9.1 Exercises

1. Given the parametric equations $x = 5 + t$ and $y = \sqrt{t}/(t-2)$, construct a table of the points (x, y) that result from integer t-values from 0 to 6, and then sketch the curve.

2. Given the parametric equations $x = (\tan\theta)/2$ and $y = \cos^2\theta + 3$, construct a table of the points (x, y) that result from the values $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6,$ and π. Using these points, sketch the graph of the equations.

3–9 A straightforward way of parametrizing the graph of a function $y = f(x)$ is with the equations $x = t$ and $y = f(t)$. Use this technique to construct parametric equations defining the graph of the given equation.

3. $y = -x^2 - 5$

4. $x^2 + \dfrac{y^2}{4} = 1$

5. $x = y^2 + 4$

6. $x = 4y - 6$

7. $y = |x-1|$

8. $x = 2(y-3)$

9. $y^2 = 1 - x^2$

10–23 Sketch the curve defined by the parametric equations by eliminating the parameter.

10. $x = 3(t+1), \quad y = 2t$

11. $x = \sqrt{t-2}, \quad y = 3t - 2$

12. $x = 1 + t, \quad y = \dfrac{t-3}{2}$

13. $x = |t+3|, \quad y = t - 5$

14. $x = \dfrac{t}{4}, \quad y = t^2$

15. $x = \dfrac{t}{t+2}, \quad y = \sqrt{t}$

16. $x = \sqrt{t+3}, \quad y = t + 3$

17. $x = \dfrac{2}{|t-3|}, \quad y = 2t - 1$

18. $x = \cos\theta, \quad y = 2\sin\theta$

19. $x = 3\sin\theta - 1, \quad y = \dfrac{\cos\theta}{2}$

20. $x = 1 - \sin\theta, \quad y = \sin\theta - 1$

21. $x = 2\cos\theta, \quad y = 3\cos\theta$

22. $x = 2\sin\theta + 2, \quad y = 2\cos\theta + 2$

23. $x = \sin\theta, \quad y = 4 - 3\cos\theta$

24–29 Match the parametrization with its graph (labeled A–F).

24. $x = 10\cos\theta, \quad y = \theta$

25. $x = 2 + 3t, \quad y = \dfrac{t^3 + 1}{4}$

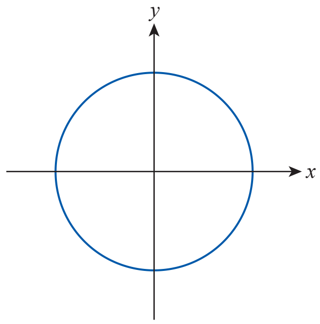
26. $x = 1 + 4t, \quad y = 3 + 2t$

27. $x = t\cos t, \quad y = t\sin t$

28. $x = 4 - 2t^2, \quad y = t^3 - 9t$

29. $x = 4\sin\theta, \quad y = 4\cos\theta$

A.



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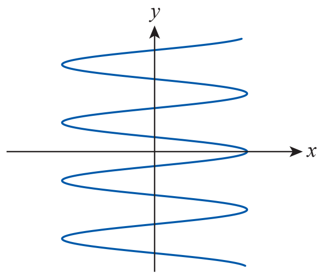

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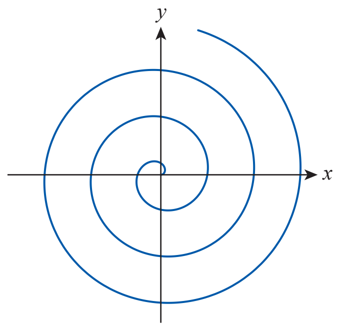

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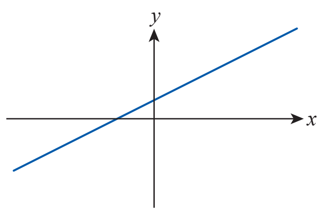

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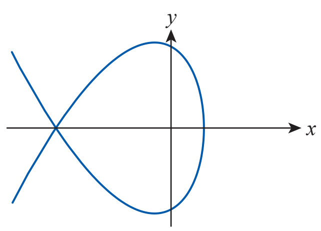
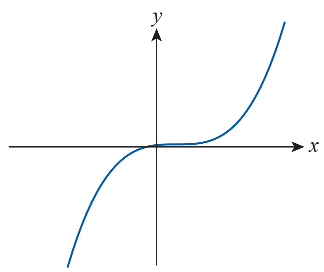


F.



30–33 Sketch the curve defined by the parametric equations and indicate the orientation of the curve using arrows.

