Example 9 Using the Direct Comparison Test

Determine whether $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.01}}\, dx$ converges.

Solution

For $x \geq 1$, we have the following.

$$\sqrt{x^2 - 0.01} \leq x$$
$$\frac{1}{\sqrt{x^2 - 0.01}} \geq \frac{1}{x}$$

You will show in Exercise 17 that the integral $\int_1^{\infty} (1/x)\, dx$ diverges, so $\int_1^{\infty} \left(1/\sqrt{x^2 - 0.01}\right) dx$ diverges as well.

7.7 Exercises

1–8 Decide whether the given integral is an improper integral. If so, explain why and identify its type. (Do not evaluate the integral.)

1. $\int_1^{\infty} 2^{-x}\, dx$

2. $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$

3. $\int_0^{1} x^{-1/3}\, dx$

4. $\int_{-1}^{1} \frac{1}{x^2}\, dx$

5. $\int_0^{1} \frac{1}{(x+1)^{4/5}}\, dx$

6. $\int_{-\infty}^{-10} \frac{1}{x}\, dx$

7. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

8. $\int_2^{\infty} \frac{1}{(x-2)^{3/2}}\, dx$

9–16 Use the definitions from this section to write the given improper integral in terms of limits. (Do not evaluate the integral.)

9. $\int_2^{\infty} \frac{1}{x^2 - 1}\, dx$

10. $\int_0^{1} \frac{1}{x^2 - 1}\, dx$

11. $\int_{-\infty}^{-1} \frac{-3}{x}\, dx$

12. $\int_{-\infty}^{\infty} \frac{2}{\sqrt{x^2 + 2}}\, dx$

13. $\int_{-\infty}^{0} e^x\, dx$

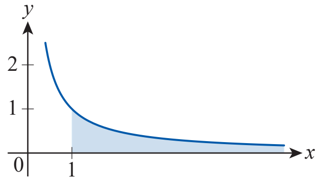
14. $\int_0^{2} \frac{1}{(x-1)^2}\, dx$

15. $\int_0^{\infty} \frac{2}{x}\, dx$

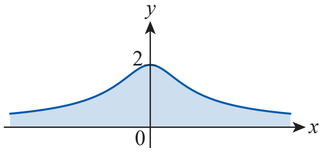
16. $\int_{-\infty}^{\infty} \frac{5}{x^2}\, dx$

17–20 Determine whether the improper integral pictured is convergent or divergent. If it is convergent, find its value.

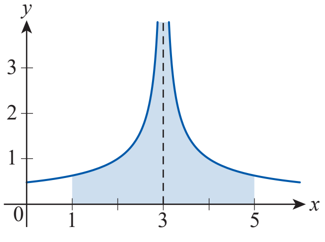
17. $\int_1^{\infty} \frac{dx}{x}$



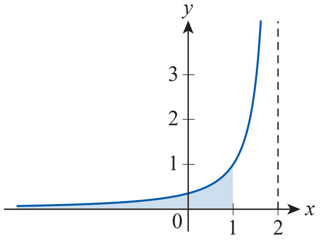
18. $\int_{-\infty}^{\infty} \frac{2}{\sqrt{1+x^2}}\, dx$



19. $\int_1^{5} \frac{dx}{(x-3)^{2/3}}$



20. $\int_{-\infty}^{1} \frac{dx}{(2-x)^{3/2}}$



21–64 Identify the type of the improper integral and determine whether it is convergent or divergent. If it is convergent, find its value.

