,y)$ is integrable and nonnegative on the bounded region R, then $\iint_R f(x,y)\,dA \ge 0$.

77. Use the Domination Property to show that if $f(x,y)$ is bounded on a region R of area A, that is, if $m \le f(x,y) \le M$ for some real numbers m, M for all $(x,y) \in R$, then

$$m \cdot A \le \iint_R f(x,y)\,dA \le M \cdot A.$$

78.* Let $F(x,y)=1$ if at least one (or both) of x and y are irrational and $F(x,y)=0$ otherwise. Prove that F is not integrable over any bounded region R in $\mathbb{R}^2$.

79. Let $R=[a,b]\times[c,d]$, and suppose that the second-order partials of $f(x,y)$ are continuous on an open region containing R.

a. Use the Fundamental Theorem of Calculus to derive a formula for $\iint_R f_{xy}(x,y)\,dA$.

(**Hint:** Write the double integral as an iterated integral and use the Fundamental Theorem in two steps.)

b. Letting $f(x,y)=x^3y-2y^2+5xy^4$ and noting that $f_{xy}(x,y)=20y^3+3x^2$ (see Exercise 41 of Section 13.3), use part a. to evaluate $\int_{-2}^{2}\int_{1}^{3} f_{xy}(x,y)\,dx\,dy$.

80. Let $R=[a,b]\times[c,d]$ and suppose $f(x,y)$ can be decomposed into a product of a function of x and a function of y, that is, $f(x,y)=g(x)\cdot h(y)$. Show that

$$\iint_R f(x,y)=\left[\int_a^b g(x)\,dx\right]\cdot\left[\int_c^d h(y)\,dy\right].$$

81–86 Use Exercise 80 to provide a second solution to the given exercise.

81. Exercise 5

82. Exercise 9

83. Exercise 10

84. Exercise 13

85. Exercise 14

86. Exercise 15

87–90. Use the Riemann sum definition to prove the properties of double integrals. (For a refresher on the proofs of the one-variable case, see Section 5.2.)

Concept Check

91–94 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

91. If $f(x)$ is continuous, then $\int_c^d\int_a^b f(x)\,dy\,dx=(d-c)\int_a^b f(x)\,dx$.

92. If R_1 and R_2 are disjoint regions and $f(x,y)$ is continuous, then

$$\left|\iint_{R_1\cup R_2} f(x,y)\,dA\right|=\left|\iint_{R_1} f(x,y)\,dA\right|+\left|\iint_{R_2} f(x,y)\,dA\right|.$$

93. If a, b, $c>0$, then $\int_0^{\frac{c}{b}}\int_0^{\frac{c}{a}-\frac{b}{a}y}(c-ax-by)\,dx\,dy=\dfrac{abc}{6}$.

94. $\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,dy\,dx=\dfrac{4}{3}\pi R^3$

14.2 Exercises

1–12 Use double integration to find the area of the region bounded by the graphs of the given equations.

1. $y = 2x, \quad y^2 = 4x^3$
2. $\sqrt{2}y = x^2, \quad x^2 + y^2 = 4, \quad x = 0$
3. $x^2 = y, \quad x = y^2$
4. $y^2 = 4x, \quad 2x = 4 - y$
5. $y^2 = 9x, \quad y^2 = 10 - x$
6. $y = x, \quad xy = 36, \quad y = 12$
7. $2x = 8 - y, \quad y = 6x, \quad 2y = x^3$
8. $y = 2\sqrt[4]{x}, \quad y = x, \quad y = 2$
9. $y = x + 10, \quad y = 8x - x^2$
10. $x = (y-2)^2, \quad x = (y+2)^2, \quad x = 0$
11. $y = \frac{x}{4}, \quad y = 4x, \quad xy = 1$
12. $y = \frac{x^2}{2}, \quad y = \frac{1}{1+x^2}$

13–24 Find the average value of $f(x,y)$ over the region bounded by the graphs of the given equations.

13. $f(x,y) = x^2 - y; \quad y = 2x, \quad y^2 = x^3$
14. $f(x,y) = 2y, \quad y = \sqrt{3}x; \quad x^2 + y^2 = 1, \quad x = 0$
15. $f(x,y) = -x; \quad y = 2x^2, \quad 4x = y^2$
16. $f(x,y) = \frac{y^2}{2}; \quad x = y^2, \quad x = 2 - y$
17. $f(x,y) = x^2 y; \quad x^2 = 4y, \quad x^2 = 5 - y$
18. $f(x,y) = xy; \quad y = x, \quad xy = 1, \quad y = 0, \quad x = 2$
19. $f(x,y) = 5y^2; \quad x = 2 - y, \quad y = x^3, \quad x = 0$
20. $f(x,y) = xy^3; \quad y^3 = x, \quad y = x, \quad x \geq 0$
21. $f(x,y) = \sqrt{x}; \quad y = x + 4, \quad y = 6x - x^2$
22. $f(x,y) = e^{x+1}; \quad y = (x-1)^2, \quad y = (x+1)^2, \quad y = 0$
23. $f(x,y) = xy; \quad y = \frac{x}{2}, \quad y = 2x, \quad xy = 2$
24. $f(x,y) = x^2; \quad y = \frac{x^2}{4}, \quad y = \frac{1}{2+2x^2}$
25. Find the center of mass of the triangular plate of Example 3 if the density of the plate $\rho(x,y) = c$ is a constant. Compare your answer with that given in Example 3.

26–35 Find the center of mass of the plane region of varying density that is bounded by the graphs of the given equations.

26. $2y = 4 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = xy$
27. $y = 2x, \quad y = 3 - x, \quad y = 0; \quad \rho(x,y) = x + y$
28. $y = x^{3/2}, \quad y = \sqrt{x}; \quad \rho(x,y) = \sqrt{x}$
29. $y = \sqrt{1-x^2}, \quad x = 0, \quad y = 0; \quad \rho(x,y) = x + 2y$
30. $y = 4 - x^2, \quad y = 2 - x; \quad \rho(x,y) = x^2 y$
31. $y = \frac{1}{x^3}, \quad x = \frac{1}{2}, \quad x = 1, \quad y = 0; \quad \rho(x,y) = 2x^3 + y$
32. $y = x, \quad y = \sqrt{x}; \quad \rho(x,y) = y - x$
33. $xy^2 = 3, \quad y = \frac{1}{3}, \quad y = 1; \quad \rho(x,y) = xy$
34. $y = \sqrt{x}, \quad xy = 1, \quad y = 0, \quad x = 2; \quad \rho(x,y) = x^2$
35. $y = e^x, \quad x = 0, \quad y = 0, \quad x = 1; \quad \rho(x,y) = xy$

36–50 Determine I_x, I_y, and I_0 for the thin plate of constant density ρ modeled by the planar region R. Then find its corresponding radii of gyration.

36. R: The region bounded by the graph of $y = 1 - x^2$ and the x-axis
37. R: The square $[-a,a] \times [-a,a]$
38. R: The square of Exercise 37 shifted in the positive x-direction by a units
39. R: The rectangle $[-l/2, l/2] \times [-w/2, w/2]$
40. R: The rectangle of Exercise 39 shifted in the positive y-direction by $w/2$ units
41. R: An equilateral triangle of side length $2a$, with its center at the origin, and a vertex on the positive y-axis
42. R: The triangle of Exercise 41 shifted vertically upward so that its base is on the x-axis
43. R: The square of Exercise 37 with the smaller square $[-b,b] \times [-b,b]$ removed ($b < a$) (Solve this problem without using the Principle of Superposition.)
44. R: The region bounded by the graph of $y = 2 - \frac{x^2}{2}$ and the x-axis
45. R: The region bounded by the graph of $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$, and the x-axis

46. R: The region bounded by the graph of $y = \sin x$, $0 \le x \le \pi$, and the x-axis

47.* R: The region bounded by the ellipse $4x^2 + y^2 = 4$

48.* R: The region inside the ellipse $4x^2 + y^2 = 4$ and outside the circle $x^2 + y^2 = 1$ (Solve this problem without using the Principle of Superposition.)

49.* R: The region bounded by the ellipse $16x^2 + 25y^2 = 400$

50. R: The region bounded by the graphs of $y = e^x$, $x = \pm 1$, and the x-axis

51–57 As we have seen in Example 5, the Principle of Superposition is a very helpful aid in determining moments of certain objects. Another principle that simplifies the calculation of moments is called the *Parallel Axis Theorem*. It states that if we know the moment I_C of an object about an axis through its center of mass, then the moment of the same object about a parallel axis is $I = I_C + Md^2$, where d is the distance between the two axes. In these exercises, you will be asked to revisit some of the previous problems, using the Parallel Axis Theorem. To help you get started, we provided a hint in Exercise 51. You should also use the hint's approach with subsequent exercises.

51. Using the answer from Exercise 37, use the Parallel Axis Theorem to obtain a second solution to Exercise 38. (**Hint:** Note that in this case, I_x doesn't change. Can you see why? As for I_y, note that after the shift, the new axis is a units away from the old. After determining I_y with the help of the Parallel Axis Theorem, using $d = a$, I_0 is readily obtained by adding I_x and I_y.)

52. Using the answer from Exercise 39, use the Parallel Axis Theorem to obtain a second solution to Exercise 40.

53. Using the answer from Exercise 41, use the Parallel Axis Theorem to obtain a second solution to Exercise 42.

54. Using the answer from Exercise 45, use the Parallel Axis Theorem to obtain a second solution to Exercise 46.

55. Use the Parallel Axis Theorem to find I_x, I_y, I_0, and the corresponding radii of gyration for the ellipse in Exercise 49, if it is shifted to the right so that its left focus coincides with the origin.

56. Use the Parallel Axis Theorem to find I_x, I_y, I_0, and the corresponding radii of gyration for the disk in Example 4, if it is shifted upward by a units (so that it becomes tangential to the x-axis).

57. Starting with the answer you gave to Exercise 37 (and changing the notation appropriately), use the Parallel Axis Theorem to provide a second solution to Example 6.

58. Use Exercise 37 and the Principle of Superposition to provide a second solution to Exercise 43.

59. Use Exercise 47 and the Principle of Superposition to provide a second solution to Exercise 48.

60. Consider an annulus, as in Example 5, with $a_2 = 1$, $a_1 = \frac{1}{2}$, and $\rho = 1$. Furthermore, suppose that it is centered at $(a_1, 0)$. Starting with the results found for a disk in Example 4, use a combination of the Principle of Superposition and the Parallel Axis Theorem to find I_x, I_y, and I_0 for this annulus.

61. Suppose that $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ is removed from the 2-by-2 square $[-1,1] \times [-1,1]$ and that its center is shifted to the point $\left(0, \frac{1}{2}\right)$. Starting with the results of Example 6 (or Exercise 37) and assuming that the density is 1, use a combination of the Principle of Superposition and the Parallel Axis Theorem to find I_x, I_y, and I_0.

62–67 Consider the thin plates from the indicated problems with the given nonconstant densities. Determine I_x, I_y, I_0, and the corresponding radii of gyration.

62. Example 6; $\rho(x, y) = xy$

63. Exercise 36; $\rho(x, y) = y$

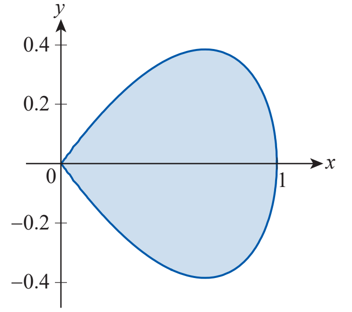
64. Exercise 45; $\rho(x, y) = y$

65. Exercise 46; $\rho(x, y) = x$

66.* Exercise 47; $\rho(x, y) = x^2$

67. Exercise 50; $\rho(x, y) = y$

68.* Determine I_x, I_y, I_0, and the corresponding radii of gyration for the thin plate of constant density ρ enclosed by the loop $y^2 = (1 - x)x^2$.



69. Suppose a thin, but solid, disk of uniform density (such as in Example 4) is rolling down an incline of height h (without slipping) in a race against the annulus of Example 5 and the square of Example 6, which is greased so it is sliding without friction (you can think of the latter as a sliding box). Which object will win, and in what order are they going to arrive at the bottom of the incline? Explain. (**Hint:** Use the fact that the initial potential energy of the object is eventually going to be shared between kinetic and rotational energies.)

Concept Check

70–73 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

70. If C is the average value of $f(x,y)$ over the region R, then $\iint_R C\,dA = \iint_R f(x,y)\,dA$.

71. If we increase the constant density of a planar object (or solid) from ρ to 2ρ, the coordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$ of its center of mass will double.

72. If we increase the constant density of a planar object (or solid) from ρ to 2ρ, then its second moments will double.

73. If we increase the constant density of a planar object (or solid) from ρ to 2ρ, then its radii of gyration will double.

14.2 Technology Exercises

74–76 Write a program for a computer algebra system or programmable calculator that uses the integral definition to find the moments of inertia I_x, I_y, I_0, and the corresponding radii of gyration. Use it to check the answers you obtained previously in the given exercise.

74. Exercise 55

75. Exercise 56

76. Exercise 59

77. Find the average value of $f(x,y)=\sqrt{y}$ on the region bounded by the graphs of $y=(x-1)^2$, $y=(x+1)^2$ and $y=0$.

In some cases, the best choice of a coordinate system might not be immediately clear. In our next example, the form of the integrand may lead us to try polar coordinates, but the shape of the region R is better suited for Cartesian coordinates. The integral is actually easier to evaluate with Cartesian coordinates, as you will see in Exercise 1, but for the sake of illustration and comparison we will use polar coordinates.

Example 4 Using a Double Integral in Polar Coordinates to Find Volume of a Solid

Let R be the triangle in the xy-plane with boundary lines $y = x$, $x = 0$, and $y = 1$. Find the volume of the solid bounded below by the region R and above by the surface $z = x^2 + y^2$.

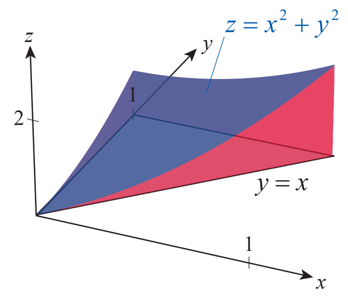


Figure 7

Solution

The solid as described is pictured in Figure 7, and the region R is shown in Figure 8. The lines $y = x$ and $x = 0$ correspond to the θ-limits $\theta = \pi/4$ and $\theta = \pi/2$, and each radial slice through R has a lower r-limit $r = 0$. Each upper r-limit corresponds to a point on the line $y = 1$, but this must be expressed in the form $r = g_2(\theta)$ for us to integrate using polar coordinates. We make the translation by noting that $y = r\sin\theta$, so

$$y = 1 \quad \Leftrightarrow \quad r\sin\theta = 1 \quad \Leftrightarrow \quad r = \csc\theta.$$

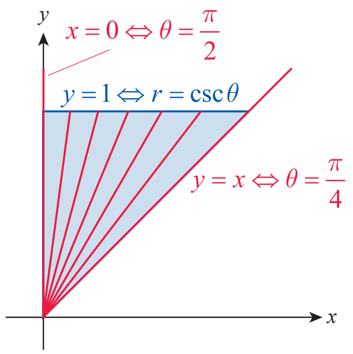


Figure 8

Putting all the pieces together, we obtain the following volume.

$$\text{Volume} = \iint_R (x^2 + y^2)\,dA = \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} r^2\,r\,dr\,d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/2}\left[r^4\right]_{r=0}^{r=\csc\theta} d\theta$$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/2}\csc^4\theta\,d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/2}\csc^2\theta\left(1 + \cot^2\theta\right)d\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/2}\csc^2\theta\,d\theta + \frac{1}{4}\int_{\pi/4}^{\pi/2}\csc^2\theta\cot^2\theta\,d\theta \qquad u = \cot\theta,\ du = -\csc^2\theta\,d\theta$$

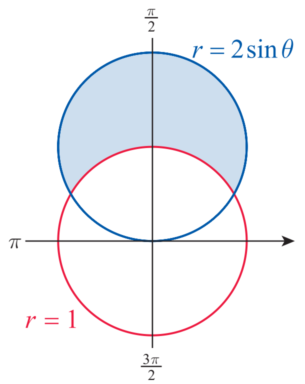
$$= \frac{1}{4}\left[-\cot\theta\right]_{\pi/4}^{\pi/2} + \frac{1}{4}\left[-\frac{1}{3}\cot^3\theta\right]_{\pi/4}^{\pi/2} = \frac{1}{3}$$

14.3 Exercises

1. Use Cartesian coordinates to determine the volume of the solid in Example 4.

2–5 Use a double integral in polar coordinates to find the area of the shaded region.

2.



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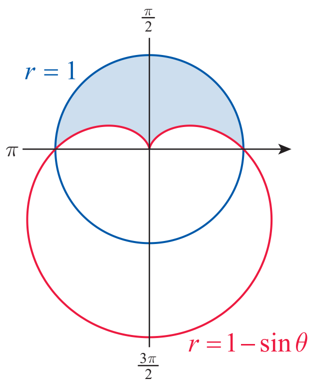

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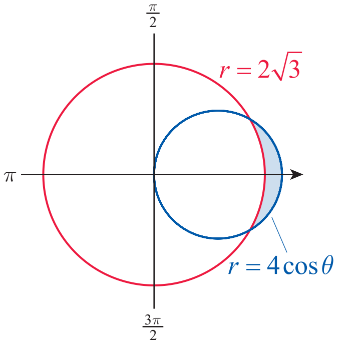
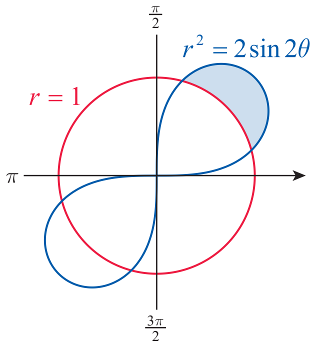


5.



6–18 Use a double integral in polar coordinates to find the area of the region R.

6. R: The region inside the circle $r = 2$

7. R: The region inside the circle $r = 6\sin\theta$

8. R: The region inside the cardioid $r = 1 - \sin\theta$

9. R: The region common to the circles $r = 6\cos\theta$ and $r = 3$

10. R: The region inside the cardioid $r = 1 + \cos\theta$ and outside the unit circle centered at the origin

11. R: The region inside the circle $r = 2$ and outside the limaçon $r = \frac{3}{2} + \sin\theta$

12. R: The region bounded by the spiral $r = 2\theta$ and the polar axis $(0 \le \theta \le \pi)$

13. R: The region inside one petal of the rose $r = 2\cos 2\theta$ and outside the circle $r = 1$

14. R: The region inside $r = 4 + 2\sin\theta$ and outside $r = 3$

15. R: The inner loop of the limaçon $1 + 2\cos\theta$

16. R: The region inside the cardioid $r = 2 - 2\sin\theta$ and outside the circle $r = 2$

17. R: The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \sin\theta$

18. R: The region inside the circle $r = 2\sin\theta$, but outside the lemniscate $r^2 = 2\cos 2\theta$ (**Hint:** Divide R into appropriate subregions and use symmetry.)

19–26 Evaluate the double integral by changing to polar coordinates. (Sketching the region of integration is helpful.)

19. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{dy\,dx}{x^2+y^2+2}$

20. $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

21. $\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)^{3/2} dx\,dy$

22. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \cos\left(x^2+y^2\right) dy\,dx$

23. $\int_{0}^{2}\int_{0}^{\sqrt{2y-y^2}} \left(x^2+y^2\right)^{5/2} dx\,dy$

24. $\int_{0}^{2a}\int_{0}^{\sqrt{2ay-y^2}} 2\,dx\,dy$

25. $\iint_R x\sqrt{x^2+y^2}\,dA$; R: The region enclosed by the first-quadrant loop of the lemniscate $r^2 = 2\sin 2\theta$

26. $\iint_R y\,dA$; R: The region inside the circle $r = 1$ and outside $r = \sin\theta$

27–30 Convert the integral into a Cartesian double integral and evaluate it.

27. $\int_{0}^{\pi/4}\int_{0}^{\sec\theta} r^2\cos\theta\,dr\,d\theta$

28. $\int_{\pi/3}^{\pi/2}\int_{0}^{2\csc\theta} r^3\sin\theta\cos\theta\,dr\,d\theta$

29. $\int_{\pi/4}^{\arctan 2}\int_{0}^{\csc\theta} r^3\sin 2\theta\,dr\,d\theta$

30. $\int_{0}^{\arctan(1/2)}\int_{0}^{2\sec\theta} r^5\sin^2(2\theta)\,dr\,d\theta + \int_{\arctan(1/2)}^{\pi/2}\int_{0}^{\csc\theta} r^5\sin^2(2\theta)\,dr\,d\theta$

31–34 Make your choice between the Cartesian and polar coordinate systems and evaluate the double integral.