30. $x=\dfrac{1}{\sqrt{t-1}}+1,\quad y=\dfrac{1}{t-1}$

31. $x=(\ln t)^2,\quad y=\dfrac{1}{\sqrt{t}},\quad 1\le t\le 5$

32. $x=e^{-t/2},\quad y=e^t,\quad t\ge 0$

33. $x=\sec^2 t,\quad y=\tan^2 t,\quad 0\le t\le \dfrac{\pi}{3}$

34. Show that the curve defined by the parametric equations $x=r\cos(nt)$ and $y=r\sin(nt)$, $0\le t\le 2\pi/n$, is a circle of radius r, centered at the origin.

35. Show that the curve defined by the parametric equations $x=a\cos(nt)$ and $y=b\sin(nt)$, $a>b$, $0\le t\le 2\pi/n$, is an ellipse with a horizontal major axis centered at the origin with respective lengths of the major and minor axes being $2a$ and $2b$. (**Hint:** Recall from precalculus that the equation of such an ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.)

36. Taking advantage of Exercise 34, along with horizontal and vertical shifts, derive the general parametric form for a circle of radius r, centered at (h,k).

37. Using Exercise 35 along with horizontal and vertical shifts, derive the general parametric form for an ellipse with a horizontal major axis and axes of lengths $2a$ and $2b$, centered at (h,k).

38–45 Find parametric equations to represent the graph described. (Answers will vary.)

38. Line, slope –2, passing through $(-5,-2)$

39. Line, passing through $(6,-3)$ and $(2,3)$

40. Line segment connecting the points $(-2,-1)$ and $(3,4)$

41. Line segment connecting the points $(-3,1)$ and $(5,-5)$

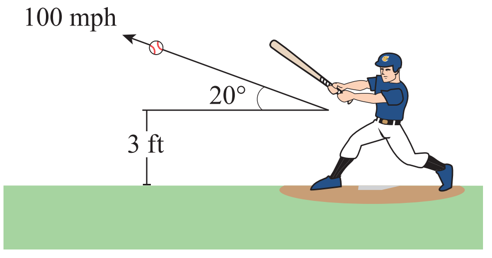
42. Circle, center $(7,-5)$, radius 4

43. Circle, center $(0,-2)$, radius 6

44. Ellipse, center $(5,-1)$, $a=3$, $b=2\sqrt{2}$, vertical major axis (**Hint:** See Exercises 35 and 37.)

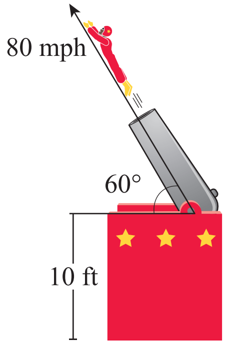
45. Ellipse, center $(0,1)$, $a=6$, $b=\sqrt{11}$, horizontal major axis (**Hint:** See Exercises 35 and 37.)

46. Suppose that a baseball is hit 3 feet above the ground, and it leaves the bat at a speed of 100 miles per hour at an angle of 20° from the horizontal. Construct parametric equations representing the path of the ball's flight, and sketch a graph of the ball's travel. (**Hint:** Supposing that the ball starts at the point $(0,3)$ and treating time t as the parameter, express the ball's x- and y-coordinates as functions of t. Do not forget to decompose the initial velocity into horizontal and vertical components!)



47. Suppose the ball in Exercise 46 has been hit toward a 10-foot-high fence that is 400 feet from home plate. Will the ball clear the fence?

48. Suppose that a circus performer is shot from a cannon at a rate of 80 mph, at an angle of 60° from the horizontal. The cannon sits on a platform 10 feet above the ground.



a. Construct parametric equations representing the performer's path as he flies through the air.

b. Sketch a graph of his flight.

c. How high is the acrobat 1.5 seconds after leaving the cannon?

d. How far from the cannon should a landing net be placed, if it is placed at ground level?

e. At what time t will the performer land in the net?

f. If a 12-foot-high wall of flames is placed 70 feet from the cannon, will he clear it unharmed?

49. On his morning paper route, John throws a newspaper from his car window 3.5 ft from the ground. The paper has an initial velocity of 10 ft/s and is tossed at an angle of $10°$ from the horizontal.

a. Construct parametric equations modeling the path of the newspaper.

b. Sketch a graph of the paper's path.

50. François shoots a basketball at an angle of $48°$ from the horizontal. It leaves his hands 7 ft from the ground with a velocity of 21 ft/s.

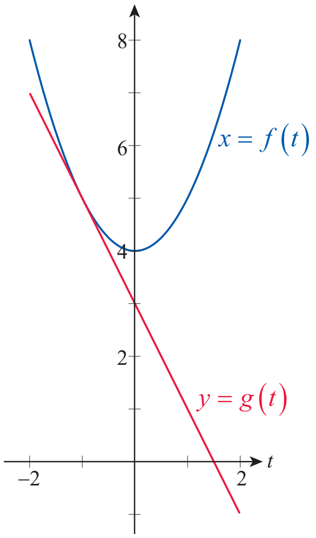
a. Construct parametric equations representing the path of the ball.

b. Sketch a graph of the basketball's flight.

c. If the goal is 15 ft away and 10 ft high, will he make the shot?

