$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2-2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta \\ &= \left[-4\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} \\ &= 4+4=8 \end{aligned}$$

Note that $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$.

9.4 Exercises

1–14 Find the slope of the line tangent to the given polar curve at the indicated point.

1. $r = 2\cos\theta; \quad \theta = \frac{\pi}{6}$

2. $r = 2\sin 2\theta; \quad \theta = \frac{\pi}{6}$

3. $r = a\theta; \quad \theta = \frac{\pi}{2}$

4. $r = a\sin\theta; \quad \theta = \frac{\pi}{6}$

5. $r = 5\sec^2\theta; \quad \theta = \frac{\pi}{4}$

6. $r = e^{\theta}; \quad \theta = \frac{\pi}{6}$

7. $r^2 = 4\csc\theta; \quad \theta = \frac{\pi}{6}$

8. $r^2 = \sin 2\theta; \quad \theta = \frac{\pi}{3}$

9. $r = \frac{1}{\theta}; \quad \theta = \frac{2\pi}{3}$

10. $r = a(1+\sin\theta); \quad \theta = \frac{2\pi}{3}$

11. $r = \ln\theta; \quad \theta = 1$

12. $r = \cos 4\theta; \quad \theta = \frac{\pi}{4}$

13. $r = \sin\frac{\theta}{4}; \quad \theta = \pi$

14. $r = 3 - 2\sin\theta; \quad \theta = \frac{\pi}{4}$

15–18 Find all points where the given polar curve has a horizontal or vertical tangent line.

15. $r = 1 + \sin\theta$

16. $r = a\cos\theta$

17. $r = a(1+\cos\theta)$

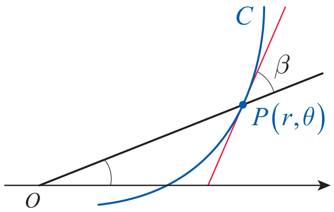
18. $r^2 = 4\cos 2\theta$

19–20 Notice that even in the case of polar curves, we still need to find dy/dx in order to determine tangents. The derivative $dr/d\theta$, while indirectly informing us about the tangent line at any given point, does not yield the slope of the tangent line. Exercises 19 and 20 shed some light on the relationship between $dr/d\theta$ and dy/dx.

19.* Suppose that the curve C is defined by the equation $r = f(\theta)$, P is a point on C, and there is a unique line tangent to C at P. Let β be the angle determined by the ray $\overrightarrow{OP}$ and the tangent at P. Prove that if $dr/d\theta$ is not equal to 0 at P, then

$$\tan\beta = \frac{r}{dr/d\theta}.$$

(**Hint:** In the formula we obtained for dy/dx in the text, divide the numerator and the denominator by $f'(\theta)\cos\theta$, and use the trigonometric identity $\tan(\alpha_1+\alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}$.)



20. Prove that if the graph of $r = f(\theta)$ passes through the pole and α is an angle such that $r = f(\alpha) = 0$, then the slope of the tangent to the graph at the pole is $\tan\alpha$; that is, the line $\theta = \alpha$ is tangent to the graph at the pole. (**Hint:** If $f'(\alpha) \neq 0$, use the formula we obtained for dy/dx in the text. Otherwise, examine $\lim_{\theta\to\alpha} \frac{\sin\theta \cdot f'(\theta)}{\cos\theta \cdot f'(\theta)}$.)

21–26 Notice that if the graph of $r = f(\theta)$ passes through the pole, we can use Exercise 20 to determine the polar equation of a tangent at the pole by solving the equation $f(\theta) = 0$. Use this observation to find all lines tangent to the given curve at the pole.

21. $r = 1 - \cos\theta$

22. $r = \sin 3\theta$

23. $r^2 = \cos 2\theta$

24. $r = \cos 4\theta$

25. $r = 4\sin\theta$

26. $r = a\sin n\theta$

27–41 Find the area enclosed by the given curve.

27. $r = 3\sin\theta$

28. $r = -2\cos\theta$

29. $r = 1 + \sin\theta$

30. $r = 4 - 4\sin\theta$

31. $r = \dfrac{3}{2} - \sin\theta$

32. $r = 3 + 2\cos\theta$

33. $r = 3 - \sin\theta$

34. $r = 3(1 + \cos\theta)$

35. $r = 2 + \cos\theta + \sin\theta$

36. $r = 2\cos 3\theta$

37. $r = \sin 6\theta$

38. $r = 6\sin\theta$

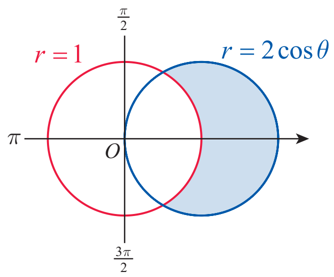
39. $r = 4\sin 4\theta$

40. $r^2 = 2\sin 2\theta$

41. $r^2 = 4\cos 2\theta$

42–45 Find the area of the shaded region.

42.