31. $\iint_R \sqrt{x^2+y^2}\,dA$; R: $x^2+y^2 \le 4$

32. $\iint_R \left(x^2+y^2\right) dA$; R: The region bounded by $y = x$, $y = x/2$, and $x = 2$

33. $\iint\limits_R e^{\sqrt{x^2+y^2}}\,dA$; R: The first-quadrant region of $x^2+y^2 \le 1$

34. $\iint\limits_R \left(x^2+y^2\right)^2 dA$; R: The region bounded by $y=0$, $x=1$, and $y=x$

35. In Section 9.4, we derived the following formula for the area A of a region bounded by $\theta = a$, $\theta = b$, and the polar equation $r = f(\theta)$.

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

Use a double integral in polar coordinates and the discussion of this section to derive the formula above.

36–44 Use double integration in polar coordinates on an appropriate region to find the volume of the solid S bounded by the given surfaces.

36. S: The solid bounded by the xy-plane and the paraboloid $z = 1 - x^2 - y^2$

37. S: The solid bounded by the xy-plane and the paraboloid $z = 16 - x^2 - y^2$

38. S: The solid bounded by $z = x + 2y + 7$ and the cylinder $x^2 + y^2 = 4$

39. S: The solid bounded by $z = 2 - \sqrt{x^2+y^2}$ and the xy-plane

40. S: The solid bounded by $z = 3x + 5y + 9$, $z = x + 2y + 3$, and the cylinder $x^2 + y^2 = 2y$

41. S: The solid bounded by $z = x + y + 8$, the xy-plane, and the cylinder $r = 1 + \sin\theta$

42. S: The solid bounded by $z = x^2 + y^2$, the xy-plane, and the cylinder $r = 2 + \cos\theta$

43. S: The solid that is common to the paraboloids $z = 2\left(x^2+y^2\right)$ and $z = 12 - x^2 - y^2$ (**Hint:** Note that calculating the curve of intersection of the two surfaces will yield the region of integration.)

44. S: The solid that is common to the paraboloids $z = 9 - 8x^2 - 8y^2$ and $z = x^2 + y^2$ (See the hint given in Exercise 43.)

45. Use double integration in polar coordinates to find the volume of the solid in the shape of an ice-cream cone bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2 + y^2 + z^2 = 8$.

46. Use double integration in polar coordinates to derive the formula for the volume of a sphere of radius R, $V = \frac{4}{3}\pi R^3$.

47. Use double integration in polar coordinates to find the volume of the solid inside the paraboloid $z = 6 - x^2 - y^2$ and above the sphere $x^2 + y^2 + z^2 = 8$.

48.* Recall the wooden toy piece from Exercise 17 of Section 6.1. We will generalize that problem as follows. Suppose a cylindrical hole of radius r is drilled through the center of a sphere of radius R. Use double integration in polar coordinates to show that the volume of the remaining ringlike solid is $V = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$.

49. Assuming constant density, use double integration in polar coordinates to find the center of mass of the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin\theta$. (**Hint:** Use the symmetry of the region.)

50. Use double integration in polar coordinates to find the center of mass of the region outside the cardioid $r = 2 - 2\sin\theta$ and inside the circle $r = 2\cos\theta$. As in Exercise 49, we assume constant density.

51. Determine the second moments I_x, I_y, and I_0 for the thin plate of constant density ρ inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

52. Determine I_x, I_y, I_0, and the corresponding radii of gyration for the quarter annulus $\left\{(x,y) \middle| 0 \le x \le 2, \sqrt{1-x^2} \le y \le \sqrt{4-x^2}\right\}$. Assume the annulus has constant density ρ.

53. Use double integration in polar coordinates to find I_x, I_y, I_0, and the corresponding radii of gyration for the disk of Example 4 of Section 14.2, if the disk is shifted upward by a units.

54. Use double integration in polar coordinates to find I_x, I_y, and I_0 directly for the annulus in Exercise 60 of Section 14.2.

55. Assuming it has constant density 1, find I_x, I_y, I_0, and the corresponding radii of gyration for the first-quadrant loop of the lemniscate $r^2 = \sin 2\theta$.

56.* Repeat Exercise 55 for the region inside the circle $r = 2$ and outside $r = 2\cos(\theta/2)$. (**Hint:** Take advantage of the symmetry of the region.)

57. By changing to polar coordinates, verify the value of the following improper double integral.

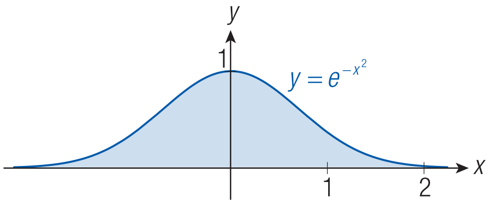
$$\int_0^\infty \int_0^\infty \frac{dy\,dx}{\left(1+x^2+y^2\right)^2} = \frac{\pi}{4}$$

(**Hint:** Integrate on the first-quadrant region of a disk of radius r, and let $r \to \infty$.)

58–59 In the next two exercises, you will be guided to use double integrals in polar coordinates to find the area under the bell curve $y = e^{-x^2}$, that is, to prove that

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

(This is an important integral not only in mathematics, but also in statistics, engineering, and physics.)



58. Use the technique of Exercise 57 to show

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}\,dx\,dy = \pi.$$

59. Evaluating J in rectangular coordinates over the square $[-a,a]\times[-a,a]$ (and letting $a \to \infty$), use Exercise 80 of Section 14.1 to show that $I^2 = J$. Conclude that $I = \sqrt{\pi}$.

Concept Check

60–63 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. If R is a square, and $f(r,\theta)$ is defined on R, then it is impossible to evaluate $\iint_R f(r,\theta)\,dA$ using polar coordinates.

61. If R is a bounded region, and $f(r,\theta) = 1$ on R, then the area A of R can be found as an iterated integral $\iint_R f(r,\theta)\,dr\,d\theta$.

62. The decision regarding which coordinate system (i.e., rectangular or polar) to use when evaluating $\iint_R f\,dA$, hinges upon the geometry (i.e., shape) of the region R.

63. Polar coordinates are suitable to determine the radii of gyration for certain planar regions.

14.3 Technology Exercises

64–66 Use a computer algebra system to evaluate the integral in both the rectangular and polar coordinate systems. Compare your answers.

64. $\int_0^1 \int_0^x \sqrt{x^2+y^2}\,dy\,dx$

65. $\int_0^1 \int_{1-y}^{2-2y} xy\,dx\,dy$

66. $\int_0^1 \int_y^{\sqrt{2-y^2}} \sqrt{x^2+y^2}\,dx\,dy$

(Note that even a CAS has a relatively hard time evaluating the integral of Exercise 66, while it is easy to determine, even just by a paper-and-pencil calculation, after converting it to polar coordinates!)

67–69 Assume that the thin plate covering the given region has a constant density of 1. Use a computer algebra system to find the moments of inertia I_x, I_y, I_0, and the corresponding radii of gyration.

67. The region bounded by $r = 1 - \sin\theta$

68. The region bounded by both $r = 1$ and $r = 1 + \sin\theta$

69. The region bounded by the inner loop of the limaçon $r = 1 + 2\cos\theta$

$$I_z = \iiint_S \left(x^2+y^2\right)\rho(x,y,z)\,dV = \rho\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\int_{x^2+3y^2+2}^{6-x^2-y^2}\left(x^2+y^2\right)dz\,dx\,dy$$

$$= \rho\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\left(4x^2-2x^4+4y^2-6x^2y^2-4y^4\right)dx\,dy$$

$$= -\frac{16\rho}{15}\int_{-1}^{1}\left(3y^4-y^2-2\right)\sqrt{2-2y^2}\,dy$$

$y=\sin\theta$
$dy=\cos\theta\,d\theta$

$$= -\frac{16\rho\sqrt{2}}{15}\int_{-\pi/2}^{\pi/2}\left(3\sin^4\theta-\sin^2\theta-2\right)\cos^2\theta\,d\theta$$

See Section 7.3.

$$= \sqrt{2}\pi\rho$$

This gives us the radius of gyration about the z-axis.

$$r_z = \sqrt{\frac{I_z}{M}} = \frac{1}{\sqrt{2}}$$

14.4 Exercises

1. Verify that
$$\iiint_S \left(3x^2y-xyz^3\right)dV = \int_1^3\int_0^2\int_{-1}^2\left(3x^2y-xyz^3\right)dy\,dz\,dx$$
yields the same result as that obtained in Example 1.

2–7 Evaluate the triple integral on the rectangular box S. (Choose a convenient order of integration.)

2. $\iiint_S xy^3z\,dV$, where $S=[-1,3]\times[0,1]\times[1,3]$

3. $\iiint_S dV$, where $S=[1,2]\times[3,4]\times[5,6]$

4. $\iiint_S \left(4xy+x^2yz^2\right)dV$, where $S=[0,1]\times[1,2]\times[-1,1]$

5. $\iiint_S \left(y^2z^2-2x^4y\right)dV$, where $S=[-1,1]\times[0,2]\times[-3,0]$

6. $\iiint_S \frac{xy}{z}\,dV$, where $S=[-1,3]\times[0,3]\times[1,e]$

7. $\iiint_S xye^z\,dV$, where $S=[1,2]\times[-1,3]\times[0,\ln 4]$

8–13 Evaluate the iterated integral.

8. $\int_0^1\int_0^{2-z}\int_0^{1-z} xy^2z\,dx\,dy\,dz$

9. $\int_0^1\int_{x^3}^{x}\int_0^{2xz} 5x\,dy\,dz\,dx$

10. $\int_0^3\int_1^2\int_0^{x+3y}(x+y)\,dz\,dy\,dx$

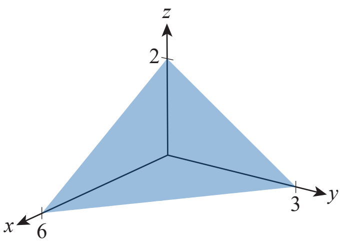
11. $\int_0^1\int_{y+1}^{3y}\int_0^{2xy}(xy)^2\,dz\,dx\,dy$

12. $\int_0^1\int_0^{4-x}\int_0^{4-x-z} dy\,dz\,dx$

13. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-y^2}} dz\,dx\,dy$

14–17 Write iterated integrals for $\iiint_S dV$ on the given solid, using the following orders of integration: **a.** $dz\,dy\,dx$, **b.** $dy\,dz\,dx$, and **c.** $dx\,dz\,dy$. Then evaluate one of them to determine the value of the integral.

14. S: The tetrahedron bounded by the coordinate planes and $x+2y+3z=6$



15. S: The cylinder bounded by $x^2+y^2=1$, $z=0$, and $z=2$

16. S: The solid bounded by the parabolic cylinder $z=1-x^2$, $z=0$, $y=0$, and $y=1$

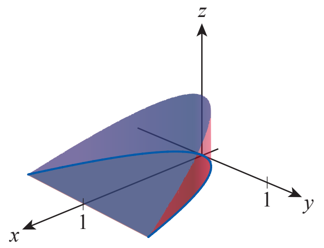
17. S: The solid bounded by $y=x^2$, $y^2=x$, $z=-1$, and $z=1$

18–36 Use a triple integral to find the volume of the solid S.

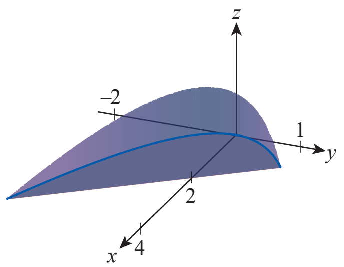
18. S: The tetrahedron bounded by the coordinate planes and the plane $\frac{x}{2}+\frac{y}{5}+\frac{z}{3}=1$

19. Generalize Exercise 18 by finding the volume of the tetrahedron S bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

20. S: The solid bounded by the parabolic cylinder $x=y^2$, and the planes $z=1-x$, $z=0$



21. S: The solid bounded by the parabolic cylinder $x=y^2$, and the planes $x+y+z=2$ and $z=0$. Use the order of integration $dz\,dx\,dy$.



22. Revisit Exercise 21, this time integrating in the order $dx\,dz\,dy$.

23. S: The solid bounded by the circular cylinders $x^2+y^2=4$ and $y^2+z^2=4$ (**Hint:** Take advantage of the symmetry of the solid.)

24. S: The solid bounded by the parabolic cylinder $5y=x^2-4$ and the planes $y=1-z$ and $z=0$

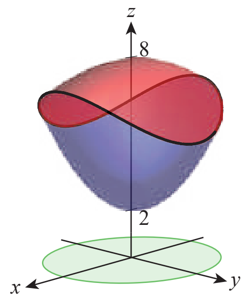
25. S: The solid bounded by the parabolic cylinder $x=1-y^2$ and the planes $x=0$, $z=0$, and $z=1-x$

26. S: The solid bounded by $z=y^2-1$, $z=0$, and $x=\pm 1$

27. S: The solid bounded by $z=x^2$ and the planes $z=4$, $y=0$, and $z=y$

28. S: The solid that is common to the paraboloids $z=9-8x^2-8y^2$ and $z=x^2+y^2$ (Exercise 44 of Section 14.3 revisited)

29. S: The solid bounded below by the surface $z=5x^2+y^2+2$ and above by the surface $z=8-x^2-y^2$. Choose the order of integration $dz\,dy\,dx$. (**Hint:** See Example 2.)



30. Repeat Exercise 29, but this time integrate in the order $dz\,dx\,dy$. (**Hint:** See Example 3.)

31. S: The solid bounded by $z=2-x^2-y^2$, $y=x^2$, $x=y^2$, and $z=0$

32. S: The solid bounded by $y=z^2$, $y=2-z^2$, $x=0$, and $x=2$

33. S: The solid bounded by $z=y^2$ and the planes $z=9-x$ and $x=0$

34. S: The solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=x+6$ (**Hint:** To make your calculations more manageable, integrate with respect to y first, and use the symmetry of the solid.)

35. S: The solid bounded by the elliptic paraboloid $z=4x^2+y^2$ and the plane $z=2y+3$ (**Hint:** Choose the order of integration carefully and use the symmetry of the solid.)

36. S: The solid bounded by the surfaces $z=2-\sqrt{4x^2+y^2}$ and $z=0$ (**Hint:** Start integrating with respect to y or x and use the symmetry of the solid.)

37. Write the triple integral over the solid of Exercise 27 in three different ways, in the orders of $dy\,dz\,dx$, $dx\,dz\,dy$, $dx\,dy\,dz$, and evaluate them. (**Note:** Quite possibly, you have already handled one of these integrals in Exercise 27.)

38. Repeat Exercise 37 for the integral in Exercise 33 using the following orders of integration: $dx\,dz\,dy$, $dy\,dz\,dx$, and $dz\,dx\,dy$.

39. Suppose there are one-variable functions g, h, and k such that $f(x,y,z)=g(x)\cdot h(y)\cdot k(z)$ and $S=[p,q]\times[r,s]\times[t,u]$. Prove the following.

$$\iiint_S f(x,y,z)\,dV = \left[\int_p^q g(x)\,dx\right]\cdot\left[\int_r^s h(y)\,dy\right]\cdot\left[\int_t^u k(z)\,dz\right]$$

(Note that this is a generalization of Exercise 80 of Section 14.1 to triple integrals.)

40–43 A solid S with variable density is given. Use a triple integral to find its mass.

40. S: The tetrahedron of Example 4, with its density at the point (x,y,z) being proportional to the point's distance from the tetrahedron's base (**Hint:** Integrate over the solid the density function $\rho(x,y,z)=k\cdot z$, where k is a constant.)

41. S: The tetrahedron bounded by $z=4-x-2y$ and the coordinate planes, with its density at the point (x,y,z) being proportional to the square of the distance from the origin (As in the previous exercise, denote the constant of proportionality k.)

42. S: The tetrahedron bounded by $z=3-2x-6y$ and the coordinate planes, with the density at any point being proportional to the sum of its coordinates

43. S: The solid upper hemisphere of radius 1 centered at the origin, with its density at the point (x,y,z) being proportional to the distance from the base

44–47 Just like we did with two-variable functions (see Section 14.2), we can define the **average value** of $f(x,y,z)$ over a solid S as follows.

$$\text{Average value of } f \text{ over } S=\frac{1}{\text{Volume}(S)}\iiint_S f(x,y,z)\,dV$$

Use the above definition to find the average value of f over S.

44. $f(x,y,z)=xyz$;
S: The cube $[0,a]\times[0,a]\times[0,a]$

45. $f(x,y,z)=\dfrac{1}{\sqrt{x}}$;
S: The cube $[0,a]\times[0,a]\times[0,a]$

46. $f(x,y,z)=xy\cos z$;
S: The cube $[0,\pi/2]\times[0,\pi/2]\times[0,\pi/2]$

47.* $f(x,y,z)=xyz$; S: The first-octant region of the sphere $x^2+y^2+z^2=R^2$

48. Finish Example 3 by showing that the value of the integral is $2\sqrt{2}\pi$.

49. **a.** Describe the solid of integration and **b.** find its centroid (assuming constant density).

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

50–62 A solid S with constant or variable density is given. Use a triple integral to find the coordinates of its center of mass.

50. S: The tetrahedron bounded by the coordinate planes and $x+y+z=2$, with constant density

51. S: The tetrahedron of Exercise 18, with constant density

52. S: The solid of Exercise 20, with constant density

53. S: The solid of Exercise 27, with constant density

54. S: The solid of Exercise 26, with constant density

55. S: The solid of Exercise 21, with constant density

56. S: The rectangular prism $[-1,1]\times[-1,1]\times[0,4]$, its density at each point is inversely proportional to the square root of the point's distance from the base

57. S: The cube of Exercise 45, its density at each point is proportional to the square of the point's distance from the origin

58. S: The tetrahedron of Exercise 50, its density at each point is proportional to the square of the point's distance from the origin

59. S: The tetrahedron of Example 4, its density at each point is proportional to the distance from the base

60.* S: The first octant of the unit sphere centered at the origin, its density at any point is proportional to the product of its distances from the coordinate planes

61. S: The solid half cylinder bounded by $z=\sqrt{1-y^2}$, $x=0$, and $x=1$, its density at any point is proportional to the square of the point's distance from the origin

62. S: The tetrahedron of Exercise 19

63. Verify that the second moments and radii of gyration for the solid of Example 4 are as follows.

$$I_x=\frac{8\rho}{15},\quad I_y=I_z=\frac{\rho}{3},\quad r_x=\frac{2}{\sqrt{5}},\quad r_y=r_z=\frac{1}{\sqrt{2}}$$

64–71 Find the center of mass and the radii of gyration of the given solid S. Assume S is made of a substance with constant density ρ.

64. $S=[-1,3]\times[0,1]\times[1,3]$, the rectangular box of Exercise 2

65. S: The cube of Exercise 45

66. S: The solid bounded by $z=4-2x-y$ and the coordinate planes

67. S: The solid bounded by $z=3-2x-6y$ and the coordinate planes

68. S: The solid bounded by $z=y^2$, the planes $z=1-x$ and $x=0$

69. S: The solid bounded by $z=y^2-1$, $z=0$, and $x==1$

70. S: The solid bounded by $z=2-x^2-y^2$, the coordinate planes, and the planes $x=1$ and $y=1$

71. S: The first-octant region of the cylinder $x^2+y^2=1$, bounded by the coordinate planes and $z=1$

72–75 Find the second moments and radii of gyration of the indicated solid.

72. The solid of Exercise 56

73. The solid of Exercise 57

74. The solid of Exercise 59

75. The solid of Exercise 58

14.4 Technology Exercises

76–79 Use a computer algebra system to find the center of mass and the radii of gyration for the given solid with nonconstant density. (Note that even though it does give nice answers, in some cases even a computer algebra system requires a relatively long time for calculating them!)

76. S: The upper hemisphere of radius 1, centered at the origin, the density at any point is inversely proportional to the square root of its distance from the base

77. S: The first octant region of the solid in Exercise 76, the density at any point is proportional to the product of its distances from the coordinate planes

78. S: The solid of Exercise 66, its density at any point being proportional to the square of the point's distance from the origin

79. S: The solid of Exercise 67, its density at any point being proportional to the square root of its distance from the base

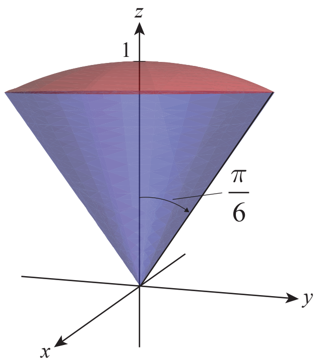


Figure 11

Example 6 Using a Triple Integral in Spherical Coordinates to Find the Volume of a Solid

Find the volume of the solid, shown in Figure 11, bounded below by the cone $\varphi = \pi/6$ and above by the sphere $\rho = 1$.