51–54 Use the given graphs of $f(t)$ and $g(t)$ to make a rough sketch of the curve defined by the parametric equations $x = f(t)$ and $y = g(t)$. (**Hint:** Plotting a few points may help.)

51.



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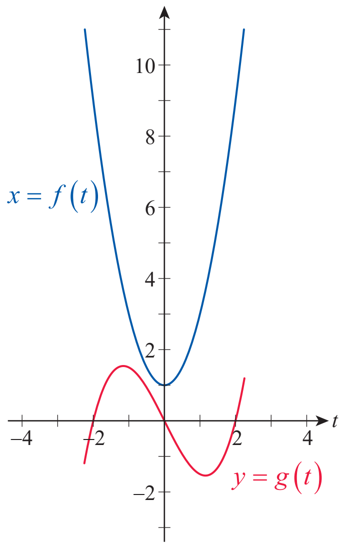

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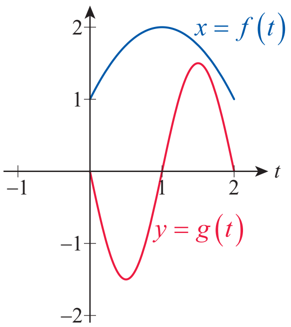

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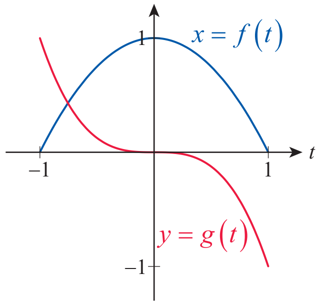


55. Verify that the three parametrizations below represent the same curve. Then come up with two parametrizations on your own. (Answers will vary.)

a. $x = t^2, \quad y = t^3, \quad -1 \le t \le 1$

b. $x = 1 - \cos^2 t, \quad y = \sin^3 t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

c. $x = \sec^2\left(\frac{\pi}{4}t\right) - 1, \quad y = \tan^3\left(\frac{\pi}{4}t\right), \quad -1 \le t \le 1$

56–57 Find all intersection points of the given parametric curves.

56. $x = 3t - 2,\ y = 3t - 1;\quad x = 2u,\ y = 4u^2 - 4u + 1$

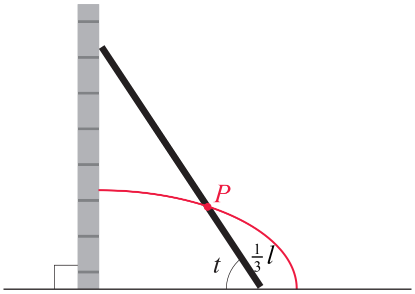
57. $x = t - 1,\ y = 3t^2 - 6t + 3;\quad x = \frac{u}{2},\ y = -\frac{u^2}{4} + 2u$

58. A wheel of radius 12 inches rolls along a flat surface in a straight line. There is a fixed point P that initially lies at the point $(0,0)$. Find parametric equations defining the cycloid traced out by P.

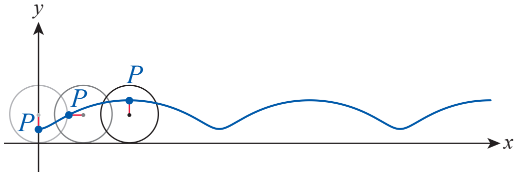
59. A ball is rolled on the floor in a straight line from one person to another. The ball has a radius of 3 cm and there is a fixed point P located on the ball. Let the person rolling the ball represent the origin. Find parametric equations defining the cycloid traced out by P.

60. Prove that the parametrizations $x_1 = \dfrac{1-t^2}{1+t^2}$, $y_1 = \dfrac{2t}{1+t^2}$, $-\infty < t < \infty$, and $x_2 = \cos t$, $y_2 = \sin t$, $-\pi < t < \pi$, represent the same curve. What is the curve?

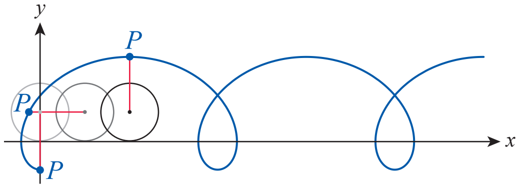
61. A ladder of length l is leaning against a wall, sliding slowly all the way down to a horizontal position. Suppose that there is a paint mark on the side of the ladder, exactly one-third of the way up from the bottom of the ladder (see point P in the figure). Assuming that the ladder started sliding from a vertical position, prove that the curve traced out by P during the slide is one-quarter of an ellipse. (**Hint:** Let t be the radian measure of the angle that the ladder makes with the horizontal, and use it as the parameter to determine the parametric equations of the curve.)