21. $\int_1^\infty \frac{1}{x^{3/2}}\,dx$

22. $\int_{-\infty}^{-1} \frac{-3}{x}\,dx$

23. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

24. $\int_{-\infty}^1 2e^x\,dx$

25. $\int_0^\infty xe^{-x}\,dx$

26. $\int_0^\infty \cos x\,dx$

27. $\int_1^2 \frac{dx}{(x-1)^3}$

28. $\int_1^2 \frac{dx}{\sqrt[3]{x-1}}$

29. $\int_0^\infty \frac{6\,dx}{x^2+9}$

30. $\int_0^\infty \frac{6\,dx}{x^2-9}$

31. $\int_0^2 \frac{dx}{(x-1)^2}$

32. $\int_2^\infty \frac{dx}{(x-2)^2}$

33. $\int_{-\infty}^\infty \frac{2e^x}{e^{2x}+1}\,dx$

34. $\int_0^2 \frac{t}{\sqrt{4-t^2}}\,dt$

35. $\int_0^\infty \frac{dt}{(t+1)\sqrt{t}}$

36. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

37. $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$

38. $\int_0^\infty e^{-\theta}\cos\theta\,d\theta$

39. $\int_0^\infty \frac{e^x}{x}\,dx$

40. $\int_0^{16} \frac{dt}{\sqrt[4]{16-t}}$

41. $\int_0^9 \frac{dt}{9-t}$

42. $\int_{-\infty}^\infty \frac{dx}{9x^2+1}$

43. $\int_4^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

44. $\int_0^\infty t^2e^{-t}\,dt$

45. $\int_e^\infty \frac{dz}{z\ln^2 z}$

46. $\int_e^\infty \frac{\ln z}{z^2}\,dz$

47. $\int_0^e \frac{\ln z}{z^2}\,dz$

48. $\int_0^e \frac{\ln z}{z}\,dz$

49. $\int_0^\infty \frac{dx}{2\sqrt{x}(x+1)}$

50. $\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$

51. $\int_{-\infty}^\infty \frac{v}{(v^2+4)^2}\,dv$

52. $\int_0^2 \frac{4v}{\sqrt{16-v^4}}\,dv$

53. $\int_{-4}^0 \frac{dx}{\sqrt{|x+2|}}$

54. $\int_1^\infty \frac{\arctan x}{1+x^2}\,dx$

55. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$

56. $\int_3^\infty \frac{dx}{x^2-2x}$

57. $\int_2^3 \frac{dx}{x^2-2x}$

58. $\int_0^\infty \frac{dx}{(x+2)(x+3)^2}$

59. $\int_0^{\pi/2} \sec x\,dx$

60. $\int_0^\infty \frac{dx}{x(x+2)^2}$

61. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

62. $\int_0^{\pi/2} \tan\theta\,d\theta$

63. $\int_1^\infty \frac{dt}{t\sqrt{3t+1}}$

64. $\int_0^\infty \frac{x-2}{x^2+x+2}\,dx$

65. Classify the integrals of the form $\int_1^\infty \left(1/x^p\right)dx$ according to convergence for all possible values of p. (**Hint:** Consider the three cases of $p>1$, $p=1$, and $p<1$.)

66. Repeat Exercise 65 for the integrals of the form $\int_0^1 \left(1/x^p\right)dx$.

67–74 Use the Direct Comparison Test to determine whether the integral converges.

67. $\int_1^\infty \frac{dx}{\sqrt{x^4+2x+3}}$

68. $\int_2^\infty \frac{\ln x}{\sqrt{x^2-1}}\,dx$

69. $\int_1^\infty \frac{dx}{e^{2x}+x^{3/2}}$

70. $\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$

71. $\int_1^\infty \frac{\ln x}{x}\,dx$

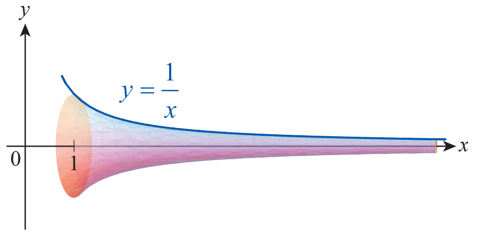
72. $\int_1^\infty \frac{\ln x}{x^{5/2}}\,dx$

73. $\int_1^\infty \frac{dx}{x^{1/2}-\frac{1}{2}}$

74. $\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}\ln x}\,dx$

75. Rotate the infinite region bounded by the graphs of $y=x^2/\sqrt{1-x^2}$, $y=0$, and $x=1$ about the y-axis. Use the method of shells to find the volume of the resulting unbounded solid.