Solution

The lower and upper limits on ρ are 0 and 1, respectively, while the φ-interval is $[0, \pi/6]$ and the θ-interval is $[0, 2\pi]$. Since we seek only the volume, we integrate the constant function 1 over S. However, don't forget to include the factor of $\rho^2 \sin\varphi$, which is part of the volume differential dV.

$$\begin{aligned} V = \iiint_S dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/6}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1} \sin\varphi \, d\varphi \, d\theta \\ &= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/6} \sin\varphi \, d\varphi \, d\theta = \frac{1}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/6} d\theta \\ &= \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right)\int_0^{2\pi} d\theta = \frac{\left(2-\sqrt{3}\right)\pi}{3} \end{aligned}$$

14.5 Exercises

1–4 Find a set of cylindrical coordinates for the point given in Cartesian coordinates.

1. $(1,1,2)$ **2.** $(1,-\sqrt{3},-1)$

3. $(-6\sqrt{3},-6,0)$ **4.** $(-3,-4,-5)$

5–8 Find the Cartesian coordinates of the point given in cylindrical coordinates.

5. $\left(2,\frac{3\pi}{4},\sqrt{2}\right)$ **6.** $(3,\pi,3)$

7. $\left(-1,\frac{\pi}{2},1\right)$ **8.** $\left(4,-\frac{\pi}{3},\frac{\pi}{2}\right)$

9–16 Write the equation in cylindrical coordinates.

9. $x^2+y^2=4x$ **10.** $x^2=2z-y^2$

11. $x^2+(y-1)^2=1$ **12.** $x^2+y^2+z^2=1$

13. $\sqrt{x^2+y^2}=3z$ **14.** $z=x-2y+1$

15. $e^{-\frac{x^2+y^2}{2}}=z$ **16.** $(x-3)^2+y^2=z+9$

17–24 Describe the graph of the equation in words, and change the equation to Cartesian coordinates.

17. $r=1$ **18.** $\theta=\frac{\pi}{6}$

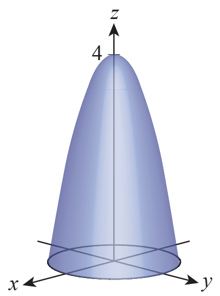
19. $2r^2=2-z^2$ **20.** $2r^2=2-z$

21. $r^2=z^2$ **22.** $r=4\cos\theta$

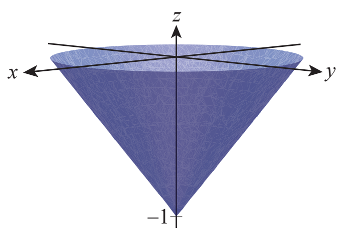
23. $z=\sqrt{1-r^2}$ **24.** $r=4\sec\theta$

25–36 Set up a triple integral in cylindrical coordinates for the volume of the solid S. Do not evaluate the integral.

25. S: The solid bounded by $z=4-(x^2+y^2)$ and the xy-plane

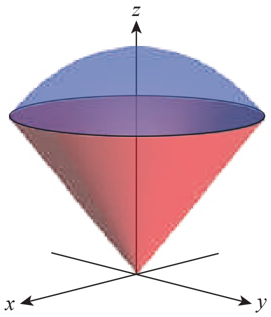


26. S: The solid bounded by $z=\sqrt{x^2+y^2}-1$ and the xy-plane



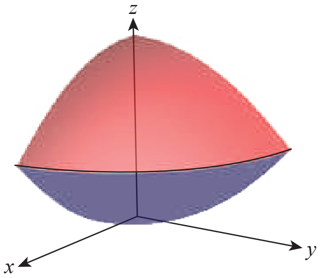
27. S: The solid bounded by $z = 4 - \sqrt{x^2 + y^2}$ and the xy-plane

28. S: The solid bounded above by $z = \sqrt{2 - x^2 - y^2}$ and below by $z = \sqrt{x^2 + y^2}$



29. S: The solid inside both $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 8$

30. S: The solid inside both $z = x^2 + y^2$ and $z = 18 - \left(x^2 + y^2\right)$ in the first quadrant



31. S: The solid bounded by $z^2 = 2 + x^2 + y^2$, $z = \sqrt{2}$, and $z = \sqrt{3}$

32.* S: The solid bounded by $z = x^2 + y^2$, $z = 1$, and $z = 4$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

33.* S: The solid bounded by $z = 2\sqrt{x^2 + y^2}$, $z = 1$, and $z = 5$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

34.* S: The solid bounded by $z^2 + 2 = x^2 + y^2$, $z = -3$, and $z = 3$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

35. S: The solid of Exercise 31 (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

36. S: The solid bounded by $z = e^{x^2 + y^2}$ and $z = e^{2 - x^2 - y^2}$

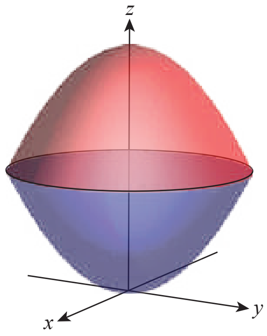
37. Show that if the solid of Example 2 has constant density, the third coordinate of its center of mass is $\bar{z} = \frac{2}{3}$.

38. Use cylindrical coordinates to verify the formula for the volume of a right circular cone of radius R and height h, $V = \frac{1}{3}\pi R^2 h$.

39–53 Use the cylindrical coordinate system to determine the mass of the solid S with the given density function.

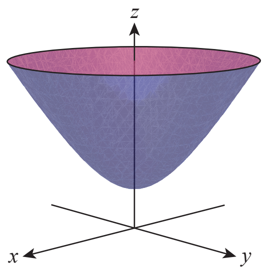
39. S is the solid bounded by $z = 1 - x^2 - y^2$ and the xy-plane, with density function $\rho(x, y, z) = z$.

40. S is the solid bounded by $z = x^2 + y^2$ and $z = 8 - \left(x^2 + y^2\right)$, with constant density ρ.



41. S is the solid upper hemisphere of radius 1 centered at the origin, with its density at the point (x, y, z) being proportional to its distance from the base. (See Exercise 43 of Section 14.4.)

42. S is the solid bounded by the upper sheet of the hyperboloid $z^2 - x^2 - y^2 = 1$ and the plane $z = \sqrt{5}$, with constant density ρ.



43. S is the solid bounded by $z = e^{\sqrt{x^2 + y^2}}$ and the plane $z = e$, with density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

44. S is the solid bounded by $x^2 + y^2 = 1$, the xy-plane, and $z = e$, with density function $\rho(x, y, z) = e^{-x^2 - y^2}$.

45. S is the solid outside $z = 1 - \sqrt{x^2 + y^2}$, bounded by the xy-plane, $z = 1$, and $x^2 + y^2 = 1$, with density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

46. S is the solid of Exercise 28, with constant density ρ.

47. S is the solid of Exercise 28, with density function $\rho(x, y, z) = z$.

48. S is the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density function $\rho(x, y, z) = e^{\sqrt{x^2 + y^2}}$.

49. S is the solid bounded by the cylinder $x^2+(y-1)^2=1$, the xy-plane, and the paraboloid $z=x^2+y^2$; its density at any point (x,y,z) is proportional to the square of the point's distance from the z-axis.

50. S is the solid bounded by the cylinder $(x-1)^2+z^2=1$, the xz-plane, and the paraboloid $y=x^2+z^2$, with density function $\rho(x,y,z)=x$. (**Hint:** Integrate with respect to y first; then define and use the polar coordinates $x=r\cos\theta$ and $z=r\sin\theta$.)

51. S is the solid bounded by $z=\sqrt{x^2+y^2}$ and $z=\left(x^2+y^2\right)^{3/2}$, with density function $\rho(x,y,z)=\sqrt{x^2+y^2}e^z$.

52. S is the solid bounded by $z=1/\sqrt{x^2+y^2}$, $2\left(x^2+y^2\right)=1$, $x^2+y^2=1$, and the xy-plane, with density function $\rho(x,y,z)=\sqrt{x^2+y^2}e^{\sqrt{x^2+y^2}}$.

53. S is a solid sphere of radius 2, with a cylindrical hole of radius 1 drilled into it along one of its diameters. Its density at any of its points is equal to the square of the distance from the origin.

54–64 Use the cylindrical coordinate system to find the center of mass of the solid S with the given density function.

54. S: The solid bounded by the paraboloids $z=x^2+y^2$ and $z=3-2x^2-2y^2$, with constant density

55. S: The solid inside $x^2+y^2=4$, outside $x^2+y^2=1$, and bounded by the paraboloid $z=16-x^2-y^2$ and the xy-plane, with constant density

56. S: The upper hemisphere of radius R centered at the origin, with constant density

57. S: The upper hemisphere of radius 1 centered at the origin, if its density is proportional to the square of the distance from the origin

58. S: The hemisphere of Exercise 56, if its density is proportional to the distance from the z-axis

59. S: The right circular cylinder of radius 1 and height 2, its base centered at the origin in the xy-plane, with density function $\rho(x,y,z)=e^{-z}$

60. S: The solid inside the paraboloid $z=6-x^2-y^2$ and outside the sphere $x^2+y^2+z^2=8$, with constant density

61. S: The right circular cone of radius R and height h with constant density ρ

62. S: The cone of Exercise 61 with the density of the cone being proportional to the distance from the cone's axis of symmetry

63. S: The cone of Exercise 61 with the density being proportional to the distance from the base

64.* S: The first octant of the unit sphere centered at the origin, its density at any point is proportional to the product of its distances from the coordinate planes (See Exercise 60 of Section 14.4.)

65.* Use cylindrical coordinates to give a second solution to Exercise 61 of Section 14.4. (See the hint given in Exercise 50.)

66–68 Use cylindrical coordinates to find the first and/or second moments of the solid, as indicated.

66. Find the second moment about the z-axis and the corresponding radius of gyration for the solid of Exercise 53. (Suppose it is centered at the origin, with the "hole" in vertical position.)

67. Find the moment of inertia about the z-axis and the corresponding radius of gyration for the cylinder $x^2+y^2=1$, bounded by $z=0$ and $z=1$, if it has constant density ρ.

68. Repeat Exercise 67 for the solid of Exercise 55.

69–76 Find the indicated quantities for the solid from an earlier exercise in this section.

69. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 39

70. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 40

71. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 41

72. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 43

73. The second moment I_z and radius of gyration r_z for the solid of Exercise 55

74.* The second moment I_z and radius of gyration r_z for the solid of Exercise 59

75. The second moment I_z and radius of gyration r_z for the solid of Exercise 42, with density being proportional to the distance from the xy-plane

76. The second moment I_z and radius of gyration r_z for the solid of Exercise 60

77–80 Find a set of spherical coordinates for the point given in Cartesian coordinates.

77. $\left(\sqrt{3},1,2\right)$

78. $(1,0,1)$

79. $\left(0,-1,-\sqrt{3}\right)$

80. $\left(-2,2,2\sqrt{2}\right)$

81–84 Find the Cartesian coordinates of the point given in spherical coordinates.

81. $\left(\sqrt{2},0,\frac{\pi}{4}\right)$

82. $\left(4,\frac{\pi}{4},\frac{\pi}{3}\right)$

83. $\left(3,-\frac{\pi}{2},\frac{\pi}{2}\right)$

84. $\left(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4}\right)$

85–92 Change the equation into spherical coordinates.

85. $x^2+y^2+z^2=4$

86. $z=0$

87. $x^2+y^2-z^2=0$

88. $x^2+y^2+(z-2)^2=4$

89. $y=x$

90. $z=1$

91. $x^2+y^2=4$

92. $x+y+z=1$

93–100 Describe the graph of the equation in words, and change the equation to Cartesian coordinates.

93. $\rho=1$

94. $\theta=\frac{\pi}{3}$

95. $\varphi=\frac{\pi}{4}$

96. $\varphi=\frac{\pi}{6}$

97. $\varphi=\frac{\pi}{2}$

98. $\rho=2\sec\varphi$

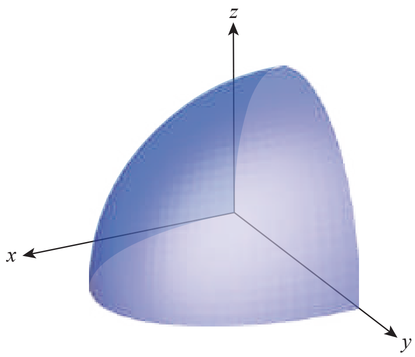
99. $\rho^2=9\csc^2\varphi$

100. $\tan^2\varphi=1$

101–105 Set up a triple integral in spherical coordinates for the volume of the solid *S*. Do not evaluate the integral.

101. *S*: The upper hemisphere of the sphere with radius 2, centered at the origin

102. *S*: The first-octant portion of the sphere $x^2+(y-1)^2+z^2=1$



103. *S*: The first-octant portion of the sphere $x^2+y^2+(z-3)^2=9$

104. The "ice-cream cone" of Exercise 45 of Section 14.3, bounded by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=8$

105. The first-octant portion of the "wedge" of the sphere $x^2+y^2+z^2=25$, bounded by the planes $y=\sqrt{3}x$ and $x=\sqrt{3}y$

106–131 Use spherical coordinates to solve the exercise.

106. Evaluate the integral you set up in Exercise 101.

107. Find the volume of the "wedge" of Exercise 105.

108. Derive the formula for the volume of a sphere of radius R, $V=\frac{4}{3}\pi R^3$.

109. Derive the formula for the volume of a right circular cone of radius r and height h, $V=\frac{1}{3}\pi r^2 h$. (**Hint:** Place an inverted cone in the coordinate system with its vertex at the origin and its axis coinciding with the positive z-axis.)

110. Find the mass of the half ball of Exercise 101 if its density is proportional to the distance from its center.

111. Find the volume of the "ice-cream cone" of Exercise 104. (Compare with Exercise 45 of Section 14.3).

112. Find the volume of the solid below the cone $z=\sqrt{x^2+y^2}$ and inside the unit hemisphere $z=\sqrt{1-x^2-y^2}$.

113. Repeat Exercise 112 for the solid below the cone $z=\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+(z-2)^2=4$, $z\geq 0$.

114. Use Exercise 112 to find the rectangular equation of the cone that divides the upper unit hemisphere into two parts of equal volume.

115. Determine the centroid of the half ball of Exercise 56.

116. Find the centroid of the solid of Exercise 104, assuming it has constant density.

117. Find **a.** the mass and **b.** the centroid of the "ice-cream cone" that is the solid common to the sphere $x^2 + y^2 + z^2 = 4z$ and the cone $x^2 + y^2 = z^2$, if it has constant density ρ.

118. Find the mass of the upper unit hemisphere centered at the origin if its density at the point (x, y, z) is proportional to the distance from its base.

119.* Generalize Exercises 111 and 116 (along with Example 6) by showing that, assuming constant density, the mass and centroid of the "ice-cream cone" bounded by the cone $\varphi = \alpha$ and the sphere of radius R are, respectively, $M = \frac{2}{3}\pi R^3(1 - \cos\alpha)\rho$ and $\left(0, 0, \frac{3}{8}R(1 + \cos\alpha)\right)$.

120. Find the moment of inertia about the z-axis of the solid of Exercise 104. Assume constant density.

121. Find the second moment I_z and radius of gyration r_z of a solid ball of radius R and constant density ρ, if it is centered at the origin.

122. Find I_z and r_z for the solid of Exercise 112.

123. Find the moment of inertia about one of the coordinate axes of the spherical shell bounded by the spheres $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 3$, if its density at any point is proportional to the distance from the origin. (**Hint:** Use the symmetry of the solid.)

124. Find a second solution (one that utilizes spherical coordinates) to Exercise 61.

125.* Find a second solution (one that utilizes spherical coordinates) to Exercise 62.

126.* Find a second solution (one that utilizes spherical coordinates) to Exercise 63.

127. **a.** Describe the solid of integration; then find **b.** its centroid, **c.** its moments of inertia about the coordinate axes, and **d.** its radii of gyration (assuming constant density).

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

128. **a.** Describe the solid of integration and **b.** find its volume by evaluating the integral.

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\sin\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

129.* **a.** Describe the solid of integration and **b.** find its volume by evaluating the integral.

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{1+\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

130. Evaluate $\iiint_S \sin\left(x^2 + y^2 + z^2\right)^{3/2} dV$, where S is the unit ball centered at the origin.

131. Evaluate $\iiint_S \frac{dV}{\sqrt{x^2 + y^2 + z^2}}$, where S is the first-octant portion of the solid inside the unit sphere and between the cones $\varphi = \pi/6$ and $\varphi = \pi/3$.

14.5 Technology Exercises

132. Set up an integral for the mass of the solid in Example 3 in Cartesian coordinates, integrating in the order of $dz\,dy\,dx$, and use a computer algebra system to evaluate it.

133. Repeat Exercise 132, but this time use spherical coordinates.

134. Find the mass and center of mass of the hemisphere of Exercise 56, if its density is proportional to the square of the distance from the origin.

135. Use spherical coordinates to find the center of mass of the first octant of the unit sphere centered at the origin, if its density at any point is proportional to the product of the point's distances from the coordinate planes.

136–143 Use a computer algebra system to evaluate the integral in the indicated exercise.

136. Exercise 53

137. Exercise 58

138. Exercise 72

139. Exercise 76

140. Exercise 124

141. Exercise 125

142. Exercise 126

143. Exercise 129

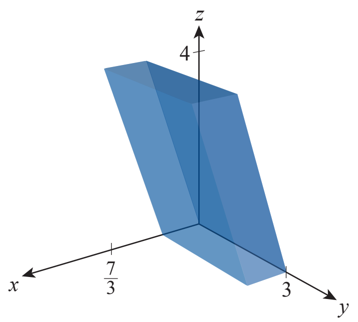


Figure 6

Solution

As in Example 2, the first step is to decide on a change of variables to try. On the basis of the integrand, and also guided by the form of two of the six planes, the assignment $u = 3x - z$ appears promising. This will allow two of the faces of the parallelepiped to be expressed as $u = 0$ and $u = 3$. In Exercise 41, you will show that if we let $v = z/2$ and $w = y/3$, then

$$\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| = 2$$

and

$$\iiint_R \left(\frac{3x - z}{2} + \frac{y}{3}\right) dV = \int_{w=0}^{w=1} \int_{v=0}^{v=2} \int_{u=0}^{u=3} \left(\frac{u}{2} + w\right)(2)\, du\, dv\, dw = 15.$$

The solid S corresponding to the limits, in xyz-space, appears in Figure 6.

14.6 Exercises

1–4 Evaluate the given determinant.

1. $\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ **2.** $\begin{vmatrix} 1 & 6 \\ -4 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 0 \\ 1 & -5 & 2 \end{vmatrix}$ **4.** $\begin{vmatrix} 2 & -2 & -1 \\ 3 & 2 & 3 \\ -4 & 1 & -1 \end{vmatrix}$

5–13 Find the Jacobian of the given transformation.

5. $x = 2u + v, \quad y = v - u$ **6.** $x = v, \quad y = 4u + \frac{v}{2}$

7. $x = 4uv, \quad y = u + 2v$ **8.** $x = u + 2v^2, \quad y = uv$

9. $x = u^2, \quad y = \frac{v}{u}$ **10.** $x = e^{-u}, \quad y = ve^{u}$

11. $x = uv, \quad y = (u + 1)v$

12. $x = e^{v} \cos u, \quad y = e^{v} \sin u$

13. $x = u \cos\varphi + v \sin\varphi, \quad y = u \sin\varphi - v \cos\varphi$

14. A transformation $T(u, v) = (au + cv, bu + dv)$ is called **linear**, where a, b, c, and d are constants. (See Exercises 5 and 6.) Find a general formula for the Jacobian of a linear transformation.

15–19 Consider the parallelogram P in the xy-plane bounded by the lines $y = x + 2$, $y = x - 4$, $y = 1 - 3x$, and $y = 5 - 3x$. We can identify a linear transformation $T(u, v)$ mapping a rectangle in the uv-plane onto P as follows. Rewrite the equations of the lines as $y - x = 2$, $y - x = -4$, $y + 3x = 1$, and $y + 3x = 5$, respectively. Then perform the change of variables $u = y - x$, $v = y + 3x$. Solve for x and y to obtain T (Exercise 15), and note that the preimage of P under T is the rectangle $-4 \le u \le 2$, $1 \le v \le 5$. In Exercises 16–19, you will be asked to follow these steps to identify the indicated linear transformation.

15. By solving the above system for x and y, find $T(u, v)$, as suggested by the above directions.

Find a linear transformation $T(u, v)$ that maps a rectangular region onto the given parallelogram P.

16. P is bounded by $2y = 1 - x$, $2y = 3 - x$, $y = 3x$, and $y = 3x + 4$.

17. P is bounded by $y = \frac{3}{2}x + 2$, $y = \frac{3}{2}x + 4$, $y = 1 - \frac{1}{4}x$, and $y = 4 - \frac{1}{4}x$.

18. P is bounded by $y = 1 - 2x$, $y = 5 - 2x$, $y = 3x - 2$, and $y = 3x + 3$.

19. P is bounded by $y = 2x$, $y = 2x + 4$, $y = -2x$, and $y = -2x - 2$.

20–25 Use a change of variables in order to integrate on an appropriate rectangle. (See Exercises 16–19.)

