8.1 Exercises

1–6 Rewrite the first-order differential equation in the form $dy/dx = f(x, y)$. (Do not attempt to solve the equation.)

1. $\sqrt{x}\,\dfrac{dy}{dx} = \sqrt{y}$

2. $dy = 3y\,dx + e^{2x}\,dx$

3. $\sqrt{1-y^2}\,dx = \dfrac{dy}{x}$

4. $\dfrac{\tan y}{x} = \cos x \dfrac{dx}{dy}$

5. $xy = y - x^2 y'$

6. $3x^2 y = xe^x - xy'$

7–12 Verify that the function (or family of functions) is a solution of the given differential equation.

7. $xy' = 5y;\quad y = Cx^5$

8. $y' - \dfrac{3x^2}{y} = 0;\quad y = \sqrt{2x^3+5}$

9. $x^2\,dy - dx = 0;\quad y = 6 - \dfrac{1}{x}$

10. $3y' + 2xy = 0;\quad y = Ce^{-x^2/3}$

11. $y' + y = e^{5x};\quad y = Ce^{-x} + \dfrac{e^{5x}}{6}$

12. $x\,dy - (y - x)\,dx = 0;\quad y = Cx - x\ln x$

13–20 Solve the differential equation with the given initial condition.

13. $y' = x^2 + \cos x;\quad y(0) = 5$

14. $y' = \dfrac{1}{x^2+1} - 2;\quad y(0) = 0$

15. $xy' = 1 + x;\quad y(1) = 0$

16. $y'\sqrt{x} = \sqrt{x} + 1;\quad y(1) = 5$

17. $dx = x^2\,dy;\quad y(2) = 3$

18. $\sqrt{1-x^2}\,y' = x;\quad y(0) = 2$

19. $\sqrt{1-x^2}\,y' = 1;\quad y(0) = 2$

20. $y' = \dfrac{1}{x^2+x};\quad y(1) = -4$

21–28 Determine whether the differential equation is separable. (Do not solve the equation.)

21. $y' = (2x-1)e^y$

22. $y' = y^2 + 4$

23. $xy' + 4y^2 = 0$

24. $y' = x + y$

25. $y' = \sqrt{x}\,y - ye^x$

26. $y' = (x+y)^2$

27. $y' = 3x^2 y - x\sqrt{y}$

28. $x^2\,dy - (y^2 + yx)\,dx = 0$

29–40 Solve the separable differential equation.

29. $y' = 2x(y^2+1)$

30. $y' = y(x^2+4)$

31. $y' = 3xy + 6x$

32. $y' = 9y^2x^2$

33. $dy - 3x^2 y\,dx = 0$

34. $y' = \sec y \sec^2 x$

35. $y'\sqrt{4-x^2} = y$

36. $y' = \sqrt{\dfrac{1-y^2}{x}}\,e^{\sqrt{x}}$

37. $e^{1/x}\dfrac{dx}{dt} = -2tx^2$

38. $(x+2)\,dy = (x+5)\,dx$

39. $dy = \dfrac{y+6}{x}\,dx$

40. $x^2 y\,dy = (y^2+1)\,dx$

41–43 Solve the given separable equation, treating y as the independent variable and solving for $x = x(y)$.

41. $x\,dy - (y+1)\,dx = 0$

42. $dx - 3(x^2+1)\,dy = 0$

43. $y' = \dfrac{2x}{y+2}$

44–53 Solve the given initial value problem.

44. $y' = -\dfrac{x}{y};\quad y(1) = 1$

45. $y' = \dfrac{x^2-1}{y^2};\quad y(0) = 2$

46. $y' = \dfrac{4x^3+2x}{2y};\quad y(2) = 5$

47. $y' = e^{2x-5y};\quad y(0) = 0$

48. $x^3 y' = (x-1)y^2;\quad y(1) = -1$

49. $y'\sec x = y^2;\quad y\left(\dfrac{\pi}{2}\right) = \dfrac{1}{3}$

50. $\dfrac{dy}{dx} = 4x^3 e^{-y};\quad y(0) = 1$

51. $\dfrac{y'}{x} = y^2 + y;\quad y(0) = -2$

52. $y' = \dfrac{1+y^2}{1+x^2};\quad y(1) = -2$

53. $\dfrac{dy}{1+\sin t} = 2e^{-y}\cos t\,dt;\quad y(0) = 0$

54. A 50-gallon tank is filled with brine (water nearly saturated with salt; used as a preservative) holding 12 pounds of salt in solution. A salt solution containing 0.5 pounds of salt per gallon is added to the tank at the rate of 1 gallon per minute. The contents of the tank are continuously and thoroughly mixed and drained out at the same rate. What is the amount of salt in the tank after an hour?

55. A tank contains 2000 gallons of diesel fuel. A fuel mixture containing a lubricity additive is pumped into the tank through two inlets. The mixture flowing in through the first inlet contains 0.48 oz of additive per gallon and is being pumped in at a rate of 25 gal/min. Meanwhile, the mixture being allowed in by the second inlet at a rate of 10 gal/min contains 10.4 oz of additive per gallon. The mixture in the tank is continuously and thoroughly mixed and drained out at the rate of 35 gal/min. If there should be 16 oz of additive for every 120 gallons of diesel fuel, how long will it take to reach the right mixture?