76. If the infinite region between the graph of $y=1/x$ and the x-axis ($x\ge 1$) is revolved about the x-axis, we obtain the solid nicknamed *Gabriel's horn* (see figure). Use improper integrals to show that Gabriel's horn has a finite volume and infinite surface area. Note that this means that, at least theoretically, Gabriel's horn can be filled with a finite amount of paint, but it would take an infinite amount to paint its surface! Can you find a mathematical explanation for this conclusion? (**Hint:** To show that the surface area is infinite, use the Direct Comparison Test.)



Gabriel's Horn

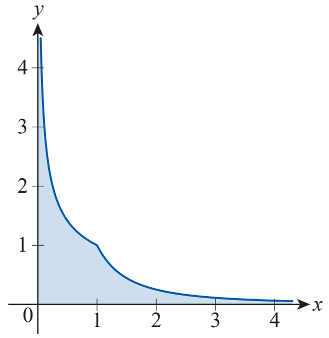
77. Show that the process in Exercise 76 results in a solid of finite volume when revolving about the x-axis, but in a solid of infinite volume when revolving about the y-axis.

78. Find the area between the graph of $y = 2/(x^2 - 1)$ and its horizontal asymptote for $x \geq 2$.

79. Find the total area between the graph of $y^2 = x^2/(16 - 16x^2)$ and its vertical asymptotes. (**Hint:** Be sure to consider both branches of the graph.)

80.* The figure below shows the graph of the following piecewise-defined function.

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ \dfrac{1}{x^2} & \text{if } 1 < x < \infty \end{cases}$$



Show that $\int_0^\infty f(x)\,dx = 3.$

81. Sketch the graph of $x^{2/3} + y^{2/3} = 1$ (hypocycloid with four cusps) and find its perimeter.

82. For $a > 0$, find a formula for the integral

$$\int_1^\infty \frac{dx}{x\sqrt{x^2 + a^2}}.$$

83. In probability theory, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is called the density function of the standard normal distribution. Show that its mean is 0 and its variance is 1; that is,

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx = 0 \text{ and } \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1.$$

(**Hint:** For the first integral, adapt your solution to Exercise 36. For the second, use integration by parts along with L'Hôpital's Rule.)

84. Find the function in the one-parameter family $y = c/(x^2 + 1)$ so that

$$\int_{-\infty}^{\infty} \frac{c}{x^2 + 1}\,dx = 1.$$

(The function you just found is another probability density function, called the *Cauchy density function*.)

85–87 In certain cases, substitution can help turn an improper integral into a proper one. For example, as part of your solution to Exercise 35, substituting $u = \sqrt{t}$ you may find

$$\int_0^1 \frac{dt}{(t+1)\sqrt{t}} = 2\int_0^1 \frac{du}{u^2 + 1},$$

the latter being a proper integral.

Use the above idea to turn the given improper integral into a proper one and evaluate.

85. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{x}\sqrt{1-x}}$

86. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

87. $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ (**Hint:** Substitute $u = 1/x$.)

88–93 The Laplace transform of a function $f(t)$, denoted $L\{f(t)\}$, is defined by the improper integral

$$L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

as long as it converges. The Laplace transform is very useful in physics and engineering, most notably, for solving certain linear ordinary differential equations.

Find the Laplace transform of the given function (we assume s is appropriately restricted so that the Laplace transform converges).

88. $L\{1\}$

89. $L\{t\}$

90. $L\{t^2\}$

91. $L\{e^{at}\}$

92. $L\{\sin kt\}$

93. $L\{\cos kt\}$

94. Recognizing a pattern from Exercises 88–90, conjecture and use induction to prove a general formula for $L\{t^n\}$ $(n \in \mathbb{N})$. (**Hint:** You may want to calculate $L\{t^3\}$ to firm up your conjecture.)

95–97 Suppose you invest money at an annual interest rate of r, which is compounded continuously, with a goal of having N dollars in t years. The amount you invest today to achieve that goal is called the present value (denoted PV) of the N dollars that is still t years out in the future. This present value can be calculated by the formula

$$PV = Ne^{-rt},$$

since investing PV dollars today will yield $(PV)e^{rt} = (Ne^{-rt})e^{rt} = N$ dollars in t years.