20. Find the area of the parallelogram in Exercise 16. (**Hint:** Start with $A = \iint_P dA$, and change variables, so you can integrate on the rectangle $[1,3]\times[0,4]$.)

21. Evaluate $\iint_P (2y + x)\,dA$ on P of Exercise 17.

22. Evaluate $\iint_P (x - 2y)^2\,dA$ on P of Exercise 18.

23. Evaluate $\iint_P \frac{y - 3x}{4y + 2x}\,dA$ on P of Exercise 16.

24. Evaluate $\iint_P \cos(x + y)\,dA$ on P of Exercise 18.

25. Evaluate $\iint_P \frac{e^{y-3x}}{x + 2y}\,dA$ on P of Exercise 16. (Note that a change of variables is necessary here, since you can't find an antiderivative in Cartesian coordinates, no matter what the order of integration is.)

26. Use a change of variables to evaluate $\iint_R \frac{x}{y + 2x}\,dA$, where R is the region bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$.

27. Let R be the region bounded by the coordinate axes and the line $x + 2y = 2$. Use the method of Example 2 to evaluate $\iint_R \frac{x + 2y}{x + y}\,dA$.

28. Use a change of variables to evaluate $\iint_R \frac{2x + y}{8x + 6y}\,dA$, where R is the region bounded by the coordinate axes and the line $y + 2x = 2$.

29. Consider the transformation $x = au$, $y = bv$ $(a, b > 0)$. Show that it maps the interior of the circle $u^2 + v^2 = 1$ onto that of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Determine the Jacobian and use a change of variables to prove the area formula for the ellipse, $A = \pi ab$.

30–33 Let T be the coordinate transformation $x = uv$, $y = v/u$, assuming $u, v > 0$.

30. Find the Jacobian for the above transformation T.

31. Show that the T-images of vertical lines $u = a$ are lines through the origin, while horizontal lines $v = b$ are mapped onto branches of hyperbolas. (Use the observations $xy = v^2$ and $x/y = u^2$.)

32. Find the T-image of the uv-rectangle $[1,\sqrt{2}]\times[1,\sqrt{2}]$. (See Exercise 31.)

33. If S denotes the T-image of the rectangle in Exercise 32, use a change of variables to evaluate $\iint_R 2x^2 y\,dy\,dx$.

34. Use a change of variables to evaluate $\iint_R xy^3\,dy\,dx$, where R is the region in the xy-plane bounded by the horizontal lines $y = 1$, $y = 3$, and the hyperbolas $y = 1/x$ and $y = 6/x$. (**Hint:** Consider the coordinate transformation $x = u/v$, $y = v$.)

35. Solve Exercise 34 without changing variables and compare your answers.

36. Generalize Exercise 14 to find a formula for the Jacobian J of a linear transformation in the three-variable case.

37–38 Find the Jacobian of the indicated coordinate transformation.

37. $x = u + 2v, \quad y = 2u + v - w, \quad z = 2v + w$

38. $x = u^2 v, \quad y = 2uvw, \quad z = u(1 + v)w$

39. Use a Jacobian to derive the formulation of triple integrals in cylindrical coordinates.

40.* Starting with the formulation of triple integrals in cylindrical coordinates and by finding a coordinate transformation T from spherical to cylindrical coordinates along with its Jacobian, provide another derivation of the formula for triple integrals in spherical coordinates. (**Hint:** See Example 3 for guidance.)

41. Verify that for the change of variables in Example 4, $\left|\partial(x,y,z)/\partial(u,v,w)\right| = 2$ and thus,

$$\iiint_R \left(\frac{3x - z}{2} + \frac{y}{3}\right) dV = \int_{w=0}^{w=1}\int_{v=0}^{v=2}\int_{u=0}^{u=3} \left(\frac{u}{2} + w\right)(2)\,du\,dv\,dw$$
$$= 15.$$

42–44 Use a change of variables to evaluate the given integral over the solid R.

42. Evaluate $\iiint_R \frac{4y + 2x - z}{4}\,dV$, where R is the solid bounded by the planes $x = 0$, $x = 2$, $z = 0$, $z = 3$, $4y = z$, and $4y = z + 6$.

43. Evaluate $\iiint_R \left(\frac{2x-y}{2} + \frac{3x+5z}{6} \right) dV$, where R is the solid bounded by the planes $z = 0$, $z = 2$, $y = 2x$, $y = 2x - 8$, and $z = 3x$, $z = 3x - 6$.

44. Evaluate $\iiint_R x(2z-y)e^{y/2}\, dV$, where R is the solid bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $y = 2z$, and $y = 2z - 1$.

45. Use a change of variables to provide a second solution to Exercise 47 of Section 14.2. (**Hint:** Using the notation of Exercise 29, notice that after transforming variables according to $x = u$, $y = 2v$, you will be able to integrate on a disk. Perform another change of variables to polar coordinates to finish the problem.)

46. Following the hint given in Exercise 45, solve Exercise 49 of Section 14.2 by changing variables twice.

47. Use the approach of Exercise 45 to derive the formula for the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

48.* After determining the second moment of the ellipsoid of Exercise 47 about the z-axis, show that its radius of gyration about the same is $r_z = \sqrt{\frac{1}{5}\left(a^2 + b^2\right)}$. Can you find formulas for r_x and r_y?

15.1 Exercises

1–6 Identify the real-life function as a vector field or a scalar field.

1. Fluid pressure in a swimming pool

2. Velocity of the wind inside a hurricane

3. Electromagnetic force around a coil

4. Air temperature near a working hair dryer

5. Velocity of water flowing through a pipe

6. CO concentration around a barbecue grill

7–10 Evaluate the vector field at the given point.

7. $\mathbf{F}(x,y)=\langle x+y,-2x\rangle$; $(1,-1)$

8. $\mathbf{F}(x,y)=\langle xe^{x},xy^{2}+1\rangle$; $(2,1)$

9. $\mathbf{F}(x,y,z)=\left\langle z,1,\sqrt{x^{2}+y^{2}+z^{2}}\right\rangle$; $(3,4,0)$

10. $\mathbf{F}(x,y,z)=\left\langle x-z,\dfrac{xy}{z^{2}},yz\right\rangle$; $(1,1,2)$

11–14 Match the given two-dimensional (planar) vector field with its graph (labeled A–D).

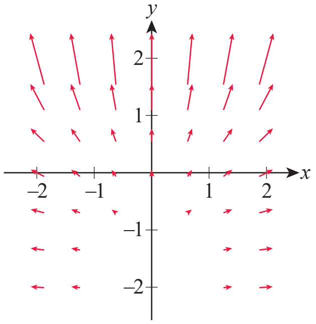
11. $\mathbf{F}(x,y)=\langle y,-x\rangle$

12. $\mathbf{F}(x,y)=\langle x,-y\rangle$

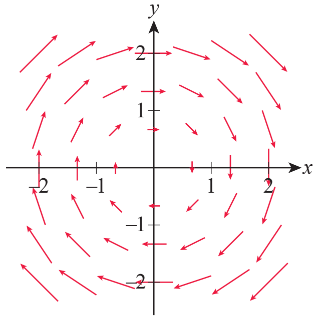
13. $\mathbf{F}(x,y)=\langle x,e^{y}\rangle$

14. $\mathbf{F}(x,y)=\dfrac{\langle x-1,y\rangle}{\left|\langle x-1,y\rangle\right|}$

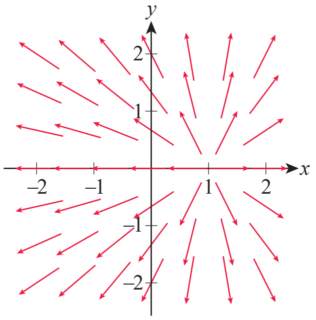
A.



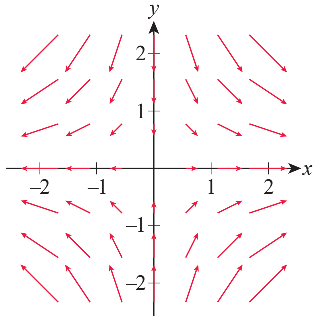
B.



C.



D.



15–23 Sketch the vector field **F** by hand, using a sufficient number of representative vectors.

15. $\mathbf{F}(x,y)=\langle 1,0\rangle$

16. $\mathbf{F}(x,y)=\langle 0,y\rangle$

17. $\mathbf{F}(x,y)=\langle -4x,0\rangle$

18. $\mathbf{F}(x,y)=\dfrac{\langle -y,x\rangle}{\left|\langle -y,x\rangle\right|}$

19. $\mathbf{F}(x,y)=\dfrac{\langle x,-y\rangle}{\left|\langle x,-y\rangle\right|}$

20. $\mathbf{F}(x,y)=\langle 1,y-1\rangle$

21. $\mathbf{F}(x,y)=\langle y+2,1-x\rangle$

22. $\mathbf{F}(x,y)=\langle x,y^{2}\rangle$

23. $\mathbf{F}(x,y)=\dfrac{\langle x,y\rangle}{\left|\langle x,y\rangle\right|}$

24–27 Match the given three-dimensional vector field with its graph (labeled A–D).

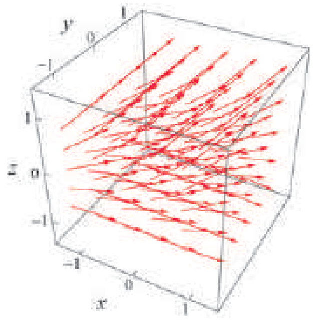
24. $\mathbf{F}(x,y,z)=\langle 1,0,1\rangle$

25. $\mathbf{F}(x,y,z)=\langle 1,1,z\rangle$

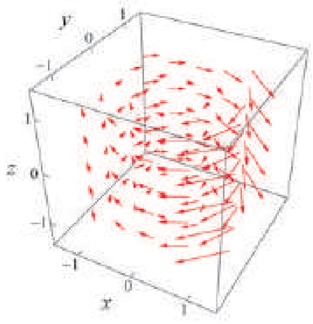
26. $\mathbf{F}(x,y,z)=\langle x,y,1\rangle$

27. $\mathbf{F}(x,y,z)=\dfrac{\langle y,-x,0\rangle}{2-x}$

A.



B.



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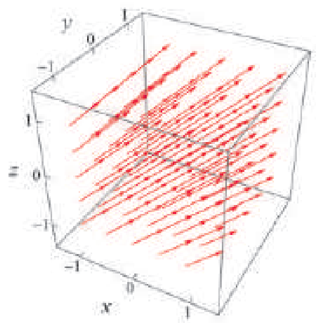
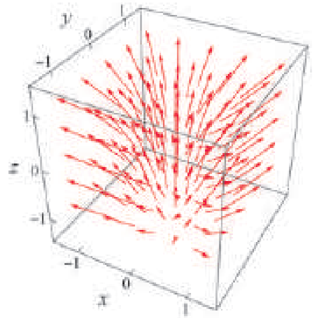


D.



28–36 Determine ∇f for the given scalar field f.

28. $f(x,y)=\dfrac{x^2y}{2}+xy^3$

29. $f(x,y)=\ln\sqrt{x^2+y^2}$

30. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

31. $f(x,y)=y^2\tan^{-1}x$

32. $f(x,y)=ye^{x+y}$

33. $f(x,y,z)=ze^{xyz}$

34. $f(x,y,z)=y^2z-2x^2z^2$

35. $f(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

36. $f(x,y,z)=\dfrac{xy}{y-z}$

37. By determining a potential function f, show that the electric field $\mathbf{F}(x,y,z)=\dfrac{\varepsilon Qq}{r^3}\mathbf{r}$ is conservative (i.e., $\mathbf{F}=\nabla f$ for some potential function f).

38. Prove that if the vector field $\mathbf{F}=\langle P,Q,R\rangle$ is conservative (i.e., $\mathbf{F}=\nabla f$ for some potential function f), then $\mathbf{F}$ satisfies what is sometimes called the *Component Test*, that is,

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}.$$

(As we will see in Section 15.7, if the domain of $\mathbf{F}$ satisfies certain regularity conditions, then the converse of the above statement also holds. **Hint:** Start by noting that $P=\partial f/\partial x$ and $Q=\partial f/\partial y$. Use this to write both $\partial P/\partial y$ and $\partial Q/\partial x$ as mixed partials of f, and use Clairaut's Theorem (see Section 13.3) to obtain the first statement. Use a similar approach to prove the last two equalities.)

39. Formulate and prove a result analogous to the one in Exercise 38 for the vector field $\mathbf{F}=\langle P,Q\rangle$.

40–47 Use Exercises 38 and 39 to decide whether the given vector field is conservative.

40. $\mathbf{F}(x,y)=\langle y^2,2xy\rangle$ **41.** $\mathbf{F}(x,y)=\langle y,xy\rangle$

42. $\mathbf{F}(x,y)=\langle xe^y,ye^x\rangle$

43. $\mathbf{F}(x,y)=\langle y^2\cos(xy^2),2xy\cos(xy^2)\rangle$

44. $\mathbf{F}(x,y,z)=\langle 1,xy,0\rangle$

45. $\mathbf{F}(x,y,z)=\langle y^2z,2xyz,xy^2\rangle$

46. $\mathbf{F}(x,y,z)=\dfrac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$

47. $\mathbf{F}(x,y,z)=\left\langle \dfrac{z}{\sqrt{xy}},-\dfrac{x}{\sqrt{xy}},\sqrt{xyz}\right\rangle$

48–53 Consider $\mathbf{F}(x,y)=\langle 12xy,6x^2+2\rangle$, a conservative vector field. We can find a potential function f by integrating, as follows. First, note that we must have $\partial f/\partial x=12xy$, so we integrate with respect to x to obtain $f(x,y)=6x^2y+g(y)$, where g is a function of y (or a constant). On the other hand, we know $\partial f/\partial y=6x^2+2$, so differentiating with respect to y the function f we obtained above, we see that $6x^2+2=\partial f/\partial y=6x^2+g'(y)$, that is, it must be the case that $g'(y)=2$. This implies $g(y)=2y+C$, hence $f(x,y)=6x^2y+2y+C$ is a potential function for $\mathbf{F}$ under any choice of the constant C.

Use the above technique to find a potential function for the given conservative vector field. (In Exercises 52–53, generalize to three variables.)

48. $\mathbf{F}(x,y)=\langle 4y^3,12xy^2\rangle$

49. $\mathbf{F}(x,y)=\langle y^2e^{xy^2},2xye^{xy^2}\rangle$

50. $\mathbf{F}(x,y)=\langle 3x^2-2y^2,2y-4xy\rangle$

51. $\mathbf{F}(x,y)=\left\langle \dfrac{x}{\sqrt{x^2+y^2}},\dfrac{y}{\sqrt{x^2+y^2}}\right\rangle$

52. $\mathbf{F}(x,y,z)=\langle y^2,2xy,2z\rangle$

53. $\mathbf{F}(x,y,z)=\langle 2x+z,2z-1,x+2y\rangle$

54. Using the notation in Example 3, let $\mathbf{r}=\langle x,y\rangle$ and $r=|\langle x,y\rangle|$, and describe the vector field

$$\mathbf{e}_r=\frac{\langle x,y\rangle}{|\langle x,y\rangle|}=\frac{\mathbf{r}}{r}$$

as well as its three-dimensional analogue (also denoted $\mathbf{e}_r$). See Example 1 for guidance.

55. Show that $\nabla r=\mathbf{e}_r$ holds both in two and three dimensions. (Note that you have already provided a constructive existence proof for the potential function of $\mathbf{e}_r$ in the two-dimensional case in Exercise 54.)

56. Find formulas for ∇r^2 and ∇r^3.

57. Find and prove a formula for ∇r^n $(n\in\mathbb{N})$.

58. Show that the vector field $\mathbf{r}/r^2$ is conservative by determining a potential function. (Handle both the planar and three-dimensional cases.)

59. Repeat Exercise 58 to find a potential function for the vector field $\mathbf{r}/r^3$. (Compare your finding with the results in Example 6 and Exercise 37.)

60.* Generalize Exercise 59 by determining a formula for the potential function of the vector field $\mathbf{r}/r^n$, $n\in\mathbb{N}$. Prove your assertion.

61–62 The *flow lines* of a vector field $\mathbf{F}$ are "paths aligned with $\mathbf{F}$," more precisely, paths whose *velocity field* is $\mathbf{F}$ when followed by a particle. Even more precisely, a flow line is a path $\mathbf{r}(t)$ such that $\mathbf{r}'(t)=\mathbf{F}(\mathbf{r}(t))$. In these exercises, you will determine the flow lines of vector fields.

61. By visualizing its graph, try to predict what the flow lines of the vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$ might look like (see Exercise 11). Then use a differential equation to determine the equations of these flow lines. (**Hint:** Note that if $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ is a flow line of $\mathbf{F}$, then $dx/dt=y$ and $dy/dt=-x$. Conclude that $dy/dx=-x/y$ and solve the differential equation. For a refresher on separable differential equations, see Section 8.1.)

62. Repeat Exercise 61 for the vector field of Exercise 12.

15.2 Exercises

1. Determine $\int_C xy^2\,ds$, where C is the first-quadrant portion of the circle $x^2+y^2=4$ traversed counterclockwise. (See Example 1 for a hint on parametrization.)

2. Show that you obtain the same answer as that in Exercise 1 if you parametrize the quarter circle as $x=t,\ y=\sqrt{4-t^2},\ 0\le t\le 2$.

3–7 Evaluate the indicated line integral.

3. $\int_C (xy+1)\,ds$, where C is the lower semicircle $y=-\sqrt{9-x^2}$, traversed counterclockwise

4. $\int_C y\,ds$, where C is the graph of $y=\sqrt{x}$, traversed from $(0,0)$ to $(1,1)$

5. $\int_C (2x+y)\,ds$, where C is the line segment from the origin to the point $(1,\sqrt{3})$, followed by the arc of the circle $x^2+y^2=4$ traversed counterclockwise from $(1,\sqrt{3})$ to $(-2,0)$

6. $\int_C (x+3y)\,ds$, where C is the line segment from $(0,1)$ to $(-2,2)$, followed by the arc of the circle $x^2+y^2=8$ traversed clockwise from $(-2,2)$ to $(2\sqrt{2},0)$

7. $\int_C \sqrt{1+18xy}\,ds$, where C is the graph of $y=2x^3$, $0\le x\le 1$

8. If a piece of wire is bent into the semicircle $y=\sqrt{16-x^2}$ and its density function is $\rho(x,y)=2x^2+y^2$, find the mass and center of mass of the wire. (See Example 1 for a hint on parametrization.)

9. Repeat Exercise 8 for a wire that is bent into the upper semicircle of radius R, $y=\sqrt{R^2-x^2}$, so that the density at any point is proportional to the distance from the line $y=R$.

10. Find the mass of the wire in Exercise 8 by using the parametrization $x=4\cos(t^2),\ y=4\sin(t^2)$, $0\le t\le\sqrt{\pi}$. Verify that you obtain the same answer as in Exercise 8.

11. Find the mass of the wire bent into a parabolic arc $y=\sqrt{x}, 1\le x\le 3$, if its density is $\rho(x,y)=2x/y$.

12–17 Evaluate the indicated line integral.

12. $\int_C (x+2y+z^2)\,ds$, where C is the line segment joining the origin and the point $(1,2,3)$

13. Integrate the function given in Exercise 12 along the path that is the line segment joining the origin with the point $(1,2,0)$, followed by the segment from $(1,2,0)$ to $(1,2,3)$. Is your answer equal to that given for Exercise 12?

14. $\int_C (2x+yz)\,ds$, where C is the line segment joining $(1,2,0)$ and $(3,4,1)$

15. $\int_C (y+16)\,ds$, where C can be parametrized as $\mathbf{r}(t)=\left\langle \frac{1}{2}t^2, 2t, \frac{8}{3}t^{3/2}\right\rangle,\ 0\le t\le 2$

16. $\int_C (3x^2y+z)\,ds$, where C is the helix $\mathbf{r}(t)=\langle \cos t, \sin t, t/2\rangle$, for $0\le t\le 8\pi$

17. $\int_C ye^z\,ds$, where C is the helix given in Exercise 16

18. Use a line integral to find the length of the helix given in Exercise 16.

19. Find the mass of the helix given in Exercise 16 if its density is proportional to the distance from the xy-plane.

20. Find the center of mass of the helix given in Exercise 19.

21. Evaluate $\int_C x\,ds$ on the curve C parametrized by $\mathbf{r}(t)=\langle \cos t, \sin t, \cos t\rangle,\ \ 0\le t\le \pi/2$.

22. Determine the moments of inertia and radii of gyration for the object in Example 3.

23. Determine the mass and the center of mass of the V-shaped object in Example 3 if its density is proportional to the distance from the xy-plane.

24. Determine the moments of inertia and radii of gyration for the object in Exercise 23.

25. Find the mass of the spring that is defined by $\mathbf{r}(t)=\langle t, 3\cos t, 3\sin t\rangle,\ \ 0\le t\le 4\pi$, if its density function is $\rho(x,y,z)=x/2$.