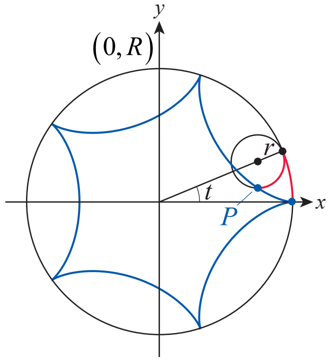
62. One way to generalize the cycloid is to consider the curve traced by a point P on a fixed radius (a "spoke") in the circle of Example 4. Generalize the argument of Example 4 to prove that if P is b units ($b < a$) from the center, then the parametric equations of the resulting trochoid (also called *curtate cycloid*) are $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$. (Note that the case of $a = b$ yields the equations we obtained in Example 4.)



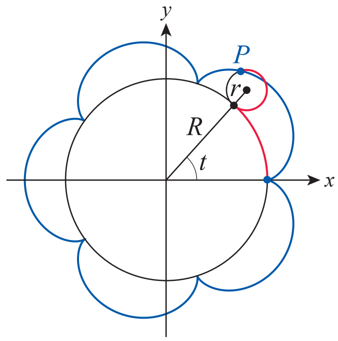
63. Repeat Exercise 62 for the case $a < b$ (imagine each spoke extending an appropriate length beyond the circumference of the circle; the resulting trochoid is called a *prolate cycloid*).



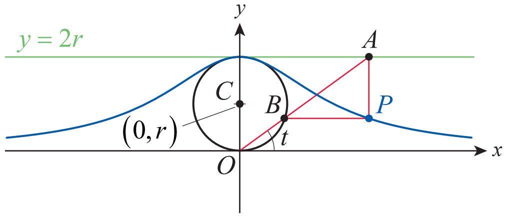
64.* Derive the parametrization of the hypocycloid seen in the Technology Note by using the angle t in the figure as a parameter. (**Hint:** Since the circle rolls without slipping, you may use the equality of the lengths of the two red arcs in the figure.)



65.* The *epicycloid* is the curve traced out by a fixed point P on a circle of radius r as it rolls without slipping on the outside of a larger circle with radius R. Using the technique of Exercise 62, derive the parametric equations of the epicycloid.



66.* The famous curve in the figure below is called the *witch of Agnesi* and is derived as follows. Suppose that a circle of radius r is centered at $(0, r)$ and the line $y = (\tan t)x$ intersects the horizontal line $y = 2r$ and the circle at the points A and B, respectively (B is not the origin). The curve is then traced out by the point P, which is the intersection of the horizontal segment through B and the vertical segment through A. Use t as a parameter to derive the parametric equations for the witch of Agnesi. (**Hint:** If O is the origin, and C is the center of the circle, start by examining the isosceles triangle OCB.)



67. Eliminate the parameter to find the Cartesian equation of the witch of Agnesi in Exercise 66.

Concept Check

68–71 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

68. If $x = f(t)$ and $y = g(t)$ are both quadratic functions, then the parametric curve defined by $x = f(t),\ y = g(t)$ is a parabola or a parabolic arc.

69. The parametric equations $x = t,\ y = t^{2/3} - 2$ and $x = 8t,\ y = 4t^{2/3} - 2$ have the same graph.

70. The graph of $x = t^3, y = t^6$ is the prototypical parabola.

71. The graph of parametric equations can either be represented in the form $y = f(x)$ (i.e., y as a function of x), or in the form $x = g(y)$ (i.e., x as a function of y).

9.1 Technology Exercises

72–78 Use a graphing utility to sketch the given curve for various values of k and explore how the value of k affects the shape of your graph.

72. $x = 2t - k\sin t, \quad y = 2 - k\cos t$ (trochoid)

73. $x = 2\cos t + k\cos\frac{2}{3}t, \quad y = 2\sin t - k\sin\frac{2}{3}t$ (hypotrochoid)

74. $x = 2kt - 4t^3, \quad y = 3t^4 - kt^2$ (swallowtail catastrophe curve)

75. $x = \frac{3kt}{1+t^3}, \quad y = \frac{3kt^2}{1+t^3}$ (folium of Descartes)

76. $x = \cos(t - \cos(kt)), \quad y = \cos(kt)$

77. $x = 2t - \cos(kt), \quad y = t^2 - \sin(kt)$

78. $x = \frac{k^2\cos(kt)}{t^2 + k^2}, \quad y = \frac{k^2\sin(kt)}{t^2 + k^2}$

79. Use technology and your parametrizing skills to display the following picture on your screen.