56. To freshen the air, a small window is opened in a room initially containing 0.12% carbon dioxide. Fresh air with 0.04% carbon dioxide is pouring in at a rate of $6 \text{ m}^3/\text{min}$, and we assume that the uniform mixture is leaving the room at the same rate. If the dimensions of the room in meters are $4 \times 6 \times 3$, how long will it take to cut the initial carbon dioxide content down to half?

57. The cane sugar in fruit juice converts into dextrose under certain conditions. At any time, the rate of this process is proportional to the amount of cane sugar that is yet to be converted. If 100 grams of cane sugar is added to a certain fruit juice and we know that 12 grams are converted into dextrose during the first hour, how much dextrose will be present in the juice after 3 hours?

58. Suppose that an ice cube melts so that its volume $V(t)$ decreases at a rate proportional to its surface area.

a. Find a differential equation satisfied by $V(t)$.

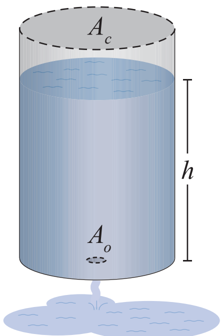
b. If an ice cube of side length 1 inch loses a third of its volume in 2 minutes, use your model to predict how long it will take for it to completely melt away. (Consider the cube melted away when your model predicts less than 1 percent remaining.)

59–61 Suppose a container is filled with fluid to a height of h. According to Torricelli's Law, when viscosity and friction are ignored, the speed v of efflux of the fluid through a small, sharp-edged opening through the bottom of the container equals the speed that the fluid would acquire when falling freely from a height of h, as follows.

$$v = \sqrt{2hg}$$

59.* If A_c is the horizontal cross-sectional area of a vertical cylindrical tank, while A_o denotes the area of the hole at the bottom of the tank, prove that the rate at which the fluid level is falling in the container is described by the following differential equation.

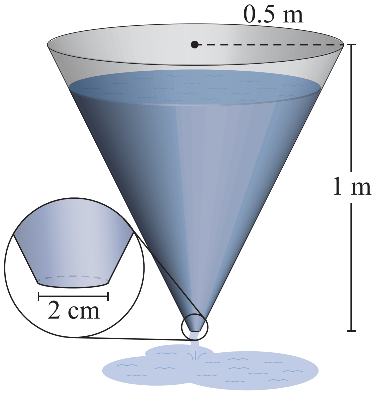
$$\frac{dh}{dt} = -\frac{A_o}{A_c}\sqrt{2hg}$$



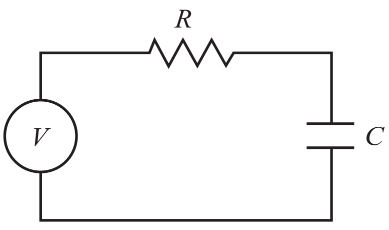
(**Hint:** First find the rate of fluid leaving the tank dV/dt; then use the fact that $V = A_c h$.)

60. If a cubic tank of side length 1 m is initially full of water but is draining through a circular orifice of diameter 2 cm that is on the bottom of the tank, what is the water level in the tank 2 minutes later? (**Hint:** Use the formula from Exercise 59.)

61.* Answer the question of Exercise 60 if the container is an inverted right circular cone of height 1 m and base radius 0.5 m. The opening on the bottom is the same as that in Exercise 60. (**Hint:** First, convince yourself that even though $A_c = A_c(h)$ now depends on *h*, the formula $dh/dt = -(A_0/A_c)\sqrt{2hg}$ remains in effect. Next, adapt your solution for Exercise 60. Ignore the geometrical change to the cone caused by the presence of the orifice.)



62.* The figure below shows a series circuit containing a resistor and a capacitor (this is called an RC circuit). Find a differential equation for the charge $q(t)$ if the impressed voltage *V* on the circuit is constant. (**Hint:** Use Ohm's Law, as well as the fact that the voltage drop on the capacitor is $\frac{1}{C}q$. For a statement of Ohm's Law, see the discussion preceding Example 5 in Section 8.2.)



63. Suppose that the air resistance encountered by a falling body is proportional to its velocity *v*.

a. Use Newton's Second Law of Motion (see Section 3.7) to find a differential equation satisfied by a falling body of mass *m*.

b. Solving your equation, find a formula for the terminal velocity of a body if it is falling from rest.

64. When starting from rest, the acceleration of a sailboat is proportional to the difference between the boat's velocity and that of the wind. Suppose that two minutes after starting from rest in 18 mph wind, a sailboat is moving at 8 mph.

a. Find a differential equation satisfied by the boat's velocity function.

b. Find the boat's velocity function.

c. How fast is the boat moving 4 minutes after it starts?

65–73 Determine the orthogonal trajectories of the family of curves, where *a* is an arbitrary nonzero constant. (If technology is available, sketch several curves from both families and visually check orthogonality.)

65. $y = ax^2$

66. $y = ax^3$

67. $y = \frac{ax}{x+2}$

68. $y = ae^x$

69. $y = \frac{x}{ax+1}$

70. $y = \frac{1}{a+x}$

71. $x^2 + ay^2 = 1$

72. $y = a\cos x$

73. $y = \frac{a}{1+x^2}$

Concept Check

74–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. A differential equation is an equation that relates an unknown function and at least one of its derivatives.

75. The equation $x^3y' + 1 = xy - x + y$ is separable.

76. A separable equation in the variables *x* and *y* always has a solution in the form $y = f(x)$.

77. This is the first time in this text that we are solving initial value problems.