26–30 Evaluate the line integral of the vector field along the given curve.

26. $\int_C x^2 y\,dx + \left(y^2 - x^2\right)dy$, where C can be parametrized as $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$

27. $\int_C \left(x^2 y + xy^2\right)dx + x^3\,dy$, where C is the unit circle centered at the origin

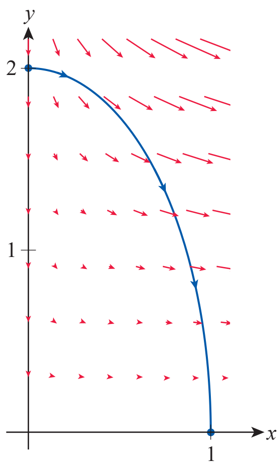
28. $\int_C y\,dx + xy\,dy$, where **a.** $C = C_1$: the line segment joining the origin with $(1,1)$, **b.** $C = C_2$: the parabola $y = x^2$ joining the origin with $(1,1)$, and **c.** $C = C_3$: the parabola $x = y^2$ joining the origin with $(1,1)$. Do your answers differ?

29. Repeat Exercise 28 for the line integral $\int_C 2xy\,dx + x^2\,dy$. Compare your answers.

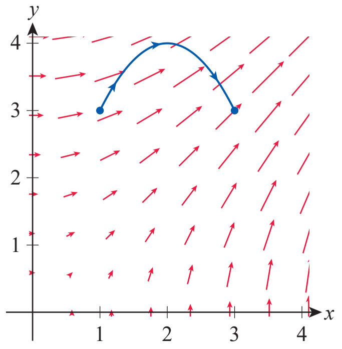
30. $\int_C \left(x^2 + y\right)dx + xy\,dy$ along the closed path $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from the origin to $(1,0)$, C_2 is the line segment from $(1,0)$ to $(2,1)$, and C_3 is the straight path from $(2,1)$ back to the origin

31. Find $\int_C 2xy\,dx + x^2\,dy$ along the path given in Exercise 30.

32. Determine the work done by the force field in Example 4 if it moves a particle along the elliptical path $y = \sqrt{4 - 4x^2}$ from $(0,2)$ to $(1,0)$.

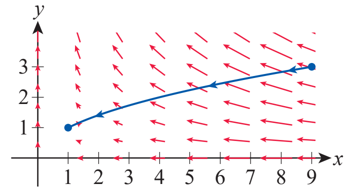


33. Determine the work done by the force field $\mathbf{F}(x,y) = \langle y, x \rangle$ if it moves a particle along the parabolic arc $y = 4x - x^2$ from the point $(1,3)$ to $(3,3)$.



34. By parametrizing the path given in Exercise 33 oriented from $(3,3)$ toward $(1,3)$, find the work done on the particle by the force field when it travels along this "reverse path." Explain your findings.

35. Determine the work done by the force field $\mathbf{F}(x,y) = \langle -x, y \rangle$ if it moves a particle along the graph of $y = \sqrt{x}$ from the point $(9,3)$ to $(1,1)$.



36. Determine the work done by the force field $\mathbf{F}(x,y) = \langle 2, 3x \rangle$ if it moves a particle counterclockwise around the ellipse parametrized as $\mathbf{r}(t) = \langle 3 + 3\cos t, 2 + 2\sin t \rangle$, $0 \le t \le 2\pi$.

37–40 Evaluate the indicated line integral.

37. $\int_C xy\,dx + z\,dy + (x + z)\,dz$, where C can be parametrized as $\mathbf{r}(t) = \langle t^2, t, t^3 \rangle$, $0 \le t \le 1$

38. The integral of Exercise 37 along the line segment from the origin to $(2,2,0)$, followed by the segment from $(2,2,0)$ to $(4,6,2)$

39. $\int_C \left(x - y^2\right)dx + \left(y - z^2\right)dy + \left(z - x^2\right)dz$, where C is the "twisted cube" $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

40. $\int_C y\,dx + z\,dy + x^2\,dz$, where C is parametrized by $\mathbf{r}(t) = \langle t^2, t, e^t \rangle$, $0 \le t \le 1$

41. Determine the work done by the force field $\mathbf{F}(x,y,z) = \langle x, y, 1 \rangle$ on a particle that moves up the helix given in Exercise 16.

42. Determine the work done by the force field $\mathbf{F}(x,y,z)=\langle x,z,y\rangle$ on a particle that moves along the curve $\mathbf{r}(t)=\langle t^2,\cos t,\sin t\rangle,\ 0\le t\le\pi.$

43. Find the work done by the force field $\mathbf{F}=-\dfrac{k\mathbf{r}}{r^3}$ as it moves a particle from the point $(0,0,4)$ to $(0,3,4)$ along a straight-line curve. (For well-known force fields of this type, see Examples 3 and 4 of Section 15.1.)

44. Determine the work done by the force field $\mathbf{F}(x,y,z)=\langle y,x+z,y\rangle$ on a particle that moves around the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in a counterclockwise direction (when viewed from the point $(1,1,1)$).

45–48 A fluid's velocity field is given by $\mathbf{F}(x,y,z)$. Determine the fluid's flow along the indicated curve $\mathbf{r}(t)$.

45. $\mathbf{F}(x,y,z)=\langle 1,2xz,4y\rangle;\quad \mathbf{r}(t)=\langle 1,t,t^2\rangle;\quad 0\le t\le 3$

46. $\mathbf{F}(x,y,z)=\langle xy,yz,zx\rangle;\quad \mathbf{r}(t)=\langle 1,2t,3t\rangle;\quad 0\le t\le 2$

47. $\mathbf{F}(x,y,z)=\langle -2y,y-x,3\rangle;$
$\mathbf{r}(t)=\langle 4\cos t,2\sin t,t\rangle;\quad 0\le t\le\pi$

48. $\mathbf{F}$ is the gradient of the potential function $f(x,y,z)=x^2+y^2+z^2$, and $\mathbf{r}$ is the helix given in Exercise 16.

49. **a.** Verify that $f(x,y,z)=\frac{1}{2}x^2y^2z$ is a potential function for the following vector field.
$\mathbf{F}(x,y,z)=\left\langle xy^2z,x^2yz,\frac{1}{2}x^2y^2\right\rangle$

b. Assuming that $\mathbf{F}$ is the velocity field of a fluid, find the circulation of the fluid around the unit circle $\mathbf{r}(t)=\langle\cos t,0,\sin t\rangle,\ 0\le t\le 2\pi.$

50–53 In physics, the electrostatic potential at a point P resulting from a single point charge q is calculated using the formula

$$V(P)=\frac{\varepsilon q}{r_P},$$

where r_P is the distance between P and the point charge q, and ε is Coulomb's constant (see Example 4 of Section 15.1). The value of ε is approximately $8.988\times10^9\ \mathrm{Nm^2/C^2}$. We note that Coulomb's constant is often used in the form $\varepsilon=1/(4\pi\varepsilon_0)$, where $\varepsilon_0\approx 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$. The constant ε_0 is called the *permittivity constant*. (We assume here that the potential at an "infinitely distant" point is zero.)

Among their many uses, line integrals enable us to calculate the electrostatic potential around a continuously charged curve, according to the following formula.

Suppose a curve C has continuous charge distribution given by its charge density function $q(x,y,z)$. The electrostatic potential at a point P is then obtained from

$$V(P)=\varepsilon\int_C\frac{q(x,y,z)}{r_P(x,y,z)}\,ds,$$

where $r_P(x,y,z)$ is the distance between (x,y,z) and P. We will use the above formula in Exercises 50–53.

50. Suppose the quarter circle in Example 4 has charge density $q(x,y,z)=\dfrac{1-y}{10^7}$ coulomb per meter (C/m). Find the electrostatic potential at the point $(0,0,1)$.

51. Repeat Exercise 50 if the charge density is $q(x,y,z)=\dfrac{xy}{10^5}\ \mathrm{C/m}.$

52. Suppose the segment of the x-axis between the origin and the point $(2,0,0)$ has charge density $q(x,y,z)=\dfrac{x}{10^4}\ \mathrm{C/m}.$ Find the electrostatic potential at the point $(0,0,1)$.

53.* Find the potential at the origin of the electric field created by the helix $\mathbf{r}(t)=\langle\cos t,\sin t,t\rangle,\ 0\le t\le 2\pi,$ with charge density $q(x,y,z)=\dfrac{1-z}{10^6}\ \mathrm{C/m}.$

15.2 Technology Exercises

54–59 Use technology to help with your calculations. Express your answers as decimal approximations.

54. Find the mass and the centroid of the thin wire given by $y=9-x^2,\ -3\le x\le 3,$ if it has constant density ρ.

55. Determine the three moments of inertia and radii of gyration for the wire in Exercise 54.

56. Repeat Exercise 54 if the wire has density $\rho(x,y)=x^2y.$

57. Repeat Exercise 55 if the wire has density $\rho(x,y)=x^2y.$

58. Determine the three moments of inertia and radii of gyration for the spring given in Exercise 25.

59. Find the electrostatic potential at $(1,0,0)$ if the wire in Exercise 54 has charge density $q(x,y,z)=\dfrac{y}{10^6}\ \mathrm{C/m}.$

15.3 Exercises

1. Use the Component Test to verify that the vector field **F** in Example 5 is conservative.

2–5 Verify that V is a potential function for **F** and determine the line integral along the indicated curve.

2. $\mathbf{F}(x,y)=\langle y\sin y, xy\cos y+x\sin y\rangle$; $V(x,y)=xy\sin y$; $\mathbf{r}(t)=\left\langle t,\frac{\pi}{2}t\right\rangle$, $0\le t\le 1$

3. $\mathbf{F}(x,y)=\left\langle e^y\cos x+\ln y, \frac{x}{y}+e^y\sin x\right\rangle$; $V(x,y)=e^y\sin x+x\ln y$; $\mathbf{r}(t)=\left\langle \frac{\pi}{2}+\frac{\pi}{2}t, 1+(e-1)t\right\rangle$, $0\le t\le 1$

4. $\mathbf{F}(x,y,z)=\langle -yz\sin x, z\cos x, y\cos x\rangle$; $V(x,y,z)=yz\cos x$; $\mathbf{r}(t)=\left\langle \frac{\pi}{2}(t-1), \frac{\pi}{2}t, t\right\rangle$, $0\le t\le 1$

5. $\mathbf{F}(x,y,z)=\left\langle 2xy+e^{z^2}, x^2, 2xze^{z^2}\right\rangle$; $V(x,y,z)=x^2y+xe^{z^2}$; $\mathbf{r}(t)=\langle t,t^2,t^3\rangle$, $0\le t\le 1$

6–21 Either find a potential function for **F** or state that a potential function does not exist. The latter implies that **F** is not conservative.

6. $\mathbf{F}(x,y)=\langle 2x, 2+6y\rangle$

7. $\mathbf{F}(x,y)=\langle 3x^2+2y^2, 4xy+3y^2\rangle$

8. $\mathbf{F}(x,y)=\langle x^2y, 3x\rangle$

9. $\mathbf{F}(x,y)=\langle x^2+y, 2xy+3y^2\rangle$

10. $\mathbf{F}(x,y)=\left\langle x^2+\ln y, \frac{x}{y}+2y^3\right\rangle$

11. $\mathbf{F}(x,y)=\left\langle e^x+y\cos x, \frac{1}{y}+\sin x\right\rangle$

12. $\mathbf{F}(x,y)=\langle e^x+\sin y, e^y+\cos x\rangle$

13. $\mathbf{F}(x,y)=\langle 4x^3y-y^5, x^4-5xy^4\rangle$

14. $\mathbf{F}(x,y)=\left\langle -\frac{2y}{x^3}, \frac{1}{x^2}+\frac{1}{\sqrt{y}}\right\rangle$

15. $\mathbf{F}(x,y,z)=\langle 2xyz, x^2z, x^2y\rangle$

16. $\mathbf{F}(x,y,z)=\langle yz, xz-z\sin y, xy+\cos y\rangle$

17. $\mathbf{F}(x,y,z)=\langle x^2y, 2xz, z^3\rangle$

18. $\mathbf{F}(x,y,z)=\left\langle \frac{zy^2}{x}, 2zy\ln x, -\frac{y}{x^2}\right\rangle$

19. $\mathbf{F}(x,y,z)=\langle \tan z, 2yz, y^2+x\sec^2 z\rangle$

20. $\mathbf{F}(x,y,z)=\langle y^2+ze^{xz}, 2xy-2z, xe^{xz}-2y\rangle$

21. $\mathbf{F}(x,y,z)=\left\langle x\cos z+z, z+\sin y, -\frac{x^2}{2}\sin z+x\right\rangle$

22. For the vector field **F** in Example 3, show that $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=2$ and $\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\pi-\frac{2}{3}$, meaning that **F** is not conservative.

23–26 Show that the line integral is not path independent by finding two different values for the integral along two different paths connecting A and B. (Answers may vary.)

23. $\int_C y\,dx+4\,dy$; $A(1,0)$, $B(0,1)$

24. $\int_C xy\,dx+2y\,dy$; $A(0,0)$, $B(2,4)$

25. $\int_C y\,dx-2z\,dy+2\,dz$; $A(-5,0,0)$, $B(3,4,0)$

26. $\int_C xy\,dx+z\,dy+(x+z)\,dz$; $A(0,0,0)$, $B(1,1,1)$

27–32 Show that the force field **F** is conservative, and use this fact to determine the work done by **F** in moving an object from A to B.

27. $\mathbf{F}(x,y)=\langle 3y, 3x-2y\rangle$; $A(-1,0)$, $B(5,3)$

28. $\mathbf{F}(x,y)=\langle x-y^2, -2xy\rangle$; $A(1,4)$, $B(3,-2)$

29. $\mathbf{F}(x,y)=\langle e^y-2xy, x(e^y-x)\rangle$; $A(-4,1)$, $B(0,0)$

30. $\mathbf{F}(\mathbf{r})=\frac{k\mathbf{r}}{r^3}$; $A(-2,1,-2)$, $B(6,0,-8)$

31. $\mathbf{F}(x,y,z)=\langle e^y\cos x-yz, e^y\sin x-xz, -xy\rangle$; $A(\pi,1,2/\pi)$, $B(\pi/2,0,0)$

32. $\mathbf{F}(x,y,z)=\langle \tan y, x\sec^2 y-z, -y\rangle$; $A(0,0,4/\pi)$, $(2,\pi/4,0)$

33. Find an "easier" solution (one that uses the Fundamental Theorem for Line Integrals) for Exercise 29 of Section 15.2.

34. Repeat Exercise 33, this time for Exercise 44 of Section 15.2.

35. a. Show that the vector field

$$\mathbf{F}(x,y)=\left\langle -\frac{y}{|\langle x,y\rangle|^2},\frac{x}{|\langle x,y\rangle|^2}\right\rangle$$

satisfies the Component Test, but is not conservative. Does this contradict the Component Test? (Such a vector field is sometimes called a *rotation field*. **Hint:** To show that **F** is not conservative, calculate a line integral along a circle centered at the origin.)

b. Verify that $\mathbf{F}(x,y)=\nabla\left(-\arctan\frac{x}{y}\right)$. Reconcile this observation with part a.

36. In this exercise, we will consider the vector field

$$\mathbf{F}(x,y)=\left\langle \frac{x}{|\langle x,y\rangle|^2},\frac{y}{|\langle x,y\rangle|^2}\right\rangle,$$

with the usual notation $P(x,y)=x/|\langle x,y\rangle|^2$ and $Q(x,y)=y/|\langle x,y\rangle|^2$.

a. Show that $\partial P/\partial y=\partial Q/\partial x$ throughout the domain of **F**.

b. Explain why **F** does not satisfy the conditions of the Component Test.

c. Show that **F** is conservative. (See Exercise 58 of Section 15.1.) Is this a contradiction with part b.?

37–42 Decide whether the specified region D is simply connected.

37. $D=\{(x,y)\mid x^2+y^2<1\}$

38. $D=\{(x,y)\mid 0<x^2+y^2<1\}$

39. $D=\{(x,y)\mid |x|+|y|<1\}$

40. $D=\{(x,y)\mid x^2\le y\le 2x^2\}$

41. $D=\{(x,y,z)\mid 1<x^2+y^2+z^2<3\}$

42. $D=\{(x,y,z)\mid 1<z^2<3\}$

43–46 In physics, the Law of Conservation of Energy states that if an object moves under the influence of a conservative force field, then the sum of its potential and kinetic energies (the energies resulting from the object's position and motion, respectively) remains constant. For example, in case of an object falling under the influence of gravity only, the kinetic energy it gains as a result of increasing speed equals the loss in potential energy stemming from loss of altitude.

In Exercises 43–46, you will use the Fundamental Theorem for Line Integrals to derive this law.

43. Suppose an object of mass m is under the influence of a conservative force field **F**. If $\mathbf{F}=\nabla f$, the potential energy of the object is defined as

$$E_p(x,y,z)=-f(x,y,z).$$

Show that the work W done by **F** in moving the object from point A to point B along a smooth curve is

$$W=E_p(A)-E_p(B).$$

44. Referring to Exercise 43, suppose the path of the object is parametrized by $\mathbf{r}(t)$ so that $\mathbf{r}(a)=A$ and $\mathbf{r}(b)=B$. Show that W can be written as

$$W=\int_a^b \mathbf{F}\cdot\mathbf{v}(t)\,dt,$$

where $\mathbf{v}(t)$ is the velocity of the object.

45. Use Newton's Second Law $(\mathbf{F}=m\mathbf{v}'(t))$ along with Exercise 44 to show that W can be expressed as

$$W=\frac{m}{2}\int_a^b \frac{d}{dt}\left(|\mathbf{v}(t)|^2\right)dt.$$

46. Use Exercise 45 and the fact that the kinetic energy of an object of mass m and speed $v=|\mathbf{v}|$ is $E_k=\frac{1}{2}mv^2$ to conclude that

$$W=E_k(B)-E_k(A).$$

Consequently, using Exercise 43,

$$E_p(A)+E_k(A)=E_p(B)+E_k(B).$$

47. Use the Law of Conservation of Energy to derive the formula for the velocity of impact of an object falling from height h under the influence of gravity only, $v_{imp}=\sqrt{2hg}$. (**Hint:** Since we are ignoring all other forces, the initial potential energy $E_p=mgh$ turns entirely into kinetic energy.)

48.* A proton is moving along the z-axis in the positive direction at a speed of $2\cdot 10^6$ m/s (assume units are meters in our coordinate system). At the origin, it encounters an electric force field $\mathbf{E}(x,y,z)=\langle 0,0,1600z\rangle$ N/C (newtons per coulomb). Use the Law of Conservation of Energy to find the proton's speed at the point $(0,0,6)$. (**Hint:** For the mass and charge of a proton, use the approximate data $m=1.6726\times 10^{-27}$ kg and $q=1.6\times 10^{-19}$ C, respectively. The potential energy of the proton at (x,y,z) is $E_p(x,y,z)=qV(x,y,z)$, where V is the electric potential.)

49. Suppose $f(x,y)$ is a harmonic function. Prove

$$\int_C \left(\frac{\partial f}{\partial y}dx-\frac{\partial f}{\partial x}dy\right)=0$$

along any smooth closed curve C in $\mathbb{R}^2$. (For a refresher on the definition of harmonic functions, see Exercise 98 of Section 13.3.)

Concept Check

50–54 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

50. If $\mathbf{F}$ is path independent on an open connected region D, then $\int_C \mathbf{F}\cdot d\mathbf{r}=0$ for every path C in D.

51. If the components of $\mathbf{F}=\langle P,Q\rangle$ have continuous first partials and $\partial P/\partial y=\partial Q/\partial x$ throughout an open connected region D, then $\mathbf{F}=\langle P,Q\rangle$ is conservative on D.

52. If $\mathbf{F}$ is continuous on an open connected region D and every line integral $\int \mathbf{F}\cdot d\mathbf{r}$ is path independent, then $\mathbf{F}$ is conservative.

53. If $\mathbf{F}$ is conservative on an open connected region D, then every line integral $\int \mathbf{F}\cdot d\mathbf{r}$ is path independent in D.

54. The domain of the vector field $\mathbf{F}(\mathbf{r})=\dfrac{k\mathbf{r}}{r^3}$ is not simply connected; therefore, it cannot be conservative.

15.4 **Exercises**

1–10 Find the divergence of the vector field **F** and evaluate the divergence at the given points.