Using Riemann sums, it is straightforward to derive the formula for the present value of an annuity, a terminating income stream of fixed payments over a finite time period, say T years:

$$PV = \int_0^T A(t)e^{-rt}\,dt,$$

where $A(t)$ is the amount paid out annually (we assume continuous payment).

In Exercises 95–97, use this formula to generate the required improper integral.

95. If the annual interest rate is 5%, find the present value of a perpetual annuity (one paying dividends forever) that continuously pays $10,000 every year.

96. How much should we invest at an annual interest rate of 4% if we want a never-ending income stream of continuous annual payments of $500?

97. Suppose we expect an investment to generate profits at $p(t) = 6500\sqrt{t}e^{0.015t}$ dollars annually forever (t stands for the number of years elapsed). If the annual interest rate is 6%, find the present value of this income stream.

98. Use an improper integral to find the work done in propelling a 500 kg satellite out of Earth's gravitational field. (**Hint:** Approximate the radius of Earth by 6371 km. For the magnitude of the force of gravity at great altitudes, see Exercise 49 in Section 6.5.)

99. If an object is leaving the surface of Earth with a velocity v_0 big enough that its kinetic energy $E_{kin} = \frac{1}{2}mv_0^2$ is equal to the work required to propel it out of Earth's gravitational field, then the object will never return, but rather travel "infinitely far away" into outer space. Use Exercise 98 to find the value of the escape velocity on the surface of Earth. (**Hint:** As in Exercise 98, approximate the radius of Earth by 6371 km. See also Exercise 55 of Section 6.5 and the Chapter 6 Project.)

100.* The *gamma function*, which after an argument shift becomes an extension of the factorial function $f(n) = n!$, is defined as follows.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

Note that here x can be any real (or even complex) number. The gamma function is especially important in the fields of combinatorics, probability, and statistics.

Prove each of the following statements.

a. The above improper integral converges for all $n > 0$. (**Hint:** Show that for $n > 0$, $0 < \frac{x^{n-1}}{e^x} \le \frac{1}{x^2}$ for all appropriately large x-values and use the Direct Comparison Test.)

b. $\Gamma(1) = 1$

c. $\Gamma(n+1) = n\Gamma(n)$
(**Hint:** Use integration by parts.)

d. $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$
(**Hint:** Use mathematical induction.)

Concept Check

101–104 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

101. If $\int_1^\infty f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) = L$ with $L \ne 0$ or $\lim_{x\to\infty} f(x)$ doesn't exist.

102. If $f(x)$ is continuous on $[1,\infty)$, positive, and decreasing, and if for any $M > 0$ there is a $b > 1$ such that $\int_1^b f(x)\,dx > M$, then $\int_1^\infty f(x)\,dx$ diverges.

103. The integral $\int_1^\infty \frac{\ln(x^3)}{x}\,dx$ diverges.

104. For any positive $a \in \mathbb{R}$, $\int_a^\infty \frac{dx}{x^{1+a}}$ converges.

7.7 Technology Exercises

105–106 The function, defined in terms of an improper integral, is important for its applications within or outside of mathematics, for example in number theory, statistics, probability, physics, or engineering. Use a graphing utility to sketch the graph and observe important features. (Answers will vary.)

105. $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$ (the error function)

106. $\operatorname{Li}(x) = \int_2^x \frac{dt}{\ln t}$ (the logarithmic integral function)

107–108 Attempt to use a computer algebra system to evaluate the improper integral (from Exercises 67 and 74) and see what happens. (**Note:** Theoretical results such as the Direct Comparison Test are extremely useful, even when powerful mathematical software is at our disposal. As we mentioned during our discussion in the text, sometimes concluding the fact of convergence is more important than the actual value of the integral; and the Direct Comparison Test provides a fast and trouble-free way to do just that.)

107. $\int_1^\infty \frac{dx}{\sqrt{x^4 + 2x + 3}}$

108. $\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}\ln x}\, dx$