We have seen this situation arise before; since the integral we are trying to evaluate appears on both sides of the equation, we can solve for it.

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln\left|\sec x + \tan x\right|$$
$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x\tan x + \ln\left|\sec x + \tan x\right|\right) + C$$

7.3 Exercises

1–36 Evaluate the given indefinite or definite integral involving powers of sines and cosines.

1. $\int \sin^2 x\cos x\,dx$

2. $\int \sin x\cos^3 x\,dx$

3. $\int \cos^5 x\,dx$

4. $\int \cos x\sin x\,dx$

5. $\int \sin^4 x\,dx$

6. $\int \sin^{-4} x\cos x\,dx$

7. $\int_0^{\pi/3} \cos^{-2} x\sin x\,dx$

8. $\int_0^{\pi/2} \sin^3 2t\,dt$

9. $\int_{-\pi}^{\pi} \cos^6\left(\frac{\theta}{2}\right)d\theta$

10. $\int \frac{2\cos 4\alpha}{\sqrt{\sin 4\alpha}}\,d\alpha$

11. $\int \sin^7 x\cos^8 x\,dx$

12. $\int \sin^2 x\cos^2 x\,dx$

13. $\int \sin^4\left(\frac{\theta}{4}\right)\cos^4\left(\frac{\theta}{4}\right)d\theta$

14. $\int_0^{\pi} \sin^5 x\,dx$

15. $\int_0^{\pi} \sin^2 x\,dx$

16. $\int \cos^3 3t\sin^4 3t\,dt$

17. $\int \sqrt{\cos x}\sin^3 x\,dx$

18. $\int_0^{3\pi} \sin^3\left(\frac{t}{3}\right)\cos^2\left(\frac{t}{3}\right)dt$

19. $\int \sin^3 x\cos^5 x\,dx$

20. $\int_0^{\pi/2} \sin^7 x\,dx$

21. $\int_{\pi^2/4}^{\pi^2} \frac{8\cos^2\sqrt{x}\sin^2\sqrt{x}}{\sqrt{x}}\,dx$

22. $\int \frac{\sin 2x + \sin^3 x}{\cos x}\,dx$

23. $\int 16\sin^2 x\cos^4 x\,dx$

24. $\int \cos^6 x\,dx$

25. $\int \sin^7 x\cos^3 x\,dx$

26. $\int \sin x\sin 5x\,dx$

27. $\int_0^{\pi/2} \cos 2x\cos 3x\,dx$

28. $\int_0^{\pi} \cos(-4x)\sin 6x\,dx$

29. $\int \sin 8x\sin(-7x)\,dx$

30. $\int \sqrt{\frac{1+\cos 2\theta}{2}}\,d\theta$

31. $\int_0^{\pi/3} \frac{\sin 3x}{\sqrt{\cos x}}\,dx$

32. $\int_0^{\pi/2} \sqrt{1-\cos x}\,dx$

33. $\int \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

34. $\int \frac{\cos x}{\sqrt{1-\cos x}}\,dx$

35. $\int \sin x\sqrt{1-\sin x}\,dx$

36. $\int \frac{\cos^2 x}{\sqrt{1+\sin x}}\,dx$

(**Hint:** In Exercises 35 and 36, use the identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.)

37–57 Evaluate the given indefinite or definite integral involving powers of tangents and secants. Note that the integrals involving cotangents and cosecants can be handled by rules analogous to those discussed in this section.

37. $\int \tan^5 x\,dx$

38. $\int \cot^4 x\,dx$

39. $\int \sec^4 x\,dx$

40. $\int \tan^3 x\sec^3 x\,dx$

41. $\int \frac{\tan\theta}{\sqrt{\sec\theta}}\,d\theta$

42. $\int \frac{\cot^5 x}{\csc x}\,dx$

43. $\int \tan x\sec^6 x\,dx$

44. $\int \frac{\sec^2 x}{\sqrt[3]{\tan x}}\,dx$

45. $\int \csc^4 t\cot^{3/2} t\,dt$

46. $\int \tan^4 x\sec^4 x\,dx$

47. $\int \tan^2 x\sec x\,dx$

48. $\int \csc^3 x\,dx$

49. $\int_0^{\pi/3} \tan^5 x\sec^3 x\,dx$

50. $\int_{\pi/3}^{2\pi/3} \csc x\cot^2 x\,dx$

51. $\int_{\pi/6}^{\pi/2} \csc^4 x\,dx$

52. $\int \frac{\csc^4 z}{\cot z}\,dz$

53. $\int \frac{\sec^4 2\alpha}{\cot^3 2\alpha}\,d\alpha$

54. $\int \sec^6 3\beta\cot^3 3\beta\,d\beta$

55. $\int \csc^4 4x\cot 4x\,dx$

56. $\int_{\sqrt[3]{\pi/4}}^{\sqrt[3]{\pi/2}} s^2\csc^2\left(s^3\right)\cot^4\left(s^3\right)ds$

57. $\int_0^{\pi/4} \frac{\cot^2 t - 1}{\csc^2 t}\,dt$

58–63 The given integral does not directly fit any of the cases discussed in this section. Use trigonometric identities and familiar integration rules to evaluate it.

58. $\int \sin x\cot 2x\,dx$

59. $\int x\sec^2 x\,dx$

60. $\int \sec^3 x\cot x\,dx$

61. $\int z\tan^2 2z\,dz$

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62. $\int \csc^4 x \cos^3 x\, dx$

63. $\int \frac{\sin 3x}{\sec x} dx$

64. Verify the following reduction formula for $m, n \in \mathbb{N}$. (**Hint:** Use integration by parts.)

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx$$

65. Use Exercises 3, 24, and 64 to evaluate the following integrals.

a. $\int \sin^2 x \cos^5 x\, dx$

b. $\int \sin^2 x \cos^6 x\, dx$

66. Find the area of the region between the x-axis and the graph of $f(x) = \sin^2 x \cos^3 x$ from $x = 0$ to $x = \pi/2$.

67. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \tan x + \cot x$, $y = 0$, $x = \pi/6$, and $x = \pi/3$ about the x-axis.

68. Repeat Exercise 67 for the graphs of $y = \cos x + \sec x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$.

69. Find the centroid of the region bounded by the graphs of $y = x + \sin x$, $y = 0$, and $x = \pi$.

70. A particle is starting from the origin and moving along the x-axis so that its velocity at t seconds is $v(t) = \pi \tan^2(\pi t/18)\sin(\pi t/18)$ units per second $(0 \le t \le 8)$. Find its position at $t = 6$ seconds.

71–73 Use the product-to-sum identities of this section to verify the given formula for $m, n \in \mathbb{N}$.

71. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$

72. $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$

73. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$

Concept Check

74–77 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. $\int \sin^2 x\, dx - x = C - \int \cos^2 x\, dx$

75. $\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \cos^2 x\, dx$

76. According to the text, the best way to evaluate $\int \sec^4 x \tan^4 x\, dx$ is by using integration by parts.

77. According to the text, the best way to evaluate $\int \sec^3 x \tan^4 x\, dx$ is by using integration by parts.

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