1. $\mathbf{F}(x,y)=\langle 5x^3-2xy,4xy^2\rangle$; $A(2,1)$, $B(1,-3)$

2. $\mathbf{F}(x,y)=\langle xe^{xy},y^2\cos x\rangle$; $A(0,-1)$, $B(\pi/2,2/\pi)$

3. $\mathbf{F}(x,y,z)=\langle x^2,3xz^2,-2yz\rangle$; $A(2,-1,0)$, $B(1,1,4)$

4. $\mathbf{F}(x,y,z)=\left\langle \frac{x^2}{2},\frac{y^2}{2},\frac{z^2}{2}\right\rangle$; $A(-1,3,0)$, $B(5,-7,2)$

5. $\mathbf{F}(x,y,z)=\langle 2yz,2xz,2xy\rangle$; $A(1,1,1)$, $B(2,-3,4)$

6. $\mathbf{F}(x,y,z)=\langle xy^2,xz+2y,z^2\rangle$; $A(2,0,-2)$, $B(4,9,0)$

7. $\mathbf{F}(x,y,z)=\langle x+y^3,e^y,z^4\rangle$; $A(1,0,-3)$, $B(2,1,4)$

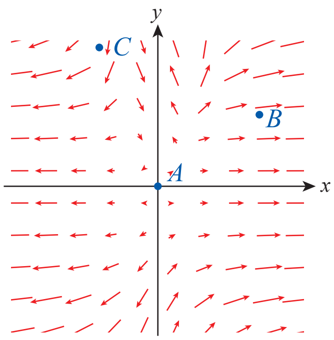
8. $\mathbf{F}(x,y,z)=\langle \cos(xy^2),x,2y-z\rangle$; $A(-2,0,1)$, $B(\pi/2,5,5)$

9. $\mathbf{F}(x,y,z)=\langle e^{xyz},e^{xyz},e^{xyz}\rangle$; $A(1,0,1)$, $B(e,1,-1)$

10. $\mathbf{F}(x,y,z)=\left\langle x^2y,xy-z^2,\frac{2z^2}{x^2+y^2}\right\rangle$; $A(0,1,3)$, $B\left(-1,1,\frac{1}{2}\right)$

11–20. Find the curl of each vector field given in Exercises 1–10.

21. Use the figure to decide whether the divergence of the vector field at the given points is positive, negative, or zero.



22. You may have noticed by now that for a vector field $\mathbf{F}=\langle P,Q,R\rangle$, those terms in P involving only x do not affect the curl; neither do those terms in Q involving only y. A similar statement holds for R and z. For example,

$$\operatorname{curl}\left\langle x^2\cos x, y^4+z, \frac{1}{\sqrt{1+z^2}}+y^2\right\rangle=\operatorname{curl}\langle 0,z,y^2\rangle.$$

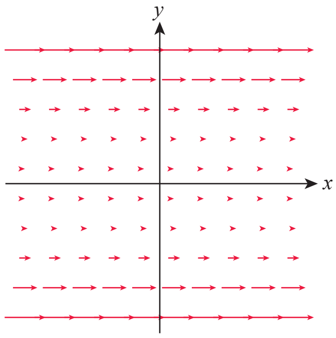
Prove the above assertions.

23–24 Use the "shortcut" suggested by Exercise 22 to evaluate the indicated curl.

23. $\operatorname{curl}\left\langle y+\ln(x^2+1),y^3-\sin y,e^{\sin z}+1\right\rangle$

24. $\operatorname{curl}\left\langle z-\ln\sqrt{1+x^2},x^2+\arctan(y^2),\frac{z}{1+z^2}\right\rangle$

25. Determine the curl of $\mathbf{F}(x,y)=\langle y^2,0\rangle$ and provide a physical interpretation of the result (as done in Example 2). See the figure below for guidance.



26–29 Find $\operatorname{curl}(\operatorname{curl}\mathbf{F})$ for the given vector field.

26. $\mathbf{F}(x,y,z)=\langle x,-y,xyz\rangle$

27. $\mathbf{F}(x,y,z)=\langle xy^2,xz,-4xz^2\rangle$

28. The vector field given in Exercise 3

29. The vector field given in Exercise 6

30–37 Decide whether or not the expression has meaning. If an expression does have meaning, state whether it is a vector field or a scalar field.

30. $\nabla\times(\nabla\cdot\mathbf{F})$

31. $\nabla\cdot(\nabla\cdot\mathbf{F})$

32. $\nabla\cdot(\nabla\times\mathbf{F})$

33. $\nabla(\nabla\cdot\mathbf{F})$

34. $\nabla(\nabla\times\mathbf{F})$

35. $\nabla\cdot(\nabla f)$

36. $\nabla\times(\nabla f)$

37. $\nabla(\nabla f)$

38–47 Supposing that all of the appropriate partial derivatives of the vector fields **F** and **G** are continuous, prove the given statement. (Assume that f and g are at least twice differentiable scalar fields; a, b denote constants.)

38. $\nabla\times(\nabla f)=\mathbf{0}$

39. $\nabla\cdot(\nabla\times\mathbf{F})=0$

40. $\nabla\cdot(a\mathbf{F}+b\mathbf{G})=a(\nabla\cdot\mathbf{F})+b(\nabla\cdot\mathbf{G})$

41. $\nabla\times(a\mathbf{F}+b\mathbf{G})=a(\nabla\times\mathbf{F})+b(\nabla\times\mathbf{G})$

42. $\nabla\times(\nabla f+\nabla\times\mathbf{F})=\nabla\times(\nabla\times\mathbf{F})$

43. $\nabla\cdot(\mathbf{F}\times\mathbf{G})=(\nabla\times\mathbf{F})\cdot\mathbf{G}-\mathbf{F}\cdot(\nabla\times\mathbf{G})$

44. $\nabla\cdot(f\mathbf{F})=f(\nabla\cdot\mathbf{F})+(\nabla f)\cdot\mathbf{F}$

45. $\nabla\times(f\mathbf{F})=f(\nabla\times\mathbf{F})+(\nabla f)\times\mathbf{F}$

46. $\nabla\cdot(\nabla f\times\nabla g)=0$

47. $\nabla\cdot(\nabla fg)=f\nabla\cdot(\nabla g)+g\nabla\cdot(\nabla f)+2(\nabla f)\cdot(\nabla g)$

48–51 Verify Green's Theorem by demonstrating the equality $\oint_C P\,dx+Q\,dy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$, where $P(x,y)=xy^2$, $Q(x,y)=2x$, and C is a smooth closed curve enclosing the region R as specified.

48. R: The triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$

49. R: The square $[-1,1]\times[-1,1]$

50. R: The region bounded by the graphs of $y=x^2$ and $y=x$

51. R: The unit circle (disk) centered at the origin

52. Parametrize curve C given in Example 4 and calculate the line integral of **F** around C to show $\oint_C y^2\,dx+3xy\,dy=\frac{1}{3}$.

53–62 Use Green's Theorem to evaluate the line integral. (Pay attention to the orientation of the path as indicated by the integration symbol.)

53. $\oint_C xy\,dx+(2x+y)\,dy$, where C is the boundary of the region between the graphs of $y=1-x^2$ and the x-axis, oriented counterclockwise

54. $\oint_C 4xy\,dx+\ln(x^2+y^2)\,dy$, where C is the cardioid $r=1+\cos\theta$, oriented counterclockwise

55. $\oint_C 2y^2\,dx-xy\,dy$, where C is the graph of $y=\sqrt{x}$ from the origin to $(1,1)$, followed by $y=x^2$ from $(1,1)$ back to $(0,0)$

56. $\oint_C x^2y\,dx+(x^3+y^2)\,dy$, where C is the boundary of the region bounded by the x-axis, the graphs of $y=x^3$ and $x=2$, oriented counterclockwise

57. $\oint_C 4x^2y^3\,dx-x^3y\,dy$, where C is the boundary of the region bounded by the coordinate axes and the line $y=2-2x$, oriented clockwise

58. $\oint_C\left[\cos(x^2)-2y\right]dx+\left[4x+\ln(y^4)\right]dy$, where C is the unit circle centered at the origin, with positive orientation

59. $\oint_C\left(y^3-\sqrt{x^2+2}\right)dx+(3xy^2-\cos y)\,dy$, where C is defined as the border of a map of Louisiana contained in the first quadrant (Assume that the border is a smooth, simple, closed curve.)

60. $\oint_C e^x\,dx+y^2e^x\,dy$, where C is the triangle with vertices $(0,0)$, $(1,2)$, and $(5,0)$, oriented counterclockwise

61. $\oint_C\left[y+2\tan^{-1}(x/4)\right]dx-\left(e^{x^2+1}+2\sec y\right)dy$, where C is the boundary of the region between the graphs of $y=|x|$ and $y=2-|x|$, with positive orientation

62. $\oint_C xe^y\,dx+xy\,dy$, where C is the boundary of the region between the graphs of $x=y^2$ and $x-y=2$, with clockwise orientation

63–67 In these exercises, we will revisit some exercises from Section 15.2. Use Green's Theorem to provide a second solution and verify that your answers agree.

63. Determine $\int_C(x^2y+xy^2)\,dx+x^3\,dy$, where C is the unit circle centered at the origin. (See Exercise 27 of Section 15.2.)

64. Determine $\int_C y\,dx+xy\,dy$, where C is the parabola $y=x^2$ joining the origin with $(1,1)$, followed by the line segment from $(1,1)$ back to the origin. (See Exercise 28 of Section 15.2.)

65. Find the line integral $\int_C 2xy\,dx+x^2\,dy$ on the curve given in Exercise 64. (See Exercise 29 of Section 15.2.)

66. Find $\int_C \left(x^2+y\right)dx + xy\,dy$ along the closed path $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from the origin to $(1,0)$, C_2 is the line segment from $(1,0)$ to $(2,1)$, and C_3 is the straight path from $(2,1)$ back to the origin. (See Exercise 30 of Section 15.2.)

67. Find $\int_C 2xy\,dx + x^2\,dy$ along the path given in Exercise 66. (See Exercise 31 of Section 15.2.)

68. Determine the work done by the force field $\mathbf{F}(x,y) = \langle x+y, 2xy\rangle$ if it moves a particle clockwise around the triangle with vertices $(0,0)$, $(2,0)$, and $(1,1)$.

69. Determine the work done by the force field given in Exercise 68 as it moves a particle clockwise around the square $[-1,1]\times[-1,1]$.

70. Determine the work done by the force field $\mathbf{F}(x,y) = \langle xy, x^2-y^2\rangle$ as it moves a particle counterclockwise around the unit circle centered at the origin.

71. Use Green's Theorem to determine the work done by the force field $\mathbf{F}(x,y) = \langle 2, 3x\rangle$ if it moves a particle counterclockwise around the ellipse parametrized as $\mathbf{r}(t) = \langle 3+3\cos t, 2+2\sin t\rangle$, $0 \le t \le 2\pi$. (See Exercise 36 of Section 15.2.)

72–75 Evaluate the outward flux of the vector field **F** across the specified path *C*.

72. $\mathbf{F}(x,y) = \langle xy, 2x+y\rangle$, C is the boundary of the region between the graphs of $y = 1 - x^2$ and the x-axis

73. $\mathbf{F}(x,y) = \langle x^2y, x^3+y^2\rangle$, C is the boundary of the region bounded by the x-axis and the graphs of $y = x^3$ and $x = 2$

74. $\mathbf{F}(x,y) = \langle 4x^2y^3, -x^3y\rangle$, C is the boundary of the region bounded by the coordinate axes and the line $y = 2 - 2x$

75. $\mathbf{F}(x,y) = \langle e^x, e^xy^2\rangle$, C is the triangle with vertices $(0,0)$, $(1,2)$, and $(5,0)$

76–84 Suppose that R is a region bounded by a positively oriented, piecewise smooth, simple closed planar curve C. Recall that the area of R is the double integral of $f(x,y)=1$ over R; that is, $A = \iint_R 1\cdot dA$. With Green's Theorem in mind, if we choose P and Q so that $\frac{dQ}{dx} - \frac{dP}{dy} = 1$, then the area of R can be found from a line integral:

$$A = \iint_R 1\cdot dA = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA = \oint_C P\,dx + Q\,dy.$$

In Exercises 76–84, use this observation to prove the area formula.

76. Show that if R and C are as above, then the area A of R is

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \oint_C x\,dy = -\oint_C y\,dx.$$

77. Use Exercise 76 to verify the area formula of the circle $x^2 + y^2 = r^2$.

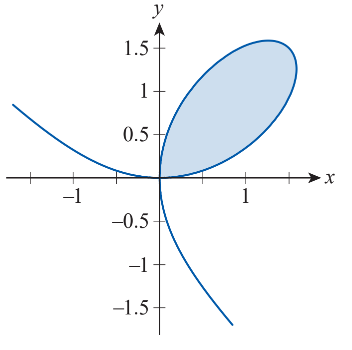
78. Use Exercise 76 to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

79. Use Exercise 76 to find the area of the region bounded by the y-axis and the lines $3y = x$ and $y = 4 - x$.

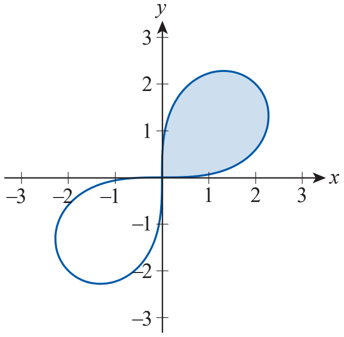
80. Use Exercise 76 to find the area of the region bounded by the graphs of $y = x^2$ and $y = x + 2$.

81. Use Exercise 76 to find the area of the region bounded by the graphs of $y = 9 - x^2$ and $3y = x^2 - 9$.

82.* The graph of the equation $x^3 + y^3 = 3xy$ is called the folium of Descartes. Use Exercise 76 to find the area of its loop. (**Hint:** Choose $t = y/x$ to obtain the parametrization $x = 3t/\left(1+t^3\right)$ and $y = 3t^2/\left(1+t^3\right)$. Use the Quotient Rule to obtain $d(y/x) = (x\,dy - y\,dx)/x^2$; then use Exercise 76. Note that you obtain half the area of the loop by integrating on the interval $0 \le t \le 1$.)



83.* Follow the hint provided for Exercise 82 to obtain the area of the first-quadrant loop of the lemniscate $\left(x^2+y^2\right)^2 = 16xy$. (Note that by symmetry, doubling your answer would yield the total area enclosed by the lemniscate.)



84. Use Exercise 76 to obtain the following convenient area formula for the polygon with vertices $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. Show that the area A of such a polygon is

$$A=\frac{1}{2}\sum_{k=1}^{n}\left(x_k y_{k+1}-x_{k+1}y_k\right),$$

with the convention that $(x_1,y_1)=(x_{n+1},y_{n+1})$.

(**Hint:** Start by letting S be one side of the polygon, say the one connecting (x_1,y_1) with (x_2,y_2), and show that $\frac{1}{2}\int_S x\,dy - y\,dx = \frac{1}{2}(x_1y_2-x_2y_1)$. Then use Exercise 76.)

85–88 Use the formula from Exercise 84 to find the area of the given polygon.

85. The square with vertices $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$

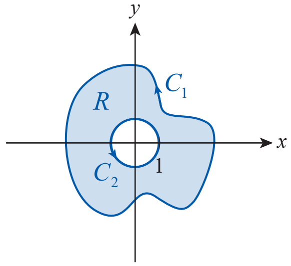
86. The triangle with vertices $(0,0)$, $(2,4)$, and $(0,4)$

87. The hexagon with vertices $(0,0)$, $(2,0)$, $(4,2)$, $(3,4)$, $(1,4)$, and $(0,2)$

88. The (nonconvex) pentagon with vertices $(1,0)$, $(3,0)$, $(5,7)$, $(2,2)$, and $(0,3)$

89. Evaluate $\oint_C \left(3x^3-y\right)dx+\left(y^2+4x\right)dy$ on the ellipse given in Exercise 78.

90. Let C_2 be the unit circle centered at the origin, let $C=C_1\cup C_2$ be the boundary of region R with the indicated orientations as in the figure below, and let $\mathbf{F}(x,y)=\left\langle 4x-3y, 3x+y^2\right\rangle$. Assuming that the area of R is 4π square units, use Green's Theorem to evaluate $\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}$. (**Hint:** Determine $\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$ and use the fact that by Green's Theorem, it equals $\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}-\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}$.)



91. In Exercise 35 of Section 15.3, you showed that for the vector field $\mathbf{F}$ in Example 6, $\oint_{C_r}\mathbf{F}\cdot d\mathbf{r}=2\pi$, where C_r is a circle of radius r around the origin (you were free to choose the radius in that problem, but $r=1$ is the natural choice). Combine that result with Example 6 of this section to show that $\oint_C\mathbf{F}\cdot d\mathbf{r}=2\pi$ for every simple closed path C enclosing the origin.

92. For the vector field given in Exercise 91, show that $\oint_C\mathbf{F}\cdot d\mathbf{r}=0$ for any simple closed path C that is disjoint from, and does not enclose the origin.

93. Use the technique illustrated in Example 6 to find $\oint_C\mathbf{F}\cdot d\mathbf{r}$ for the vector field

$$\mathbf{F}(x,y)=\frac{\langle x,y\rangle}{\left(x^2+y^2\right)},$$

where C is a simple closed path enclosing the origin.

94. Use Green's Theorem to show that the area of a region R enclosed by the simple closed curve C in polar coordinates is

$$A=\frac{1}{2}\oint_C r^2\,d\theta.$$

95. Use Green's Theorem to show that the coordinates of the centroid of a region R of constant density ρ and area A, enclosed by the simple closed curve C can be found as

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy, \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx.$$

(**Hint:** Start by determining $M_x = \iint_R y\rho\,dA$ and $M_y = \iint_R x\rho\,dA$.)

96.* Use Green's Theorem to show that the second moments I_x and I_y of the region R given in Exercise 95 are

$$I_x = -\frac{1}{3}\oint_C \rho y^3\,dx \quad \text{and} \quad I_y = \frac{1}{3}\oint_C \rho x^3\,dy.$$

97. Use Exercise 95 to find the centroid of the half disk bounded by the upper semicircle $y = \sqrt{R^2 - x^2}$ and the x-axis.

98. Use Exercise 96 to find the second moments about the coordinate axes of the half disk given in Exercise 97, assuming constant density ρ.

99. Use Green's Theorem to provide a proof of the Component Test. If D is an open and simply connected region and $\mathbf{F} = \langle P, Q\rangle$ is a vector field with continuous first partials, then $\partial P/\partial y = \partial Q/\partial x$ throughout D implies $\mathbf{F}$ is conservative.

(**Hint:** Show that **F** is path independent by using Green's Theorem and demonstrating that for a simple closed path C enclosing the region R,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

Extend this result to general closed paths.)

Concept Check

100–103 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

100. The divergence of a vector field at a point P is a vector pointing in the direction of greatest outflow from a small rectangular neighborhood.

101. The curl of a vector field is a vector field.

102. The curl of a planar (i.e., two-dimensional) vector field is undefined.

103. In applications of Green's Theorem, clockwise orientation is negative.

```
In[6]:= ParametricPlot3D[{x, y, z}, {r, 0, 1.38}, {θ, 0, 2 Pi},
          PlotPoints → 100, Mesh → None, Boxed → False,
          Axes → False, PlotRange → All, ViewPoint → {1, 1, 10}]
Out[6]=
```

Figure 11b

15.5 Exercises

1–4 Describe the surface with a vector function of two parameters. (Answers will vary.)

1. The graph of $y = \frac{1}{3}x$, $0 \le x \le 6$, revolved about the x-axis

2. The graph of $z = 1/x$, $0 \le x \le 8$, revolved about the x-axis

3. The graph of $x = 1 - z^2$, $-1 \le z \le 1$, revolved about the z-axis

4. The graph of $z = \cos y$, $-\pi/2 \le y \le \pi/2$, revolved about the y-axis

5–13 Match the parametric surface with its graph (labeled A–I).

5. $\mathbf{r}(s,t) = \langle 3\cos s, 3\sin s, t \rangle$, $0 \le s \le 2\pi$, $0 \le t \le 8$

6. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 2s \rangle$, $0 \le s \le 5$, $0 \le t \le 2\pi$

7. $\mathbf{r}(s,t) = \langle \cos 2t \sin s, \cos s \cos 2t, \sin t \rangle$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$

8. $\mathbf{r}(s,t) = \left\langle s, \dfrac{\cos t}{s^2}, \dfrac{\sin t}{s^2} \right\rangle$, $\dfrac{1}{3} \le s \le 3$, $0 \le t \le 2\pi$

9. $\mathbf{r}(s,t) = \langle 2t - s, 1 + 2s + t, 1 + s - 3t \rangle$, $0 \le s \le 5$, $0 \le t \le 5$

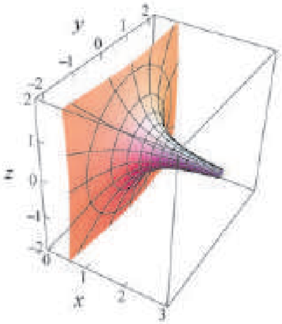
10. $\mathbf{r}(s,t) = \langle s\sin t, s\cos t, \cos s \rangle$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$

11. $\mathbf{r}(s,t) = \langle \sin s \cos t, \sin s \sin t, \cos s \rangle$, $0 \le s \le \pi/2$, $0 \le t \le 2\pi$

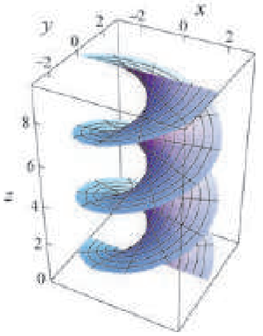
12. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, t \rangle$, $-3 \le s \le 3$, $0 \le t \le 3\pi$

13. $\mathbf{r}(s,t) = \langle \cos s \sin t, 2\sin s \sin t, \cos t \rangle$, $0 \le s \le \pi$, $0 \le t \le 2\pi$

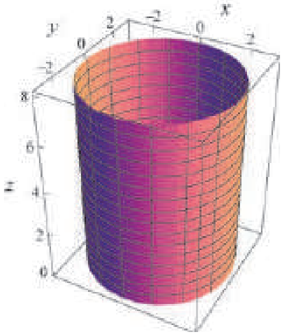
A.