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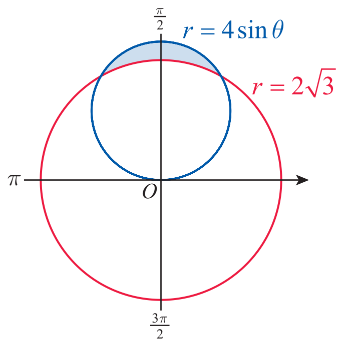

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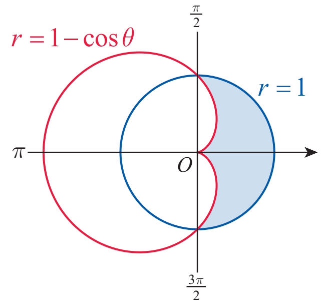

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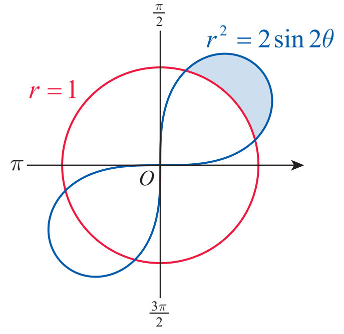

46–49 Find the area of the specified region.

46. The innermost loop of the spiral $r = 3\theta$ bounded by the polar axis and $\theta = \pi/2$

47. The inner loop of the limaçon $1 + 2\sin\theta$

48. The region common to the circle $r = 1$ and a petal of $r = 2\sin 3\theta$

49. The region inside $r = 3 + 2\sin\theta$ but outside $r = 2$

50–57 Find the arc length of the given polar curve.

50. $r = 3\sin\theta,\quad 0 \le \theta \le \pi$

51. $r = 3\theta,\quad 0 \le \theta \le 4\pi$

52. $r = 3e\theta,\quad 0 \le \theta \le \pi$

53. $r = \csc\theta,\quad \dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3}$

54.* $r = 1 - \sin\theta,\quad 0 \le \theta \le 2\pi$

55. $r = \theta^2,\quad 0 \le \theta \le 2\pi$

56.* $r = \sin^2\theta,\quad 0 \le \theta \le \pi$

57.* $r = 1 + \sin\theta,\quad 0 \le \theta \le \pi$

58. Use our derivation of the arc length formula for polar curves along with the surface area formula of Section 9.2 to arrive at the following formulas for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \le \theta \le b$, where f' is continuous on $[a,b]$ and the curve is traced out only once over the interval.

$$A = 2\pi \int_a^b r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(rotation about the polar axis)

$$A = 2\pi \int_a^b r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(rotation about $\theta = \pi/2$)

59–62 Use Exercise 58 to find the surface area of the solid generated by rotating the given curve as described.

59. $r = \sin\theta$, rotated about the polar axis

60. $r = \sin\theta$, rotated about $\theta = \pi/2$

61. The spiral $r = 3e\theta$, $0 \le \theta \le \pi$, rotated about the polar axis

62. $r = 2(1 + \sin\theta)$, $0 \le \theta \le \pi/2$, rotated about $\theta = \pi/2$

63. Show that the values of dy/dx at the tips of the petals of the curve in Example 2 are as claimed.

64.* Prove that if P is a point on the cardioid $r = a(1 - \cos\theta)$, then the smaller of the two angles determined by $\overrightarrow{OP}$ and the tangent at P is one-half the angle determined by $\overrightarrow{OP}$ and the polar axis.

65. A point O inside a polar curve is called *equichordal* if every chord passing through O has the same length. (An obvious example is the center of a circle of radius r.) Prove that the pole is an equichordal point of the limaçon $r = a + \cos\theta$ (assume $a \ge 1$).

66. Find the polar equation of the circle whose diameter has the same length as the chords in the limaçon of Exercise 65, and prove that the areas of the limaçon and the circle are not equal. (As a consequence, we see that the existence of chords of equal length in every direction through a common point is insufficient to determine areas of regions.)

67. Suppose the polar curve $r = f(\theta)$, $a \le \theta \le b$, has length L and encloses an area of A square units. Prove that for any constant k, the curve $r = kf(\theta)$, $a \le \theta \le b$, has length $|k|L$ and encloses an area of k^2A square units.

9.4 Technology Exercises

68–72 Use a graphing utility to sketch the given curve. Then use the integration capabilities of your technology to approximate its arc length.

68. $r = 2 + \cos 2\theta$

69. $r = \theta\cos\theta$, $0 \le \theta \le 3\pi$

70. $r = 3 - 2\sin 3\theta$

71. The inner loop of the limaçon $r = 1 + 2\sin\theta$

72. The 3-petaled rose $r = \sin 3\theta$

73–76 Use a graphing utility to approximate the surface area of the described solid of revolution. (**Hint:** See Exercise 58.)

73. $r = 3e\theta$, $0 \le \theta \le \pi$, rotated about the polar axis

74. $r^2 = \sin 2\theta$, rotated about the polar axis

75. The outer loop of the limaçon $r = 1 + 2\sin\theta$, rotated about $\theta = \pi/2$

76. $r = 4 - 4\cos 2\theta$, rotated about $\theta = \pi/2$