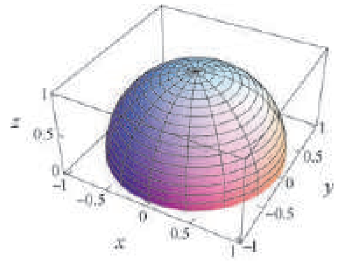
B.



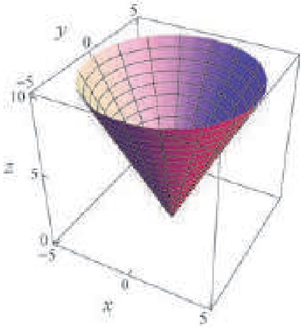
C.



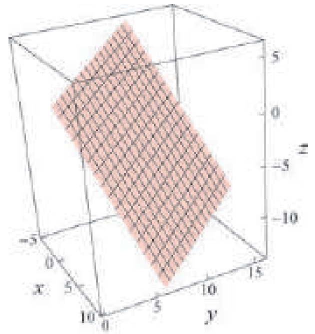
D.



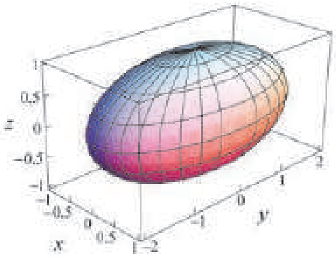
E.



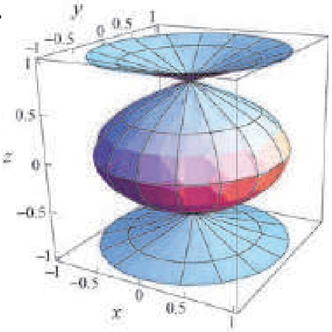
F.



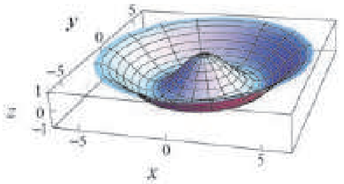
G.



H.



I.



14–20 Identify the surface by examining its grid curves.

14. $\mathbf{r}(s,t) = \langle t, \cos s, \sin s \rangle, \quad 0 \le s \le 2\pi, \quad -\infty < t < \infty$

15. $\mathbf{r}(s,t) = \langle R\sin s \cos t, R\sin s \sin t, R\cos s \rangle,$
$0 \le s \le \pi/2, \quad 0 \le t \le \pi/2$

16. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s \rangle,$
$-\infty < s < \infty, \quad 0 \le t \le 2\pi$

17. $\mathbf{r}(s,t) = \langle s, t, 2s^2 + t^2 \rangle,$
$-\infty < s < \infty, \quad -\infty < t < \infty$

18. $\mathbf{r}(s,t) = \langle (3+\cos t)\cos s, (3+\cos t)\sin s, \sin t \rangle,$
$0 \le s \le 2\pi, \quad 0 \le t \le 2\pi$

19. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 4 - s^2 \rangle,$
$0 \le s \le 2, \quad 0 \le t \le 2\pi$

20. $\mathbf{r}(s,t) = \left\langle s, \frac{\cos t}{s}, \frac{\sin t}{s} \right\rangle, \quad \frac{1}{4} \le s \le 4, \quad 0 \le t \le 2\pi$

21–31 Obtain a parametrization for the indicated surface. (Answers will vary.)

21. $z = x + y$

22. $z = xy$

23. $x^2 + y^2 = 1, 0 \le z \le 1$

24. $z = \sqrt{x^2 + y^2}, z \le 2$

25. $x^2 + y^2 + z^2 = 4$

26. $z = x^2 + y^2$

27. $z = -\sqrt{1 - x^2 - y^2}$

28. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

29. $x^2 + 4y^2 = 4$

30. The intersection of $z = x^2 + y^2$ with the interior of $x^2 + y^2 = 4$

31. The portion of the sphere $x^2 + y^2 + z^2 = 4$ outside the double cone $z^2 = 3x^2 + 3y^2$

32–37 Construct an equation for the plane tangent to the surface at the indicated point.

32. $\mathbf{r}(s,t) = \langle s^2, s+t, t^2 \rangle; \quad \mathbf{r}(1,1)$

33. $\mathbf{r}(s,t) = \langle s, t, 2s^2 + t^2 \rangle; \quad \mathbf{r}(1,2)$

34. $\mathbf{r}(s,t) = \langle 2st, s^2, t^2 \rangle; \quad \mathbf{r}(1,-3)$

35. $\mathbf{r}(s,t) = \langle 2s\cos t, s\sin t, s^2 \rangle; \quad \mathbf{r}\left(2, \frac{\pi}{4}\right)$

36. $\mathbf{r}(s,t) = \left\langle 2s^2, st^2, \frac{st}{2} \right\rangle; \quad \mathbf{r}(1,2)$

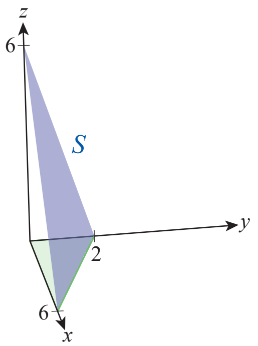
37. $\mathbf{r}(s,t) = \langle s\sin t, s^4, s\cos t \rangle; \quad \mathbf{r}\left(1, \frac{\pi}{3}\right)$

38. Parametrize the sphere of radius R as in Example 2, and show that its normal vector $\mathbf{n}(\theta,\varphi)$ is a constant multiple of $\mathbf{e}_r$ (see Exercise 54 of Section 15.1). What is that constant?

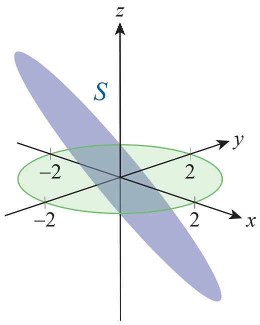
39. Verify that the parametric approach to surface area is consistent with the surface area of a solid of revolution discussed in Section 6.3. (**Hint:** Let $f(x) \geq 0$ be a continuously differentiable single-variable function defined on $[a,b]$, and rotate its graph around the x-axis. Parametrize the resulting surface of revolution as in Example 1. Calculate $\mathbf{n} = \mathbf{r}_s \times \mathbf{r}_t$ as in Example 5 and find the surface area $A = \int_0^{2\pi} \int_a^b |\mathbf{n}|\, dA$ after showing that $|\mathbf{n}| = |\mathbf{r}_s \times \mathbf{r}_t| = f(x)\sqrt{1+[f'(x)]^2}$.)

40–56 Find the area of the surface S. (Use polar coordinates wherever they simplify your calculations.)

40. S is the first-octant portion of the plane $x + 3y + z = 6$.

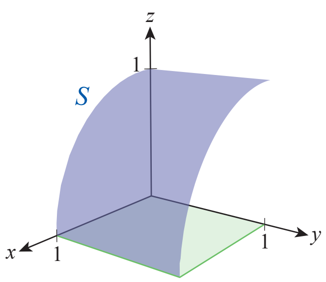


41. S is the intersection of the plane $2x + 2y + z = 0$ and the interior of the cylinder $x^2 + y^2 = 4$.



42. S is the graph of $z = 3 + 2x - y$ defined on the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

43. S is the graph of $z = \sqrt{1-x^2}$ defined on the square $[0,1]\times[0,1]$.



44. S is the graph of $z = 2x^2 + y$ defined on the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

45. S is the portion of the paraboloid $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 1-s^2\rangle$ above the xy-plane.

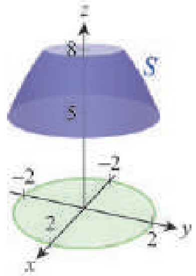
46. S is the surface of the cone $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s\rangle$ between the planes $z = 1$ and $z = 2$.

47. S is the portion of the paraboloid $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s^2\rangle$ between the planes $z = 4$ and $z = 9$.

48. S is the portion of the cone $z = \sqrt{x^2 + y^2}$ defined on the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

49. S is the graph of $z = y^2 - x^2$ defined on the first-quadrant portion of the disk of radius 2 centered at the origin.

50. S is the surface $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 9-s^2\rangle$ between the planes $z = 5$ and $z = 8$.



51. S is the portion of the paraboloid $\mathbf{r}_1(s,t) = \langle s\cos t, s\sin t, s^2/2\rangle$ between the cylinders $\mathbf{r}_2(s,t) = \langle \cos s, \sin s, t\rangle$ and $\mathbf{r}_3(s,t) = \langle 2\cos s, 2\sin s, t\rangle$.

52. S is the portion of $x^2 + y^2 = z^2 + 1$ between the xy-plane and $z = \sqrt{3}$.

53. S is the first-quadrant portion of $z + y^2 = 1$, $z \geq 0$, between the yz-plane and $x = 1$.

54. S is the torus $\mathbf{r}(s,t) = \langle (2+\cos t)\cos s, (2+\cos t)\sin s, \sin t\rangle$, $0 \leq s \leq 2\pi$, $0 \leq t \leq 2\pi$.

55. S is the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ between the planes $z = 3$ and $z = 4$.

56. S is the portion of the cylinder $x^2 + y^2 = 25$ between the planes $z = 3$ and $z = 4$. (Compare your answer to the solution of Exercise 55.)

57.* Generalize Exercise 54 to arrive at the formula for the surface area of the torus parametrized as $\mathbf{r}(s,t)=\langle (a+b\cos t)\cos s,(a+b\cos t)\sin s,b\sin t\rangle$, where $a>b>0$. (**Hint:** Consider the circle of radius b in the xz-plane, centered at $(a,0,0)$, and rotate it around the z-axis.)

58.* Generalize your observations made in Exercises 55 and 56 to prove Archimedes' famous result:

The surface area of the section of the sphere $x^2+y^2+z^2=R^2$ between the planes $z=a$ and $z=b$ equals the surface area of the corresponding section of the circumscribed cylinder $x^2+y^2=R^2$.

59.* Show that surface area is independent of parametrization; that is, prove the following statement.

Let R_1 and R_2 be regions in the plane enclosed by simple closed paths, and let $\mathbf{r}_1: R_1 \to \mathbb{R}^3$, $\mathbf{r}_2: R_2 \to \mathbb{R}^3$ be continuously differentiable, one–to–one parametrizations of the same surface, that is, $\mathbf{r}_1(R_1)=\mathbf{r}_2(R_2)$. With the usual notation $\mathbf{n}_i=(\mathbf{r}_i)_s \times (\mathbf{r}_i)_t$, $i=1,2$, prove that

$$\iint_{R_1} |\mathbf{n}_1|\,dA = \iint_{R_2} |\mathbf{n}_2|\,dA.$$

(**Hint:** Use the rules for differentiation of inverses and change of variables.)

Concept Check

60–63 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. All grid curves of a parametric surface are parallel to a coordinate plane.

61. Before we can evaluate its area, a surface S must be parametrized.

62. A parametric surface $\mathbf{r}(s,t)$ has a tangent plane at any point of smoothness $\mathbf{r}(s_0,t_0)$.

63. The area of a parametric surface is a limit of Riemann sums.

15.5 Technology Exercises

64. Describe in words the grid curves of the parametric surface given in Exercise 12; then use a graphing utility to sketch the surface. Finally, use your technology to approximate its area. (**Note:** This is an example of what are called *helicoid surfaces*.)

65. Sketch the parametric surface

$$\mathbf{r}(s,t)=\left\langle \left(\sqrt{25-t^2}-3\right)\cos s,\left(\sqrt{25-t^2}-3\right)\sin s,t\right\rangle,$$
$$0\le s\le 2\pi,\quad -4\le t\le 4,$$

using a graphing utility and verify that the graph resembles a football. Change the coefficients to make the "football" appear "thinner" or "thicker," respectively. (Carefully determine the domain for each parameter. Answers will vary.)

66. The parametric surface

$$\mathbf{r}(s,t)=\left\langle \cos t+s\cos\frac{t}{2},3\sin t+s\cos\frac{t}{2},s\sin\frac{t}{2}\right\rangle,$$
$$-\frac{1}{2}\le s\le \frac{1}{2},\quad 0\le t\le 2\pi$$

is an example of the famous *Möbius strip* (after the German mathematician August Ferdinand Möbius). Use a graphing utility to graph and examine this surface. Can you think of a way to produce such a surface using a strip of paper and tape? Notice that if you start sliding your finger along one side of the surface, you will eventually arrive back at your starting point without crossing any edges! Surfaces with this property are called *nonorientable*, or *one-sided*.

67. Graph and examine the surface

$$\mathbf{r}(s,t)=\langle (a+\sin t)\cos s,(a+\sin t)\sin s,t\rangle,$$
$$0\le s\le 2\pi,\quad 0\le t\le 2\pi$$

for several values of the parameter a. Then use your technology to find its surface area if $a=2$.

where κ is the thermal conductivity constant for the substance of the ball. From Example 4, we know that the outward unit normal vector **n** at the point (x, y, z) on the surface $x^2 + y^2 + z^2 = r^2$ is

$$\mathbf{n}(x, y, z) = \frac{1}{r}\langle x, y, z\rangle,$$

so on S,

$$\mathbf{F}\cdot\mathbf{n} = \frac{2\kappa}{r}\langle x, y, z\rangle\cdot\langle x, y, z\rangle = \frac{2\kappa}{r}\left(x^2 + y^2 + z^2\right) = \frac{2\kappa}{r}\left(r^2\right) = 2\kappa r.$$

Hence, the flow of heat per unit time across S is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 2\kappa r\iint_S d\sigma = 2\kappa r\left(4\pi r^2\right) = 8\kappa\pi r^3.$$

15.6 Exercises

1. Use the substitution suggested in Example 1 to conclude that

$$M = \iint_S \rho\,d\sigma = \frac{\pi\left(25\sqrt{5} - 11\right)}{60}.$$

2. Use the same substitution as in Exercise 1 to finish Example 2 by showing that

$$\overline{z} = \frac{M_{xy}}{M} = \frac{2\left(25\sqrt{5} + 4\right)}{7\left(25\sqrt{5} - 11\right)} \approx 0.38.$$

3. Suppose the surface S is given as the graph of a function $z = g(x, y)$, defined on a domain R. Prove that for a continuous function $f = f(x, y, z)$ defined on S, the surface integral of f over S is

$$\iint_S f\,d\sigma = \iint_R f\left(x, y, g(x, y)\right)\sqrt{1 + \left[g_x\right]^2 + \left[g_y\right]^2}\,dA.$$

(**Hint:** Parametrize S by using x and y as parameters.)

4–25 Parametrize the surface S and evaluate the indicated surface integral. (You may use the formula from Exercise 3 whenever feasible. Use polar or spherical coordinates where needed.)

4. $\iint_S 2z\,d\sigma$, where S is the first-octant portion of the plane $x + 2y + z = 4$

5. $\iint_S y\,d\sigma$, where S is the graph of $2x + y + z = 6$ above the square $[0,2]\times[0,2]$

6. $\iint_S 3x\,d\sigma$, where S is the graph of $x^2 + z = 9$ above the rectangle $[0,3]\times[0,4]$

7. $\iint_S x\,d\sigma$, where S is the first-octant portion of the plane $x + y + z = a$ $(a > 0)$

8. $\iint_S z^2\,d\sigma$, where S is the intersection of the plane $2x + 2y + z = 0$ and the interior of the cylinder $x^2 + y^2 = 1$

9. $\iint_S \left(x^2 + y^2 + z\right)d\sigma$, where S is the graph of $x + 2y + z = 2$ above the rectangle $[0,1]\times[0,2]$

10. $\iint_S 9z\,d\sigma$, where S is the surface $z = y^3$ over the rectangle $[-1,1]\times[0,1]$

11. $\iint_S (2xy + z)\,d\sigma$, where S is the graph of $2y - x + z = 4$ above the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$

12. $\iint_S (x + y + z)\,d\sigma$, where S is the graph of $z = \sqrt{1 - x^2}$ above the square $[0,1]\times[0,1]$

13. $\iint_S (z - y)\,d\sigma$, where S is the graph of $z = 2x^2 + y$ above the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$

14. $\iint_S 2z\,d\sigma$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane

15. $\iint_S \left(z + 2x^2\right)d\sigma$, where S is the surface $z = y^2 - x^2$ above the half disk bounded by $y = \sqrt{1 - x^2}$ and the x-axis

16. $\iint_S y^2\,d\sigma$, where S is the upper unit hemisphere $z=\sqrt{1-y^2-x^2}$

17. $\iint_S \left(x^2+y^2\right)d\sigma$, where S is the portion of the paraboloid $z = 2 - x^2 - y^2$ above the xy-plane

18. $\iint_S z\,d\sigma$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 3$

19. $\iint_S \frac{y^2}{1-z}\,d\sigma$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ with $\frac{3}{4} \le z \le 1$

20. $\iint_S \sin z\,d\sigma$, where S is the cylinder $x^2+y^2=1,\ 0\le z\le \pi/2$

21. $\iint_S \left(2x^2+2y^2+2z^2\right)d\sigma$, where S is the unit sphere centered at the origin; use spherical coordinates

22. $\iint_S \left(yz+y^3\right)d\sigma$, where S is the box $[0,1]\times[0,2]\times[0,3]$

23. $\iint_S \left(x^2+2xz\right)d\sigma$, where S is the tetrahedron formed by the plane $x + 2y + 2z = 4$ and the coordinate planes

24. $\iint_S z\,d\sigma$, where S is the solid bounded by the cylinder $x^2 + y^2 = 1$, the xy-plane, and the plane $2z = x + 3$

25. $\iint_S xyz\,d\sigma$, where S is the cone frustum $z=3-\sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$

26. Find the centroid of the surface in Example 1, assuming constant density.

27. Determine the mass M of the half cylinder S defined by $\mathbf{r}(s,t)=\langle 2\cos s, 2\sin s, t\rangle$, $0\le s\le\pi$, $0\le t\le 2$, if its mass density is $\rho(x,y,z)=y+z$.

28. Determine the mass M of the cone S defined by $\mathbf{r}(s,t)=\langle s\cos t, s\sin t, s\rangle$, $0\le s\le 1$, $0\le t\le 2\pi$, if its mass density is proportional to the distance from the z-axis.

29. Find the center of mass of the half cylinder given in Exercise 27.

30. Find the center of mass of the cone surface given in Exercise 28.

31. Determine the mass of the portion of the thin spherical shell $z=\sqrt{25-x^2-y^2}$ between the planes $z = 3$ and $z = 4$ if its density at any point is proportional to the distance from the xy-plane. (**Hint:** Consider spherical coordinates.)

32. Find the centroid of the thin hemispherical surface $g(x,y)=\sqrt{R^2-x^2-y^2}$ if it has constant density ρ. (See the hint given in Exercise 31.)

33. Determine the second moments about the coordinate axes of the sphere $x^2 + y^2 + z^2 = R^2$ if it has constant density ρ. (See the hint given in Exercise 31.)

34–52 Determine the indicated flux of the vector field **F** across the surface S. Unless otherwise specified, the surfaces are oriented with outward-pointing normal vectors.

34. The flux of $\mathbf{F}=\langle 0,0,c\rangle$ across the hemisphere given in Exercise 32

35. The flux of $\mathbf{F}=\langle x,y,2\rangle$ out of the solid R bounded by $3z = x^2 + y^2$ and the plane $z = 3$

36. The flux of $\mathbf{F}=\langle 2,z,y\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

37. The flux of $\mathbf{F}=\langle -y,x,2\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

38. The flux of $\mathbf{F}=\langle x,y,z\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

39. The flux of $\mathbf{F}=\langle x,y,z\rangle$ across the first-octant portion of the cylinder $x^2 + y^2 = 1$, between the planes $z = 0$ and $z = 1$

40. The flux of $\mathbf{F}=\langle x^2,xy,xz\rangle$ across the first-octant portion of the hemisphere of radius R, centered at the origin

41. The flux of $\mathbf{F}=\langle 0,0,x^2+y^2\rangle$ across the paraboloid $z = x^2 + y^2 + 1$, $1 \le z \le 10$

42. The flux of $\mathbf{F}=\langle 0,y^2,x^2+z^2\rangle$ across the hemisphere $z=\sqrt{4-x^2-y^2}$

43. The flux of $\mathbf{F}=\langle y,-x,6\rangle$ across the portion of the upper unit hemisphere centered at the origin that projects onto the disk $x^2+y^2\le\frac{1}{2}$ (Note that this can be interpreted as the upward flux through the hemisphere of a fluid with a "rotating flow.")

44. The flux of the vector field given in Exercise 43 across the portion of the paraboloid $z = 1 - x^2 - y^2$ that projects onto the disk $x^2 + y^2 \le \frac{1}{2}$ (Compare your answer with the solution of Exercise 43.)

45. The flux of the vector field given in Exercise 43 across the portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ that projects onto the disk $x^2 + y^2 \le \frac{1}{2}$ (Compare your answer with the solution of Exercise 43 or 44.)

46. The flux of $\mathbf{F} = \langle -2y, x, 2z \rangle$ out of the solid R bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 6$

47. The flux of $\mathbf{F} = \langle x, y, 2 \rangle$ out of the solid R bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

48. The flux of $\mathbf{F} = \langle 2x, y, z \rangle$ across the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane

49. The flux of $\mathbf{F} = \langle -y, x, z^2/2 \rangle$ across the cone frustum $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$

50. The flux of $\mathbf{F} = \langle x^2, 2y, yz \rangle$ out of the cube in Example 3

51. The flux of $\mathbf{F} = \langle 0, y, 2z \rangle$ out of the solid region bounded above by $z = 4 - x^2 - y^2$ and below by the plane $z = 2$

52. The flux of $\mathbf{F} = \langle x, y, 1 \rangle$ out of the solid bounded by the paraboloid $z = x^2 + y^2$ and the planes $z = 1$ and $z = 4$

53. Solve Example 6 if the temperature is inversely proportional to the square of the distance from the origin.

54. Suppose the temperature function of the solid ball $x^2 + y^2 + z^2 \le 4$ is $T(x, y, z) = 70 - 0.1x^2$ and $\kappa = 3$. Determine the heat flow out of the region.

55–57 A discussion analogous to the one given preceding Exercises 50–53 in Section 15.2 yields the formula below. If S is a thin surface with electrical charge density $q(x, y, z)$, the electrostatic potential at a point P away from the surface is obtained from the surface integral

$$V(P) = \varepsilon \iint_S \frac{q(x, y, z)}{r_P(x, y, z)} \, d\sigma,$$

where $r_P(x, y, z)$ is the distance between (x, y, z) and P.

Use the above formula in Exercises 55–57.

55.* Suppose a uniformly charged sphere of radius R and total charge Q is centered at the origin. If point P is r units from the center of the sphere, then show that

$$V(P) = \varepsilon \int_0^{\pi} \int_0^{2\pi} \frac{Q \sin\varphi}{4\pi\sqrt{R^2 + r^2 - 2rR\cos\varphi}} \, d\theta \, d\varphi.$$

(**Hint:** Because of uniform charge distribution, the charge density on the sphere is given by $q(x, y, z) = \dfrac{Q}{4\pi R^2}$. Notice also that because of radial symmetry, it suffices to pick P on one of the coordinate axes.)

56.* Use Exercise 55 to show that

$$V(P) = \frac{\varepsilon Q}{2rR}\left(|R + r| - |R - r|\right).$$

(**Hint:** Substitute $R^2 + r^2 - 2rR\cos\varphi = u$.)

57.* Use Exercises 55 and 56 to conclude that the electrostatic potential for a uniformly charged sphere r units from its center is as follows.

$$V(P) = \begin{cases} \varepsilon \dfrac{Q}{r} & \text{if } P \text{ is outside the sphere} \\ \varepsilon \dfrac{Q}{R} & \text{if } P \text{ is inside the sphere} \end{cases}$$

Note what this means is that the potential is constant inside the sphere, while outside the sphere the potential function behaves as if all of the charge Q were concentrated at the origin!

15.6 Technology Exercises

58–61 Use a graphing utility.

58. Determine the center of mass of the hemispherical surface given in Exercise 31.

59. Determine the moments of inertia of the surface given in Exercise 31.

60. Determine the mass and centroid of the thin parabolic surface $z = 1 - x^2 - y^2$, $z \ge 0$, if it has constant density ρ.

61. Find the second moments about the coordinate axes of the surface given in Exercise 58.

Assuming, as we have been, that the second partial derivatives of **F** are continuous in D, the equality of mixed partial derivatives means any one of the above statements implies $\nabla \times \mathbf{F} = \mathbf{0}$ throughout D. Incorporating the latest theorem, we can summarize the relationships between conservation and curl by

$$\mathbf{F} = \nabla f \quad \Rightarrow \quad \nabla \times \mathbf{F} = \mathbf{0} \quad \text{if } D \text{ is an open connected region,}$$
$$\nabla \times \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f \quad \text{if } D \text{ is an open and simply connected region.}$$

15.7 Exercises

1–4 Verify Stokes' Theorem by showing that the integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ and $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ are equal.

1. $\mathbf{F}(x,y,z) = \langle -2y, 2x, y+z \rangle$, where S is the upper unit hemisphere centered at the origin

2. $\mathbf{F}(x,y,z) = \langle y-z, x+z, -x+y \rangle$, where S is the hemisphere $z = \sqrt{4-x^2-y^2}$

3. $\mathbf{F}(x,y,z) = \langle -y, x, 1 \rangle$ (as in Example 1), where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane

4. $\mathbf{F}(x,y,z) = \langle y, z^2, 2x \rangle$, where S is the portion of the paraboloid $z = x^2 + y^2$ between the xy-plane and the plane $z = 9$

5. Calculate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ for the vector field $\mathbf{F}(x,y,z) = \langle -2y, 2x, y+z \rangle$, on the upper semiellipsoid S: $x^2 + y^2 + 4z^2 = 1$, $z \ge 0$, and compare your answer to the solution of Exercise 1.

6. Verify by calculation that the field of normal vectors for the upper half of the ellipsoid given in Example 2 is $\mathbf{r}_x \times \mathbf{r}_y = \left\langle \dfrac{x}{3\sqrt{9-x^2-y^2}}, \dfrac{y}{3\sqrt{9-x^2-y^2}}, 1 \right\rangle$.

7–14 Use Stokes' Theorem to evaluate the indicated line integral.

7. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -2y, x^2, 3z^2 \rangle$ and C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + 2y + z = 4$, with positive orientation when viewed from above

8. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle z^2, x^2, y^2 \rangle$ and C is the intersection of the cylinder $x^2 + y^2 = 4x$ and the plane $z = 2x$, with positive orientation when viewed from above

9. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3y, 2x, z^2 \rangle$ and C is the triangle with vertices $(4,0,0)$, $(0,2,0)$, and $(0,0,3)$, with positive orientation when viewed from above

10. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle zy^2, z^3, 9y - 2x \rangle$ and C is the triangle with vertices $(1,0,0)$, $(3,4,1)$, and $(0,0,2)$, with positive orientation when viewed from above

11. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3z, x, 2y \rangle$ and C is the boundary of the disk $x^2 + y^2 \le 1$, $z = 1$, with positive orientation when viewed from above

12. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3z, x, 2y \rangle$ and C is the intersection of the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$, with positive orientation when viewed from above (Compare your answer to the solution of Exercise 11.)

13. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle 2x^3, 4x + y^2, e^{z^2} \rangle$ and C is the intersection of the paraboloids $2z = x^2 + y^2$ and $z = 6 - x^2 - y^2$, with positive orientation when viewed from above

14. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -xz, 2z, x - y^2 \rangle$ and C is the intersection of $z = x^2 + y^2$ and $z = 2x + 3$, with negative orientation when viewed from above

15–22 Use Stokes' Theorem to evaluate the surface integral.

15. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle -y, x, xyz \rangle$ and S is the upper hemisphere $z = \sqrt{1-x^2-y^2}$, oriented with an upward-pointing unit normal vector field

16. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle 4z, -3x, 2y \rangle$ and S is the portion of the paraboloid $z = x^2 + y^2$ with $0 \le z \le 9$, oriented with a downward-pointing unit normal vector field

17. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle x - y^2, x^4, z^2 \rangle$ and S is the triangle determined by the first-octant portion of the plane $x + y + z = 1$, oriented with an upward-pointing unit normal vector field

18. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle 2y^2, -z, 4x \rangle$ and S is the triangle determined by the first-octant portion of the plane $5x + y + 2z = 10$, oriented with an upward-pointing unit normal vector field

19. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle -2yz, 3xz, 2z^3 \rangle$ and S is the cone frustum $z = \sqrt{x^2 + y^2}$, $1 \le z \le 2$, oriented with an inward-pointing unit normal vector field

20. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle yz, -2xz, x^2 y \rangle$ and S is the portion of the cylinder $x^2 + y^2 = 4$, $2 \le z \le 4$, oriented with an outward-pointing unit normal vector field

21. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle y \sin(z^2), xy^2, xz + y \rangle$ and S is the portion of the paraboloid $x = y^2 + z^2$, $0 \le x \le 4$, oriented with an inward-pointing unit normal vector field

22. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where
$\mathbf{F} = \langle 3yx^2, x \ln(z^4), y \cos(xz^2) - 2x \rangle$ and S is the hemisphere $y = \sqrt{4 - x^2 - z^2}$, oriented with an inward-pointing unit normal vector field

23. Suppose, as in Exercise 3 of Section 15.6, that the (piecewise smooth) surface S is the graph of a function $z = g(x, y)$, defined on a domain R, and that S has a piecewise smooth boundary C. Prove that in this case, Stokes' Theorem takes the following form.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \langle -g_x, -g_y, 1 \rangle\, dA$$

24–26 Assume that both f and g have continuous second-order partial derivatives, and that both the surface S and its boundary C meet the conditions of Stokes' Theorem. Verify the statement by using Stokes' Theorem.

24. $\oint_C (f \nabla f) \cdot \mathbf{T}\, ds = 0$

25. $\oint_C (f \nabla g) \cdot \mathbf{T}\, ds = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma$

26. $\oint_C (f \nabla g + g \nabla f) \cdot \mathbf{T}\, ds = 0$

27. Let S be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 1$. If $\mathbf{F}$ is a vector field with continuous partials in an open region containing S, show that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

28.* Prove the following generalization of Exercise 27. If the vector field $\mathbf{F}$ and the closed surface S satisfy the conditions of Stokes' Theorem, then

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

29.* The force field $\mathbf{F}$ is called a *central force* if it points directly away from, or toward, a point called the *center*, and its magnitude depends only on the distance from the center. In addition, we assume that this dependency is continuously differentiable, or in symbols, $\mathbf{F} = f(r)\mathbf{r}$, where the single-variable function f is continuously differentiable everywhere except possibly at zero. Show that if such a force is moving an object around a closed path that doesn't enclose the origin, then the total work done by the force is zero. (**Hint:** Show that the curl of the force field is zero.)

30. Consider the vector field

$$\mathbf{F} = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right\rangle$$

and show that while $\nabla \times \mathbf{F} = \mathbf{0}$, $\oint_C \mathbf{F} \cdot d\mathbf{r} \ne 0$ on a circle in the xy-plane centered at the origin. Does this contradict the theorem stating that $\nabla \times \mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$ is conservative?

15.7 Technology Exercises

31. Write a program on a computer algebra system that accepts a vector field, the parametrizations of a surface S and its boundary C, and returns both integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ and $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$. Use it to check your answers to Exercises 1–4.

15.8 Exercises

1–4 Verify the Divergence Theorem by showing the equality of the integrals $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ and $\iiint_D \nabla\cdot\mathbf{F}\,dV$ for the given vector field **F** on the solid *D*.

1. $\mathbf{F}(x,y,z)=\langle -y,x,z\rangle$, where D is the unit ball centered at the origin
2. $\mathbf{F}(x,y,z)=\langle x,2y,3z\rangle$, where D is the tetrahedron bounded by the coordinate planes and $z=1-x-y$
3. $\mathbf{F}(x,y,z)=\langle xz,xy,yz\rangle$, where D is the cube given in Example 1b
4. $\mathbf{F}(x,y,z)=\langle xz,3yz,z^2\rangle$, where D is the solid bounded by the paraboloid $z=4-x^2-y^2$ and the xy-plane

5–18 Use the Divergence Theorem to find the flux of the vector field **F** over the surface of the given solid *D*. Consider cylindrical or spherical coordinates where appropriate.

5. $\mathbf{F}(x,y,z)=\langle 2x-y^2,zx-y,z+e^{xy}\rangle$, where D is the solid ball of radius 2, centered at the origin
6. $\mathbf{F}(x,y,z)=\langle 3x-yz,2y+e^{xz},\cos 2y-2z\rangle$, where D is the solid bounded by the parabolic cylinder $z=1-y^2$, the xy-plane, and the planes $x=0$ and $x=1$
7. $\mathbf{F}(x,y,z)=\langle x,3y,z-y^2\rangle$, where D is the solid spherical shell $4\le x^2+y^2+z^2\le 9$
8. $\mathbf{F}(x,y,z)=|\langle x,y,z\rangle|^2\langle x,y,z\rangle$, where D is the cube given in Example 1b
9. $\mathbf{F}(x,y,z)=\langle 3x-yz,2y+e^{xz},\cos 2y-2z\rangle$, where D is the solid cylinder bounded by $x^2+y^2=1$, the xy–plane, and the plane $z=2$
10. $\mathbf{F}(x,y,z)=\langle x^3,y^3,z^3\rangle$, where D is the solid cylinder given in Exercise 9
11. $\mathbf{F}(x,y,z)=\langle x^3,y^3,z^3\rangle$, where D is the solid bounded by the paraboloid $z=1-x^2-y^2$ and the xy-plane
12. $\mathbf{F}(x,y,z)=\langle y-2x,e^{2zx},z^2-\tan^{-1}(xy)\rangle$, where D is the tetrahedron with vertices at the origin, $(2,0,0)$, $(0,4,0)$, and $(0,0,1)$
13. $\mathbf{F}(x,y,z)=\langle 6xy^2,6x^2y,x\cos(y^2)\rangle$, where D is the solid bounded by the paraboloid $z=4-x^2-y^2$ and the plane $z=3$
14. $\mathbf{F}(x,y,z)=\langle x-3z,y,4z^2\rangle$, where D is the solid inside the cylinder $x^2+y^2=1$, between the planes $z=1$ and $z=4-x$
15. $\mathbf{F}(x,y,z)=\langle xy-\sqrt{y^2+z^2},2y+e^{z^3},3z-x^2\cot y\rangle$, where D is the solid inside the cylinder $x^2+y^2=1$, between the planes $z=1$ and $z=x+4$
16. $\mathbf{F}(x,y,z)=\langle 2x^2,ze^x,y-xz\rangle$, where D is the solid cylindrical shell $1\le x^2+y^2\le 2$ between the xy-plane and $z=4$
17. $\mathbf{F}(x,y,z)=\langle 3y^2z,y^3/3,x^2z\rangle$, where D is the portion of the solid cone $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$

18.* $\mathbf{F}(x,y,z)=\langle x^3,y^3,z^3\rangle$, where D is the solid hemisphere $z=\sqrt{1-x^2-y^2}$

19. Verify the Divergence Theorem for the vector field $\mathbf{F}(x,y,z)=\langle z^2,3y^2,2yz\rangle$ and the solid bounded by the cylinder $x^2+z^2=1$, the xz-plane, and $y=2$.
20. If $\mathbf{F}(x,y,z)=\langle z^2\tan y,3xe^z,y\sin 2x^2\rangle$, S, and D satisfy the conditions of the Divergence Theorem, prove
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma=0.$$
21. Use the Divergence Theorem to find the flux of the vector field $\mathbf{F}(x,y,z)=\langle x^3y,y^2-\sin z^2,xe^z\rangle$ over the surface of the box $[0,4]\times[0,2]\times[0,3]$.
22. Suppose the space region D and its boundary S with unit normal field $\mathbf{n}$ meet the conditions of the Divergence Theorem, and let $\mathbf{F}(x,y,z)=\langle x,y,z\rangle$. Use the Divergence Theorem to show that the volume of D, $V(D)$ is equal to the following surface integral.
$$V(D)=\frac{1}{3}\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$
23. Use Exercise 22 to verify the formula for the volume of the box with side lengths a, b, and c, respectively. (**Hint:** Position the box appropriately in the coordinate system and apply the Divergence Theorem along with Exercise 22.)
24. Repeat Exercise 23 for the ball of radius R.

25. Repeat Exercise 23 for the right circular cone of radius R and height h.

26. If S is a surface such as in Exercise 22, use the Divergence Theorem to show that the volume of the solid D bounded by S can also be found as

$$V(D) = \iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy.$$

27. Let Q be a single point charge at the origin, as in our discussion that follows Example 2. Show that $\nabla \cdot \mathbf{E} = 0$ for the electric field $\mathbf{E}$ created by Q.

28. Mimic our discussion following Example 2 to prove the following, slightly more general version of Gauss' Law.

If S is a closed surface satisfying the hypotheses of the Divergence Theorem, k is a constant and $\mathbf{F} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$ is an inverse square field, then the flux of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \begin{cases} 4k\pi & \text{if } S \text{ encloses the origin.} \\ 0 & \text{otherwise} \end{cases}$$

29.* Recall the uniformly charged sphere of radius R and total charge Q from Exercises 55–57 of Section 15.6. In this problem, you will provide a "second solution" to the aforementioned Exercise 57 as follows. Use Gauss' Law to show that the electric field $\mathbf{E}$ due to the sphere is

$$\mathbf{E} = \begin{cases} \dfrac{\varepsilon Q \mathbf{e}_r}{r^2} & \text{if } R < r \\ 0 & \text{if } r < R. \end{cases}$$

(Recall $\mathbf{e}_r$ from Exercise 54 of Section 15.1.
Hint: Note that $\mathbf{E}$ is a central force as in Exercise 29 of Section 15.7. Using the notation $\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r$, show that the flux across a sphere of radius r is $4\pi r^2 E(r)$ and use Gauss' Law. Finally, note that if $r < R$, the total charge enclosed by any sphere of radius r is zero, and hence so is $\mathbf{E}$.)

30. Use the Divergence Theorem to provide a second solution to Exercise 28 of Section 15.7. (**Hint:** See Exercise 39 of Section 15.4.)

31.* Prove that if $\mathbf{F}$ has continuous partial derivatives, then its divergence at a point P can be obtained from the formula

$$\nabla \cdot \mathbf{F}(P) = \lim_{R \to 0} \frac{3}{4\pi R^3} \iint_{S_R(P)} \mathbf{F} \cdot \mathbf{n}\,d\sigma,$$

where $S_R(P)$ is the sphere of radius R centered at P. (**Hint:** Use the Divergence Theorem; then argue that the divergence function "assumes its average value" somewhere in the closed ball bounded by $S_R(P)$. For a refresher on average value, see Section 5.2.)

32–35 Prove the identity, assuming the scalar-valued functions f and g have continuous partial derivatives at least through the second order, and S and D satisfy the hypotheses of the Divergence Theorem. Note that $\nabla^2 f$ stands for $f_{xx} + f_{yy} + f_{zz}$, i.e., $\nabla^2 f = 0$ means that f satisfies Laplace's equation (see Section 13.3). Recall also that $D_n f$ stands for the directional derivative of f in the direction of $\mathbf{n}$. (For a review of directional derivatives, see Section 13.5.)

32.* $\iint_S D_{\mathbf{n}} f\,d\sigma = \iiint_D \nabla^2 f\,dV$

33.* $\iint_S f D_{\mathbf{n}} g\,d\sigma = \iiint_D \left(f\nabla^2 g + \nabla f \cdot \nabla g\right) dV$
(This is called *Green's first identity*. **Hint:** See Exercise 44 in Section 15.4.)

34.* $\iint_S f D_{\mathbf{n}} f\,d\sigma = \iiint_D |\nabla f|^2\,dV$, if f satisfies Laplace's equation on D. (**Hint:** You may give a direct proof, or use Exercise 33.)

35.* $\iint_S \left(f D_{\mathbf{n}} g - g D_{\mathbf{n}} f\right) d\sigma = \iiint_D \left(f\nabla^2 g - g\nabla^2 f\right) dV$ (This is called *Green's second identity*. **Hint:** Use Green's first identity from Exercise 33 twice.)

Chapter 15
Review Exercises

1–4 Match the given planar vector field $\mathbf{F}(x,y)$ with its graph (labeled A–D).

1. $\mathbf{F}(x,y) = \langle 1, y \rangle$ **2.** $\mathbf{F}(x,y) = \left\langle \frac{x}{2}, y \right\rangle$ **3.** $\mathbf{F}(x,y) = \langle 2y, -x \rangle$ **4.** $\mathbf{F}(x,y) = \left\langle -x, \frac{y}{|\langle x,y \rangle|} \right\rangle$